THE BROUWER DEGREE ASSOCIATED TO CLASSICAL EIGENVALUE PROBLEMS AND APPLICATIONS TO NONLINEAR SPECTRAL THEORY

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Dedicated to the memory of the outstanding mathematician Andrzej Granas, whose contribution to non-linear analysis has deeply inspired our research

ABSTRACT. Thanks to a connection between two completely different topics, the classical eigenvalue problem in a finite dimensional real vector space and the Brouwer degree for maps between oriented differentiable real manifolds, we were able to solve, at least in the finite dimensional context, a conjecture regarding global continuation in nonlinear spectral theory that we formulated in some recent papers. The infinite dimensional case seems nontrivial, and is still unsolved.

1. INTRODUCTION

Consider the *nonlinear eigenvalue problem*

$$(1.1) \qquad \begin{cases} L\mathbf{v} + sN(\mathbf{v}) = \lambda\mathbf{v}, \\ \mathbf{v} \in \mathbf{S}, \end{cases}$$

where s and λ are real parameters, $L\colon \mathbb{R}^k \to \mathbb{R}^k$ is a linear operator, and $N\colon \mathbf{S} \to \mathbb{R}^k$ is a continuous map defined on the unit sphere of $\mathbb{R}^k$.

One can regard (1.1) as a nonlinear perturbation of the classical eigenvalue problem $L\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq 0$.

By a *solution* of (1.1) we mean any triple $(s, \lambda, \mathbf{v}) \in \mathbb{R} \times \mathbb{R} \times \mathbf{S}$ which verifies the system. Solution triples having $s = 0$ are called *trivial*. They correspond to pairs $(\lambda, \mathbf{v})$, hereafter called *eigenpoints (of L)*, in which $\lambda \in \mathbb{R}$ is an eigenvalue of L and $\mathbf{v} \in \mathbf{S}$ is one of the associated unit eigenvectors.

Let $\Sigma \subset \mathbb{R} \times \mathbb{R} \times \mathbf{S}$ denote the set of the solutions of (1.1). In a recent paper [5] we proved that *if $\lambda_* \in \mathbb{R}$ is a simple eigenvalue of L and $\mathbf{v}_*$ is one of the two corresponding unit eigenvectors, then the connected component of Σ containing $z_* = (0, \lambda_*, \mathbf{v}_*)$ is either unbounded or includes a trivial solution $z^* = (0, \lambda^*, \mathbf{v}^*)$ different from z_*.* Thus, in the second alternative, the eigenvector $\mathbf{v}^*$ must be different from $\mathbf{v}_*$. Although, according to the statement, one could have $\lambda_* = \lambda^*$ and, necessarily, $\mathbf{v}^* = -\mathbf{v}_*$.

Denote by $\mathcal{E} \subset \mathbb{R}^2$ the projection of Σ into the $s\lambda$-plane. In [4] we proved that *if λ_* is a simple eigenvalue of L, then the connected component of $\mathcal{E}$ containing $(0, \lambda_*)$ is either unbounded or includes a pair $(0, \lambda^*)$ with $\lambda^* \neq \lambda_*$.*

In [5], supported by this fact, as well as by the above result regarding Σ, we formulated the following conjecture that, because of the inadequateness of the topological tools utilized in that article, we were not able to prove or disprove.

Conjecture 1.1. *Let $\mathbf{v}_* \in \mathbf{S}$ be a unit eigenvector of L corresponding to a simple eigenvalue λ_*. Then, the connected component of Σ containing $(0, \lambda_*, \mathbf{v}_*)$ is either unbounded or includes a triple $(0, \lambda^*, \mathbf{v}^*)$ with $\lambda^* \neq \lambda_*$.*

The purpose of this paper is to obtain a Rabinowitz-type continuation result, Theorem 4.8, which implies the positive answer to the above conjecture, and whose proof is now possible thanks to a special link (which, as far as we know, seems new in the literature) between two completely different topics in mathematics, one belonging to algebra and one to topology: namely, classical eigenvalue problems and Brouwer degree theory.

As we shall see, this link depends on the fact that the eigenpoints of L are the zeros of the C^∞ map

$$\Psi_L \colon \mathbb{R} \times \mathbf{S} \to \mathbb{R}^k, \quad (\lambda, \mathbf{v}) \mapsto L\mathbf{v} - \lambda\mathbf{v},$$

whose domain, the cylinder $\mathbb{R} \times \mathbf{S}$, is an orientable, k-dimensional, smooth manifold embedded in $\mathbb{R} \times \mathbb{R}^k$.

Thanks to the Brouwer degree, once one of the two possible orientations of the cylinder $\mathbb{R} \times \mathbf{S}$ has been chosen (as we shall see one of them may be considered "natural"), the map Ψ_L allows us to assign an integer, 1 or -1, to any eigenpoint $p_* = (\lambda_*, \mathbf{v}_*)$ of L in which λ_* is a simple eigenvalue. This integer, which we denote by L-$\deg(p_*)$ and call *L-degree of p_**, can merely be detected by observing the characteristic polynomial of L (Theorem 3.7), and has two crucial properties from which the positive answer to the Conjecture 1.1 springs.

The first one is a consequence of Theorem 4.7: *If all the trivial solutions contained in a bounded connected component of Σ correspond to simple eigenvalues, then the sum of the L-degrees of the associated eigenpoints is zero.*

The second one states that the (nonzero) L-degree of an eigenpoint $p_* = (\lambda_*, \mathbf{v}_*)$, corresponding to a simple eigenvalue λ_*, is the same as that of its "twin brother" $\bar{p}_* = (\lambda_*, -\mathbf{v}_*)$; as it should be, since the characteristic polynomial of L ignores which one of the two unit eigenvectors, $\mathbf{v}_*$ or $-\mathbf{v}_*$, one considers.

These two facts imply that the connected component of Σ containing the trivial solution $(0, \lambda_*, \mathbf{v}_*)$, if bounded, must include at least another trivial solution different from $(0, \lambda_*, -\mathbf{v}_*)$.

A pioneer work regarding the persistence of the unit eigenvectors of a perturbed eigenvalue problem is due to R. Chiappinelli [7], who examined a problem like (1.1) in the context of a real Hilbert space H (instead of merely $\mathbb{R}^k$), obtaining the so-called *local persistence property* of a unit eigenvector $\mathbf{v}_*$ and the corresponding eigenvalue λ_*. Namely, under the assumptions that $L \colon H \to H$ is a self-adjoint bounded operator, that $\lambda_* \in \mathbb{R}$ is a simple isolated eigenvalue of L, and that $N \colon \mathbf{S} \to H$ is a Lipschitz continuous map defined on the unit sphere of H, he proved the existence of two Lipschitz functions, $\varepsilon \mapsto \mathbf{v}_\varepsilon \in \mathbf{S}$ and $\varepsilon \mapsto \lambda_\varepsilon \in \mathbb{R}$,

defined on neighborhood $(-\delta, \delta)$ of $0 \in \mathbb{R}$ and satisfying the following properties:

$$\mathbf{v}_0 = \mathbf{v}_*, \ \lambda_0 = \lambda_* \quad \text{and} \quad L\mathbf{v}_\varepsilon + \varepsilon N(\mathbf{v}_\varepsilon) = \lambda_\varepsilon \mathbf{v}_\varepsilon, \ \forall \varepsilon \in (-\delta, \delta).$$

Nonlinear eigenvalue problems are related to *nonlinear spectral theory* (see [2]) and find applications to differential equations (see e.g. the recent survey [9] and references therein).

Further results regarding the local persistence of eigenvalues, as well as unit eigenvectors, have been obtained in [3, 8, 10, 11, 12, 13] also in the case in which the eigenvector λ_* is not necessarily simple. In this framework, a natural question arose: can one prove a sort of "global persistence" of eigenvalues and unit eigenvectors? This was the object of the recent papers [4, 5, 6]. As it is of this one; in fact, our main result (Theorem 4.8) ensures the "global persistence" of a trivial solution $(0, \lambda_*, \mathbf{v}_*)$ under the weak assumption that the algebraic multiplicity of the eigenvalue λ_* is odd. Example 5.5 shows that, in Theorem 4.8, the oddness of the algebraic multiplicity of λ_* is crucial.

A possible extension of our main result to the context of real Hilbert spaces seems reasonable, although nontrivial. A byproduct of the extension would be the positive solution to a question posed in [6], which is analogous to Conjecture 1.1, but in the infinite dimensional context. This will be the subject of future investigations.

2. Notation and preliminaries

Here we introduce some notation, some terminology, and some (more or less) known concepts the we shall need later.

2.1. **Notation and terminology.** Let E and F be two normed, finite dimensional, real vector spaces. By $\mathcal{L}(E, F)$ we shall mean the normed space of the linear operators from E into F. The space $\mathcal{L}(E, E)$ will simply be represented by the symbol $\mathcal{L}(E)$. For the norm of a linear operator $L \in \mathcal{L}(E, F)$ will use the common notation $\|L\|$. The standard inner product of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k$ will be denoted by $\langle \mathbf{v}, \mathbf{w} \rangle$. Thus, the Euclidean norm of an element $\mathbf{v} \in \mathbb{R}^k$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

Sometimes, for the sake of simplicity, we shall use the same symbol for a function and for its restriction to a different domain or codomain, or both of them. In this case, however, the new sets will be evident from the context. Observe that, according to the formal definition of a function as a triple of sets, domain, codomain and graph, the first set (and, consequently, the third) may be empty.

Notation 2.1. Let X, Y and Z be metric spaces, $f \colon X \times Y \to Z$ a continuous map, and x a given element of X. By $f_x \colon Y \to Z$ we shall denote the map $y \mapsto f(x, y)$, called the *partial map of f at x*. Moreover, if D is a subset $X \times Y$, by D_x we shall mean the set $D_x = \{y \in Y : (x, y) \in D\}$, called *x-slice of D*. In this case, by $f_x \colon D_x \to Z$ we shall denote the *partial map of f at x, relative to D*; that is, the restriction to the (possibly empty) slice D_x of the above partial map $f_x \colon Y \to Z$. If no set D is mentioned, we shall assume $D = X \times Y$, so that $D_x = Y$.

Hereafter, the boundary and the closure of a subset A of a metric space X will be denoted by ∂A and $\bar{A}$, respectively.

Definition 2.2. Let X be a locally compact metric space, C a closed subset of X, and K a compact subset of C. Given an open subset U of X, we shall say that U *isolates K (among C)* or that U *is an isolating neighborhood of K (among C)*, if

- $\overline{U}$ is compact;
- $U \cap C = K$;
- $\partial U \cap C = \emptyset$.

In this case, K, apart of being compact, is open in C. Notice that any such a subset of C can be isolated by a convenient neighborhood. In this case we shall say that it is an *isolated subset of C*.

2.2. **Orientation of a finite-dimensional real vector space.** Let E be a finite-dimensional real vector space. We recall that an *orientation* of E is one of the two equivalence classes of ordered bases of E, where two ordered bases are equivalent if the linear transformation that takes one onto the other has positive determinant. The space E is *oriented* when one of the two classes, say $\mathcal{B}$, has been chosen. In this case an ordered basis is said to be *positive* if it belongs to $\mathcal{B}$ and *negative* otherwise. For example, the standard (ordered) basis of $\mathbb{R}^k$ is positive for the *standard orientation* of $\mathbb{R}^k$.

Observe that if a (finite dimensional real) vector space E is a direct sum $E_1 \oplus E_2$ of two oriented subspaces, then, an orientation of E, called *compatible (with the splitting)*, may be obtained by the (ordered) union of two positive bases, the first of E_1 and the second of E_2. Analogously, if E_1 and E_2 are oriented spaces, so are (in a natural way) the subspaces $\bar{E}_1 = E_1 \times \{\mathbf{0}\}$ and $\bar{E}_2 = \{\mathbf{0}\} \times E_2$ of the product $E_1 \times E_2$. Since $E_1 \times E_2 = \bar{E}_1 \oplus \bar{E}_2$, the above argument shows that the orientations of E_1 and E_2 determine a *compatible orientation* of the product $E_1 \times E_2$. From this facts one can deduce the following

Remark 2.3. *Let E_1 and E_2 be two finite-dimensional real vector spaces. Then, the orientations of two of the three spaces E_1, E_2 and $E_1 \times E_2$ determine a compatible orientation on the third one. The same holds when the three spaces are E_1, E_2 and E, with $E = E_1 \oplus E_2$.*

Let $L \colon E \to F$ be an isomorphism between oriented finite-dimensional real vector spaces. The operator L is said to be *orientation preserving [reversing]* if it transforms positive bases of E into positive [negative] bases of F. The *sign* of L, $\mathrm{sign}(L)$, is $+1$ if L is orientation preserving, and -1 if it is orientation reversing. Notice that, if $F = E$, then, no matter what is the orientation of E, $\mathrm{sign}(L)$ and $\det(L)$ are canonically defined and $\mathrm{sign}(L) = \mathrm{sign}\det(L)$.

Remark 2.4. *Let $L_1 \colon E_1 \to F_1$ and $L_2 \colon E_2 \to F_2$ be two isomorphisms between oriented, finite dimensional, real vector spaces. Then, the linear operator*

$$L_1 \times L_2 \colon E_1 \times E_2 \to F_1 \times F_2, \quad (\mathbf{v}_1, \mathbf{v}_2) \mapsto (L_1\mathbf{v}_1, L_2\mathbf{v}_2)$$

is orientation preserving if and only if L_1 and L_2 are both orientation preserving or both orientation reversing. More explicitly one has

$$\mathrm{sign}(L_1 \times L_2) = \mathrm{sign}(L_1)\,\mathrm{sign}(L_2).$$

An analogous assertion holds when $F_1 \times F_2$ is replaced by $F = F_1 \oplus F_2$.

2.3. **Elementary notions on Differentiable Topology.** By a *(differentiable) manifold* we shall mean a smooth (i.e. of class C^∞), boundaryless, real differentiable manifold, embedded in some Euclidean space.

Given a manifold $\mathcal{M}$ and given $p \in \mathcal{M}$, the tangent space of $\mathcal{M}$ at p will be denoted by $T_p(\mathcal{M})$. An *orientation* on $\mathcal{M}$ is a "continuous" map ω, which to

any $p \in \mathcal{M}$ associates an orientation $\omega(p)$ of $T_p(\mathcal{M})$. We refer to [17] for details regarding the notion of oriented manifolds.

If $f \colon \mathcal{M} \to \mathcal{N}$ is a C^1 map between two manifolds and $p \in \mathcal{M}$, the differential of f at p will be written as df_p. This is a linear operator from $T_p(\mathcal{M})$ into $T_{f(p)}(\mathcal{N})$.

Remember that, if $f \colon \mathcal{M} \to \mathcal{N}$ is a C^1 map, an element $p \in \mathcal{M}$ is said to be a *regular point* (of f) if df_p is surjective. Non-regular points are called *critical (points)*. The *critical values* of f are the elements of the target manifold $\mathcal{N}$ which lie in the image $f(C)$ of the set C of critical points. Any $q \in \mathcal{N}$ which is not in $f(C)$ is a *regular value*. Therefore, in particular, any element of $\mathcal{N}$ which is not in the image of f is a regular value.

The well-known Sard's Lemma implies that *the set of regular values of a smooth map $f \colon \mathcal{M} \to \mathcal{N}$ between two manifolds is dense in $\mathcal{N}$.*

We recall the following result (see e.g. [17]).

Theorem 2.5 (Regularity of the level set)**.** *Let $f \colon \mathcal{M} \to \mathcal{N}$ be a smooth map between two manifolds of dimensions m and n, respectively.*

If $q \in \mathcal{N}$ is a regular value for f, then $f^{-1}(q)$, if nonempty, is a manifold of dimension $m - n$. Moreover, given $p \in f^{-1}(q)$, one has $T_p(f^{-1}(q)) = \operatorname{Ker} df_p$.

2.4. **The Brouwer degree for maps between oriented manifolds.** We will use a convenient extension, which belongs to the folklore, of the classical Brouwer degree for maps between real finite dimensional oriented manifolds of the same dimension.

We recall, first, some basic notions regarding the classical case, in which the domain manifold is compact or, more generally, the map is "observed" on a relatively compact open subset of the domain. More details can be found, for example, in [15, 17, 18, 19].

Let $f \colon \mathcal{M} \to \mathcal{N}$ be a continuous map between two oriented manifolds of the same dimension. Given a value $q \in \mathcal{N}$ and an open subset U of $\mathcal{M}$, one says that the triple (f, U, q) is *admissible (for the Brouwer degree)* provided that U isolates $U \cap f^{-1}(q)$ among $f^{-1}(q)$; that is, U is relatively compact in $\mathcal{M}$ and $q \notin f(\partial U)$.

The Brouwer degree is a (special) function that to any admissible triple (f, U, q) assigns an integer, denoted by $\deg(f, U, q)$ and called the *(Brouwer) degree of f in (the observed set) U with target q.* Roughly speaking, $\deg(f, U, q)$ is an algebraic count of the solutions in U of the equation $f(p) = q$. In fact, one of the properties of this integer valued function is given by the

Computation Formula. If (f, U, q) is admissible, f is smooth, and q is a regular value for f in U, then

$$\deg(f, U, q) \;=\; \sum_{p \in f^{-1}(q) \cap U} \operatorname{sign}(df_p).$$

This formula is actually the basic definition of the Brouwer degree, and the integer associated to any admissible triple (g, U, r) is defined by

$$\deg(g, U, r) := \deg(f, U, q),$$

where f and q satisfy the assumptions of the Computation Formula and are, respectively, "sufficiently close" to g and r. It is known that this is a well-posed definition.

For other fundamental properties of the degree that we shall use later, such as Additivity, Excision and Homotopy Invariance, we refer to [18]. Here, we just recall the following useful consequence of the Additivity Property:

Existence Property. If (f, U, q) is admissible and $\deg(f, U, q) \neq 0$, then $f^{-1}(q) \cap U$ is nonempty.

A special important case regarding the Brouwer degree is when $\mathcal{M}$ is compact and $\mathcal{N}$ is connected. In this case (see, for example, [17]) the map $q \in \mathcal{N} \mapsto \deg(f, \mathcal{M}, q)$ is constant. Thus, taking into account that the observed set is the whole domain of f, one can simply write $\deg(f)$ instead of $\deg(f, \mathcal{M}, q)$.

A more special, and very interesting case (that we will need later), is when f is a selfmap of a not necessarily oriented manifold $\mathcal{M}$. In fact, we have the following

Remark 2.6. *If f is a selfmap acting on a compact, connected and orientable manifold M, then the integer $\deg(f)$ is well defined and does not depend on the chosen orientation of $\mathcal{M}$ (assuming that it is the same for domain and codomain).*

It belongs to the folklore the fact that the degree may be extended to triples (f, V, q), from now on called *weakly admissible*, in which V is any open subset of $\mathcal{M}$ with the unique requirement that $f^{-1}(q) \cap V$ is compact. In this case one puts

$$\deg_w(f, V, q) := \deg(f, U, q),$$

where U is any open subset of V which isolates $f^{-1}(q) \cap V$ among $f^{-1}(q)$.

The Excision Property of the classical Brouwer degree shows that this definition is well posed, even if some properties of the classical degree do not hold anymore in the extended domain of the weakly admissible triples. One of these is the continuity of the map $q \in \mathcal{N} \setminus f(\partial U) \mapsto \deg(f, U, q)$; another one is the interesting Boundary Dependence Property which is valid for the admissible triples when the target manifold is a vector space.

If we identify $\mathbb{C}$ with $\mathbb{R}^2$, a special weakly admissible triple is $(P, \mathbb{C}, q)$, where P is a non-constant complex polynomial and $q \in \mathbb{C}$. In this case $\deg_w(P, \mathbb{C}, q)$ does not depend on q. Thus, one can simply write $\deg_w(P)$ instead of $\deg_w(P, \mathbb{C}, q)$. One can check that this integer, called *topological degree of P*, is the same as the algebraic degree. Consequently, the Existence Property implies the surjectivity of any non-constant polynomial.

In the context of the weakly admissible triples one has the following folk result that we shall need in the sequel.

Lemma 2.7 (Generalized Homotopy Invariance)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two oriented manifolds of the same dimension. Given an interval $\mathcal{J} \subseteq \mathbb{R}$, an open subset W of $\mathcal{J} \times \mathcal{M}$, a value $q \in \mathcal{N}$, and a continuous map $H \colon W \to \mathcal{N}$, suppose that the subset $H^{-1}(q)$ of W is compact. Then, the function $s \in \mathcal{J} \mapsto \deg_w(H_s, W_s, q)$ is constant.*

The easy proof of Lemma 2.7 can be performed by showing that the integer valued function $s \mapsto \deg_w(H_s, W_s, q)$ is locally constant. In fact, this property is a straightforward consequence of the definition of the extended degree and the classical Homotopy Invariance Property.

2.5. **On the sign-jump at a simple eigenvalue of an endomorphism.**

Definition 2.8. Given a real polynomial P and a real root λ_* of P, the integer

$$\lim_{\varepsilon \to 0^+} \big(\operatorname{sign} P(\lambda_* + \varepsilon) - \operatorname{sign} P(\lambda_* - \varepsilon) \big) \in \{2, -2, 0\}$$

will be called the *sign-jump of P at λ_**.

In the next sections we will consider the continuous real function

$$\mathcal{P}\colon \mathcal{L}(\mathbb{R}^k)\times\mathbb{R}\to\mathbb{R}, \quad (L,\lambda)\mapsto \det(L-\lambda I),$$

where I is the identity in $\mathbb{R}^k$. Therefore, according to Notation 2.1, given any operator $L\in\mathcal{L}(\mathbb{R}^k)$, the partial map $\mathcal{P}_L$ of $\mathcal{P}$ at L is the real characteristic polynomial of L.

Recall that $\lambda_*\in\mathbb{R}$ is a simple eigenvalue of an endomorphism $L\in\mathcal{L}(\mathbb{R}^k)$ if and only if $\operatorname{Ker}(L-\lambda_*I)=\mathbb{R}\mathbf{v}_*$ for some $\mathbf{v}_*\notin\operatorname{Img}(L-\lambda_*I)$. In this case, putting $T=L-\lambda_*I$, the restriction $\widehat{T}\colon\operatorname{Img}T\to\operatorname{Img}T$ of T is an automorphism (*associated to T*). Thus, its determinant is well defined and non-zero. The following result shows that this determinant is positive if and only if the sign-jump at λ_* of characteristic polynomial $\mathcal{P}_L$ is negative.

Lemma 2.9. *Let $\lambda_*\in\mathbb{R}$ be a simple eigenvalue of an endomorphism $L\in\mathcal{L}(\mathbb{R}^k)$. Then the sign-jump at λ_* of $\mathcal{P}_L$ equals $-2\operatorname{sign}(\widehat{T})$, where $\widehat{T}\in\mathcal{L}(\operatorname{Img}T)$ is the automorphism associated to $T=L-\lambda_*I$.*

Proof. It is enough to prove that the sign-jump at $\lambda^*=0$ of the characteristic polynomial $\mathcal{P}_T$ of T equals $-2\operatorname{sign}(\widehat{T})$. Since the eigenvalue $\lambda^*=0$ of T is simple, one gets the splitting $\mathbb{R}^k=\operatorname{Ker}T\oplus\operatorname{Img}T$ and the matrix representation

$$T-\lambda I=\begin{pmatrix} T_{11}-\lambda I_{11} & T_{12} \\ T_{21} & T_{22}-\lambda I_{22} \end{pmatrix}=\begin{pmatrix} -\lambda I_{11} & 0 \\ 0 & \widehat{T}-\lambda I_{22} \end{pmatrix}.$$

Therefore, $\mathcal{P}_T(\lambda)=\det(T-\lambda I)=-\lambda\det(\widehat{T}-\lambda I_{22})$.

The assertion now follows recalling the definition of sign-jump and observing that $\det(\widehat{T}-\lambda I_{22})\to\det\widehat{T}$ as $\lambda\to 0$. $\qquad\square$

3. The eigenvalue problem and the associated Brouwer degree

Consider the eigenvalue problem

$$(3.1)\qquad\qquad \begin{cases} L\mathbf{v}=\lambda\mathbf{v}, \\ \mathbf{v}\in\mathbf{S}, \end{cases}$$

where $\lambda\in\mathbb{R}$, $L\in\mathcal{L}(\mathbb{R}^k)$, and $\mathbf{S}$ is the unit sphere of $\mathbb{R}^k$.

An element $(\lambda,\mathbf{v})$ of the cylinder $\mathbb{R}\times\mathbf{S}$ will be called an *eigenpoint of L* if it satisfies the equation $L\mathbf{v}=\lambda\mathbf{v}$. So that, the second element $\mathbf{v}$ is a *unit eigenvector of L* corresponding to the (real) *eigenvalue λ*.

The set of the eigenpoints of L will be denoted by $\mathcal{S}$. Thus, given any $\lambda\in\mathbb{R}$, the *λ-slice* $\mathcal{S}_\lambda=\{\mathbf{v}\in\mathbf{S}:(\lambda,\mathbf{v})\in\mathcal{S}\}$ of $\mathcal{S}$ coincides with $\mathbf{S}\cap\operatorname{Ker}(L-\lambda I)$.

Observe that $\mathcal{S}_\lambda$ is nonempty if and only if λ is a real eigenvalue of L. In this case $\mathcal{S}_\lambda$ will be called the *eigensphere of L corresponding to λ*. In fact, it is a sphere whose dimension equals the geometric multiplicity of λ minus one. In particular, if λ is simple or, more generally, if its geometric multiplicity is one, then $\mathcal{S}_\lambda$ has only two elements: the two unit eigenvectors of L, one opposite to the other.

If $\lambda\in\mathbb{R}$ is an eigenvalue of L, the nonempty set $\{\lambda\}\times\mathcal{S}_\lambda$ will be called the *eigenset of L corresponding to λ* or, briefly, the *λ-eigenset of L*. Notice that the set

of all the eigenpoints of L is given by

$$\mathcal{S} = \bigcup_{\lambda \in \mathbb{R}} \{\lambda\} \times \mathcal{S}_\lambda.$$

It is convenient to regard $\mathbb{R} \times \mathbf{S}$ as the subset of the space $\mathbb{R} \times \mathbb{R}^k$ satisfying the equation $g(\lambda, \mathbf{v}) = 1$, where $g \colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is defined by $g(\lambda, \mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle$. The differential $dg_p \in \mathcal{L}(\mathbb{R} \times \mathbb{R}^k, \mathbb{R})$ of g at a point $p = (\lambda, \mathbf{v})$ is given by $(\dot{\lambda}, \dot{\mathbf{v}}) \mapsto 2\langle \mathbf{v}, \dot{\mathbf{v}} \rangle$. Therefore, the set of the critical points of g is the λ-axis $\mathbf{v} = 0$ and, consequently, the number 1 is a regular value for g. This shows that $\mathbb{R} \times \mathbf{S}$ is a smooth manifold of codimension one in $\mathbb{R} \times \mathbb{R}^k$ and, given any $p = (\lambda, \mathbf{v}) \in g^{-1}(1)$, the tangent space of $\mathbb{R} \times \mathbf{S}$ at p is the kernel of dg_p. Namely,

$$T_{(\lambda, \mathbf{v})}(\mathbb{R} \times \mathbf{S}) = \left\{ (\dot{\lambda}, \dot{\mathbf{v}}) \in \mathbb{R} \times \mathbb{R}^k : \langle \mathbf{v}, \dot{\mathbf{v}} \rangle = 0 \right\} = \mathbb{R} \times \mathbf{v}^\perp = (0, \mathbf{v})^\perp \subset \mathbb{R} \times \mathbb{R}^k.$$

Observe that the eigenpoints of L are the zeros of the smooth map

$$\Psi_L \colon \mathbb{R} \times \mathbf{S} \to \mathbb{R}^k, \quad (\lambda, \mathbf{v}) \mapsto L\mathbf{v} - \lambda\mathbf{v}.$$

Actually, as we shall see, it is convenient to define

$$\Psi \colon \mathcal{L}(\mathbb{R}^k) \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k, \quad (T, \lambda, \mathbf{v}) \mapsto T\mathbf{v} - \lambda\mathbf{v},$$

so that, according to Notation 2.1, Ψ_L may be regarded as the partial map of Ψ at L, relative to $D = \mathcal{L}(\mathbb{R}^k) \times \mathbb{R} \times \mathbf{S}$.

Since $\mathbb{R} \times \mathbf{S}$ is an orientable real differentiable manifold of the same dimension as the oriented space $\mathbb{R}^k$, once we give an orientation to it, it makes sense to consider the Brouwer degree (with target $\mathbf{0} \in \mathbb{R}^k$), $\deg(\Psi_L, U, \mathbf{0})$, of Ψ_L on any relatively compact open set U such that $\partial U \cap \Psi_L^{-1}(\mathbf{0}) = \emptyset$.

Since $\mathbb{R}$ has the standard orientation (given by 1 as a basis), in order to orient $\mathbb{R} \times \mathbf{S}$, it is sufficient to choose one of the two orientations of $\mathbf{S}$. We prefer the natural one, induced by regarding $\mathbf{S}$ as the boundary of the oriented unit disk $\mathbf{D}$ of $\mathbb{R}^k$. Namely, given any $\mathbf{v} \in \mathbf{S}$, as a basis of the one-dimensional subspace $\mathbb{R}\mathbf{v}$ of $\mathbb{R}^k$ we take the unit vector $\nu(\mathbf{v}) = \mathbf{v}$ which points outside $\mathbf{D}$. Consequently, the orientation of the tangent space $T_\mathbf{v}(\mathbf{S}) = \mathbf{v}^\perp$ is the one compatible with the splitting $\mathbb{R}\mathbf{v} \oplus \mathbf{v}^\perp$ of the oriented space $\mathbb{R}^k$ (see Remark 2.3).

In order to simplify some statements, it is convenient to introduce the following notion of L-degree, which is well-posed thanks to the Excision Property of the Brouwer degree.

Definition 3.1. Let $K \subset \mathbb{R} \times \mathbf{S}$ be an isolated set of eigenpoints of an operator $L \in \mathcal{L}(\mathbb{R}^k)$. By the L-*degree of* K we mean the integer $L\text{-deg}(K) = \deg(\Psi_L, U, \mathbf{0})$, where $U \subset \mathbb{R} \times \mathbf{S}$ is any isolating neighborhood of K among $\Psi_L^{-1}(\mathbf{0})$. In particular, if $p \in \Psi_L^{-1}(\mathbf{0})$ is such that the differential $d(\Psi_L)_p$ is invertible, then p is isolated among $\Psi_L^{-1}(\mathbf{0})$ and $L\text{-deg}(p) = \text{sign}(d(\Psi_L)_p)$.

Let $p_* = (\lambda_*, \mathbf{v}_*)$ be an eigenpoint of $L \in \mathcal{L}(\mathbb{R}^k)$ corresponding to a simple eigenvalue. In [5] it was shown that $\Psi_L \colon \mathbb{R} \times \mathbf{S} \to \mathbb{R}^k$ maps diffeomorphically a neighborhood of p_* in $\mathbb{R} \times \mathbf{S}$ onto a neighborhood of $\mathbf{0}$ in $\mathbb{R}^k$. This implies that the L-degree of p_* is either 1 or -1. The following result, whose importance is crucial for the remaining part of this paper, shows how $L\text{-deg}(p_*)$ can be detected by the characteristic polynomial of L.

Lemma 3.2. *Let λ_* be a simple real eigenvalue of an endomorphism L of $\mathbb{R}^k$ and $p_* = (\lambda_*, \mathbf{v}_*)$ any one of the two corresponding eigenpoints. Then the L-degree of p_* is one-half of the sign-jump at λ_* of the characteristic polynomial of L.*

Proof. As pointed out in [5], and easy to check, the differential

$$d(\Psi_L)_{p_*} \colon T_{p_*}(\mathbb{R} \times \mathbf{S}) \to \mathbb{R}^k, \quad (\dot{\lambda}, \dot{\mathbf{v}}) \mapsto (L - \lambda_* I)\dot{\mathbf{v}} - \dot{\lambda}\mathbf{v}_*$$

of Ψ_L at p_* is an isomorphism. Therefore, $L\text{-}\deg(p_*) = \operatorname{sign}(d(\Psi_L)_{p_*})$.

Hence, by Lemma 2.9, it is sufficient to show that $d(\Psi_L)_{p_*}$ and the automorphism $\widehat{T} \in \mathcal{L}(\operatorname{Img} T)$ associated to $T = L - \lambda_* I$ have opposite signs. As λ_* is simple, we have the splitting

$$\mathbb{R}^k = \operatorname{Ker} T \oplus \operatorname{Img} T = \mathbb{R}\mathbf{v}_* \oplus T(\mathbf{v}_*^\perp).$$

Since the line $\mathbb{R}\mathbf{v}_*$ is oriented (by the vector $\mathbf{v}_*$), so is, according with the standard orientation of $\mathbb{R}^k$, any of its complement (such as $\mathbf{v}_*^\perp$ and $T(\mathbf{v}_*^\perp)$), as well as the quotient space $\mathbb{R}^k/\mathbb{R}\mathbf{v}_*$, which is canonically isomorphic to any complement of $\mathbb{R}\mathbf{v}_*$. This implies that the automorphism $\widehat{T}$ is orientation preserving if and only if so is the restriction $\widetilde{T} \colon \mathbf{v}_*^\perp \to T(\mathbf{v}_*^\perp)$ of T.

Thus, it is enough to prove that $\widetilde{T}$ and $d(\Psi_L)_{p_*}$ have opposite signs. To this purpose, observe that the domain $T_{p_*}(\mathbb{R} \times \mathbf{S})$ of the differential $d(\Psi_L)_{p_*}$ is the product $\mathbb{R} \times T_{v_*}(\mathbf{S}) = \mathbb{R} \times \mathbf{v}_*^\perp$. Moreover,

$$d(\Psi_L)_{p_*} \colon \mathbb{R} \times \mathbf{v}_*^\perp \to \mathbb{R}^k = \mathbb{R}\mathbf{v}_* \oplus T(\mathbf{v}_*^\perp)$$

is the sum of two maps: the first one, call it Λ, acts from $\mathbb{R}$ to $\mathbb{R}\mathbf{v}_*$, and is given by $\dot{\lambda} \mapsto -\dot{\lambda}\mathbf{v}_*$; the second one is $\widetilde{T}$. Therefore, according to Remark 2.4, one has $\operatorname{sign}(d(\Psi_L)_{p_*}) = \operatorname{sign}(\Lambda)\operatorname{sign}(\widetilde{T})$. Since Λ is orientation reversing, we finally get $\operatorname{sign}(d(\Psi_L)_{p_*}) = -\operatorname{sign}(\widetilde{T})$. $\qquad\square$

As a consequence of Lemma 3.2 we get

Lemma 3.3. *Let λ_* be a simple real eigenvalue of $L \in \mathcal{L}(\mathbb{R}^k)$. Then the L-degree of the λ_*-eigenset is the sign-jump at λ_* of the characteristic polynomial of L. Consequently, it is either 2 or -2.*

Proof. Since λ_* is simple, the eigenset $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ is made up of two isolated eigenpoints. Thus, because of the Additivity Property of the Brouwer degree, its L-degree is the sum of the L-degrees of the two eigenpoints, and the assertion follows from Lemma 3.2. $\qquad\square$

The following remark belongs to the folklore, and the easy proof is left to the reader.

Remark 3.4. *Let $f \colon X \times K \to Y$ be a continuous map between metric spaces, and let $q \in Y$ be a "target point". Assume that K is compact. Then, the set of the elements $x \in X$ for which q lies in the image $f_x(K)$ of the partial map $f_x \colon K \to Y$ is closed.*

We recall that, if an operator $T \in \mathcal{L}(\mathbb{R}^k)$ has no eigenpoints in the boundary of a bounded open subset U of $\mathbb{R} \times \mathbf{S}$, then $\deg(\Psi_T, U, \mathbf{0})$ is well defined. The next result regards the continuous dependence on T of this integer.

Theorem 3.5. *Let U be a bounded open subset of the cylinder $\mathbb{R} \times \mathbf{S}$. Then the set $\mathcal{U}$ of the operators $T \in \mathcal{L}(\mathbb{R}^k)$ without eigenpoints in the boundary of U is open in $\mathcal{L}(\mathbb{R}^k)$. Moreover, the map $T \in \mathcal{U} \mapsto \deg(\Psi_T, U, \mathbf{0})$ is locally constant.*

Proof. Since the boundary ∂U of U is a compact set, the first assertion follows directly from Remark 3.4. Consequently, the map $T \in \mathcal{U} \mapsto \deg(\Psi_T, U, \mathbf{0})$ is well defined. To prove that it is locally constant, take any $L \in \mathcal{U}$ and let $\mathcal{V} \subseteq \mathcal{U}$ be a convex neighborhood of L. Given $T \in \mathcal{V}$, it is enough to show that $\deg(\Psi_T, U, \mathbf{0}) = \deg(\Psi_L, U, \mathbf{0})$. To this purpose, consider the homotopy $H \colon [0,1] \times \overline{U} \to \mathbb{R}^k$, defined by

$$(t, (\lambda, \mathbf{v})) \mapsto (L + t(T - L))\mathbf{v} - \lambda\mathbf{v},$$

and notice that, because of the convexity of $\mathcal{V}$, $L + t(T - L) \in \mathcal{V}$ for all $t \in [0,1]$. This implies that the partial map $H_t \colon \overline{U} \to \mathbb{R}^k$, $t \in [0,1]$, never vanishes on ∂U. Thus, because of the Homotopy Invariance Property of the degree, one has

$$\deg(H_0, U, \mathbf{0}) = \deg(H_1, U, \mathbf{0}),$$

and the assertion follows from the equalities $H_0 = \Psi_L$ and $H_1 = \Psi_T$. $\square$

Theorem 3.6. *Given $L \in \mathcal{L}(\mathbb{R}^k)$, let (a, b) be a bounded real interval such that neither a nor b are eigenvalues of L. Then*

$$\deg(\Psi_L, (a, b) \times \mathbf{S}, \mathbf{0}) = \operatorname{sign} \mathcal{P}_L(b) - \operatorname{sign} \mathcal{P}_L(a),$$

where $\mathcal{P}_L$ is the characteristic polynomial of L. In particular, if λ_* is any real eigenvalue of L, then the L-degree of the λ_*-eigenset is the sign-jump at λ_* of $\mathcal{P}_L$.

Proof. We need to prove only the first assertion: the last one is a straightforward consequence of this, keeping in mind the definitions of L-degree (Definition 3.1) and sign-jump (Definition 2.8).

If (a, b) contains only simple eigenvalues (or no eigenvalues), the first assertion follows from the Additivity Property of the Brouwer degree and Lemma 3.3.

Otherwise, because of the continuity of the map

$$\mathcal{P} \colon \mathcal{L}(\mathbb{R}^k) \times \mathbb{R} \to \mathbb{R}, \quad (T, \lambda) \mapsto \det(T - \lambda I),$$

there exists an open neighborhood $\mathcal{V}$ of L in $\mathcal{L}(\mathbb{R}^k)$ such that

$$\operatorname{sign} \mathcal{P}_T(b) - \operatorname{sign} \mathcal{P}_T(a) = \operatorname{sign} \mathcal{P}_L(b) - \operatorname{sign} \mathcal{P}_L(a),$$

for all $T \in \mathcal{V}$.

Applying Theorem 3.5 with $U = (a, b) \times \mathbf{S}$, we may also assume that $\mathcal{V}$ is contained in $\mathcal{U}$ and that

$$\deg(\Psi_T, (a, b) \times \mathbf{S}, \mathbf{0}) = \deg(\Psi_L, (a, b) \times \mathbf{S}, \mathbf{0}),$$

for all $T \in \mathcal{V}$.

The first assertion now follows from the fact that in $\mathcal{V}$ there are operators having only simple eigenvalues. $\square$

The following result extends Lemma 3.2 to the case in which the geometric (but not necessarily the algebraic) multiplicity of an eigenvalue is one.

Theorem 3.7. *Let $p_* = (\lambda_*, \mathbf{v}_*)$ be an isolated eigenpoint of an endomorphism L of $\mathbb{R}^k$. Then, the L-degree of p_* is one-half of the sign-jump at λ_* of the characteristic polynomial of L.*

Proof. Since p_* is isolated, the geometric multiplicity of λ_* is one, and the eigenpoint p_* has a twin brother, $\bar{p}_* = (\lambda_*, -\mathbf{v}_*)$. Consequently, the λ_*-eigenset of L is $\{p_*, \bar{p}_*\}$, and its L-degree, according to Theorem 3.6, is the sign-jump at λ_* of the characteristic polynomial $\mathcal{P}_L$ of L. Therefore, because of the Additivity Property of the Brouwer degree, the L-degree of $\{p_*, \bar{p}_*\}$ is the sum of the L-degrees of p_*

and $\bar{p}_*$. Since we do not have any preference, we are incline to believe that the L-degree of each one of these points is as in the assertion. It remains to prove that this is true.

Let (a, b) be an isolating interval of λ_*. Thus, the L-degree of $\{p_*, \bar{p}_*\}$, which is the sign-jump at λ_* of $\mathcal{P}_L$, equals $\operatorname{sign} \mathcal{P}_L(b) - \operatorname{sign} \mathcal{P}_L(a)$.

Now, consider the open subset $U = (a, b) \times (\mathbf{S}^+ \cup \mathbf{S}^-)$ of $\mathbb{R} \times \mathbf{S}$, where $\mathbf{S}^+$ is the open hemisphere $\{\mathbf{v} \in \mathbf{S} : \langle \mathbf{v}, \mathbf{v}_* \rangle > 0\}$ and $\mathbf{S}^-$ is the opposite one. This set U is clearly an isolating neighborhood of the λ_*-eigenset $\{p_*, \bar{p}_*\}$ of L. Thus, its L-degree, $\operatorname{sign} \mathcal{P}_L(b) - \operatorname{sign} \mathcal{P}_L(a)$, coincides with $\deg(\Psi_L, U, \mathbf{0})$, which, because of the Additivity Property of the degree, equals $\deg(\Psi_L, U^+, \mathbf{0}) + \deg(\Psi_L, U^-, \mathbf{0})$, where $U^+ = (a, b) \times \mathbf{S}^+$ and $U^- = (a, b) \times \mathbf{S}^-$ are isolating neighborhoods of p_* and $\bar{p}_*$, respectively. Hence, it remains to show that $\deg(\Psi_L, U^+, \mathbf{0}) = \deg(\Psi_L, U^-, \mathbf{0})$.

As in the proof of Theorem 3.6, if $T \in \mathcal{L}(\mathbb{R}^k)$ is sufficiently close to L, then

$$(3.2) \quad \deg(\Psi_L, U^+, \mathbf{0}) = \deg(\Psi_T, U^+, \mathbf{0}) \quad \text{and} \quad \deg(\Psi_L, U^-, \mathbf{0}) = \deg(\Psi_T, U^-, \mathbf{0}).$$

We may assume that T has only simple eigenvalues, so that its eigenset is made up of isolated eigenpoints of T whose T-degree, according to Lemma 3.2, is either 1 or -1, and any eigenpoint (of T) in U^+ has a twin brother in U^- with the same T-degree. This, because of the Additivity Property of the degree, implies that $\deg(\Psi_T, U^+, \mathbf{0}) = \deg(\Psi_T, U^-, \mathbf{0})$.

The assertion now follows from the equalities (3.2). $\qquad \square$

4. The perturbed eigenvalue problem and global continuation

Consider the following perturbed eigenvalue problem:

$$(4.1) \qquad \begin{cases} L\mathbf{v} + sN(\mathbf{v}) = \lambda\mathbf{v}, \\ \mathbf{v} \in \mathbf{S}, \end{cases}$$

where s, λ are real parameters, $L \in \mathcal{L}(\mathbb{R}^k)$, and $N \colon \mathbf{S} \to \mathbb{R}^k$ is a continuous map defined on the unit sphere of $\mathbb{R}^k$.

An element $(s, \lambda, \mathbf{v}) \in \mathbb{R} \times \mathbb{R} \times \mathbf{S}$ is said to be a *solution (triple)* of (4.1) if it satisfies the equation $L\mathbf{v} + sN(\mathbf{v}) = \lambda\mathbf{v}$. With a slight abuse of terminology, the last element $\mathbf{v}$ is said to be a *unit eigenvector* corresponding to the *eigenpair* (s, λ).

We will denote by Σ the set of solution triples of (4.1) and by $\mathcal{E}$ its projection into the $s\lambda$-plane, which is the set of the eigenpairs.

Remark 4.1. *If N is defined on the whole space $\mathbb{R}^k$ and it is linear, then*

$$\mathcal{E} = \big\{ (s, \lambda) \in \mathbb{R}^2 : \det(L + sN - \lambda I) = 0 \big\}.$$

Notice that Σ is the set of zeroes of the continuous map

$$\Phi \colon \mathbb{R} \times \mathbb{R} \times \mathbf{S} \to \mathbb{R}^k, \quad (s, \lambda, \mathbf{v}) \mapsto L\mathbf{v} - \lambda\mathbf{v} + sN(\mathbf{v}).$$

Therefore, $\Sigma = \Phi^{-1}(\mathbf{0})$ is a closed subset of $\mathbb{R} \times \mathbb{R} \times \mathbf{S}$.

The subset $\mathcal{E}$ of $\mathbb{R}^2$ is closed as well. This is a consequence of Remark 3.4 with $X = \mathbb{R}^2$, $K = \mathbf{S}$, $Y = \mathbb{R}^k$, $q = \mathbf{0}$ and $f = \Phi$.

The following remark shows, in particular, that, if the space $\mathbb{R}^k$ is odd dimensional, then $\mathcal{E}$ is unbounded. Consequently, so is Σ.

Remark 4.2. *Let $\mathbb{R}^k$ be odd dimensional. Then, if $\bar{\lambda} > \|L\|$, the points $(0, \bar{\lambda})$ and $(0, -\bar{\lambda})$ belong to different connected components of $\mathbb{R}^2 \setminus \mathcal{E}$. Consequently $\mathcal{E}$ is unbounded.*

To prove Remark 4.2, consider the set $C(\mathbf{S})$ of the continuous selfmaps of $\mathbf{S}$ endowed with the compact open topology and, for any $p = (s, \lambda) \in \mathbb{R}^2 \setminus \mathcal{E}$, define $h_p \in C(\mathbf{S})$ by $h_p(\mathbf{v}) = r(L\mathbf{v} + sN(\mathbf{v}) - \lambda\mathbf{v})$, where $r \colon \mathbb{R}^k \setminus \{\mathbf{0}\} \to \mathbf{S}$ denotes the radial retraction. Assume, by contradiction, that $p^- = (0, -\bar{\lambda})$ and $p^+ = (0, \bar{\lambda})$ belong to the same connected component, say W, of $\mathbb{R}^2 \setminus \mathcal{E}$. Since W is open, it is actually path connected. Thus, p^- and p^+ are the extremes of a path $\gamma \colon [0, 1] \to W$. Consequently, one gets a homotopy $t \mapsto h_{\gamma(t)}$ joining h_{p^-} with h_{p^+}. But this is impossible since, as one can easily check, h_{p^-} and h_{p^+} are, respectively, homotopic to the identity and the antipodal map, which, notoriously, belong to different homotopy classes, $\mathbf{S}$ being even dimensional.

Observe that Φ_0, the partial map of Φ at $s = 0$, coincides with the function Ψ_L defined in Section 3. Moreover, Σ_0, the slice of Σ at $s = 0$, is the same as the set $\mathcal{S}$ of the eigenpoints of L. Notice that the 0-slice $\mathcal{E}_0$ of $\mathcal{E}$ is the set of the real eigenvalues of L and

$$\Sigma_0 = \bigcup_{\lambda \in \mathcal{E}_0} \{\lambda\} \times (\mathbf{S} \cap \mathrm{Ker}(L - \lambda I)).$$

The solution triples of the type $(0, \lambda, \mathbf{v})$ are called *trivial*. Therefore, the (possibly empty) set of these distinguished elements is $\{0\} \times \Sigma_0$. Analogously, an eigenpair (s, λ) is said to be *trivial* if $s = 0$. Thus, $\{0\} \times \mathcal{E}_0$ is the set of the trivial eigenpairs.

Once one has a well-defined notion of solution of a given problem (as in the case of (4.1)) and a distinguished subset of solutions (called *trivial*), one may consider the natural notion of bifurcation point, provided that the set of solutions has a topology. Therefore, regarding problem (4.1), we give the following

Definition 4.3. A trivial solution of (4.1) is a *bifurcation point* (for problem (4.1)) if it belongs to the closure of the set of nontrivial solutions.

Identifications between trivial solutions and corresponding aliases are frequent in bifurcation theory. Thus, if $(0, \lambda_*, \mathbf{v}_*)$ is a bifurcation point in the sense of Definition 4.3, we may equivalently call "bifurcation point" its *alias* $(\lambda_*, \mathbf{v}_*)$. Actually, if it is clear that we are referring to the specific eigensphere $\mathcal{S}_{\lambda_*}$, we may simply call "bifurcation point" the vector $\mathbf{v}_*$. Obviously, this is a significative information only when $\mathcal{S}_{\lambda_*}$ has positive dimension. Regarding the case $\dim \mathcal{S}_{\lambda_*} > 0$, in [10] a necessary condition as well as some sufficient conditions for a vector $\mathbf{v}_* \in \mathcal{S}_{\lambda_*}$ to be a bifurcation point are given.

Our next result, Theorem 4.5, regards the existence of bifurcation points for problem (4.1). In its proof we will use the following point-set topology result, which is particularly suited to our purposes and is deduced from general results by C. Kuratowski (see [16], Chapter 5, Vol. 2). We also recommend [1] for a helpful paper on connectivity theory.

Lemma 4.4 ([14])**.** *Let C be a compact subset of a locally compact metric space X. Assume that every compact subset of X containing C has nonempty boundary. Then $X \setminus C$ contains a connected set whose closure in X is non-compact and intersects C.*

Theorem 4.5. *Let Ω be an open subset of $\mathbb{R} \times \mathbb{R} \times \mathbf{S}$ and let*

$$\Omega_0 = \{(\lambda, \mathbf{v}) \in \mathbb{R} \times \mathbf{S} : (0, \lambda, \mathbf{v}) \in \Omega\}$$

denote the slice of Ω at $s = 0$. Assume that the degree of Ψ_L in Ω_0, $\deg_w(\Psi_L, \Omega_0, \mathbf{0})$, is (defined and) different from zero. Then Ω has a connected set of nontrivial solutions of (4.1) whose closure in Ω is non-compact and contains at least one trivial *solution*.

Proof. It is enough to prove that the assumptions of Lemma 4.4 are satisfied for

$$X = \Phi^{-1}(\mathbf{0}) \cap \Omega \quad \text{and} \quad C = \{0\} \times X_0,$$

where X_0 is the 0-slice of X, so that C is the set of the trivial solutions of (4.1) which are contained in Ω.

Notice that X is locally compact, being open in the closed subset $\Phi^{-1}(\mathbf{0})$ of the finite dimensional manifold $\mathbb{R} \times \mathbb{R} \times \mathbf{S}$. Moreover, the subset C of X is compact, since its 0-slice C_0 coincides with X_0, which is the set $\Psi_L^{-1}(\mathbf{0}) \cap \Omega_0$, whose compactness is ensured by the assumption that $\deg_w(\Psi_L, \Omega_0, \mathbf{0})$ is well defined. Thus, it remains to show that every compact subset of X containing C has nonempty boundary in X.

Assume the contrary. Then, there exists a compact subset K of X, containing C, whose boundary in X is empty. This means that K is a relatively open subset of X. That is, there exists an open subset W of Ω such that $X \cap W = K$.

Observe that K coincides with $\Phi^{-1}(\mathbf{0}) \cap W$. Therefore, its compactness ensures that the integer $\deg_w(\Phi_s, W_s, \mathbf{0})$ is well defined for any $s \in \mathbb{R}$ (recall Notation 2.1). Moreover, because of Lemma 2.7, this integer does not depend on $s \in \mathbb{R}$.

Since K is compact, there exists $\check{s} \in \mathbb{R}$ for which the solution set $K_{\check{s}} \subset W_{\check{s}}$ is empty. Consequently, because of the Existence Property of the degree, one has $\deg_w(\Phi_{\check{s}}, W_{\check{s}}, \mathbf{0}) = 0$, which implies $\deg_w(\Phi_s, W_s, \mathbf{0}) = 0$ for all $s \in \mathbb{R}$. This contradicts the assumption $\deg_w(\Psi_L, \Omega_0, \mathbf{0}) \neq 0$, since, as we will show, one has $\deg_w(\Phi_0, W_0, \mathbf{0}) = \deg_w(\Psi_L, \Omega_0, \mathbf{0})$.

To see this, observe first that Ψ_L coincides with the partial map Φ_0 of Φ, so that $\deg_w(\Psi_L, \Omega_0, \mathbf{0}) = \deg_w(\Phi_0, \Omega_0, \mathbf{0})$. On the other hand, taking into account that the subset X_0 of Ω_0 is actually contained in W_0, we have $\deg_w(\Phi_0, \Omega_0, \mathbf{0}) = \deg_w(\Phi_0, W_0, \mathbf{0})$. Therefore, we get the contradiction $\deg_w(\Phi_0, W_0, \mathbf{0}) \neq 0$, which implies the assertion. $\qquad\square$

Recall that the slice Σ_0 of Σ coincides with the set $\mathcal{S}$ of the eigenpoints of L. Therefore, $\{0\} \times \Sigma_0$ is the set of the trivial solutions of (4.1).

Corollary 4.6. *Let U be an open subset of $\mathbb{R} \times \mathbf{S}$. Assume that $\deg_w(\Psi_L, U, \mathbf{0})$ is (defined and) different from zero. Then there exists a connected set of nontrivial solutions of (4.1) whose closure meets $\{0\} \times \Sigma_0$ in $\{0\} \times U$ and is either unbounded or intersects $\{0\} \times \Sigma_0$ outside $\{0\} \times U$.*

Proof. Define the open subset Ω of $\mathbb{R} \times \mathbb{R} \times \mathbf{S}$ by removing the elements of $\{0\} \times \Sigma_0$ which are not in $\{0\} \times U$. Then, apply Theorem 4.5. $\qquad\square$

Observe that Σ_0 is a finite union of connected components, each of them consisting of an isolated point (when the geometric multiplicity of the corresponding eigenvalue is 1) or a λ-eigenset with λ having geometric multiplicity bigger than 1. Thus, each one of these components is clopen in Σ_0. Therefore, if $\mathcal{C}$ is a connected component of Σ, then its 0-slice $\mathcal{C}_0$ is clopen in Σ_0 and, consequently, its L-degree, L-$\deg(\mathcal{C}_0)$, is well defined.

Theorem 4.7. *Let $\mathcal{C}$ be a bounded connected component of the set Σ of the solutions of* (4.1). *Then, the L-degree of its 0-slice, $L\text{-deg}(\mathcal{C}_0)$, is zero.*

Proof. If $\mathcal{C}_0$ is empty, the assertion is true. If this is not the case, assume, by contradiction, that $L\text{-deg}(\mathcal{C}_0) \neq 0$, and let U be an open subset of $\mathbb{R} \times \mathbf{S}$ which isolates $\mathcal{C}_0$ among Σ_0. Then, by definition of L-degree, we get $\deg(\Psi_L, U, \mathbf{0}) = L\text{-deg}(\mathcal{C}_0) \neq 0$. Therefore, recalling that the closure of a connected set is connected, because of Corollary 4.6 the compact component $\mathcal{C}$ contains a connected set of nontrivial solutions of (4.1) whose closure meets $\{0\} \times \Sigma_0$ outside $\{0\} \times U$. This contradicts the assumption that U contains $\mathcal{C}_0$, and the assertion follows. $\qquad\square$

In [5] we conjectured that if z_* is a trivial solution of (4.1) corresponding to a simple eigenvalue λ_* of L, then the connected component of Σ containing z_*, if bounded, includes a trivial solution z^* corresponding to an eigenvalue λ^* different from λ_*. Theorem 4.8 below, which is our main result, provides more than a positive answer to our conjecture: what counts to get the assertion is that the algebraic multiplicity of λ_* is odd, no matter what is its geometric multiplicity.

Theorem 4.8. *Let $\mathbf{v}_*$ be a unit eigenvector of L corresponding to an eigenvalue λ_* with odd algebraic multiplicity. Then, in the set Σ of the solution triples of* (4.1), *the connected component containing $(0, \lambda_*, \mathbf{v}_*)$ is either unbounded or includes a trivial solution $(0, \lambda^*, \mathbf{v}^*)$ with $\lambda^* \neq \lambda_*$.*

Proof. We may suppose that the connected component $\mathcal{C}$ of Σ containing the trivial solution $(0, \lambda_*, \mathbf{v}_*)$ is bounded.

Thus, we have to show that the slice $\mathcal{C}_0$ of $\mathcal{C}$ has at least one eigenpoint which does not belong to the eigenset $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$. Assuming, by contradiction, $\mathcal{C}_0 \subseteq \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$, we have two possibilities:

- $\mathcal{C}_0$ coincides with $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$;
- $\mathcal{C}_0$ is strictly contained in $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ (which is possible only when $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ is disconnected, and therefore made up of two twin brothers).

We claim that none of the alternatives may occur, since, because of the oddness of the algebraic multiplicity of λ_*, in both cases the L-degree of $\mathcal{C}_0$ would be nonzero, contradicting Theorem 4.7. In fact, in the first case, because of Theorem 3.6, the L-degree of $\mathcal{C}_0$ would be either 2 or -2; while, in the second alternative, $\mathcal{C}_0$ would coincide with the singleton $\{(\lambda_*, \mathbf{v}_*)\}$ and, according to Theorem 3.7, the L-degree would be 1 or -1.

The conclusion is that $\mathcal{C}_0 \nsubseteq \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$, and the assertion is established. $\qquad\square$

5. Examples

In this last section we will present some examples illustrating the assertion of Theorem 4.8. In particular, Example 5.5 will show that, in this theorem, the assumption that the algebraic multiplicity of the eigenvalue λ_* is odd cannot be removed.

For the sake of simplicity, we consider only perturbed eigenvalue problems in which the dimension of the space $\mathbb{R}^k$ is "very low" (1, 2 or 3). Some of these problems concern linear equations, and in any case it is possible to find explicitly the set of solutions. We start from the five examples in [5] - the paper in which Conjecture 1.1 was formulated - and we reinterpret all of them in the new light of

the results of this work. Furthermore, we add Example 5.6, in which the multiplicity of the eigenvalue λ_* is three.

As in the previous sections, Σ and $\mathcal{E}$ will denote, respectively, the set of solutions and the set of eigenpairs of the given problem.

The first example concerns a linear problem in $\mathbb{R}^2$. Here the operator L has two simple real eigenvalues, and the trivial solutions are four: two for each eigenvalue. The set of solution triples Σ is a smooth curve, diffeomorphic to a circle, which contains all the trivial solutions, and the projection of Σ onto $\mathcal{E}$ is a double covering map.

Example 5.1. In $\mathbb{R}^2$, consider the problem

(5.1)
$$\begin{cases} x_1 - sx_2 &= \lambda x_1, \\ -x_2 + sx_1 &= \lambda x_2, \\ x_1^2 + x_2^2 &= 1. \end{cases}$$

Here both L and N are linear: respectively,

$$L\colon (x_1, x_2) \mapsto (x_1, -x_2) \text{ and } N\colon (x_1, x_2) \mapsto (-x_2, x_1).$$

The operator L has two simple eigenvalues, $\lambda_* = -1$ and $\lambda^* = 1$, with two corresponding pairs of antipodal unit eigenvectors:

$$\pm\mathbf{v}_* = \pm(0,1) \quad \text{and} \quad \pm\mathbf{v}^* = \pm(1,0).$$

A simple computation (compare also with Remark 4.1) shows that the set $\mathcal{E}$ of the eigenpairs of (5.1) is the unit circle $s^2 + \lambda^2 = 1$ in the $s\lambda$-plane. So, $\mathcal{E}$ can be parametrized as $(\sin t, \cos t)$, with $t \in [0, 2\pi]$.

Now, to find the set Σ of the solution triples, observe that for any fixed $t \in [0, 2\pi]$ the kernel of the linear operator

$$L + (\sin t)N - (\cos t)I$$

is a straight line in the (x_1, x_2)-plane which meets the unit sphere $\mathbf{S}$ in the pair of antipodal unit eigenvectors

$$\pm(\cos(t/2), \sin(t/2)).$$

Therefore, Σ is a bounded subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ which can be expressed parametrically as

$$\gamma\colon [0, 4\pi] \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2, \quad \gamma(t) = \Big(\sin t, \cos t, \big(\cos(t/2), \sin(t/2) \big) \Big),$$

that is, Σ is a simple, regular, closed curve which meets all the four trivial solutions of (5.1) for $t = 0, \pi, 2\pi, 3\pi$. Incidentally, we observe that the projection of Σ onto $\mathcal{E}$ is a double covering map, and the above parametrization γ of Σ is the lifting of the curve

$$\sigma\colon [0, 4\pi] \to \mathcal{E}, \quad \sigma(t) = (\sin t, \cos t),$$

with the initial condition $\gamma(0) = \big(0, 1, (1,0)\big)$.

Finally, observe that the L-degree of the slice Σ_0 of Σ is zero. In fact, Σ_0 is made up of four isolated eigenpoints and, as shown by Lemma 3.2, two of them have L-degree 1 and the others have L-degree -1. This agrees with Theorem 4.7. Moreover, the bounded and connected set Σ satisfies the assertion of Theorem 4.8.

The next example concerns a linear problem in $\mathbb{R}^2$ which is very similar to the previous one; in fact, the unperturbed problem is the same, while the perturbing operator N differs from the one in Example 5.1 only for a sign in the first component. In spite of this, the set of solutions is drastically different: it is composed of four unbounded components, each of them containing only one trivial solution.

Example 5.2. In $\mathbb{R}^2$, consider the problem

(5.2)
$$\begin{cases} x_1 + sx_2 = \lambda x_1, \\ -x_2 + sx_1 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

As in the previous example, both L and N are linear: respectively,

$$L\colon (x_1, x_2) \mapsto (x_1, -x_2) \text{ and } N\colon (x_1, x_2) \mapsto (x_2, x_1).$$

The operator L is the same as in Example 5.1. So, in particular, the four trivial solutions of (5.2) coincide with those of (5.1); namely,

$$\big(0, -1, (0, \pm 1)\big) \quad \text{and} \quad \big(0, 1, (\pm 1, 0)\big).$$

Here, on the other hand, the set $\mathcal{E}$ of the eigenpairs of (5.2) is unbounded: it is the hyperbola $\lambda^2 - s^2 = 1$ in the $s\lambda$-plane. The two branches of $\mathcal{E}$, the lower and the upper ones, can be represented parametrically as

$$(s_*(t), \lambda_*(t)) = (\sinh t, -\cosh t) \quad \text{and} \quad (s^*(t), \lambda^*(t)) = (\sinh t, \cosh t),$$

with $t \in \mathbb{R}$.

To find Σ, let us first consider the lower branch of $\mathcal{E}$. For any fixed $t \in \mathbb{R}$, the kernel of the linear operator

$$L + s_*(t)N - \lambda_*(t)I$$

is the straight line containing the pair of opposite (not necessarily unit) vectors

$$\pm \mathbf{w}_*(t) = \pm(-\sinh t, 1 + \cosh t)$$

and, similarly, concerning the upper branch, the kernel of

$$L + s^*(t)N - \lambda^*(t)I$$

is the straight line containing

$$\pm \mathbf{w}^*(t) = \pm(1 + \cosh t, \sinh t).$$

Thus, for example, a parametrization of the component of Σ containing the trivial solution $\big(0, 1, (1, 0)\big)$ is

$$t \in \mathbb{R} \mapsto \big(\sinh t, \cosh t, \mathbf{v}^*(t)\big),$$

where $\mathbf{v}^*(t) = \mathbf{w}^*(t)/\|\mathbf{w}^*(t)\|$. In this way we find that Σ is made up of four components: the other three can be parametrized in an analogous way.

Obviously, each of these components satisfies the assertion of Theorem 4.8.

In conclusion, the set Σ of the solution triples has four unbounded connected components, each of them diffeomorphic to $\mathbb{R}$ and containing one and only one trivial solution, being 1 or -1 the L-degree of the corresponding eigenpoint. This shows that in Theorem 4.7 the assumption that the component $\mathcal{C}$ is bounded cannot be removed.

In the following example the space is again $\mathbb{R}^2$ and N is nonlinear (actually, is constant). Here, as in Examples 5.1 and 5.2, L has two different real eigenvalues, so that the trivial solutions are four. The set Σ is the union of a topological circle $\mathcal{C}$ and two straight lines. In spite of the fact that $\mathcal{C}$ contains only two trivial solutions with the *same* eigenvalue, Σ connects all the four trivial solutions, compatibly with Theorem 4.8.

Example 5.3. In $\mathbb{R}^2$, consider the problem

(5.3)
$$\begin{cases} x_1 + s & = \lambda x_1, \\ 2x_2 & = \lambda x_2, \\ x_1^2 + x_2^2 & = 1. \end{cases}$$

Here we have

$$L\colon (x_1, x_2) \mapsto (x_1, 2x_2) \text{ and } N\colon (x_1, x_2) \mapsto (1, 0).$$

The operator L has two simple eigenvalues, $\lambda_* = 1$ and $\lambda^* = 2$, with corresponding unit eigenvectors, $\pm\mathbf{v}_* = \pm(1, 0)$ and $\pm\mathbf{v}^* = \pm(0, 1)$. So, the four trivial solutions of (5.3) are

$$\big(0, 1, (\pm 1, 0)\big) \quad \text{and} \quad \big(0, 2, (0, \pm 1)\big).$$

Regarding Σ, notice that any solution $\big(s, \lambda, (x_1, x_2)\big)$ of (5.3) must verify either $x_2 = 0$ (and, consequently, $x_1 = \pm 1$) or $\lambda = 2$ (and, therefore, $x_1 = s$ with $|s| \leq 1$). In the first case, with $x_2 = 0$, we get two straight lines in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$:

$$\ell_1 = \big\{ \big(s, 1 - s, (-1, 0)\big) : s \in \mathbb{R} \big\} \quad \text{and} \quad \ell_2 = \big\{ \big(s, 1 + s, (1, 0)\big) : s \in \mathbb{R} \big\},$$

lying in the two different planes of equations $(x_1, x_2) = (-1, 0)$ and $(x_1, x_2) = (1, 0)$, respectively. In the second case, we observe that the set of solutions having $\lambda = 2$ can be represented as follows:

$$\mathcal{C} = \big\{ \big(\sin t, 2, (\sin t, \cos t) \big) : t \in [0, 2\pi] \big\}.$$

The set $\mathcal{C}$ is diffeomorphic to a circle and contains two - and only two - of the four trivial solutions, but both with the same eigenvalue $\lambda^* = 2$. If $\mathcal{C}$ was a connected component of Σ, this would be in contrast with Theorem 4.8. However, Σ, which is the union of ℓ_1, ℓ_2 and $\mathcal{C}$, is connected, since

$$\ell_1 \cap \mathcal{C} = \big\{ \big(-1, 2, (-1, 0) \big) \big\} \quad \text{and} \quad \ell_2 \cap \mathcal{C} = \big\{ \big(1, 2, (1, 0) \big) \big\}.$$

Therefore, the connected component of Σ containing any of the four trivial solutions is Σ itself, which satisfies the assertion of Theorem 4.8.

The simplest example that one can conceive is when the space is one-dimensional. In this case the unit sphere $\mathbf{S}$ is $\{-1, 1\}$ and, whatever is N, the set Σ consists of two unbounded connected components.

Example 5.4. Let $\lambda_* \in \mathbb{R}$ be given and, in $\mathbb{R}$, consider the problem

(5.4)
$$\begin{cases} \lambda_* x + sN(x) & = \lambda x, \\ x & = \pm 1, \end{cases}$$

where $N\colon \{-1, 1\} \to \mathbb{R}$ is arbitrary. Given any $s \in \mathbb{R}$, one has two solutions:

$$(s, \lambda_* + sN(1), 1) \quad \text{and} \quad (s, \lambda_* - sN(-1), -1).$$

Thus Σ is composed of two straight lines:

$$\big\{ \big(s, \lambda_* + sN(1), 1\big) : s \in \mathbb{R} \big\} \quad \text{and} \quad \big\{ \big(s, \lambda_* - sN(-1), -1\big) : s \in \mathbb{R} \big\},$$

contained in two different planes of $\mathbb{R}^3$. This is compatible with Theorem 4.8.

The following example in $\mathbb{R}^3$ shows that, in Theorem 4.8, the assumption that the algebraic multiplicity of the eigenvalue λ_* is odd cannot be omitted.

Example 5.5. In $\mathbb{R}^3$, consider the problem

(5.5)
$$\begin{cases} sx_2 &= \lambda x_1, \\ 2x_1 - sx_1 &= \lambda x_2, \\ 2x_3 + sx_1 &= \lambda x_3, \\ x_1^2 + x_2^2 + x_3^2 &= 1. \end{cases}$$

Here both L and N are linear: respectively,

$$L\colon (x_1, x_2, x_3) \mapsto (0, 2x_1, 2x_3) \quad \text{and} \quad N\colon (x_1, x_2, x_3) \mapsto (x_2, -x_1, x_1).$$

The operator L has two real eigenvalues: $\lambda_* = 0$, with geometric multiplicity 1 and algebraic multiplicity 2; and $\lambda^* = 2$, which is simple. So, (5.5) has four trivial solutions given by

$$\big(0, 0, (0, \pm 1, 0)\big) \quad \text{and} \quad \big(0, 2, (0, 0, \pm 1)\big).$$

It is not difficult to show (compare with Remark 4.1) that the set $\mathcal{E}$ of the eigenpairs has two connected components: the circle $(s - 1)^2 + \lambda^2 = 1$ and the straight line $\lambda = 2$. In particular, notice that the disconnected set $\mathcal{E}$ is unbounded, as it should be, according to Remark 4.2. Consequently, Σ is as well unbounded and disconnected.

Now, let us investigate the solution set Σ. To this purpose, observe that the bounded connected component of $\mathcal{E}$ can be parametrized as

$$(s(t), \lambda(t)) = (1 - \cos t, \sin t), \quad \text{with} \quad t \in [0, 2\pi].$$

For any fixed $t \in [0, 2\pi]$, the kernel of the linear operator

$$L + s(t)N - \lambda(t)I$$

is spanned by the (nonzero) vector

$$\mathbf{w}(t) = \big(\sin(t/2), \cos(t/2), c(t) \big),$$

where $c(t)$ is defined by $2c(t) + s(t)\sin(t/2) = \lambda(t)c(t)$, in order to satisfy the third equation of (5.5). This implies that the connected component $\mathcal{C}$ of Σ containing the trivial solution $z_* = \big(0, 0, (0, 1, 0)\big)$ can be parametrized as follows:

$$\sigma\colon [0, 4\pi] \to \mathbb{R} \times \mathbb{R} \times S^2, \quad \sigma(t) = \big(1 - \cos t, \sin t, \mathbf{v}(t)\big),$$

where $\mathbf{v}(t) = \mathbf{w}(t)/\|\mathbf{w}(t)\|$. That is, $\mathcal{C}$ is diffeomorphic to S^1 - in particular, $\mathcal{C}$ is bounded - and contains both the trivial solutions corresponding to the eigenvalue $\lambda_* = 0$: $\big(0, 0, (0, 1, 0)\big)$ for $t = 0$ (or, equivalently, for $t = 4\pi$) and $\big(0, 0, (0, -1, 0)\big)$ for $t = 2\pi$. Notice that the other two trivial solutions, the ones corresponding to $\lambda^* = 2$, do not belong to $\mathcal{C}$. Therefore, the slice $\mathcal{C}_0$ at $s = 0$ is the λ_*-eigenset of L.

Since the algebraic multiplicity of λ_* is even, the sign-jump at λ_* of the characteristic polynomial of L is zero. Consequently, due to Theorem 3.6, the L-degree of the λ_*-eigenset is zero, which shows that the bounded component $\mathcal{C}$ satisfies the statement of Theorem 4.7.

However, $\mathcal{C}$ does *not* satisfy the assertion of Theorem 4.8. Hence, in this result, the assumption that the algebraic multiplicity of λ_* is *odd* cannot be removed.

The other two components of Σ, the ones containing the trivial solutions corresponding to the simple eigenvalue $\lambda^* = 2$, do satisfy the assertion of Theorem 4.8. In fact, they are the straight lines

$$\big\{(s, 2, (0,0,1)) : s \in \mathbb{R}\big\} \quad \text{and} \quad \big\{(s, 2, (0,0,-1)) : s \in \mathbb{R}\big\}.$$

Finally, we observe that the projection of $\mathcal{C}$ onto the circle $(s-1)^2 + \lambda^2 = 1$ is a double covering map, and the above parametrization σ of $\mathcal{C}$ is the lifting of the curve $t \in [0, 4\pi] \mapsto (1 - \cos t, \sin t)$ with initial condition $\sigma(0) = z_* = \big(0, 0, (0, 1, 0)\big)$.

We close with an example in $\mathbb{R}^3$ in which the linear operator L has an eigenvalue λ_* with algebraic multiplicity three and geometric multiplicity two. The λ_*-eigensphere is one-dimensional and contains two bifurcation points.

Example 5.6. In $\mathbb{R}^3$, consider the problem

(5.6)
$$\begin{cases} x_1 + x_3 + sx_1 &= \lambda x_1, \\ x_2 - sx_3 &= \lambda x_2, \\ x_3 + sx_2 &= \lambda x_3, \\ x_1^2 + x_2^2 + x_3^2 &= 1. \end{cases}$$

Here one has

$$L\colon (x_1, x_2, x_3) \mapsto (x_1 + x_3, x_2, x_3) \quad \text{and} \quad N\colon (x_1, x_2, x_3) \mapsto (x_1, -x_3, x_2).$$

The operator L has only one eigenvalue: $\lambda_* = 1$, with algebraic multiplicity 3 and geometric multiplicity 2. Applying Remark 4.1 we get

$$\mathcal{E} = \big\{(s, \lambda) \in \mathbb{R}^2 : (1 - \lambda + s)\big((1-\lambda)^2 + s^2\big) = 0\big\}.$$

Therefore, $\mathcal{E}$ is the straight line in the $s\lambda$-plane of equation $\lambda = 1 + s$. This line includes the unique trivial eigenpair $(0, \lambda_*) = (0, 1)$ corresponding to the eigensphere

$$\mathcal{S}_{\lambda_*} = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, \ x_3 = 0\big\}.$$

One can check that Σ contains the straight lines

$$\Sigma_- = \big\{\big(s, 1 + s, (-1, 0, 0)\big) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 : s \in \mathbb{R}\big\}$$

and

$$\Sigma_+ = \big\{\big(s, 1 + s, (1, 0, 0)\big) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 : s \in \mathbb{R}\big\}.$$

Consequently, $\mathcal{S}_{\lambda_*}$, which is the unit circle in the plane $x_3 = 0$, contains two (aliases of) bifurcation points: $(-1, 0, 0)$ and $(1, 0, 0)$.

Notice that Σ is the union of three connected sets: Σ_-, Σ_+, and the circle

$$\{0\} \times \{1\} \times \mathcal{S}_{\lambda_*} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3,$$

which intersects both Σ_- and Σ_+. Thus, Σ is connected and, being unbounded, the assertion of Theorem 4.8 is satisfied.

References

[1] Alexander J.C., *A primer on connectivity*, Lecture Notes in Math., Vol. 886, Springer Verlag, Berlin-New York, 1981, 455–483.

[2] Appell J. - De Pascale E. - Vignoli A., *Nonlinear spectral theory*, de Gruyter, Berlin, 2004.

[3] Benevieri P. - Calamai A. - Furi M. - Pera M.P., *On the persistence of the eigenvalues of a perturbed Fredholm operator of index zero under nonsmooth perturbations*, Z. Anal. Anwend. **36** (2017), no. 1, 99–128.

[4] Benevieri P. - Calamai A. - Furi M. - Pera M.P., *Global continuation of the eigenvalues of a perturbed linear operator*, Ann. Mat. Pura Appl. **197** (2018), no. 4, 1131–1149.

[5] Benevieri P. - Calamai A. - Furi M. - Pera M.P., *Global continuation in Euclidean spaces of the perturbed unit eigenvectors corresponding to a simple eigenvalue*, Topol. Methods in Nonlinear Anal., **55** (2020), no. 1, 169–184.

[6] Benevieri P. - Calamai A. - Furi M. - Pera M.P., *Global persistence of the unit eigenvectors of perturbed eigenvalue problems in Hilbert spaces*, Z. Anal. Anwend., **39** (2020), no. 4, 475–497.

[7] Chiappinelli R., *Isolated Connected Eigenvalues in Nonlinear Spectral Theory*, Nonlinear Funct. Anal. Appl. **8** (2003), no. 4, 557–579.

[8] Chiappinelli R., *Approximation and convergence rate of nonlinear eigenvalues: Lipschitz perturbations of a bounded self-adjoint operator*, J. Math. Anal. Appl. **455** (2017), no. 2, 1720–1732.

[9] Chiappinelli R., *What do you mean by "nonlinear eigenvalue problems"?*, Axioms **7** (2018), Paper no. 39, 30 pp.

[10] Chiappinelli R. - Furi M. - Pera M.P., *Normalized Eigenvectors of a Perturbed Linear Operator via General Bifurcation*, Glasg. Math. J. **50** (2008), no. 2, 303–318.

[11] Chiappinelli R. - Furi M. - Pera M.P., *Topological persistence of the normalized eigenvectors of a perturbed self-adjoint operator*, Appl. Math. Lett. **23** (2010), no. 2, 193–197.

[12] Chiappinelli R. - Furi M. - Pera M.P., *Persistence of the normalized eigenvectors of a perturbed operator in the variational case*, Glasg. Math. J. **55** (2013), no. 3, 629–638.

[13] Chiappinelli R. - Furi M. - Pera M.P., *Topological Persistence of the Unit Eigenvectors of a Perturbed Fredholm Operator of Index Zero*, Z. Anal. Anwend. **33** (2014), no. 3, 347–367.

[14] Furi M. - Pera M.P., *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory*, Pacific J. Math. **160** (1993), no. 3, 219–244.

[15] Hirsch M.W., *Differential Topology*, Graduate Texts in Math., Vol. 33, Springer Verlag, Berlin, 1976.

[16] Kuratowski C., *Topology*, Vol. 2, Academic Press, New York, 1968.

[17] Milnor J.W., *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.

[18] Nirenberg L., *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New York, 1974.

[19] Outerelo E. - Ruiz J.M., *Mapping degree theory*, Graduate Studies in Mathematics, **108**, American Mathematical Soc., Providence, RI; Real Soc. Matemática Española, Madrid, 2009.

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