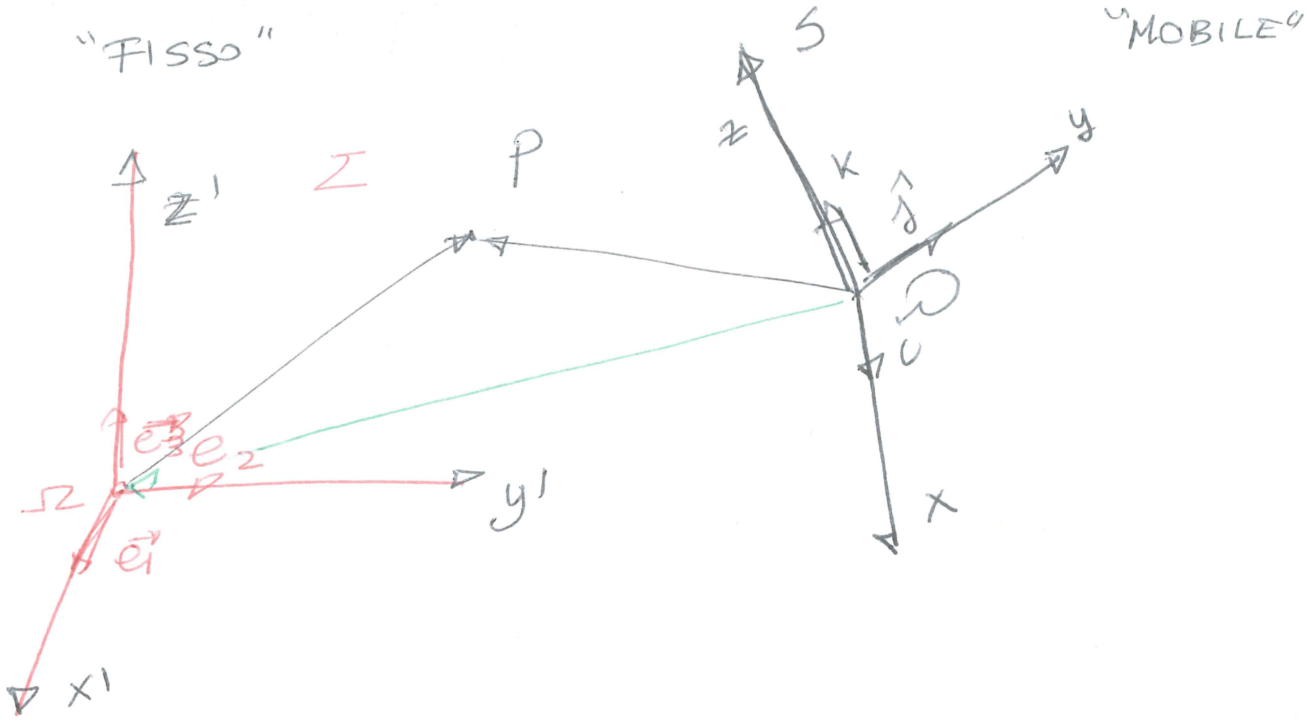
CINEMATICA RELATIVA: MATRICI DI ROTAZIONE



P può essere rapp. da 2 vettori $(P-O)_S$ $(P-\Omega)_\Sigma$

$$(P-\Omega)\Big|_\Sigma = \vec{e_1}\, x'_p + \vec{e_2}\, y'_p + \vec{e_3}\, z'_p$$

$$(P-O)\Big|_S = x\,\hat{\imath} + y\,\hat{j} + z\,\hat{k}$$

Relazione geometrica (indip. da osservatori)

$$\underset{\text{esprimo in } S}{(P-O)} = \underbrace{(P-\Omega)+(\Omega-O)}_{\text{esprimo in } \Sigma}$$

NOTA $\quad (O-\Omega)\Big|_\Sigma = x'_O\, \vec{e_1} + y'_O\, \vec{e_2} + z'_O\, \vec{e_3}$

$$(P-\Omega)\Big|_{\Sigma} + (\Omega - O)\Big|_{\Sigma} = (x' - x'_0)\hat{e}_1 + (y' - y'_0)\hat{e}_2 + (z' - z'_0)\hat{e}_3$$

$$\parallel$$

$$(P-O)\Big|_{S} = x\hat{i} + y\hat{j} + z\hat{k}$$

molt. scal per $\hat{i}$ ...

$$x = (x' - x'_0)\underbrace{\boxed{\hat{e}_1 \cdot \hat{i}}}_{\alpha_1} + (y' - y'_0)\underbrace{\boxed{\hat{e}_2 \cdot \hat{i}}}_{\alpha_2} + (z' - z'_0)\underbrace{\boxed{\hat{e}_3 \cdot \hat{i}}}_{\alpha_3}$$

$$y = (x' - x'_0)\underbrace{\boxed{\hat{e}_2 \, \hat{j}}}_{\beta_1} + (y' - y'_0)\underbrace{\boxed{\hat{e}_2 \, \hat{j}}}_{\beta_2} + (z' - z'_0)\underbrace{\boxed{\hat{e}_3 \, \hat{j}}}_{\beta_3}$$

$$z = (x - x'_0)\underbrace{\boxed{\hat{e}_1 \, k}}_{\gamma_1} + (y' - y'_0)\underbrace{\boxed{\hat{e}_2 \cdot k}}_{\gamma_2} + (z' - z'_0)\underbrace{\boxed{\hat{e}_3 \, \hat{k}}}_{\gamma_3}$$

In forma matriciale

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}}_{\equiv} \begin{pmatrix} x' - x'_0 \\ y' - y'_0 \\ z' - z'_0 \end{pmatrix}$$

A matrice dei coseni direttori o matrice di rotazione

I coeff. di A sono della forma

$\hat{e} \cdot \hat{i} = \cos\theta_{\hat{e}\hat{i}}$

Gli elementi della matrice non sono tutti indip.

- 9 coeff con 6 relazioni

NOTA

$\hat{i} = i_1 \vec{e}_1 + i_2 \vec{e}_2 + i_3 \vec{e}_3$

base ortonormale

posso sempre esprimere un vettore così

proc. come prima $i_1 = \hat{i} \cdot \hat{e}_1 \quad i_2 = \hat{i} \cdot \vec{e}_2$

$i_3 = \hat{i} \cdot \vec{e}_3$

quindi vale la proiezione

$\hat{i} = (\hat{i} \cdot \hat{e}_1)\hat{e}_1 + (\hat{i} \cdot \hat{e}_2)\hat{e}_2 + (\hat{i} \cdot \vec{e}_3)\hat{e}_3 = \Sigma(\hat{i} \cdot \hat{e}_j)\hat{e}_j$

in generale $\vec{v} = \Sigma(\vec{v} \cdot \hat{e}_i)\hat{e}_i$

uso questo e noto

$1 = |\hat{i}|^2 = i_1^2 + i_2^2 + i_3^2 = (\hat{i} \cdot \hat{e}_1)^2 + (\hat{i} \cdot \hat{e}_2)^2 + (\hat{i} \cdot \hat{e}_3)^2$

$= \underbrace{\sum_{j=1}^{3} (\hat{i} \cdot \hat{e}_j)^2}_{\Sigma (\alpha_i)^2 = 1} = 1$

$$\left.\begin{array}{l} |\hat{i}|^2 = 1 \Rightarrow \sum(\alpha_i)^2 = 1 \\ |\hat{j}|^2 = 1 \Rightarrow \sum(\beta_i)^2 = 1 \\ |\hat{k}|^2 = 1 \Rightarrow \sum(\gamma_i)^2 = 1 \end{array}\right\} \text{3 relazioni}$$

$$\hat{i}\cdot\hat{j} = 0$$

$$\hat{i} = \hat{e}_1(\hat{i}\cdot\hat{e}_1) + \hat{e}_2(\hat{i}\cdot\hat{e}_2) + \hat{e}_3(\hat{i}\cdot\hat{e}_3)$$

$$\hat{j} = \hat{e}_1(\hat{j}\cdot\hat{e}_1) + \hat{e}_2(\hat{j}\cdot\hat{e}_2) + \hat{e}_3(\hat{j}\cdot\hat{e}_3)$$

$$\hat{i}\cdot\hat{j} = \underbrace{(\hat{i}\cdot\hat{e}_1)}_{\alpha_1}\underbrace{(\hat{j}\cdot\hat{e}_1)}_{\beta_1} + \underbrace{(\hat{i}\cdot\hat{e}_2)}_{\alpha_2}\underbrace{(\hat{j}\cdot\hat{e}_2)}_{\beta_2} + \underbrace{(\hat{i}\cdot\hat{e}_3)}_{\alpha_3}\underbrace{(\hat{j}\cdot\hat{e}_3)}_{\beta_3} = 0$$

$$\sum_{i=1}^{3} \alpha_i \beta_i = 0$$

analogo per $\hat{i}\cdot\hat{k} = 0$ $\hat{j}\cdot\hat{k} = 0$

TOTALE di 6 relazioni

Calcoliamo

$$A^T A = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} =$$

$A A^T =$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} =$$

$$\begin{pmatrix} \Sigma(\alpha_i)^2 & \Sigma \alpha_i \beta_i & \Sigma \alpha_i \gamma_i \\ \Sigma \alpha_i \beta_i & \Sigma(\beta_i)^2 & \Sigma \beta_i \gamma_i \\ \Sigma \alpha_i \gamma_i & \Sigma \beta_i \gamma_i & \Sigma \gamma_i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

quindi $\quad A A^T = I \Rightarrow$

$$A = (A^T)^{-1} \Rightarrow A^t = A^{-1} \quad \boxed{\text{mat. ortogonale}}$$

$$1 = \det(A A^T) = \det(A)\det(A^T) = \det(A)^2$$

$\Rightarrow \det(A) = \pm 1 \quad \Rightarrow$ A è un'isometria

conserva lunghezze e angoli

Infine possiamo facilmente invertire le relaz. fra i coeff.

$$\begin{pmatrix} x' - x'_0 \\ y' - y'_0 \\ z' - z'_0 \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^t \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Matrici ortogonali ~~mantengono~~ conservano modulo e prodotto scalare

Sia A ortogonale

$$(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot \underbrace{\mathbb{I}}_{\substack{\text{mat. identica} \\ \times \vec{v}}}\vec{v}) = (\vec{u}\cdot A^t A\vec{v})$$

$$\underline{A}\vec{v} = \vec{v}'$$

$$(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot A^t\vec{v}') = \sum_j u_j\,(\underline{A}^t\vec{v}')_j$$

detto $\vec{\eta} = \underline{A}^t\vec{v}' \Rightarrow \eta_j = \sum_k A^t_{jk} v'_k$

$$\Rightarrow \vec{u}\cdot\vec{v} = \sum_{kj} u_j \underbrace{A^t_{jk}}_{= A_{kj}} v'_k = \sum_{kj} \underbrace{A_{kj} u_j}_{\equiv u'_k} v'_k$$

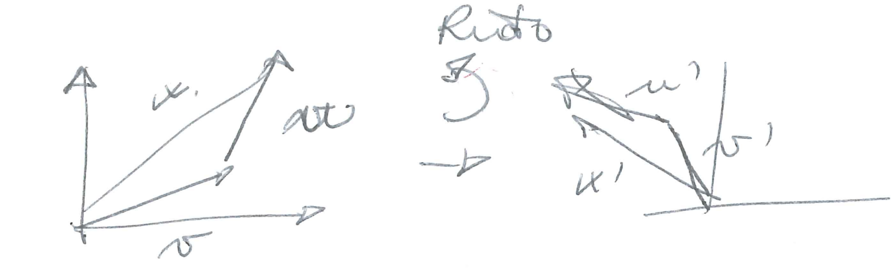
$$= u'_k v'_k = (\vec{u}'\cdot\vec{v}')$$

Rotazione di un vettore: descrizione matriciale

In generale $\underline{A}\,\underline{x} = \underline{y}$ associo a $x \to y$

operatore lineare: Ogni operat. lineare si esprime in questo modo

Le rotazioni sono operatori lineari



$$R(w) = R(v+u) = R(u) + R(v)$$

Notazione matriciale

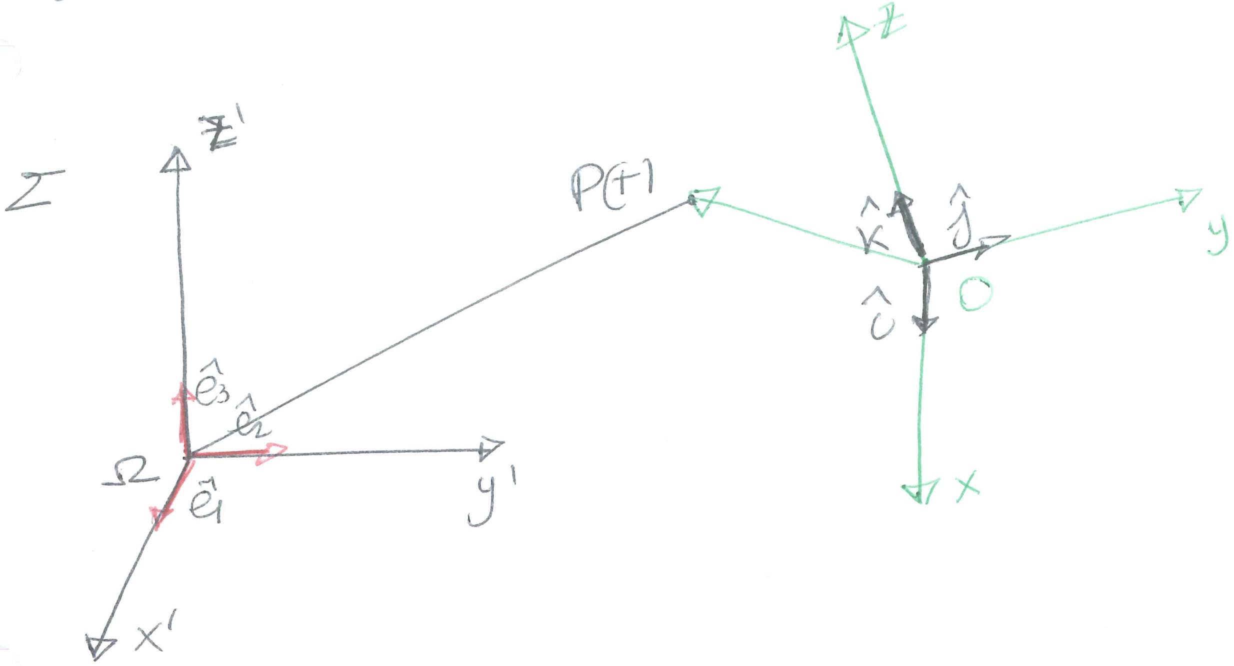
$$\underbrace{\vec{u}\,' = \underline{A}\,\vec{u}}_{\text{componente } i}$$

$\vec{u} = (u_1, u_2 \ldots u_n)$ componente di u

$u_i' = \sum A_{ij} u_j$ (dalla def. di prodotto matrice)

$$\begin{pmatrix} u_1' \\ u_2' \\ u_i' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \ldots \\ & & \\ & & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_i \\ u_n \end{pmatrix}$$

Velocità assoluta e relativa



$$(P-\Omega) = x'\hat{e}_1 + y'\hat{e}_2 + z'\hat{e}_3$$

$$(P-O) = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

Calcolo della velocità del punto

$$\vec{v}_{P-\Omega}\Big|_{\Sigma} \doteq \frac{d}{dt}(P-\Omega)\Big|_{\Sigma} = \dot{x}'\hat{e}_1 + \dot{y}'\hat{e}_2 + \dot{z}'\hat{e}_3$$

$$\vec{v}_{P-O}\Big|_{S} \doteq \frac{d}{dt}(P-O)\Big|_{S} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{k}$$

NOTA: In generale $\vec{v}_{P-\Omega}\Big|_{S} \neq \vec{v}_{P-O}\Big|_{\Sigma}$

ex P punto fisso rispetto a $\Sigma \Rightarrow \vec{v}_{P-\Omega}\Big|_{\Sigma} = 0$

ma $\vec{v}_{P-O}\Big|_{S} \neq 0$

FORMULA di POISSON

Vale la seguente relazione dato $\vec{u}$ un vettore generico

$$\left.\frac{d\vec{u}}{dt}\right|_{\Sigma} = \left.\frac{d\vec{u}}{dt}\right|_{S} + \vec{\omega} \wedge \vec{u}$$

Da intendersi come la def. dell'esistenza di un vettore $\vec{\omega}(t)$ che lega le 2 derivate

$\vec{\omega}$ è unico ed indipendente da $\vec{u}$

$\vec{\omega}$: velocità angolare (del sist. di rif.)

Dimo:

$\vec{u} = u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}$ lo scrivo in S

$$\left.\frac{du}{dt}\right|_{S} = \frac{du_x}{dt}\hat{\imath} + \frac{du_y}{dt}\hat{\jmath} + \frac{du_z}{dt}\hat{k}$$

($\frac{du_x}{dt} = \dot{u}_x$)

ora lo guardo da Σ

$$\left.\frac{du}{dt}\right|_{\Sigma} = \frac{d}{dt}(u_x \hat{\imath}) + \frac{d}{dt}(u_y \hat{\jmath}) + \frac{d}{dt}(u_z \hat{k})$$

$$= \dot{u}_x \hat{\imath} + u_x \left.\frac{d\hat{\imath}}{dt}\right|_{\Sigma} + \dot{u}_y \hat{\jmath} + \left.\frac{d\hat{\jmath}}{dt}\right|_{\Sigma} u_y$$

$$+ \dot{u}_z \hat{k} + u_z \left.\frac{d}{dt}\hat{k}\right|_{\Sigma}$$

$$\left.\frac{d\vec{u}}{dt}\right|_{\Sigma} = \left.\frac{d\vec{u}}{dt}\right|_{S} + \underbrace{\omega_x \boxed{\left.\frac{d\hat{\imath}}{dt}\right|_{\Sigma}} + \omega_y \boxed{\left.\frac{d\hat{\jmath}}{dt}\right|_{\Sigma}} + \omega_z \boxed{\left.\frac{d\hat{k}}{dt}\right|_{\Sigma}}}_{}$$

Semplifico la notazione e sopprimo Σ ①

uso che $\frac{d\hat{v}}{dt} \perp \hat{v}$

$$\frac{d\hat{\imath}}{dt} = \alpha_1 \hat{\jmath} + \alpha_2 \hat{k}$$

$$\frac{d\hat{\jmath}}{dt} = \alpha_3 \hat{k} + \boxed{\alpha_4} \hat{\imath} = \alpha_3 \hat{k} - \alpha_1 \hat{\imath}$$

$$\frac{d\hat{k}}{dt} = \boxed{\alpha_5} \hat{\imath} + \boxed{\alpha_6} \hat{\jmath} = -\alpha_2 \hat{\imath} - \alpha_3 \hat{\jmath}$$

Utilizzo $\hat{\imath} \cdot \hat{\jmath} = 0 \Rightarrow \frac{d\hat{\imath}}{dt} \cdot \hat{\jmath} + \hat{\imath} \frac{d\hat{\jmath}}{dt} = \alpha_1 + \alpha_4 = 0$

$$\boxed{\alpha_1 = -\alpha_4}$$

$$\hat{\jmath} \cdot \hat{k} = 0 \Rightarrow \frac{d\hat{\jmath}}{dt} k + \hat{\jmath} \frac{d\hat{k}}{dt} = \alpha_3 + \alpha_6 = 0$$

$$\boxed{\alpha_6 = -\alpha_3}$$

$$\hat{\imath} \cdot \hat{k} = 0 \Rightarrow \frac{d\hat{\imath}}{dt} \cdot \hat{k} + \hat{\imath} \frac{dk}{dt} = \alpha_2 + \alpha_5 = 0$$

$$\boxed{\alpha_5 = -\alpha_2}$$

Ottengo (1) $\Rightarrow$

$$u_x(\alpha_1\hat{j}+\alpha_2\hat{k})+u_y(\alpha_3\hat{k}-\alpha_1\hat{i})+$$

$$+u_z(-\alpha_2\hat{i}-\alpha_3\hat{j})=$$

$$=\hat{i}(-\alpha_1 u_y-\alpha_2 u_z)+\hat{j}(\alpha_1 u_x-\alpha_3 u_z)$$

$$+\hat{k}(\alpha_2 u_x+\alpha_3 u_y)=$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha_3 & -\alpha_2 & \alpha_1 \\ u_x & u_y & u_z \end{vmatrix} = \vec{\omega}\wedge\vec{u}$$

$$\vec{\omega} = \alpha_3\hat{i}-\alpha_2\hat{j}+\alpha_1\hat{k}$$

$$\equiv \omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k}$$

Ho dimost. il T. di P.

$$\left.\frac{d\vec{u}}{dt}\right|_{\Sigma} = \left.\frac{du}{dt}\right|_{S} + \vec{\omega}\wedge\vec{u}$$

Ritorniamo al caso originario in cui cerchiamo la velocità di 1 pt. fra i 2 sist. di rif.

$$(P-\Omega)+(\Omega-O)=(P-O)$$

deriviamo tutto rispetto a Σ

$$\frac{d}{dt}\left[(P-\Omega)+(\Omega-O)\right]\Big|_{\Sigma}=\frac{d}{dt}(P-O)\Big|_{\Sigma}=v_{(P-O)}\Big|_{\Sigma}$$

$$\vec{v}_P\Big|_{\Sigma} \qquad -\vec{v}_O\Big|_{\Sigma}$$

Ris.

$$\vec{v}_P\Big|_{\Sigma}=v_O\Big|_{\Sigma}+v_{(P-O)}\Big|_{\Sigma}=v_O\Big|_{\Sigma}+v_P\Big|_{S}+\vec{\omega}\wedge(P-O)$$

Formula di conversione delle velocità

$$\boxed{\underbrace{\vec{v}_P\Big|_{\Sigma}=v_O\Big|_{\Sigma}}_{\text{v. "assoluta"}}+\vec{\omega}\wedge(P-O)+\underbrace{v_P\Big|_{S}}_{\text{v. "relativa"}}}$$

quindi

$$\omega_x = \frac{d\hat{\imath}}{dt} \cdot \hat{k} = 0$$

$$\omega_y = \frac{d\hat{\jmath}}{dt} \hat{k} = 0$$

$$\omega_z = \dot{\varphi}$$

Formula velocità angolare x moti piani

$$\Rightarrow \quad \vec{\omega} = \dot{\varphi} \hat{k}$$

↳ derivata dell'angolo di rotazione del rif. mobile (S) rispetto a fisso (Σ)

Ottengo

$$v_P \big|_\Sigma = v_O \big|_\Sigma + \dot{\varphi} \hat{k} \wedge (P-O) + v_P \big|_S$$

$$= \dot{x}'_O \hat{e}_1 + \dot{y}'_O \hat{e}_2 + \dot{\varphi} \hat{k} \wedge (x_p \hat{\imath} + y_p \hat{\jmath}) +$$

$$+ \dot{x}_p \hat{\imath} + \dot{y}_p \hat{\jmath}$$

$$= \dot{x}'_O \hat{e}_1 + \dot{y}'_O \hat{e}_2 + \dot{\varphi} (+\hat{\jmath}_p x_p - \hat{\imath} y_p) + \dot{x}_p \hat{\imath} + \dot{y}_p \hat{\jmath}$$

$$= \dot{x}'_O \hat{e}_1 + \dot{y}'_O \hat{e}_2 + \hat{\imath} (\dot{x}_p - y_p \dot{\varphi}) + \hat{\jmath} (\dot{y}_p + x_p \dot{\varphi})$$

volendo posso esprimere $\hat{\imath}, \hat{\jmath}$ attraverso $\hat{e}_1, \hat{e}_2$ con le formule prec.

Dalla definizione si ricava

$$\omega_x = \alpha_3 = \frac{d\hat{j}}{dt}\,\hat{k}$$

$$\omega_y = -\alpha_2 = \frac{d\hat{k}}{dt}\cdot\hat{i}$$

$$\omega_z = \alpha_1 = \frac{d\hat{i}}{dt}\cdot\hat{j}$$

La velocità angolare $\vec{\omega}$ è un vettore, posso applicare

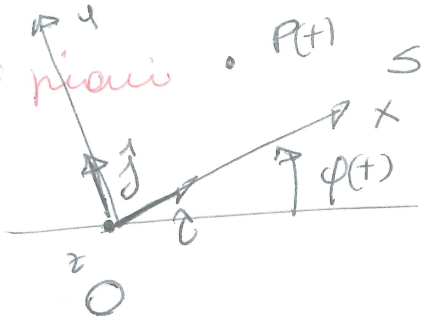
Poisson

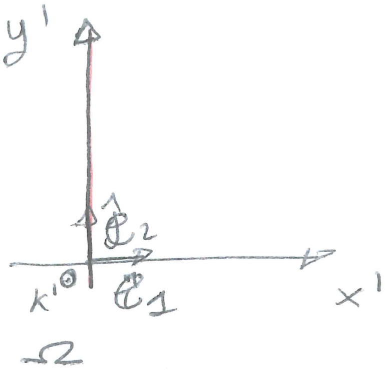
$$\left.\frac{d\vec{\omega}}{dt}\right|_{\Sigma} = \left.\frac{d\vec{\omega}}{dt}\right|_{S} + \cancel{\vec{\omega}\wedge\vec{\omega}}^{\,\to 0}$$

↳ accelerazione angolare

La posso calcolare indiffer. in qualsiasi sistema di rif.

Esempio; caso moti piani





Sia dato $(P-O) = x(t)\,\hat{\imath} + y(t)\,\hat{j}$

$$(O-\Omega) = x_o^{\Sigma}(t)\,\hat{e}_1 + y_o^{\Sigma}(t)\,\hat{e}_2$$

Formula del moto relativo

$$v_P\big|_{\Sigma} = v_O\big|_{\Sigma} + \vec{\omega}\wedge(P-O) + v_P\big|_S$$

Devo determinare $\hat{\omega}$

Dalla def.
$$\omega_x = \frac{d\hat{j}}{dt}\Big|_{\Sigma}\cdot\hat{k}$$
$$\omega_y = \frac{d\hat{k}}{dt}\Big|_{\Sigma}\cdot\hat{\imath}$$
$$\omega_z = \frac{d\hat{\imath}}{dt}\Big|_{\Sigma}\cdot\hat{j}$$

$$\hat{\imath} = \hat{e}_1\cos(\varphi(t)) + \hat{e}_2\sin(\varphi(t))$$
$$\hat{j} = \hat{e}_2\cos(\varphi(t)) - \hat{e}_1\sin(\varphi(t))$$

$$\frac{d\hat{\imath}}{dt}\Big|_{\Sigma} = \hat{e}_1(-\sin(\varphi)\dot{\varphi}) + \hat{e}_2\cos(\varphi)\dot{\varphi} = \dot{\varphi}\,\hat{j}$$

$$\frac{d\hat{j}}{dt}\Big|_{\Sigma} = \hat{e}_2(-\sin(\varphi)\dot{\varphi}) - \hat{e}_1\cos(\varphi)\dot{\varphi} = \dot{\varphi}(-\hat{\imath})$$

Esempio

sia $\quad x_0'(t) = 1+t$

$y_0'(t) = 1$

$$(P-O) = \underbrace{\operatorname{sen}(\omega_1 t)}_{x_P}\,\hat{\imath} + \underbrace{\cos(\omega_1 t)}_{y_P}\,\hat{\jmath}$$

$$\varphi = \omega_0 t$$

$$\vec{v}_P\Big|_\Sigma = \hat{e}_1 + \hat{\imath}\left(\omega_1 \cos(\omega_1 t) - \cos(\omega_1 t)\,\omega_0\right)$$
$$+ \hat{\jmath}\left(-\omega_1 \operatorname{sen}(\omega_1 t) + \operatorname{sen}(\omega_1 t)\,\omega_0\right)$$
$$= \hat{e}_1 + \hat{\imath}\cos(\omega_1 t)\left(\omega_1 - \omega_0\right)$$
$$+ \hat{\jmath}\operatorname{sen}(\omega_1 t)\left(\omega_0 - \omega_1\right)$$

Sostit. esp. $\hat{\imath}$, e $\hat{\jmath}$

$\rightarrow$

$$= \hat{e}_1 + \hat{e}_1 \cos(\omega_0 t)\cos(\omega_1 t)(\omega_0 - \omega_1) +$$
$$\hat{e}_2 \sin(\omega_0 t)\cos(\omega_1 t)(\omega_0 - \omega_1) +$$
$$\hat{e}_2 \cos(\omega_0 t)\operatorname{sen}(\omega_1 t)(\omega_0 - \omega_1) +$$
$$- \hat{e}_1 \operatorname{sen}(\omega_0 t)\operatorname{sen}(\omega_1 t)(\omega_0 - \omega_1)$$

$$= \hat{e}_1 + \hat{e}_1(\omega_0 - \omega_1)\big(\underbrace{\cos(\omega_0 t)\cos(\omega_1 t) - \operatorname{sen}(\omega_0 t)\operatorname{sen}(\omega_1 t)}_{\cos((\omega_0+\omega_1)t)}\big)$$
$$+ \hat{e}_2(\omega_0 - \omega_1)\big(\underbrace{\operatorname{sen}(\omega_0 t)\cos(\omega_1 t) + \cos(\omega_0 t)\operatorname{sen}(\omega_1 t)}_{\sin((\omega_0+\omega_1)t)}\big)$$