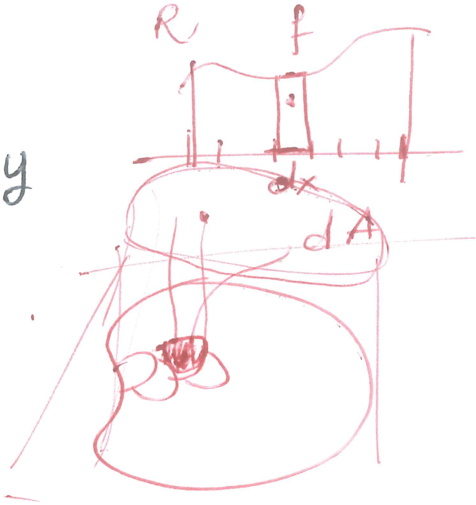
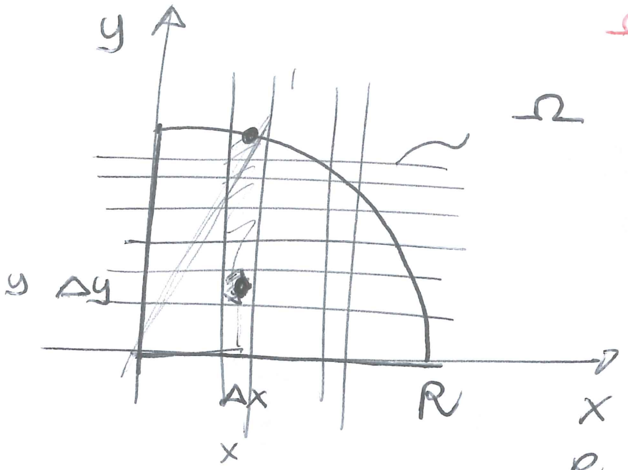
Integrazione R^2

In coord cart: $I = \int_\Omega f(x,y)\, dx\, dy$

$I = \int_\Omega f \, dA$





$$I \simeq \sum_{ij} f(x_i, y_j) \underbrace{\Delta x_i \Delta y_j}_{\Delta A_i}$$

$$\int_\Omega f(x,y)\, dx\, dy = \int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \; f(x,y)$$

coord polari !

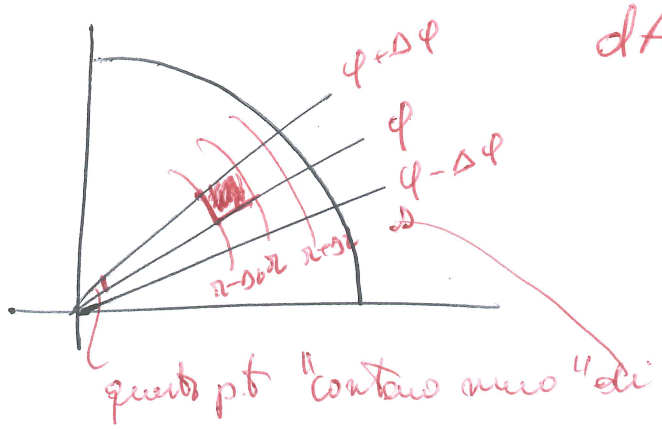
$\longrightarrow$ $\int_0^R dr \int_0^{\pi/2} d\varphi \; f(r,\varphi)\, |J|$

mnemonico $\quad dx\, dy = dr\, d\varphi \left| \frac{\partial(x,y)}{\partial(r,\varphi)} \right|$

Perché c'è J ! $\quad \boxed{dA = dx\,dy \neq dr\,d\varphi}$

1° ordine

$$dA = dr \cdot r\, d\varphi = \underbrace{r}_{|J|}\, d\varphi\, dr$$



In coord sferiche

$$\int_{\Omega} f\,dx\,dy\,dz = \int_{\Omega} f(r^2 \sin\theta)\,d\varphi\,d\theta\,dr$$

espresso in cartes.

volume della sfera $f = 1$

$$\int_{\Omega} \cancel{f}\,dx\,dy\,dz = \int_0^R dr \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi\; r^2 \sin\theta$$

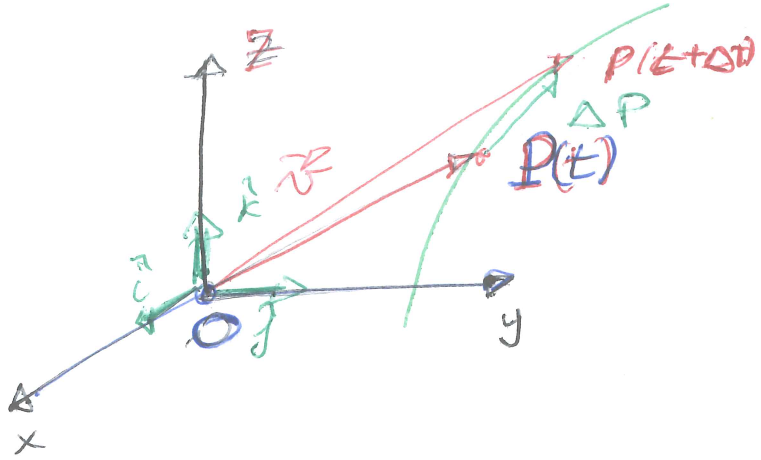
$$= 2\pi \int_0^R r^2\,dr \int_0^{\pi} \sin\theta\,d\theta =$$

$$= 2\pi \left(\frac{r^3}{3}\Big|_0^R \right) \left(-\cos\theta \Big|_0^{\pi} \right) =$$

$$= 2\pi \frac{R^3}{3} (1+1) = \frac{4}{3}\pi R^3$$

TRAIETTORIA e velocità

Osservo un pt da una terna "fissa" S

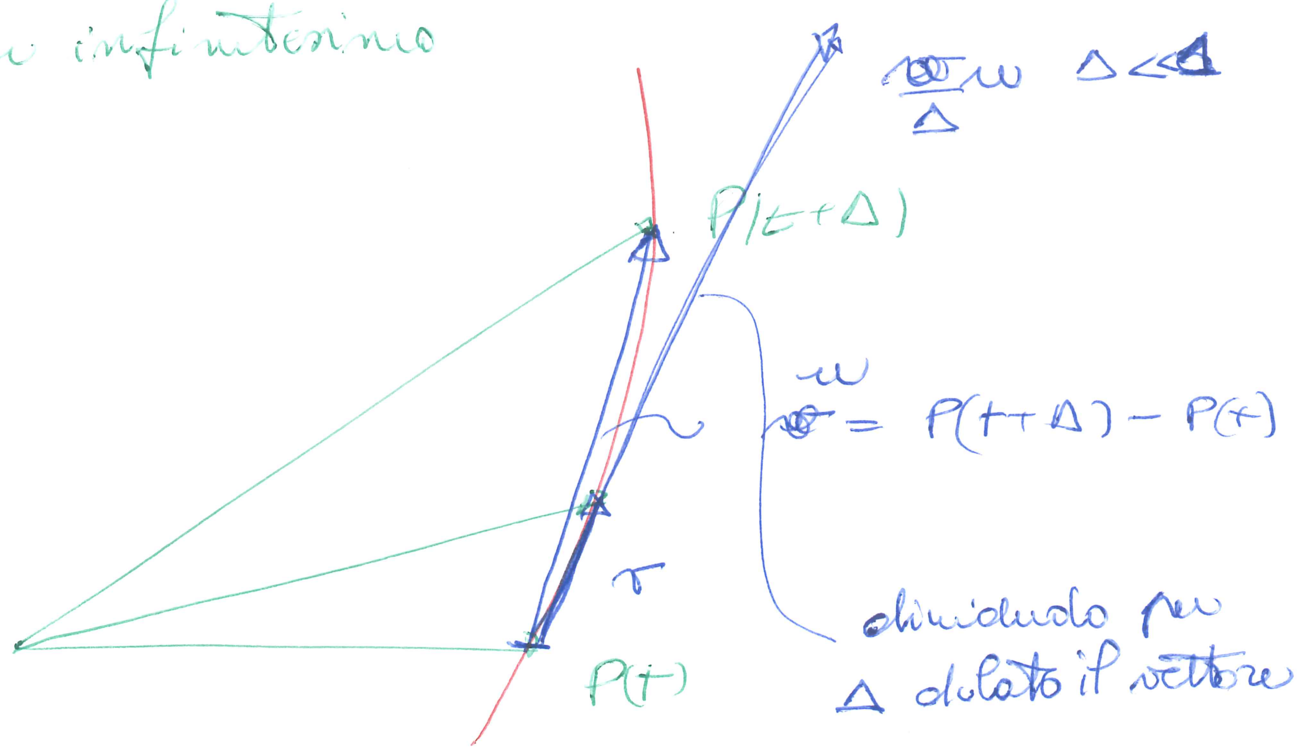


$\vec{v}(t) = (P(t) - O)$

Significato geometrico della derivata

$$\frac{\Delta P}{\Delta t} \overset{(t)\Delta}{=} \frac{P(t+\Delta) - P(t)}{\Delta t} \xrightarrow[\Delta t \to 0]{\lim} \frac{0}{0}$$

vettore a norma che va a zero almeno un infinitesimo



se il limite è ben definito ottengo un vettore (modulo $\neq 0$) tangente alla traiettoria

improprio

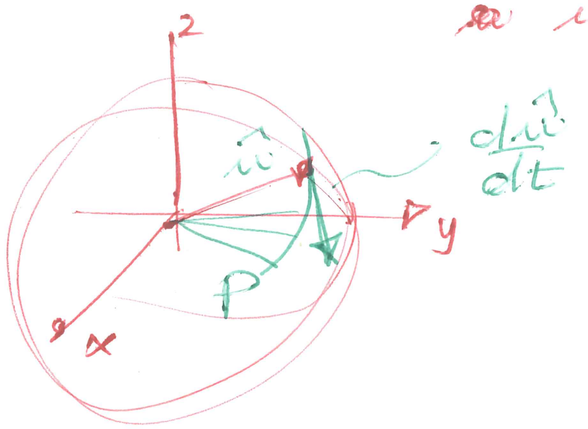
$$\vec{v} \doteq \frac{d(P(t)-O)}{dt} = \frac{d\,P(t)}{dt}$$

↓
velocità

Proprietà generale caso particolare $d\hat{v}$

traiettoria individuata da un versore $\hat{u}(t)$

se il pt si muove su una sfera



$|P(t)-O| = |\hat{u}(t)| = 1$ costante

derivo risp. al tempo

$$\frac{d}{dt}|P(t)-O|^2 = 0 = \frac{d}{dt}|\hat{u}(t)|^2 =$$

$$\frac{d}{dt}(\hat{u}(t)\cdot\hat{u}(t)) = \frac{d\hat{u}(t)}{dt}\cdot\hat{u}(t) +$$

$$\hat{u}(t)\cdot\frac{d\hat{u}}{dt} = 2\,\frac{d\hat{u}(t)}{dt}\cdot\hat{u}(t)$$

$$\Rightarrow \frac{d\hat{u}}{dt}\cdot\hat{u} = 0 \Rightarrow \hat{u} \perp \frac{d\hat{u}}{dt}$$

come si calcola?

I) metodo semplice: base fissa

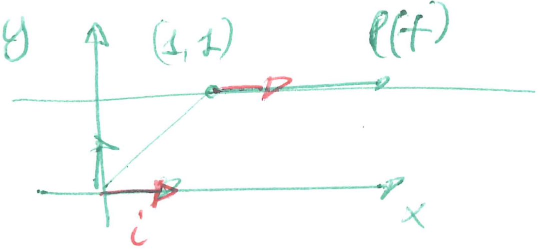
$$(P-O) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$\vec{v}_P = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k} \qquad \text{nota.. } \frac{d(x\hat{i})}{dt} + \ldots$$

posso derivare la $\vec{v}$ e ottengo $\vec{a}$

$$\vec{a}_P = \frac{d}{dt}\vec{v}_P = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k}$$

Esempi in $\mathbb{R}^2$



$$P(t) \equiv P(t) - O = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$x(t) = 1 + t$$

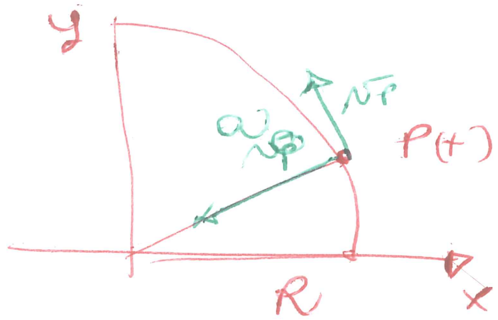
$$y(t) = 1$$

$$\vec{v}_P(t) = \frac{d}{dt}\left((1+t)\,\hat{i} + \hat{j}\right) = \hat{i}$$

nota $\quad u_P(t) = (P-O) = (1+t)\,\hat{i} + \hat{j}$

$$\vec{u}_P \cdot \frac{du}{dt} = 1 + t \neq 0$$

Moto circolare



costante

$$P(t) \Rightarrow \begin{cases} \cancel{x(t)} = R\cos(\omega_0 t) \\ y(t) = R\sin(\omega t) \end{cases}$$

$$\vec{r}_P = R\left(\cos(\omega_0 t)\,\hat{i} + \sin(\omega t)\,\hat{j}\right)$$

$$\vec{v}_P = \frac{d}{dt}\vec{r} = R\omega\underbrace{\left(-\sin(\omega t)\,\hat{i} + \cos(\omega t)\,\hat{j}\right)}$$

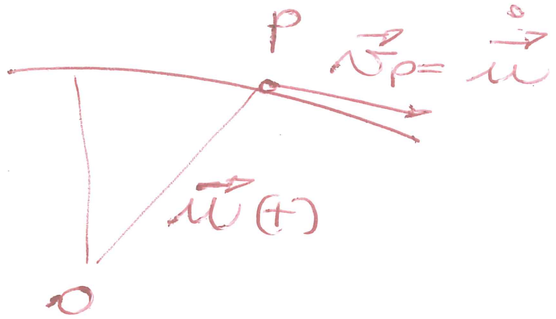
$t = 0 \quad \hat{j} \quad \uparrow$

$\omega t = \frac{\pi}{4} \quad -\hat{i} + \hat{j} \quad \nwarrow$

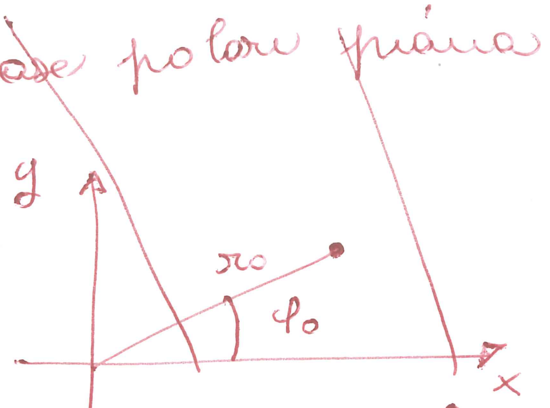
$\omega t = \frac{\pi}{2} \quad -\hat{i} \quad \leftarrow$

$$\vec{a} = \frac{d\,\vec{v}_P}{dt} = R\omega^2\underbrace{\left(-\cos(\omega t)\,\hat{i} - \sin(\omega t)\,\hat{j}\right)}_{= -\vec{r}_P}$$

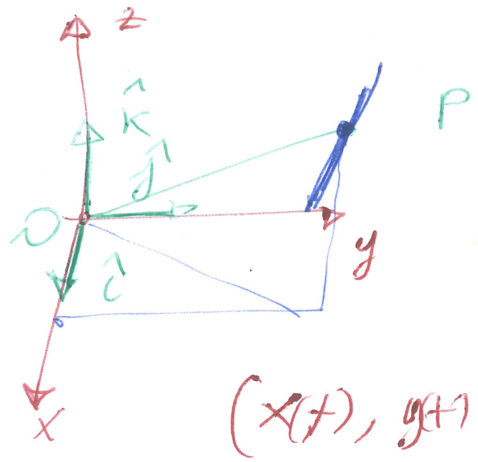
Utilizzo di traiettorie per ~~...~~ def. versori e basi locali



Base polare piana



Caso cartesiano (banale)



$(P-O) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

$(x(t), y(t), z(t))$ arbitrari

<u>SCELGO</u>
$$\begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 \end{cases}$$
retta parallela a asse x e passante per (x_0, y_0, z_0)

$\vec{v}_P = \dfrac{d}{dt}\left((x_0+t)\hat{i} + y_0\hat{j} + z_0\hat{k}\right) = \hat{i}$ ho estratto il versore $\hat{i}$

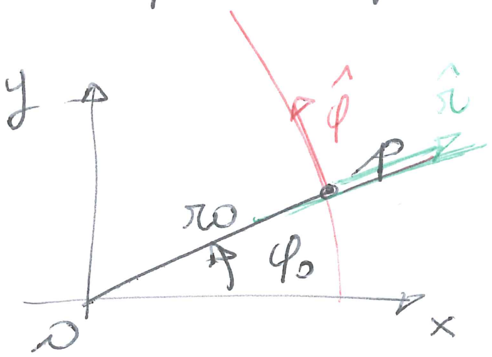
IDEA: ~~cost~~ considerando traiettorie ~~formate~~ ~~ottenute~~ mantenendo costanti tutte le coord. eccetto 1 che varia linearmente col tempo

$$\begin{cases} x_i = x_{i,0} \\ x_{\bar{i}} = x_{\bar{i},0} + t \end{cases}$$

e derivando posso costruire un'ennupla di vettori indipendenti.

Questa costruzione dipende dalla parametrizz. ovvero dal sistema di coord scelto (vi si adatta)

Caso polare piano



P_0: $\overset{(r,\varphi)}{\begin{cases} r = r_0 \\ \varphi = \varphi_0 \end{cases}}$

considero la traiett. (1) $\begin{cases} r = r_0 + t \\ \varphi = \varphi_0 \end{cases}$

(2) $\begin{cases} r = r_0 \\ \varphi = \varphi_0 + t \end{cases}$

Le derivate formano 2 vettori

conviene fare i calcolo in coord. cartesiane

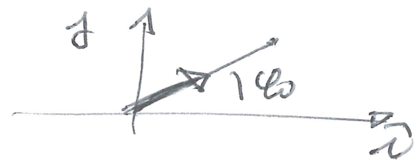
$$x = r\cos\varphi$$

$$y = r\,\mathrm{sen}\,\varphi$$

$$\Rightarrow \quad ① \Rightarrow \begin{cases} x(t) = (r_0 + t)\cos\varphi_0 \\ y(t) = (r_0 + t)\,\mathrm{sen}\,\varphi_0 \end{cases} \equiv (P-O)$$

$$(P_1 - O) = (r_0 + t)\cos\varphi_0\,\hat{\imath} + (r_0 + t)\,\mathrm{sen}\,\varphi_0\,\hat{\jmath}$$

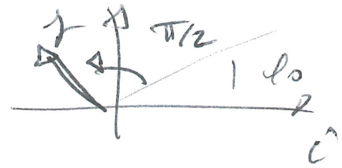
$$\vec{v}_{P_1} = \frac{d(P-O)}{dt}\bigg|_{t=0} = \cos(\varphi_0)\,\hat{\imath} + \mathrm{sen}\,\varphi_0\,\hat{\jmath} = \hat{r}$$



$$② \Rightarrow \begin{cases} x(t) = r_0\cos(\varphi_0 + t) \\ y(t) = r_0\,\mathrm{sen}(\varphi_0 + t) \end{cases}$$

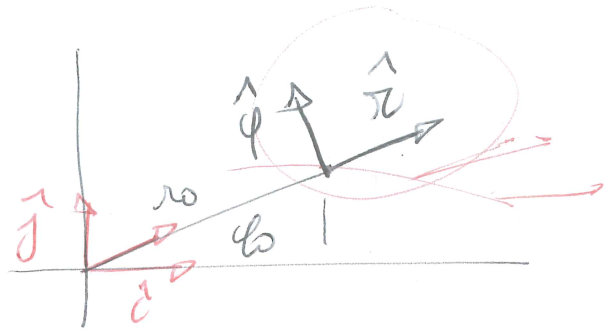
$$(P_2 - O) = r_0\left(\cos(\varphi_0 + t)\,\hat{\imath} + \mathrm{sen}(\varphi_0 + t)\,\hat{\jmath}\right)$$

$$\Rightarrow v_{P_2}\Big|_{t=0} = r_0\left(-\mathrm{sen}(\varphi_0)\,\hat{\imath} + \cos(\varphi_0)\,\hat{\jmath}\right) = \hat{\varphi}$$



Normalizzato $\frac{v_{P_2}\big|_{t=0}}{|v_{P_2}|} = -\sin\varphi_0\,\hat{i} + \cos\varphi_0\,\hat{j}$

$= \hat{\varphi}$



base locale dipende dal punto in cui la calcolo

Ho ottenuto

$\hat{r} = \cos(\varphi)\,\hat{i} + \sin(\varphi)\,\hat{j}$

$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos(\varphi)\,\hat{j}$

$P(r(t), \varphi(t))$

Posso scrivere $(P-O) = r\,\hat{r}$

Se $P = P(t) \Rightarrow (P-O) = r(t)\,\hat{r}(t)$

posso calcolare velocità e acc. in questa rappresent.

$v_P = \frac{d}{dt}(P-O) = \frac{d}{dt}\left(r(t)\,\hat{r}(t)\right) =$

$\frac{dr}{dt}\,\hat{r} + r\frac{d}{dt}\hat{r} = \dot{r}\,\hat{r} + r\boxed{\frac{d}{dt}\hat{r}}$

$\frac{dr}{dt} \equiv \dot{r}$, $\quad \frac{d}{dt}\hat{r} \neq 0$

conviene calcolarlo in coord cart.

$$\frac{d}{dt}\hat{r} = \frac{d}{dt}\left(\cos\varphi\,\hat{\imath} + \operatorname{sen}\varphi\,\hat{\jmath} \right) = \quad \to \text{con } \varphi = \varphi(t)$$

$$\frac{d}{dt}(\cos\varphi\,\hat{\imath}) + \frac{d}{dt}(\operatorname{sen}\varphi\,\hat{\jmath}) =$$

$$= -\operatorname{sen}\varphi\,\dot{\varphi}\,\hat{\imath} + \cos\varphi\,\dot{\varphi}\,\hat{\jmath} = \dot{\varphi}\,\underbrace{(-\operatorname{sen}\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath})}_{\hat{\varphi}}$$

$$= \dot{\varphi}\,\hat{\varphi} \qquad \frac{d}{dt}\hat{r} = \dot{\varphi}\,\hat{\varphi}$$

Stesso calcolo da $\frac{d}{dt}\hat{\varphi} = -\dot{\varphi}\,\hat{r}$

La derivaz. ruota il versore in senso antiorario di angolo $\pi/2$

$$v_P = \frac{d}{dt} r\hat{r} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$$

Accelerazione

$$a_P = \frac{d}{dt} v_P = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\varphi}\hat{\varphi}) =$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi} + \frac{d}{dt}(r\dot{\varphi})\hat{\varphi} + r\dot{\varphi}(-\dot{\varphi}\hat{r})$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi} + \hat{\varphi}(\dot{r}\dot{\varphi} + r\ddot{\varphi}) - r\dot{\varphi}^2\hat{r}$$

$$= \hat{r}(\ddot{r} - r\dot{\varphi}^2) + \hat{\varphi}(2\dot{r}\dot{\varphi} + r\ddot{\varphi})$$

Moto circolare $\quad r = r_0$

$\varphi = \varphi(t) \overset{*}{\sim} \varphi_0 + t\,\omega_0$

$$v_P = r_0 \dot{\varphi}\, \hat{\varphi} \xrightarrow{*} r_0\, \omega_0\, \hat{\varphi}$$

$$a_P = \hat{r}\,(-r_0 \dot{\varphi}^2) + \hat{\varphi}\,(+ r \ddot{\varphi}) \xrightarrow{*}$$

$$\hat{r}\,(-r_0\, \omega_0^2)$$