Compito 26 GIU. 19

Un'asta materiale omogenea di massa $m$ e lunghezza $l$ è incernierata in un suo estremo $A$ in un piano orizzontale. L'altro estremo $B$ è attratto da due punti $C$ e $D$ allineati con $A$ e situati a uguale distanza $d$ da parti opposte rispetto ad $A$ mediante due forze elastiche di costanti $k_1$ e $k_2$

1) Scrivere le equazioni di moto dell'asta.

2) Determinare le posizioni di equilibrio e discuterne la stabilità in funzione di $k_1$ e $k_2$.

3) Determinare il periodo delle piccole oscillazioni per tutte le posizioni di equilibrio stabile.

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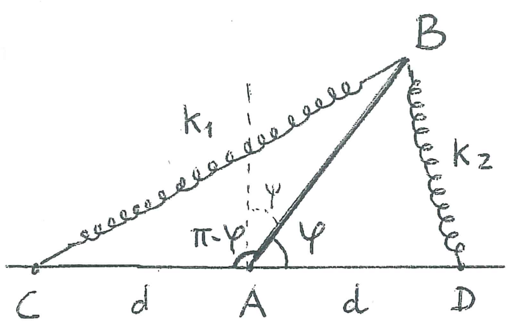


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1) $T = \boxed{\frac{1}{2}\,\frac{1}{3} m l^2 \dot{\varphi}^2}$

$\frac{1}{2} I_0 \omega^2$

$$U = -\frac{k_1}{2}(B-C)^2 - \frac{k_2}{2}(B-D)^2$$

$$(B-C)^2 = d^2 + l^2 - 2ld\cos(\pi-\varphi) = d^2 + l^2 + 2ld\cos\varphi$$

$$(B-D)^2 = d^2 + l^2 - 2ld\cos\varphi$$

$\psi + \varphi = \frac{\pi}{2}$

$\psi = \frac{\pi}{2} - \varphi$

$\cos\psi = \operatorname{sen}\varphi$

$$\mathcal{L} = \frac{1}{6} m l^2 \dot{\varphi}^2 - \frac{1}{2}k_1(d^2 + l^2 + 2ld\cos\varphi) - \frac{1}{2}k_2(d^2 + l^2 - 2ld\cos\varphi)$$

$$= \frac{1}{6} m l^2 \dot{\varphi}^2 - \frac{1}{2}(k_1+k_2)d^2 - \frac{1}{2}(k_1+k_2)l^2 - (k_1-k_2)ld\cos\varphi$$

$$= \frac{1}{6} m l^2 \dot{\varphi}^2 - k_1 ld\cos\varphi + k_2 ld\cos\varphi$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{1}{3} m l^2 \dot{\varphi} \qquad \frac{\partial\mathcal{L}}{\partial\varphi} = (k_1-k_2)ld\operatorname{sen}\varphi$$

$$\Rightarrow \quad \frac{1}{3} m l^2 \ddot{\varphi} - (k_1-k_2)ld\operatorname{sen}\varphi = 0$$

$$\frac{1}{3} m l^2 \ddot{\psi} - (k_1-k_2)ld\cos\psi = 0$$

2) $U = (k_2-k_1)ld\cos\varphi \qquad U' = -(k_2-k_1)ld\operatorname{sen}\varphi = 0$

$(k_2-k_1)ld$

$$\begin{cases} \varphi = 0 \\ \varphi = \pi \end{cases} \quad \text{Equilibrio}$$

$$\ddot{\varphi} + \frac{3(k_2-k_1)d}{ml}\operatorname{sen}\varphi = 0$$

$$U'' = (k_1-k_2)ld\cos\varphi$$

$$\ddot{\psi} - \frac{3(k_2-k_1)d}{ml}\cos\psi = 0$$

$$U''(\varphi) = (k_1 - k_2)\,\ell d \cdot \cos\varphi$$

$$U''(0) = (k_1 - k_2)\,\ell d \quad \begin{cases} > 0 & \text{se } k_1 > k_2 \\ < 0 & \text{se } k_1 < k_2 \end{cases}$$

$$U''(\pi) = -(k_1 - k_2)\,\ell d \quad \begin{cases} > 0 & \text{se } k_2 > k_1 \\ < 0 & \text{se } k_2 < k_1 \end{cases}$$

$\varphi = 0$ è max se $k_1 < k_2$ STABILE; min se $k_1 > k_2$ INSTABILE

$\varphi = \pi$ è MAX se $k_2 < k_1$ STABILE; min se $k_2 > k_1$ INSTABILE

Se $k_1 = k_2$ EQUILIBRIO INDIFFERENTE

P.O. intorno a $\varphi = 0$ $\boxed{k_2 > k_1}$

$$\mathcal{L} = \frac{1}{6} m\ell^2 \dot\varphi^2 + (k_2 - k_1)\,\ell d \cos\varphi \qquad \cos\varphi = 1 - \frac{\varphi^2}{2}$$

$$\mathcal{L}_{p.o.} = \frac{1}{6} m\ell^2 \dot\varphi^2 - \frac{k_2 - k_1}{2}\,\ell d\, \varphi^2$$

$$\Rightarrow \quad \frac{1}{3} m\ell^2 \ddot\varphi + (k_2 - k_1)\,\ell d\,\varphi = 0 \qquad \ddot\varphi + \frac{3(k_2 - k_1)d}{m\ell}\varphi = 0 \qquad k_2 > k_1$$

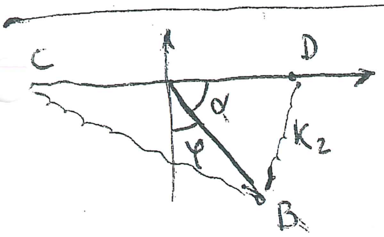
$$T = \frac{2\pi}{\omega} = \frac{2\pi\sqrt{m\ell}}{\sqrt{3(k_2 - k_1)d}}$$

P.O. intorno a $\varphi = \pi$ $\boxed{k_2 < k_1}$

$$\mathcal{L} = \frac{1}{6} m\ell^2 \dot\varphi^2 - (k_1 - k_2)\,\ell d \cos\varphi \qquad \cos\varphi = -1 + \frac{(\varphi - \pi)^2}{2}$$

$$\mathcal{L}_{p.o.} = \frac{1}{6} m\ell^2 \dot\varphi^2 - (k_1 - k_2)\,\ell d \frac{(\varphi - \pi)^2}{2} \qquad \psi = \varphi - \pi$$

$$\Rightarrow \quad \frac{1}{3} m\ell^2 \ddot\varphi + (k_1 - k_2)\,\ell d (\varphi - \pi) = 0 \qquad \ddot\psi + \frac{3(k_1 - k_2)d}{m\ell}\psi = 0$$




$$T = \frac{1}{6} m\ell^2 \dot\varphi^2 \qquad U = (k_2 - k_1)\,\ell d \,\mathrm{sen}\,\varphi$$

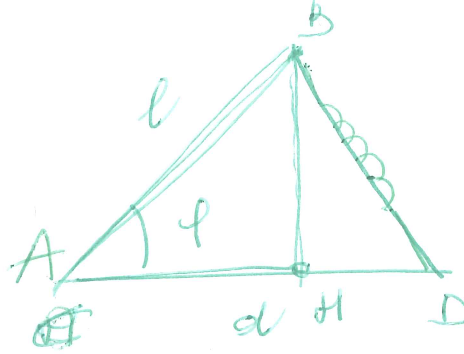
$$\mathcal{L} = \frac{1}{6} m\ell^2 \dot\varphi^2 + (k_2 - k_1)\,\ell d \,\mathrm{sen}\,\varphi$$

$$\frac{1}{3} m\ell^2 \ddot\varphi - (k_2 - k_1)\,\ell d \cos\varphi = 0$$

$$\ddot\varphi - \frac{3d(k_2 - k_1)}{m\ell}\cos\varphi = 0$$

$$\ddot\alpha + \frac{3d(k_2 - k_1)}{m\ell}\,\mathrm{sen}\,\varphi = 0$$

COMP. 26 - GIU 19 FRQ: asta con molle




$|B-D|^2=$

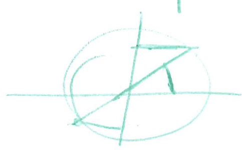
$|B-H|^2= \ell \sin\varphi$

$|H-D| = d - \ell\cos\varphi$

$|B-D|^2 = \ell^2 \sin^2\varphi + (d-\ell\cos\varphi)^2 =$

$\ell^2\sin^2\varphi + \ell^2\cos^2\varphi - 2d\ell\cos\varphi + d^2$

$= \ell^2+d^2-2d\ell\cos\varphi$ T. CARNOT

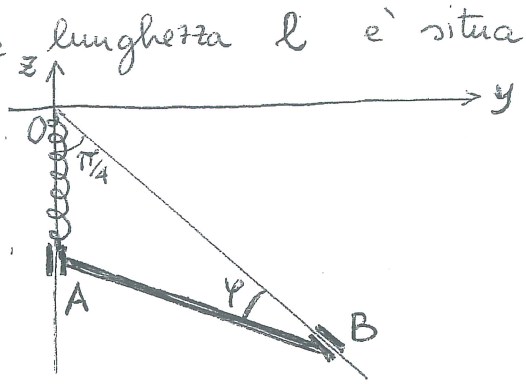


$|B-D|^2 = \ell^2+d^2-2d\ell\cos\varphi$

$|C-B|^2 = \ell^2+d^2-2d\ell\cos(\pi-\varphi)$

$= \ell^2+d^2+2d\ell\cos(\varphi)$

Un'asta omogenea, pesante, di massa M e lunghezza $\ell$ è situata nel piano verticale, con l'estremo A vincolato alla retta liscia $y=0$ e l'estremo B vincolato alla retta liscia $y=-z$.
Nel punto A è applicata la forza elastica $\underline{F} = -c(A-O)$, $c>0$.
Scelto il parametro $\varphi$, in figura, scrivere l'energia cinetica dell'asta. Determinare le posizioni di equilibrio dell'asta. Infine discuterne la stabilità.

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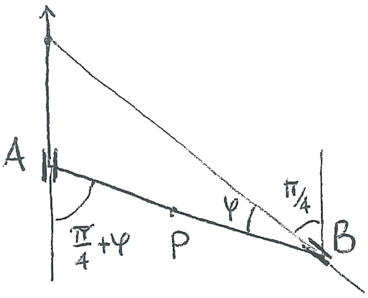
Quota di A: Ponendo $2L=\ell$

$$\frac{z_A}{\sin\varphi} = \frac{2L}{\sin\pi/4} \Rightarrow z_A = \frac{2L}{\sqrt2/2}\sin\varphi = 2\sqrt2 L\sin\varphi = \sqrt2\,\ell\sin\varphi$$

$$z_A = -2\sqrt2 L\sin\varphi$$




$$P_0 = L\sin\left(\frac{\pi}{4}+\varphi\right)\underline{j} - \left(2\sqrt2 L\sin\varphi + L\cos\left(\frac{\pi}{4}+\varphi\right)\right)\underline{k}$$
$$= L\frac{\sqrt2}{2}(\sin\varphi+\cos\varphi)\underline{j} - \left(2\sqrt2 L\sin\varphi + \frac{L\sqrt2}{2}\cos\varphi - \frac{L\sqrt2}{2}\sin\varphi\right)\underline{k}$$
$$= \frac{\sqrt2}{2}L(\sin\varphi+\cos\varphi)\underline{j} - \frac{\sqrt2}{2}L(3\sin\varphi+\cos\varphi)\underline{k}$$
$$\dot P_0 = \frac{\sqrt2}{2}L(\cos\varphi-\sin\varphi)\dot\varphi\,\underline{j} - \frac{\sqrt2}{2}L(3\cos\varphi-\sin\varphi)\dot\varphi\,\underline{k}$$

$$\sin\left(\frac{\pi}{4}+\varphi\right) = \frac{\sqrt2}{2}(\sin\varphi+\cos\varphi)$$
$$\cos\left(\frac{\pi}{4}+\varphi\right) = \frac{\sqrt2}{2}(\cos\varphi-\sin\varphi)$$

$$\dot P_0^2 = \frac{1}{2}L^2\dot\varphi^2(10\cos^2\varphi + 2\sin^2\varphi - 8\sin\varphi\cos\varphi)$$
$$\frac{1}{8}\ell^2(2+8\cos^2\varphi - 8\sin\varphi\cos\varphi)\dot\varphi^2$$

$$T = \frac12 m\dot P_0^2 + \frac12 I_{AB}\dot\varphi^2 = \frac12 mL^2(5\cos^2\varphi + \sin^2\varphi - 4\sin\varphi\cos\varphi)\dot\varphi^2 + \frac12\frac1{12}m(4L^2)\dot\varphi^2$$
$$(1+4\cos^2\varphi - 2\sin 2\varphi)$$

con $L=\frac{\ell}{2}$
$$\boxed{T = \frac16 m\ell^2(1+3\cos^2\varphi - 3\sin\varphi\cos\varphi)\dot\varphi^2}$$

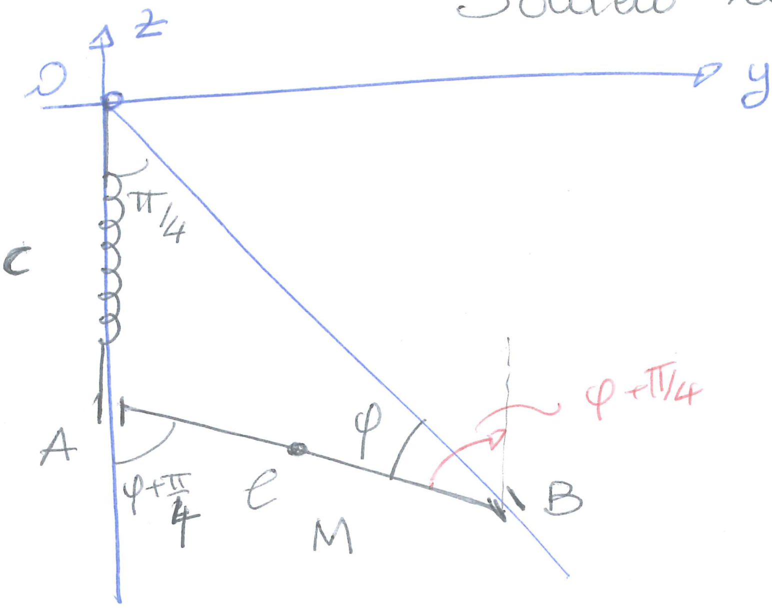
Potenziale
$$U = Mg\,\ell\frac{\sqrt2}{4}(3\sin\varphi+\cos\varphi) - c\ell^2\sin^2\varphi$$

$$U = -Mg z_{P_0} - \frac12 c(A-O)^2 = +Mg\frac{\sqrt2}{2}L(3\sin\varphi+\cos\varphi) - \frac{c}{2}(-2\sqrt2 L\sin\varphi)^2$$
$$= -4cL^2\sin^2\varphi + Mg\frac{3\sqrt2}{2}L\sin\varphi + Mg\frac{\sqrt2}{2}L\cos\varphi$$

$$\delta U = \left(-8cL^2\sin\varphi\cos\varphi + Mg\frac{3\sqrt2}{2}L\cos\varphi - Mg\frac{\sqrt2}{2}L\sin\varphi\right)\delta\varphi$$

Quindi

# Sorrew Lagrang.




$$|A-O| \equiv z_A$$

$$\frac{l}{\sin \pi/4} = \frac{|AO|}{\sin\varphi}$$

$$|z_A| = \frac{l \sin\varphi}{\sin(\pi/4)} = \sqrt{2}\, l \sin\varphi$$

$$z_A = -\sqrt{2}\, l \sin\varphi$$

$$(P-O) = \hat{j}\, \frac{l}{2} \sin\left(\varphi+\frac{\pi}{4}\right) + \hat{k}\left(-\sqrt{2}\, l \sin\varphi - \frac{l}{2}\cos\left(\varphi+\frac{\pi}{4}\right)\right)$$

$$\sin(\alpha+\beta) = \cancel{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}$$
$$\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

$$\sin\left(\varphi+\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(\sin\varphi+\cos\varphi\right)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos\left(\varphi+\frac{\pi}{4}\right) = \left[\cos\varphi - \sin\varphi\right]\frac{\sqrt{2}}{2}$$

$$(P-O) = \hat{j}\, \frac{l}{2}\frac{\sqrt{2}}{2}\left[\sin\varphi+\cos\varphi\right] + \hat{k}\Big(-\sqrt{2}\, l\sin\varphi + $$
$$+\, l\frac{\sqrt{2}}{4}\left(\cos\varphi + \sin\varphi\right)\Big)$$

$$(P-O) = \cancel{\frac{\sqrt{2}}{2}\left[\frac{\ell}{2}(\sin\varphi+\cos\varphi)-8\,\ell\cos\varphi\right]}$$

$$= \vec{\jmath}\,\frac{\ell\sqrt{2}}{4}\left[\sin\varphi+\cos\varphi\right]+$$

$$\vec{k}\,\ell\sqrt{2}\left[\underbrace{-\sin\varphi+\frac{1}{4}\sin\varphi}_{-\frac{3}{4}\sin\varphi}-\frac{1}{4}\cos\varphi\right]$$

$$v_P = \vec{\jmath}\,\frac{\ell\sqrt{2}\dot\varphi}{\cancel{4}}\left[\cos\varphi-\sin\varphi\right]+\vec{k}\,\frac{\ell\sqrt{2}}{4}\dot\varphi\left[-\frac{3}{\cancel{4}}\cos\varphi+\sin\varphi\right]$$

$$v_P^2 = \frac{\ell^2}{8}\dot\varphi^2\left[\cos^2\varphi+\sin^2\varphi-2\cos\varphi\sin\varphi+9\cos^2\varphi+\sin^2\varphi-6\cos\varphi\sin\varphi\right]$$

$$= \frac{\ell^2\dot\varphi^2}{8}\left[2+8\cos^2\varphi-8\cos\varphi\sin\varphi\right]$$

$$T_P = \frac{1}{2}Mv_P^2 = \frac{\ell^2\dot\varphi^2}{8}M\left[1+4\cos^2\varphi-4\cos\varphi\sin\varphi\right]$$

$$T_{ROTA} = \frac{1}{2}\frac{1}{12}M\ell^2\dot\varphi^2 \qquad \frac{1}{24}+\frac{1}{8}=\frac{\cancel{4}}{24}$$

$$T = \frac{\ell^2\dot\varphi^2 M}{\cancel{24}_{6}}\left[\cancel{4}^{1}+\cancel{12}^{3}\cos^2\varphi-\cancel{12}^{3}\cos\varphi\sin\varphi\right]$$

$$U = Mg\,\ell\sqrt{2}\left(-\sin\varphi + \frac{1}{4}\sin\varphi - \frac{\cos\varphi}{4}\right) + \frac{c^2}{\cancel{4}}\cancel{4}\,\ell^2\sin^2\varphi$$

$$= Mg\ell\sqrt{2}\left(-\frac{3}{4}\sin\varphi - \frac{1}{4}\cos\varphi\right) + c^2\ell^2\sin^2\varphi$$