

GLOBAL PERSISTENCE OF THE UNIT EIGENVECTORS OF PERTURBED EIGENVALUE PROBLEMS IN HILBERT SPACES

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ABSTRACT. We consider the nonlinear eigenvalue problem $Lx + \varepsilon N(x) = \lambda Cx$, $\|x\| = 1$, where ε, λ are real parameters, $L, C: G \rightarrow H$ are bounded linear operators between separable real Hilbert spaces, and $N: S \rightarrow H$ is a continuous map defined on the unit sphere of G . We prove a global persistence result regarding the set Σ of the *solutions* $(x, \varepsilon, \lambda) \in S \times \mathbb{R} \times \mathbb{R}$ of this problem. Namely, if the operators N and C are compact, under suitable assumptions on a solution $p_* = (x_*, 0, \lambda_*)$ of the unperturbed problem, we prove that the connected component of Σ containing p_* is either unbounded or meets a triple $p^* = (x^*, 0, \lambda^*)$ with $p^* \neq p_*$. When C is the identity and $G = H$ is finite dimensional, the assumptions on $(x_*, 0, \lambda_*)$ mean that x_* is an eigenvector of L whose corresponding eigenvalue λ_* is simple. Therefore, we extend a previous result obtained by the authors in the finite dimensional setting.

Our work is inspired by a paper of R. Chiappinelli concerning the local persistence property of the unit eigenvectors of perturbed self-adjoint operators in a real Hilbert space.

Dedicated to the memory of our dear friend and exceptional mathematician Alfonso Vignoli

1. INTRODUCTION

In this paper we study a *nonlinear eigenvalue problem* of the type

$$(1.1) \quad \begin{cases} Lx + \varepsilon N(x) = \lambda Cx, \\ x \in S, \end{cases}$$

where $\varepsilon, \lambda \in \mathbb{R}$, $L, C: G \rightarrow H$ are bounded linear operators between real Hilbert spaces, and $N: S \rightarrow H$ is a continuous map defined on the unit sphere of G .

Problem (1.1) can be thought as a nonlinear perturbation of the eigenvalue problem $Lx = \lambda Cx$, where, as usual, by abuse of terminology, $\lambda_* \in \mathbb{R}$ is said to be an *eigenvalue of (the equation) $Lx = \lambda Cx$* if the operator $L - \lambda_* C$ is not injective.

By a *solution* of (1.1) we mean a *triple* $(x, \varepsilon, \lambda)$ which satisfies the system, and we call (ε, λ) the *eigenpair* corresponding to the *(unit) eigenvector* x . The solutions and the eigenpairs with $\varepsilon = 0$ are said to be *trivial*. The set of all the solutions is denoted by Σ , while Σ_0 stands for its subset of the trivial ones. Obviously, Σ and Σ_0 are closed in $S \times \mathbb{R} \times \mathbb{R}$.

Date: October 3, 2021.

2010 Mathematics Subject Classification. 47J10, 47A75.

Key words and phrases. Fredholm operators, nonlinear spectral theory, eigenvalues, eigenvectors.

The first, second and fourth authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

A. Calamai is partially supported by GNAMPA - INdAM (Italy).

Recently, in [3], under the assumptions that $G = H = \mathbb{R}^n$ and that C is the identity I , we obtained a sort of “global persistence” of the solution triples of the above problem (see Corollary 3.13 below). That is, we proved that, if $\lambda_* \in \mathbb{R}$ is a simple eigenvalue of L (in the usual sense) and x_* is any one of the two corresponding unit eigenvectors, then one gets the following

Assertion 1.1. *The set $\Sigma \setminus \Sigma_0$ of the nontrivial solutions of (1.1) has a connected subset whose closure contains $p_* = (x_*, 0, \lambda_*)$ and is either unbounded or meets a trivial solution $p^* = (x^*, 0, \lambda^*)$ different from p_* .*

Observe that this assertion does not imply that λ^* is different from λ_* . However, if $\lambda^* = \lambda_*$, one necessarily has $p^* = (-x_*, 0, \lambda_*)$.

Taking into account that the closure of a connected set is connected, from Assertion 1.1 one gets that *the component of Σ containing p_* is either unbounded or meets a trivial solution $p^* \neq p_*$.*

We point out that, given any trivial solution $p_* = (x_*, 0, \lambda_*)$ of (1.1), this last statement is meaningful when (and only when) the kernel of $L - \lambda_* C$ is one dimensional, due to the fact that (only) in this case the sphere of the unit eigenvectors corresponding to λ_* is disconnected.

In this paper we extend the global persistence result obtained in [3] to the infinite dimensional setting (see Theorem 3.10 below). Namely, given a trivial solution $p_* = (x_*, 0, \lambda_*)$ of (1.1), we get Assertion 1.1 under the following assumptions:

- $\text{Ker}(L - \lambda_* C) = \mathbb{R}x_*$,
- $Cx_* \neq 0$,
- $\text{Img}(L - \lambda_* C) \oplus C(\text{Ker}(L - \lambda_* C)) = H$,
- G and H are separable,
- N and C are compact.

For the sake of simplicity, when a trivial solution $(x_*, 0, \lambda_*)$ of (1.1) satisfies the first three of the above assumptions, we say that it is a *simple solution*. This terminology is justified by the following Remark, whose easy proof is left to the reader.

Remark 1.2. Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator. Then $\lambda_* \in \mathbb{R}$ is a simple eigenvalue of L and x_* a corresponding unit eigenvector if and only if the triple $(x_*, 0, \lambda_*)$ is a simple solution of (1.1) in which $G = H = \mathbb{R}^n$ and $C = I$.

Despite the apparent similarity to the case in \mathbb{R}^n , the study of the nonlinear eigenvalue problem (1.1) in the infinite dimensional setting presents several difficulties. First of all, to adapt the topological tools used in [3] to the new context, we need the sphere S to be a differentiable manifold, and this forces us to work in Hilbert spaces, since this property of S is generally false in Banach spaces. In addition, our strategy requires that the nonlinear map N could be approximated by smooth maps, as can be done in *separable* Hilbert spaces. This is the main reason imposing us the separability assumption on G and H . Moreover, thanks to the separability, we can apply the celebrated Sard–Smale result [18] about the density of regular values of nonlinear Fredholm maps between separable Hilbert manifolds. Finally, some results require the linear operator $L - \lambda C$ to be Fredholm of index zero for any $\lambda \in \mathbb{R}$, and this, based on the first three of the above assumptions, is ensured by the compactness of C .

Coming back to Assertion 1.1, we do not know whether or not our assumptions imply that the eigenvalues λ_* and λ^* are different. Nevertheless, in [2], we tackled the problem of the global persistence of the eigenvalues (more precisely, of the eigenpairs) of (1.1) and, in particular, by means of the Leray–Schauder degree theory we obtained the following

Theorem 1.3 (Global continuation of eigenpairs). *Regarding problem (1.1), assume that the operator L is Fredholm of index zero, that C and N are compact, and that, for some $\lambda_* \in \mathbb{R}$, the kernel of $L - \lambda_* C$ is odd dimensional and the condition*

$$\text{Im}(L - \lambda_* C) + C(\text{Ker}(L - \lambda_* C)) = H$$

is satisfied.

Then, in the set of all the eigenpairs (ε, λ) of (1.1), the connected component containing $(0, \lambda_)$ is either unbounded or meets a trivial eigenpair $(0, \lambda^*)$ with $\lambda^* \neq \lambda_*$.*

We wish to point out that extending the above global continuation property, from eigenpairs (ε, λ) , which lie in $\mathbb{R} \times \mathbb{R}$, to solution triples $(x, \varepsilon, \lambda)$, which belong to $S \times \mathbb{R} \times \mathbb{R} \subseteq G \times \mathbb{R} \times \mathbb{R}$, is a nontrivial issue and requires a different approach from the one adopted in [2].

Because of Theorem 1.3, we are inclined to believe that our main result (Theorem 3.10) could be sharpened according to the following conjecture that until now we have not been able to prove or deny:

Conjecture 1.4. *Let $(x_*, 0, \lambda_*)$ be a simple solution of problem (1.1). Suppose that G and H are separable, and that N and C are compact.*

Then, the set of the nontrivial solutions of (1.1) has a connected subset whose closure contains $(x_, 0, \lambda_*)$ and is either unbounded or meets a trivial solution $(x^*, 0, \lambda^*)$ with $\lambda^* \neq \lambda_*$.*

Our investigation is mainly inspired by a paper of R. Chiappinelli [4], who obtained a “local persistence” result for problem (1.1). Namely, assuming that

- L is a self-adjoint operator defined on G ,
- $C = I$ is the identity of G ,
- $N: S \rightarrow G$ is Lipschitz continuous,
- $\lambda_* \in \mathbb{R}$ is an isolated simple eigenvalue of L ,
- x_* is any one of the two unit eigenvectors corresponding to λ_* ,

he proved that, defined in a neighborhood V of $0 \in \mathbb{R}$, there exists a G -valued Lipschitz curve $\varepsilon \mapsto x_\varepsilon$, as well as a real Lipschitz function $\varepsilon \mapsto \lambda_\varepsilon$, such that

$$Lx_\varepsilon + \varepsilon N(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon, \quad \|x_\varepsilon\| = 1,$$

for any $\varepsilon \in V$. Moreover, when $\varepsilon = 0$ one has $x_0 = x_*$ and $\lambda_0 = \lambda_*$.

The hypotheses of our Theorem 3.10 seem incompatible with the assumptions of Chiappinelli’s local persistence result, since, what in (1.1) is the compact operator C , in Chiappinelli’s case is the identity I , which is not compact when the space is infinite dimensional. Nevertheless, under some natural conditions on the operator L , our result applies. This is the case, for example, when L is compact or, more generally, when it is of the type $\lambda_* I + C$, with $\lambda_* \in \mathbb{R}$ and C compact. To see this, put $\varepsilon = -\sigma/\mu$ and $\lambda = \lambda_* + 1/\mu$, and observe that the equation $Lx + \varepsilon N(x) = \lambda x$ becomes $x + \sigma N(x) = \mu Cx$, which is as in (1.1) with the identity in place of L .

Further results regarding the local persistence of eigenvalues, as well as unit eigenvectors, have been obtained in [1, 5, 7, 8, 9, 10] in the case in which the eigenvalue λ_* is not necessarily simple. For a general review on nonlinear eigenvalue problems and applications to differential equations, see e.g. [6] and references therein.

The proof of Theorem 3.10 does not need any advanced tool (such as Leray-Schauder degree theory) and mainly requires basic concepts in Differential Topology and Functional Analysis that can be found in textbooks such as [15, 16, 17].

A crucial result in our investigation is Lemma 3.2, which concerns the unperturbed eigenvalue problem: it states that *if $(x_*, 0, \lambda_*)$ is a simple solution of (1.1), then the map $\Psi: S \times \mathbb{R} \rightarrow H$, given by $(x, \lambda) \mapsto Lx - \lambda Cx$, establishes a diffeomorphism from a neighborhood of (x_*, λ_*) in $S \times \mathbb{R}$ onto a neighborhood of $0 \in H$.*

We close the paper with some illustrating examples showing situations in which Theorem 3.10 applies. We also show that, in our main result, the hypothesis that the “starting” trivial solution is simple cannot be removed.

2. NOTATION AND PRELIMINARIES

In this section, in addition to introducing our notation, we will recall some topological and algebraic concepts that will be needed in the following (for general reference see e.g. [15, 16, 17]).

Throughout the paper, G , H and $G \times \mathbb{R} \times \mathbb{R}$ indicate real Hilbert spaces. In each one of these spaces, the inner product is denoted by $\langle \cdot, \cdot \rangle$, or by a similar symbol, such as $\langle \cdot, \cdot \rangle'$, only when a possible misunderstanding regarding the hosting space may occur. For example, the inner product of two elements $p_1 = (x_1, \varepsilon_1, \lambda_1)$ and $p_2 = (x_2, \varepsilon_2, \lambda_2)$ of $G \times \mathbb{R} \times \mathbb{R}$ is defined in the most natural way as follows:

$$\langle p_1, p_2 \rangle = \langle x_1, x_2 \rangle' + \varepsilon_1 \varepsilon_2 + \lambda_1 \lambda_2,$$

where, here, $\langle x_1, x_2 \rangle'$ is the inner product of $x_1, x_2 \in G$.

The norm in any Hilbert space will be tacitly assumed to be the standard one associated with the inner product; namely, $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

By a manifold we shall always mean a smooth (i.e. C^∞) boundaryless submanifold of a real Hilbert space. Given a manifold \mathcal{M} in a Hilbert space, say G , and given a point $p \in \mathcal{M}$, the tangent space $T_p(\mathcal{M})$ of \mathcal{M} at p will be always identified with a vector subspace of G ; so that any $v \in T_p(\mathcal{M})$ is the derivative $\gamma'(0)$ of a smooth path $\gamma: (-1, 1) \rightarrow G$ whose image is in \mathcal{M} and such that $\gamma(0) = p$.

Obviously, if \mathcal{M} and \mathcal{N} are two manifolds such that $\mathcal{M} \subseteq \mathcal{N}$ and $p \in \mathcal{M}$, then $T_p(\mathcal{M})$ is a subspace of $T_p(\mathcal{N})$.

By a smooth (i.e. C^∞) map $f: \mathcal{M} \rightarrow \mathcal{N}$ between two manifolds $\mathcal{M} \subseteq G$ and $\mathcal{N} \subseteq H$ we mean the restriction (to \mathcal{M} as domain and to \mathcal{N} as codomain) of a smooth map $\hat{f}: U \rightarrow H$ defined on an open neighborhood U of \mathcal{M} in G .

Throughout this section, \mathcal{M} and \mathcal{N} denote two manifolds embedded in G and H respectively, and f is a smooth map from \mathcal{M} into \mathcal{N} .

Given $p \in \mathcal{M}$, the differential $df_p: T_p(\mathcal{M}) \rightarrow T_{f(p)}(\mathcal{N})$ is the restriction of the Fréchet differential $d\hat{f}_p: G \rightarrow H$ of any smooth extension $\hat{f}: U \rightarrow H$ of f to an open neighborhood U of \mathcal{M} . To check that df_p is well defined, think about the tangent vectors as derivatives of smooth curves.

If f is bijective and its inverse $f^{-1}: \mathcal{N} \rightarrow \mathcal{M}$ is smooth, then f is said to be a *diffeomorphism (of \mathcal{M} onto \mathcal{N})*. One also says that *f maps \mathcal{M} diffeomorphically*

onto \mathcal{N} . It is known that f is a diffeomorphism if and only if it is a homeomorphism with df_p invertible for any $p \in \mathcal{M}$. Notice that the function $t \mapsto t^3$ from \mathbb{R} onto itself is smooth, is a homeomorphism, but not a diffeomorphism.

Here we will distinguish between differential and derivative. The first one is always a linear map; the second one, when it makes sense, is a “representative” of the differential. For example, if $f: \mathbb{R}^k \rightarrow \mathbb{R}^s$ and $p \in \mathbb{R}^k$, the derivative of f at p , $f'(p)$, is the matrix representing the differential df_p (the Jacobian matrix, in this case); if f is a curve into G and p is in the domain of f , then $f'(p)$ is a vector of G , namely $f'(p) = df_p(1)$.

The following well-known result regarding 1-dimensional manifolds will be crucial in the next section.

Theorem 2.1 (On the classification of 1-dimensional manifolds). *Any connected (boundaryless) 1-dimensional manifold is either diffeomorphic to an open real interval or to a circle.*

The following result concerning diffeomorphisms, known as the “Local Inverse Function Theorem”, will be useful in the next section.

Theorem 2.2 (On the local diffeomorphism). *Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map between two manifolds and let $p \in \mathcal{M}$.*

Then f maps diffeomorphically an open neighborhood of p in \mathcal{M} onto an open neighborhood of $f(p)$ in \mathcal{N} if and only if the differential $df_p: T_p(\mathcal{M}) \rightarrow T_{f(p)}(\mathcal{N})$ is invertible.

Let, as above, $f: \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map between two manifolds. Recall that $p \in \mathcal{M}$ is said to be a *regular point* (of f) if the differential df_p is surjective; otherwise p is called a *critical point*. An element $q \in \mathcal{N}$ is a *critical value* (of f) if $f^{-1}(q)$ does not contain critical points; otherwise q is said to be a *regular value*. Obviously, if q is not in the image of f , then it is a regular value. Notice that “points” are in the domain and “values” in the codomain.

An useful consequence of the Implicit Function Theorem is the following

Theorem 2.3 (On regularly defined finite codimensional manifolds). *Let f be a smooth map from \mathcal{M} into \mathcal{N} . Assume that \mathcal{N} is finite dimensional and that $q \in \mathcal{N}$ is a regular value for f .*

Then $f^{-1}(q)$, if nonempty, is a manifold whose codimension in \mathcal{M} is the same as the dimension of \mathcal{N} . Moreover, for any $p \in f^{-1}(q)$, one has $T_p(f^{-1}(q)) = \text{Ker } df_p$.

For example, the unit sphere S of G is a 1-codimensional submanifold of G . Moreover, given $x_* \in S$, one has $T_{x_*}(S) = (\mathbb{R}x_*)^\perp = \{\dot{x} \in G : \langle x_*, \dot{x} \rangle = 0\}$. To see this, define $f: G \rightarrow \mathbb{R}$ by $f(x) = \|x\|^2$, consider the regular value $q = 1 \in \mathbb{R}$, and apply Theorem 2.3.

Let X and Y denote metric spaces, and let $g: X \rightarrow Y$ be a continuous map. One says that g is *compact* if it sends bounded subsets of X into relatively compact subsets of Y , and that it is *locally compact* if, given any $x \in X$, the restriction of g to a convenient neighborhood of x is a compact map.

The map g is said to be *proper* if $g^{-1}(K)$ is compact for any compact subset K of Y . As one can easily check, if g is proper, then is a *closed map*, in the sense that $g(D)$ is closed in Y whenever D is closed in X .

Recall that a subset D of X is called *locally compact* if any point of D admits a neighborhood (in D) which is compact. Clearly, any (relatively) closed or (relatively) open subset of a locally compact set is locally compact. Observe that the union of two locally compact sets could not be locally compact (for example, add a point to the boundary of the open unit disk in \mathbb{C}).

Let hereafter E and F denote real Banach spaces, and let $L(E, F)$ be the Banach space of all the bounded linear operators from E into F .

Recall that $L \in L(E, F)$ is said to be a *Fredholm operator* if both its kernel, $\text{Ker } L$, and its co-kernel, $\text{coKer } L = F / \text{Im } L$, are finite dimensional. The difference of these dimensions is called the *index* of L and denoted by $\text{ind } L$.

The following are some useful properties regarding Fredholm operators:

- (1) *the image of a Fredholm operator $L \in L(E, F)$ is a closed subspace of its codomain F ;*
- (2) *the index of the Fredholm operators is stable, in the sense that, in $L(E, F)$, the set of Fredholm operators of a given index is open;*
- (3) *if $L \in L(E, F)$ is Fredholm and $C \in L(E, F)$ is compact, then $L + C$ is Fredholm with $\text{ind}(L + C) = \text{ind } L$;*
- (4) *the composition of two Fredholm operators is still Fredholm, and its index is the sum of the indices of the composite operators;*
- (5) *Fredholm operators are proper on bounded closed subsets of their domains.*

A smooth map $f: \mathcal{M} \rightarrow \mathcal{N}$ is said to be *Fredholm of index n* if so is the linear operator $df_p: T_p(\mathcal{M}) \rightarrow T_{f(p)}(\mathcal{N})$ for any $p \in \mathcal{M}$.

A fundamental result regarding Fredholm maps is the following infinite dimensional version of the well-known Sard–Brown Lemma.

Theorem 2.4 (On the density of regular values [18]). *If $f: \mathcal{M} \rightarrow \mathcal{N}$ is a Fredholm map between two second countable manifolds, then the set of its regular values is dense in \mathcal{N} .*

Regarding the notion of Fredholm map one has another useful consequence of the Implicit Function Theorem.

Theorem 2.5 (On regularly defined finite dimensional manifolds). *Let $f: \mathcal{M} \rightarrow \mathcal{N}$ be a Fredholm map and let $q \in \mathcal{N}$ be a regular value for f .*

Then $f^{-1}(q)$, if nonempty, is a manifold whose dimension is the same as the index of f . Moreover, for any $p \in f^{-1}(q)$ one has $T_p(f^{-1}(q)) = \text{Ker } df_p$.

3. RESULTS

Let G and H be two real Hilbert spaces, and consider the problem

$$(3.1) \quad \begin{cases} Lx + \varepsilon N(x) = \lambda Cx, \\ x \in S, \end{cases}$$

where $\varepsilon, \lambda \in \mathbb{R}$, $L, C: G \rightarrow H$ are bounded linear operators, and $N: S \rightarrow H$ is a continuous map defined on the unit sphere of G .

A *solution* of (3.1) is a *triple* $(x, \varepsilon, \lambda)$ which satisfies the system, and (ε, λ) is called the *eigenpair* corresponding to the (unit) *eigenvector* x . The solutions and the eigenpairs with $\varepsilon = 0$ are said to be *trivial*.

By Σ we denote the set of the solutions of (3.1) and by \mathcal{E} we designate its projection into the $\varepsilon\lambda$ -plane. Notice that \mathcal{E} coincides with the set of the eigenpairs of (3.1). By Σ_0 and \mathcal{E}_0 we mean, respectively, the sets of the trivial solutions and the trivial eigenpairs of (3.1).

In order to simplify some statements, it is convenient to introduce the following

Definition 3.1. A trivial solution $(x_*, 0, \lambda_*)$ of (3.1), as well as the corresponding eigenpair $(0, \lambda_*)$, will be called *simple* if the associated linear operator $A = L - \lambda_* C$ satisfies the following conditions:

- (1) $\text{Ker } A = \mathbb{R}x_*$,
- (2) $Cx_* \neq 0$,
- (3) $\text{Img } A \oplus C(\text{Ker } A) = H$.

To proceed with the analysis of the structure of Σ , we need a result regarding the linear eigenvalue problem $Lx = \lambda Cx$. Namely

Lemma 3.2. *Assume that $(x_*, 0, \lambda_*)$ is a simple solution of problem (3.1). Then, the map*

$$\Psi: S \times \mathbb{R} \rightarrow H, \quad (x, \lambda) \mapsto Lx - \lambda Cx,$$

sends, diffeomorphically, a neighborhood of (x_, λ_*) in $S \times \mathbb{R}$ onto a neighborhood of the origin $0 \in H$.*

Proof. Let $z_* = (x_*, \lambda_*)$ and observe that the tangent space of $S \times \mathbb{R}$ at z_* is the 1-codimensional subspace

$$T_{z_*}(S \times \mathbb{R}) = T_{x_*}(S) \times \mathbb{R} = (\mathbb{R}x_*)^\perp \times \mathbb{R}$$

of $G \times \mathbb{R}$.

Because of the (Local) Inverse Function Theorem, it is enough to prove that the differential $d\Psi_{z_*}: T_{z_*}(S \times \mathbb{R}) \rightarrow H$ of Ψ at z_* is an isomorphism.

As in the finite dimensional case (see the proof of Lemma 3.3 in [3]) one can show that $d\Psi_{z_*}$ is injective. Thus, its surjectivity will follow if we prove that it is Fredholm of index zero.

To this purpose we regard the differential $d\Psi_{z_*}: (\mathbb{R}x_*)^\perp \times \mathbb{R} \rightarrow H$ as the sum of two bounded linear operators, A and B , defined by

$$A\dot{z} = (L - \lambda_* C)\dot{x} \quad \text{and} \quad B\dot{z} = -\dot{\lambda} Cx_*,$$

with $\dot{z} = (\dot{x}, \dot{\lambda}) \in (\mathbb{R}x_*)^\perp \times \mathbb{R}$.

Concerning the operator A , observe that its kernel is the 1-dimensional subspace $\{0\} \times \mathbb{R}$ of $(\mathbb{R}x_*)^\perp \times \mathbb{R}$ and its image coincides with that of $L - \lambda_* C$, whose codimension is one. Thus A is Fredholm of index zero and, consequently, so is $d\Psi_{z_*}$, since the operator B is compact. This completes the proof. \square

Lemma 3.2 allows us to provide an alternative proof of the result of Chiappinelli mentioned in the Introduction. In fact, we get the following corollary, whose proof is omitted, being essentially the same as that given in the finite dimensional setting (see [3, Corollary 3.4]).

Corollary 3.3. *Let $(x_*, 0, \lambda_*)$ be a simple solution of problem (3.1) and assume that $N: S \rightarrow H$ is a Lipschitz continuous map. Then there exists a neighborhood $(-\delta, \delta)$ of $0 \in \mathbb{R}$ and a Lipschitz curve*

$$\varepsilon \in (-\delta, \delta) \mapsto (x(\varepsilon), \lambda(\varepsilon)) \in S \times \mathbb{R}$$

such that $(x(0), \lambda(0)) = (x_*, \lambda_*)$ and

$$Lx(\varepsilon) + \varepsilon N(x(\varepsilon)) = \lambda(\varepsilon)Cx(\varepsilon), \quad \forall \varepsilon \in (-\delta, \delta).$$

Before proving our main theorem about global persistence we need some preliminary results regarding some crucial properties of the map

$$\Phi: S \times \mathbb{R} \times \mathbb{R} \rightarrow H, \quad (x, \varepsilon, \lambda) \mapsto Lx + \varepsilon N(x) - \lambda Cx,$$

whose set of zeros, $\Phi^{-1}(0)$, coincides with Σ .

Incidentally, we observe that the function $\Psi: S \times \mathbb{R} \rightarrow H$ defined in Lemma 3.2 is the partial map of Φ corresponding to $\varepsilon = 0$. In other words, Ψ can be, and occasionally will be, identified with the restriction of Φ to the subset $Z = S \times \{0\} \times \mathbb{R}$ of the domain $S \times \mathbb{R} \times \mathbb{R}$ of Φ . Because of this identification, the set $\Sigma_0 = Z \cap \Sigma$ of the trivial solutions of (3.1) may be regarded as $\Psi^{-1}(0)$.

We point out that $S \times \mathbb{R} \times \mathbb{R}$ is a 1-codimensional submanifold of the Hilbert space $G \times \mathbb{R} \times \mathbb{R}$ and Z is a 1-codimensional submanifold of $S \times \mathbb{R} \times \mathbb{R}$.

The next lemma provides conditions on the operators L , C and N ensuring the properness of the map Φ on bounded and closed subsets of $S \times \mathbb{R} \times \mathbb{R}$.

Lemma 3.4. *Regarding problem (3.1), assume that C and N are compact, and that L is a Fredholm operator.*

Then the map Φ is proper on bounded closed sets.

Proof. Let K be a compact subset of H and D a bounded and closed subset of $S \times \mathbb{R} \times \mathbb{R}$. We need to show that $D \cap \Phi^{-1}(K)$ is a compact set. To this purpose, observe that the function Φ is the sum of three maps: \mathcal{L} , \mathcal{N} and \mathcal{C} , given by $\mathcal{L}(p) = Lx$, $\mathcal{N}(p) = \varepsilon N(x)$ and $\mathcal{C}(p) = -\lambda Cx$, where $p = (x, \varepsilon, \lambda)$.

The operator L , being Fredholm, is proper on bounded closed sets. Consequently, as one can easily check, the map \mathcal{L} has the same property. Since, by assumption, the operators C and N are compact, so are the corresponding maps \mathcal{C} and \mathcal{N} . Thus, $\mathcal{C}(D)$ and $\mathcal{N}(D)$ are relatively compact in H . Hence, the set $K - \mathcal{N}(D) - \mathcal{C}(D)$ is contained in a compact subset \mathcal{K} of H . The assertion now follows from the fact that $D \cap \Phi^{-1}(K)$ is a closed subset of the compact set $D \cap \mathcal{L}^{-1}(\mathcal{K})$. \square

The next result provides sufficient conditions for the nonlinear map Φ to be Fredholm. In order to make the statement meaningful, N is assumed to be smooth.

Lemma 3.5. *Under the assumptions of Lemma 3.4 suppose, in addition, that N is smooth and that the index of L is zero.*

Then the map Φ is Fredholm of index 1.

Proof. Let $\mathcal{L}, \mathcal{N}, \mathcal{C}: S \times \mathbb{R} \times \mathbb{R} \rightarrow H$ be as in the proof of Lemma 3.4. Since the smooth maps \mathcal{N} and \mathcal{C} are compact, so are their differentials at any $p \in S \times \mathbb{R} \times \mathbb{R}$. Therefore, recalling that $\Phi = \mathcal{L} + \mathcal{N} + \mathcal{C}$, it is enough to show that \mathcal{L} is Fredholm of index 1. To this purpose observe that \mathcal{L} is the composition of three maps: the projection $P: S \times \mathbb{R} \times \mathbb{R} \rightarrow S$, the inclusion $J: S \hookrightarrow G$ and the linear operator $L: G \rightarrow H$. The first two maps, P and J , are Fredholm of index 2 and -1 , respectively. As L has index 0, the composite map $\mathcal{L} = L \circ J \circ P$ is Fredholm of index $2 - 1 + 0 = 1$. \square

The next result is crucial in the proof of Theorem 3.10.

Lemma 3.6. *Let $p_* = (x_*, 0, \lambda_*)$ be a simple solution of problem (3.1). Assume that the operator C is compact and that N is compact and smooth.*

Then, given a neighborhood U of p_ in $Z = S \times \{0\} \times \mathbb{R}$, there exists a neighborhood V of $0 \in H$ such that, if $q \in V$ is a regular value for Φ , the set $\Phi^{-1}(q)$ contains a smooth boundaryless connected curve that intersects U and is either unbounded or diffeomorphic to a circle containing at least two points of Z .*

Proof. Notice that, if the assertion holds for the neighborhood U , then it holds as well for any W such that $U \subseteq W \subseteq Z$. Therefore, recalling that the restriction of Φ to Z can be identified with the map Ψ of Lemma 3.2, we may assume that U is so small that it is mapped by Φ diffeomorphically onto a neighborhood V of $0 \in H$. Moreover, as p_* is simple, the operator $A = L - \lambda_* C$ is Fredholm of index zero. Consequently, so is L , because of the compactness of C .

Let $q \in V$ be a regular value for Φ . Since, according to Lemma 3.5, the map Φ is Fredholm of index 1, the set $\Phi^{-1}(q)$ is a smooth (boundaryless) 1-dimensional submanifold of $S \times \mathbb{R} \times \mathbb{R}$ (see Theorem 2.5). Any component of $\Phi^{-1}(q)$ is a closed subset of $S \times \mathbb{R} \times \mathbb{R}$, which is either compact, and therefore diffeomorphic to a circle, or non-compact, and consequently diffeomorphic to the open interval $(0, 1)$, according to Theorem 2.1.

Denote by Γ the component of $\Phi^{-1}(q)$ containing the unique point $p \in U$ such that $\Phi(p) = q$. In particular the assertion that $\Gamma \cap U$ is nonempty is verified.

Now, if Γ is unbounded, the proof is completed. Suppose, on the contrary, that it is bounded. Then, due to the properness of Φ on bounded and closed sets ensured by Lemma 3.4, the component Γ is compact and, therefore, diffeomorphic to the unit circle S^1 .

Therefore, to complete the proof, it is sufficient to show that Γ intersects Z at some point different from p . Namely, it is enough to prove that the continuous function $\sigma: (x, \varepsilon, \lambda) \in \Gamma \mapsto \varepsilon \in \mathbb{R}$ vanishes at some point of $\Gamma \setminus \{p\}$.

To this purpose we will show that the disjoint open subsets

$$\Gamma_- = \sigma^{-1}((-\infty, 0)) \quad \text{and} \quad \Gamma_+ = \sigma^{-1}((0, +\infty))$$

of the connected set $\Gamma \setminus \{p\}$ are both nonempty; and this will be obvious if we show that the intersection at p between the curve Γ and the manifold Z is transversal, which implies that σ has a sign-jump at p . Indeed, the transversality is a consequence of the fact that the point p , apart of being regular for the map Φ , is as well regular for the restriction of Φ to U , due to the diffeomorphism $\Phi|_U: U \rightarrow V$. In fact, as p is regular for Φ , one has $T_p(\Gamma) = \text{Ker } d\Phi_p$ (Theorem 2.5); and this 1-dimensional space is not contained in $T_p(Z)$, since the operator $d(\Phi|_U)_p: T_p(Z) \rightarrow H$, which coincides with the restriction of $d\Phi_p$ to $T_p(Z)$, is injective (it is actually invertible).

In conclusion, the union of the open sets Γ_- and Γ_+ cannot coincide with the connected set $\Gamma \setminus \{p\}$. Therefore, the function σ vanishes at some point of $\Gamma \setminus \{p\}$; and this concludes the proof. \square

The next lemma is a Whyburn-type topological result which is crucial in the proof of Theorem 3.10.

Lemma 3.7 ([12]). *Let Y_0 be a compact subset of a locally compact metric space Y . Assume that every compact subset of Y containing Y_0 has nonempty boundary. Then $Y \setminus Y_0$ contains a connected set whose closure in Y is non-compact and intersects Y_0 .*

Recalling that Σ denotes the set of solutions of problem (3.1) and that Σ_0 is its subset of the trivial ones, the following simple result provides an example of locally compact space, needed in the proof of Theorem 3.10.

Lemma 3.8. *Assume that the operators N and C are compact and that p_* is a simple solution of problem (3.1).*

Then, the metric subspace $Y = (\Sigma \setminus \Sigma_0) \cup \{p_\}$ of Σ is locally compact.*

Proof. Observe that $\Sigma_0 = Z \cap \Sigma$, where $Z = S \times \{0\} \times \mathbb{R}$. Therefore Σ_0 is closed in the metric space $\Sigma = \Phi^{-1}(0)$, which is locally compact because of Lemma 3.4. Clearly Y coincides with $\Sigma \setminus (\Sigma_0 \setminus \{p_*\})$. Moreover $\Sigma_0 \setminus \{p_*\}$ is closed in Σ , the point p_* being isolated in the closed set Σ_0 because of Lemma 3.2. Thus, the metric space Y , as an open subset of Σ , is locally compact. \square

From Lemmas 3.7 and 3.8 we derive the following

Lemma 3.9. *Let the operators N and C be compact, and let p_* be a simple solution of problem (3.1). Assume that any compact subset of $Y = (\Sigma \setminus \Sigma_0) \cup \{p_*\}$ containing p_* has nonempty boundary in Y .*

Then $\Sigma \setminus \Sigma_0$ has a connected set whose closure in Σ contains p_ and is either unbounded or meets a trivial solution $p^* \neq p_*$.*

Proof. Because of Lemma 3.8 and our assumption, Lemma 3.7 applies to the metric pair (Y, Y_0) , $Y_0 = \{p_*\}$. Consequently, $Y \setminus Y_0 = \Sigma \setminus \Sigma_0$ has a connected subset, say \mathcal{D} , whose closure $\bar{\mathcal{D}} \cap Y$ in Y is non-compact and contains p_* (here $\bar{\mathcal{D}}$ denotes the closure of \mathcal{D} in Σ or, equivalently, in $S \times \mathbb{R} \times \mathbb{R}$).

Since p_* belongs to the connected set $\bar{\mathcal{D}}$ (actually $p_* \in \bar{\mathcal{D}} \cap Y$), it is sufficient to show that, if $\bar{\mathcal{D}}$ is bounded (hence compact, because of Lemma 3.4), then it must intersect $\Sigma_0 \setminus \{p_*\}$, and this is clearly true since otherwise $\bar{\mathcal{D}}$ would coincide with $\bar{\mathcal{D}} \cap Y$, which is non-compact. \square

We are now in a position to prove our main result regarding problem (3.1).

Theorem 3.10 (Global continuation of solution triples). *Regarding problem (3.1), assume that the spaces G and H are separable, that the operators N and C are compact, and that $p_* = (x_*, 0, \lambda_*)$ is a simple solution.*

Then, the set $\Sigma \setminus \Sigma_0$ of the nontrivial solutions has a connected subset whose closure in Σ contains p_ and is either unbounded or meets a trivial solution $p^* \neq p_*$.*

Proof. Denoting by Y the locally compact metric space $(\Sigma \setminus \Sigma_0) \cup \{p_*\}$, it is sufficient to apply Lemma 3.9 by proving that any compact subset of Y containing p_* has nonempty boundary in Y .

By contradiction, assume there exists a compact subset K of Y containing p_* whose boundary, in Y , is empty. This compact set is open in Y , therefore it is far away from its (relatively) closed complement $Y \setminus K$. Actually, it is also far away from $\Sigma \setminus K = (Y \setminus K) \cup (\Sigma_0 \setminus \{p_*\})$, this set being disjoint from $\Sigma_0 \setminus \{p_*\}$, which is closed according to Lemma 3.2. Consequently, there exists a bounded open subset W of $S \times \mathbb{R} \times \mathbb{R}$ such that $W \cap \Sigma = K$ and

$$(3.2) \quad \partial W \cap \Sigma = \emptyset.$$

Because of Lemma 3.2, we may suppose that the intersection $U = Z \cap W$ is mapped by Φ diffeomorphically onto a neighborhood V of the origin $0 \in H$.

Since G is separable, given $\delta > 0$, there exists a smooth map $N_\delta: S \rightarrow H$ with finite dimensional image and such that $\|N(x) - N_\delta(x)\| < \delta$ for all $x \in S$. In fact, a well known result in separable Hilbert spaces (see, for example, [11] and references therein) ensures the existence of a smooth $(\delta/2)$ -approximation \hat{N}_δ of N , so that the required map N_δ is obtained by composing \hat{N}_δ with the orthogonal projection of H onto the vector space spanned by a $(\delta/2)$ -net of the totally bounded set $N(S)$. Now, define $\Phi_\delta: S \times \mathbb{R} \times \mathbb{R} \rightarrow H$ by $\Phi_\delta(x, \varepsilon, \lambda) = (L - \lambda C)x + \varepsilon N_\delta(x)$. This map is a nonlinear Fredholm operator between two separable Hilbert manifolds. Therefore, the celebrated Sard–Smale result [18] implies the existence of a regular value $q_\delta \in V$ of Φ_δ such that $\|q_\delta\| < \delta$. By Lemma 3.6 we deduce that $\Phi_\delta^{-1}(q_\delta)$ has a connected subset Γ_δ that intersects U and is either unbounded or contains at least two points of Z . Since Φ maps U diffeomorphically onto V (and coincides with Φ_δ on U), one and only one of the points of Γ_δ lies in U . Therefore, in any of the two cases the connected set Γ_δ must have points outside the bounded set W and, consequently, must contain at least one point $p_\delta \in \partial W$.

Now, denoting $p_\delta = (x_\delta, \varepsilon_\delta, \lambda_\delta)$, we have

$$\|\Phi(p_\delta)\| \leq \|\Phi_\delta(p_\delta)\| + \|\Phi(p_\delta) - \Phi_\delta(p_\delta)\| = \|q_\delta\| + |\varepsilon_\delta| \|N(x_\delta) - N_\delta(x_\delta)\| \leq \delta + c\delta,$$

where $c = \sup\{|\varepsilon| : (x, \varepsilon, \lambda) \in \partial W\}$.

As a consequence we get $\inf\{\|\Phi(p)\| : p \in \partial W\} = 0$. Thus, the properness of Φ on ∂W (ensured by Lemma 3.4) implies the existence of a compact subset of ∂W in which the infimum (and therefore the minimum) of the real functional $\|\Phi\|$ is zero (to see this consider a minimizing sequence of $\|\Phi\|$ in ∂W , then add $0 \in H$ to the image of this sequence in order to get a compact subset \mathcal{K} of H , and take the set $\Phi^{-1}(\mathcal{K}) \cap \partial W$).

Hence, we obtain $\partial W \cap \Phi^{-1}(0) = \partial W \cap \Sigma \neq \emptyset$, contradicting (3.2), so that Lemma 3.9 applies. \square

Remark 3.11. Let $p_* = (x_*, 0, \lambda_*)$ be a trivial solution of problem (3.1), and assume that the connected component of Σ containing p_* , call it Γ , is bounded.

Under the assumptions of Theorem 3.10, taking into account that the closure of a connected set is still connected, one gets that

$$(3.3) \quad \Gamma \text{ meets a point } p^* = (x^*, 0, \lambda^*) \text{ different from } p_*.$$

Consequently, one necessarily has $x^* \neq x_*$, since otherwise the condition $Cx_* \neq 0$ would imply $\lambda^* = \lambda_*$, contradicting $p^* \neq p_*$. Obviously, one could have $\lambda^* = \lambda_*$, but in this case, p_* being simple, one would obtain $x^* = -x_*$.

We point out that assertion (3.3) is trivially satisfied even in most cases in which p_* is not simple. Namely, whenever $\dim(\text{Ker}(L - \lambda_* C)) > 1$, since Γ contains (and possibly coincides with) the geometric sphere

$$(\text{Ker}(L - \lambda_* C) \cap S) \times \{(0, \lambda_*)\},$$

made up of infinitely many trivial solutions, all having the same eigenvalue λ_* .

Nevertheless, when p_* is simple, Lemma 3.2 ensures that it is isolated in the set Σ_0 of the trivial solutions; consequently, assertion (3.3) implies that Γ is not contained in Σ_0 .

Under the assumptions of our main result, Theorem 1.3 ensures that, in the set \mathcal{E} of all the eigenpairs of (3.1), the connected component containing the simple eigenpair $(0, \lambda_*)$ is either unbounded or meets a trivial eigenpair $(0, \lambda^*)$ with $\lambda^* \neq$

λ_* . For this reason we are inclined to believe that Theorem 3.10 could be extended according to Conjecture 1.4.

Sometimes, a bounded linear operator $A: G \rightarrow H$ acting between real Hilbert spaces is easily recognized as Fredholm of index zero. For example, this happens when G and H have the same finite dimension, or when A is a compact linear perturbation of one of the following operators (see e.g. [17]):

- an invertible operator $L: G \rightarrow H$;
- the restriction L of a surjective operator $\hat{L}: \hat{G} \rightarrow H$ to a closed subspace G of \hat{G} , whose codimension is finite and the same as $\text{Ker } \hat{L}$ ($x \in G$ can be regarded as a sort of boundary condition).
- a self-adjoint operator $L: G \rightarrow G$ for which $0 \in \mathbb{R}$ is an isolated eigenvalue of finite multiplicity.

In these cases, in order to verify that a trivial solution of problem (3.1) is simple, it may be convenient to take into consideration the following

Proposition 3.12. *Let $(x_*, 0, \lambda_*)$ be a trivial solution of problem (3.1). Assume that $\lambda_* \in \mathbb{R}$ and $x_* \in S$ are such that*

- (0) $A = L - \lambda_* C$ is Fredholm of index zero
- (1) $\text{Ker } A = \mathbb{R}x_*$.

Then, $(x_, 0, \lambda_*)$ is simple if and only if the equation $Ax = Cx_*$ is unsolvable in G .*

Proof. Assume first that the equation $Ax = Cx_*$ has no solutions. Thus, condition (2) of Definition 3.1 is satisfied (otherwise the above equation would admit the trivial solution) and, obviously, $Cx_* \notin \text{Img } A$. Consequently, since conditions (0) and (1) imply that $\text{Img } A$ has codimension one, the one-dimensional subspace $\mathbb{R}Cx_* = C(\text{Ker } A)$ of H intersects $\text{Img } A$ transversally, and this proves that condition (3) of Definition 3.1 is as well satisfied.

Conversely, since, as already pointed out, $\text{Img } A$ has codimension one, conditions (1) and (3) of Definition 3.1 imply that $Cx_* \notin \text{Img } A$; that is, $Ax = Cx_*$ has no solutions. \square

We close this section with a **straightforward** consequence of Theorem 3.10 and **Remark 1.2** regarding the perturbed finite dimensional classical eigenvalue problem.

Corollary 3.13 (Global continuation in finite dimension [3]). *In problem (3.1), let $G = H$ be finite dimensional. Suppose that C is the identity I of G , and let $x_* \in S$ be an eigenvector of L corresponding to a simple eigenvalue λ_* .*

Then, the set $\Sigma \setminus \Sigma_0$ has a connected subset whose closure contains p_ and is either unbounded or meets a trivial solution $p^* \neq p_*$.*

4. EXAMPLES

In this last section we will see, in some examples, how Theorem 3.10 applies. In particular we will show that, in our main result, the hypothesis that the “starting” trivial solution is simple cannot be removed. In each example the real Hilbert spaces G and H , as well as the operators L , N and C , will be explicitly introduced. The norm in any Hilbert space will be the standard one associated with the inner product. By S , Σ and \mathcal{E} we shall always mean, respectively, the unit sphere of G , the set of solutions of the given problem, and the set of the corresponding eigenpairs (which, we recall, is the projection of Σ into \mathbb{R}^2). As in the previous section, Σ_0 is the subset of Σ of the trivial solutions.

We begin with an elementary example regarding a perturbed two-dimensional classical eigenvalue problem, whose set Σ is a (topological) circle and connects all the trivial solutions which, in this case, are four (two for each eigenvalue of the unperturbed problem) and all simple. As one can check, the projection of Σ onto \mathcal{E} is a double covering map.

Example 4.1. Let $G = H = \mathbb{R}^2$ and consider the problem

$$(4.1) \quad \begin{cases} x_1 + \varepsilon x_2 &= \lambda x_1, \\ -x_2 - \varepsilon x_1 &= \lambda x_2, \\ x_1^2 + x_2^2 &= 1, \end{cases}$$

in which the operators L and N are defined by sending $x = (x_1, x_2) \in G$ into $(x_1, -x_2)$ and $(x_2, -x_1)$, respectively, and C is the identity.

The operator L has two simple eigenvalues: $\lambda_* = 1$ and $\lambda^* = -1$. Consequently, one gets four trivial solutions of the above problem (all of them simple):

$$(\pm(1, 0), 0, 1) \quad \text{and} \quad (\pm(0, 1), 0, -1).$$

Notice that the set \mathcal{E} of the eigenpairs of (4.1) is the unit circle $\varepsilon^2 + \lambda^2 = 1$. Therefore, Σ is necessarily bounded.

The eigenpairs (ε, λ) can be represented, parametrically, as $(-\sin \theta, \cos \theta)$, with $\theta \in [0, 2\pi]$ and, as one can check, given θ , the kernel of the linear operator

$$L - (\sin \theta)N - (\cos \theta)C$$

is spanned by the unit vector

$$x(\theta) = (x_1(\theta), x_2(\theta)) = (\cos(\theta/2), \sin(\theta/2)).$$

Thus, Σ can be parametrized as follows:

$$\theta \in [0, 4\pi] \mapsto (x(\theta), -\sin \theta, \cos \theta),$$

and this shows that Σ is a topological circle.

As one can easily check, Σ encounters all the four trivial solutions of problem (4.1). Moreover, the set $\Sigma \setminus \Sigma_0$ of the nontrivial solutions has four connected components, each of them diffeomorphic to an open real interval and satisfying the assertion of Conjecture 1.4 (and, consequently, the thesis of Theorem 3.10).

Incidentally, we observe that the projection of Σ onto the circle \mathcal{E} is a double covering map.

The following illustrating example regards a problem in which the unperturbed equation has a unique eigenvalue. Since, as we shall see, the assumptions of Theorem 3.10 are satisfied, according to Theorem 1.3, the set \mathcal{E} of the eigenpairs is unbounded and, consequently, so is the set Σ of the solution triples.

Example 4.2. Let $H^1([0, 2\pi], \mathbb{R})$ denote the space of all the absolutely continuous functions $x: [0, 2\pi] \rightarrow \mathbb{R}$ whose derivative is in $L^2([0, 2\pi], \mathbb{R})$. This is a separable real Hilbert space with inner product

$$\langle x, y \rangle_1 = \frac{1}{2\pi} \int_0^{2\pi} (x(s)y(s) + x'(s)y'(s)) ds.$$

Our source space G is the kernel of the continuous functional

$$x \in H^1([0, 2\pi], \mathbb{R}) \mapsto x(2\pi) - x(0) = \int_0^{2\pi} x'(s) ds.$$

So that G is a closed, 1-codimensional subspace of $H^1([0, 2\pi], \mathbb{R})$. The target space H is $L^2([0, 2\pi], \mathbb{R})$ with inner product

$$\langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(s)y(s) ds.$$

Consider the problem

$$(4.2) \quad \begin{cases} x'(t) + \varepsilon \sin t = \lambda x(t), \\ x(0) = x(2\pi), \\ \langle x, x \rangle_1 = 1. \end{cases}$$

In abstract form, this can be written as

$$\begin{cases} Lx + \varepsilon N(x) = \lambda Cx, \\ x \in S, \end{cases}$$

where

- $L: G \rightarrow H$ is the derivative $x \mapsto x'$;
- $N: G \rightarrow H$ is the (constant) map defined by $N(x) = \sin(\cdot)$;
- C is the (compact) inclusion of G into H .

The unperturbed equation $Lx = \lambda Cx$ has a unique eigenvalue, $\lambda_* = 0$, whose corresponding eigenspace, the kernel of L , is the set of constant functions. Therefore, our problem has only two trivial solutions: $(\pm x_*, 0, 0)$, where $x_* \in S$ is the constant function $t \mapsto 1$.

The operator $A = L - \lambda_* C = L$ is Fredholm of index zero, since it is the composition of the inclusion $G \hookrightarrow H^1([0, 2\pi], \mathbb{R})$, which is Fredholm of index -1 , with the differential operator $x \in H^1([0, 2\pi], \mathbb{R}) \mapsto x' \in H$, which is Fredholm of index 1 (being surjective with 1-dimensional kernel). Therefore, A satisfies the conditions (0) and (1) of Proposition 3.12. Hence, the trivial solution $(x_*, 0, 0)$ is simple provided that the problem

$$\begin{cases} x'(t) = 1, \\ x(0) = x(2\pi) \end{cases}$$

has no solutions, and this is clearly true. Thus, Theorem 3.10 applies. Consequently, according to Remark 3.11, the connected component Γ of Σ containing $(x_*, 0, 0)$ is either unbounded or meets $(-x_*, 0, 0)$. As we shall see, Γ has both the properties.

Standard computations show that, given any (ε, λ) , the problem

$$(4.3) \quad \begin{cases} x'(t) + \varepsilon \sin t = \lambda x(t), \\ x(0) = x(2\pi) \end{cases}$$

has a solution

$$x(t) = \frac{\varepsilon}{1 + \lambda^2} (\lambda \sin t + \cos t),$$

which is unique if and only if $\lambda \neq 0$, and its norm is $|\varepsilon|/\sqrt{1 + \lambda^2}$.

Regarding the case $\lambda = 0$, given any ε , problem (4.3) has infinitely many solutions: $x(t) = c + \varepsilon \cos t$ ($c \in \mathbb{R}$), with norm $\sqrt{c^2 + \varepsilon^2}$.

Therefore, the set \mathcal{E} of the eigenpairs is the union of three connected sets. One is the *segment* $[-1, 1] \times \{0\}$, corresponding to the case $\lambda = 0$. The other two are the *left* and *right branches* of the hyperbola

$$\varepsilon^2 - \lambda^2 = 1.$$

Thus, \mathcal{E} is unbounded and, consequently, so is the set Σ of all the solution triples, \mathcal{E} being its projection into \mathbb{R}^2 .

Apart of being unbounded, the set \mathcal{E} is connected, since both the branches of the hyperbola have a point in common with the segment $[-1, 1] \times \{0\}$. These points are $(-1, 0)$ for the left branch and $(1, 0)$ for the right one.

Let us show that Σ is as well connected and, consequently, coincides with the component Γ containing the trivial solution $(x_*, 0, 0)$.

Obviously, Σ is the union of three sets, each of them “over” one of the following three subsets of \mathcal{E} : the segment $[-1, 1] \times \{0\}$, and the left and right branches of the hyperbola $\varepsilon^2 - \lambda^2 = 1$.

The bounded set over $[-1, 1] \times \{0\}$, say Γ_s , regards the solutions $x(t) = c + \varepsilon \cos t$, corresponding to the case $\lambda = 0$ and having norm $\sqrt{c^2 + \varepsilon^2} = 1$. Putting $\varepsilon = \sin \theta$ and $c = \cos \theta$, with $\theta \in [0, 2\pi]$, we get the following parametrization of Γ_s :

$$\theta \in [0, 2\pi] \mapsto (\cos \theta + \sin \theta \cos(\cdot), \sin \theta, 0),$$

which shows that this set is a topological circle, which meets the two trivial solutions $(\pm x_*, 0, 0)$, both simple and corresponding to the eigenvalue $\lambda_* = 0$ of L .

Notice that, if the bounded and connected set Γ_s were a component of Σ , then Γ would coincide with Γ_s , and this fact, although compatible with Theorem 3.10 (see Remark 3.11), would contradict Conjecture 1.4. We will show that this is not the case: as we shall see, Γ is unbounded, in accord with our conjecture.

The unbounded sets over the branches of the hyperbola, call them Γ_l and Γ_r , can be parametrized as follows:

$$s \in \mathbb{R} \mapsto \left(\mp \frac{1}{\sqrt{1+s^2}} (s \sin(\cdot) + \cos(\cdot)), \mp \sqrt{1+s^2}, s \right);$$

showing that they are both connected (more precisely, diffeomorphic to \mathbb{R}).

As one can check, Γ_s has the point $(-1, -1, 0)$ in common with Γ_l and the point $(1, 1, 0)$ in common with Γ_r . Thus, the set $\Sigma = \Gamma_s \cup \Gamma_l \cup \Gamma_r$ turns out to be connected. Therefore, the component Γ containing the simple solution $(x_*, 0, 0)$ coincides with the unbounded set Σ , in accord with Remark 3.11.

Finally, we observe that, in accord with Theorem 3.10, the set $\Sigma \setminus \Sigma_0$ is the union of two connected components, both unbounded, whose closure of each of them contains the two simple solutions.

The following is an example of a linear system of two coupled differential equations with periodic boundary conditions whose set Σ has a bounded component Γ , diffeomorphic to a circle and containing four trivial solutions, all of them simple.

In addition to Γ , the other components of Σ are infinitely many geometric circles contained in Σ_0 , each of them corresponding to an isolated trivial eigenpair. This shows that, in Theorem 3.10, the hypothesis that the “starting” solution $(x_*, 0, \lambda_*)$ is simple cannot be removed.

Example 4.3. Consider the following system of coupled differential equations with 2π -periodic boundary conditions:

$$(4.4) \quad \begin{cases} x_1'(t) + x_1(t) - \varepsilon x_1(t) = \lambda x_2(t), \\ x_2'(t) - x_2(t) - \varepsilon x_2(t) = -\lambda x_1(t), \\ x_1(0) = x_1(2\pi), \quad x_2(0) = x_2(2\pi). \end{cases}$$

Let $H^1([0, 2\pi], \mathbb{R}^2)$ denote the real Hilbert space of the absolutely continuous functions $x = (x_1, x_2): [0, 2\pi] \rightarrow \mathbb{R}^2$ whose derivative is in $L^2([0, 2\pi], \mathbb{R}^2)$.

In this example, the source space G is the closed subspace of $H^1([0, 2\pi], \mathbb{R}^2)$ of the functions satisfying the periodic condition $x(0) = x(2\pi)$. The target space H is $L^2([0, 2\pi], \mathbb{R}^2)$.

Denoting by $a \cdot b$ the standard dot product of two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ of \mathbb{R}^2 , the inner product of x and y in H is given by

$$\langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(t) \cdot y(t) dt,$$

while the inner product of x and y in G is

$$\langle x, y \rangle_1 = \frac{1}{2\pi} \int_0^{2\pi} (x(t) \cdot y(t) + x'(t) \cdot y'(t)) dt.$$

Since G has codimension 2 in $H^1([0, 2\pi], \mathbb{R}^2)$, the operator $L: G \rightarrow H$, given by $(x_1, x_2) \mapsto (x_1' + x_1, x_2' - x_2)$, is Fredholm of index zero. In fact, it is the restriction to G of a surjective operator defined on $H^1([0, 2\pi], \mathbb{R}^2)$ whose kernel is 2-dimensional.

The operators N and C are defined as $(x_1, x_2) \mapsto (-x_1, -x_2)$ and $(x_1, x_2) \mapsto (x_2, -x_1)$, respectively. They are compact, due to the compact inclusion

$$H^1([0, 2\pi], \mathbb{R}^2) \hookrightarrow L^2([0, 2\pi], \mathbb{R}^2).$$

As in the previous examples we seek for solutions of (4.4) in the sphere S of G . That is, we consider the problem

$$(4.5) \quad \begin{cases} Lx + \varepsilon N(x) = \lambda Cx, \\ x \in S. \end{cases}$$

The system of two coupled differential equations

$$(4.6) \quad \begin{cases} x_1'(t) + x_1(t) - \varepsilon x_1(t) = \lambda x_2(t), \\ x_2'(t) - x_2(t) - \varepsilon x_2(t) = -\lambda x_1(t) \end{cases}$$

can be represented in a matrix form as

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \varepsilon - 1 & \lambda \\ -\lambda & \varepsilon + 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where, given ε and λ , the eigenvalues of the matrix

$$M(\varepsilon, \lambda) = \begin{pmatrix} \varepsilon - 1 & \lambda \\ -\lambda & \varepsilon + 1 \end{pmatrix}$$

are $\varepsilon \pm \sqrt{1 - \lambda^2}$. Therefore, if $|\lambda| > 1$, (4.6) admits non-zero 2π -periodic solutions if and only if $\varepsilon = 0$ and $\sqrt{\lambda^2 - 1} \in \mathbb{N}$; these solutions are oscillating and, with the addition of the trivial one, they form a two-dimensional subspace of $H^1([0, 2\pi], \mathbb{R}^2)$.

While, if $|\lambda| \leq 1$, (4.6) has non-zero 2π -periodic solutions if and only if $\varepsilon^2 + \lambda^2 = 1$; these solutions are constant and, with the trivial one, constitute a one-dimensional space.

Clearly, the set \mathcal{E} of the eigenpairs of (4.5) is

$$\{(\varepsilon, \lambda) \in \mathbb{R}^2 : \varepsilon^2 + \lambda^2 = 1\} \cup \{(0, \lambda) \in \mathbb{R}^2 : \lambda = \pm\sqrt{1+n^2}, n \in \mathbb{N}\}.$$

As in the Example 4.1, the eigenpairs of the circle $\{(\varepsilon, \lambda) \in \mathbb{R}^2 : \varepsilon^2 + \lambda^2 = 1\}$ can be represented parametrically. In this case we set $(\varepsilon, \lambda) = (\cos \alpha, \sin \alpha)$, with $\alpha \in [0, 2\pi]$. Thus, as one can show, given any α , the kernel of the linear operator

$$L + (\cos \alpha)N - (\sin \alpha)C$$

is the straight line $\mathbb{R}x^\alpha$, where $x^\alpha \in G$ is the constant function

$$x^\alpha : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \mapsto x^\alpha(t) = (x_1^\alpha(t), x_2^\alpha(t)) = (\cos(\alpha/2), \sin(\alpha/2)),$$

whose norm is one, $(x^\alpha)'(t)$ being identically zero. Consequently, in the metric space Σ , the connected component Γ containing the trivial solution

$$p_* = (x_*, 0, \lambda_*) = (x^{\pi/2}, 0, 1) \in S \times \mathbb{R} \times \mathbb{R}$$

of (4.5) is diffeomorphic to a circle, as can be seen by means of the parametrization

$$\alpha \in [0, 4\pi] \mapsto (x^\alpha, \cos \alpha, \sin \alpha) \in S \times \mathbb{R} \times \mathbb{R},$$

which encounters four points of the set Σ_0 of the trivial solutions: two of them corresponding to the eigenvalue $\lambda_* = 1$ of the unperturbed operator $L - \lambda C$ and the others corresponding to $\lambda^* = -1$.

We observe that the projection of Γ onto the circle $\{(\varepsilon, \lambda) \in \mathbb{R}^2 : \varepsilon^2 + \lambda^2 = 1\}$ is a double covering map.

Let us check whether or not the trivial solution $p_* = (x_*, 0, \lambda_*)$ is simple. We have already shown that L is Fredholm of index zero. Hence, so is the operator $A = L - \lambda_* C$, as a compact linear perturbation of L . Moreover, $\text{Ker } A = \mathbb{R}x_* = \mathbb{R}x^{\pi/2}$. Therefore, A satisfies the conditions (0) and (1) of Proposition 3.12 and, consequently, p_* is simple if and only if the equation $Ax = Cx_*$ is unsolvable.

Since $Cx_* = Cx^{\pi/2}$ is the constant function $t \mapsto (\sqrt{2}/2, -\sqrt{2}/2)$, the trivial solution p_* is simple provided that the problem

$$(4.7) \quad \begin{cases} x_1'(t) + x_1(t) - x_2(t) = \sqrt{2}/2, \\ x_2'(t) + x_1(t) - x_2(t) = -\sqrt{2}/2, \\ x_1(0) = x_1(2\pi), \quad x_2(0) = x_2(2\pi). \end{cases}$$

is unsolvable; and this is clearly true since any solution $t \mapsto (x_1(t), x_2(t))$ of the first two equations cannot be periodic, the function $t \mapsto x_1(t) - x_2(t)$ being strictly increasing. Analogously, one can check that all the other three trivial solutions of Γ are simple. Moreover, $\Gamma \setminus \Sigma_0$ is the disjoint union of four arcs, each of them satisfying the assertion of Theorem 3.10.

Notice that the solutions which are not in Γ are all trivial and correspond to the eigenpairs $(0, \pm\sqrt{1+n^2})$, with $n \in \mathbb{N}$. For example, the solutions of the type $(x, 0, \sqrt{2})$ are the elements of the geometric circle

$$(\text{Ker}(L - \sqrt{2}C) \cap S) \times \{(0, \sqrt{2})\},$$

which is a connected component of Σ contained in Σ_0 . Consequently, according to Remark 3.11, none of the solutions of this circle satisfies the assertion of Theorem

3.10. The fact that they are trivial but not simple shows that, in Theorem 3.10, the assumption that the solution $(x_*, 0, \lambda_*)$ is simple cannot be removed.

We close with an elementary example showing that, in Conjecture 1.4, the assumption that $(x_*, 0, \lambda_*)$ is simple cannot be removed.

Example 4.4. Let $G = H = \mathbb{R}^2$ and consider the linear problem

$$(4.8) \quad \begin{cases} -\varepsilon x_2 = \lambda x_1, \\ -2x_1 + \varepsilon x_1 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

The operators L and N are $(x_1, x_2) \mapsto (0, -2x_1)$ and $(x_1, x_2) \mapsto (-x_2, x_1)$, respectively, and $C = I$ is the identity. The operator L has a unique eigenvalue, $\lambda_* = 0$, whose multiplicity is two, and the kernel of $L - \lambda_* I = L$ is spanned by the unit eigenvector $x_* = (0, 1)$.

The set \mathcal{E} of the eigenpairs of (4.8) is the circle $\varepsilon(\varepsilon - 2) + \lambda^2 = 0$. Therefore, Σ is necessarily bounded, and so is its connected component containing the trivial solution $p_* = (x_*, 0, \lambda_*)$. This component cannot meet any solution $p^* = (x^*, 0, \lambda^*)$ with $\lambda^* \neq \lambda_*$, since our unperturbed problem has only one eigenvalue.

Thus, at least one of the following two possibilities holds true: the Conjecture 1.4 is false or the trivial solution p_* is not simple. We are not sure about the conjecture, but of course p_* is not simple, as one can easily check, for example, by means of Proposition 3.12.

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