GLOBAL CONTINUATION IN EUCLIDEAN SPACES OF THE PERTURBED UNIT EIGENVECTORS CORRESPONDING TO A SIMPLE EIGENVALUE

PIERLUIGI BENEVIERI, ALESSANDRO CALAMAI, MASSIMO FURI, AND MARIA PATRIZIA PERA

Abstract. In the Euclidean space $\mathbb{R}^k$, we consider the perturbed eigenvalue problem $Lx + \varepsilon N(x) = \lambda x$, $\|x\| = 1$, where ε, λ are real parameters, L is a linear endomorphism of $\mathbb{R}^k$, and $N\colon S^{k-1} \to \mathbb{R}^k$ is a continuous map defined on the unit sphere of $\mathbb{R}^k$.

We prove a sort of global continuation of the *solutions* $(x, \varepsilon, \lambda)$ of this problem. Namely, under the assumption that $x_* \in S^{k-1}$ is one of the two unit eigenvectors of L corresponding to a simple eigenvalue $\lambda_* \in \mathbb{R}$, we show that, in the set of all the solutions, the connected component containing $(x_*, 0, \lambda_*)$ is either unbounded or meets a solution $(x^*, 0, \lambda^*)$ having $x^* \neq x_*$.

Our result is inspired by a paper of R. Chiappinelli regarding the local persistence property of eigenvalues and eigenvectors of a perturbed self-adjoint operator in a real Hilbert space.

1. Introduction

Let $T\colon H \to H$ be a self-adjoint bounded operator in a real Hilbert space H, and $N\colon S \to H$ a Lipschitz continuous map defined on the unit sphere of H. Consider the *nonlinear eigenvalue problem*

$$\begin{cases} Tx + \varepsilon N(x) = \lambda x, \\ x \in S, \end{cases} \tag{1.1}$$

where ε and λ are real parameters.

Under the assumption that $x_* \in S$ is an eigenvector of T corresponding to an isolated simple eigenvalue $\lambda_* \in \mathbb{R}$, Raffaele Chiappinelli in [4] deduced the so-called *local persistence property* of the unit eigenvector x_* and the eigenvalue λ_*. More precisely, he proved that, on a neighborhood $(-\delta, \delta)$ of $0 \in \mathbb{R}$, two Lipschitz functions, $\varepsilon \mapsto x_\varepsilon \in S$ and $\varepsilon \mapsto \lambda_\varepsilon \in \mathbb{R}$, are defined and satisfy the following properties:

$$x_0 = x_*,\ \lambda_0 = \lambda_* \quad \text{and} \quad Tx_\varepsilon + \varepsilon N(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon,\ \forall\, \varepsilon \in (-\delta, \delta).$$

Date: July 2, 2019.

2010 *Mathematics Subject Classification.* 47J10, 47A75, 55M25.

Key words and phrases. eigenvalues, eigenvectors, nonlinear spectral theory, transversality, intersection number.

A. Calamai is partially supported by G.N.A.M.P.A. - INdAM (Italy).

The second and fourth authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Obviously, another pair of similar functions exists if instead of x_* one considers the antipodal eigenvector $-x_*$.

Further results regarding the local persistence of eigenvalues, as well as unit eigenvectors, have been recently obtained in [2, 5, 7, 8, 9, 10] in the case in which the eigenvector λ_* is not necessarily simple. For a general review on nonlinear eigenvalue problems and applications to differential equations, see e.g. [6] and references therein.

Recently, for problems such as (1.1), in [3] a sort of "global persistence property" of the eigenvalues (but not of the unit eigenvectors) has been proved. These problems include the following one:

$$\begin{cases} Lx + \varepsilon N(x) = \lambda x, \\ x \in S^{k-1}, \end{cases} \tag{1.2}$$

where $L\colon \mathbb{R}^k \to \mathbb{R}^k$ is a linear operator and $N\colon S^{k-1} \to \mathbb{R}^k$ is a continuous map.

A triple $(x, \varepsilon, \lambda) \in S^{k-1} \times \mathbb{R} \times \mathbb{R}$ is a *solution* of (1.2) if it satisfies the equation $Lx+\varepsilon N(x) = \lambda x$. The first element x is said to be a *unit eigenvector* corresponding to the *eigenpair* (ε, λ). The set of solutions of (1.2) is denoted by Σ and the set of the eigenpairs by $\mathcal{E}$.

The solutions of the type $(x, 0, \lambda)$ will be called *trivial*, and the set of these distinguished triples will be denoted by Σ_0. So that, whenever $\Sigma_0 \neq \emptyset$, one has

$$\Sigma_0 = \bigcup_{i=1}^{s} \big((\mathrm{Ker}(L - \lambda_i I) \cap S^{k-1}) \times \{0\} \times \{\lambda_i\}\big), \tag{1.3}$$

where I is the identity in $\mathbb{R}^k$ and λ_i, $i = 1 \dots s \leq k$, are the real eigenvalues of L. Analogously, an eigenpair (ε, λ) is said to be *trivial* if $\varepsilon = 0$, and in this case λ is a real eigenvalue of L.

In the finite-dimensional setting, a consequence of a result in [3] is the following

Theorem 1.1. *If $\lambda_* \in \mathbb{R}$ is an eigenvalue of L whose geometric and algebraic multiplicities coincide and are odd, then the connected component of $\mathcal{E}$ containing $(0, \lambda_*)$ is either unbounded or includes a trivial eigenpair $(0, \lambda^*)$ different from $(0, \lambda_*)$.*

Observe that Theorem 1.1 cannot be regarded as a global version of Chiappinelli's result in the finite-dimensional context, since $\mathcal{E}$ is nothing more than the projection into the $\varepsilon\lambda$-plane of the set Σ of all the solution triples of (1.2); thus, the information on the eigenvector component is lost in the projection.

The principal motivation of this paper is to fill this gap by proving a result (Theorem 3.8) which, as we shall see (Remark 3.9), implies the following Rabinowitz-type continuation result:

Proposition 1.2. *If $x_* \in S^{k-1}$ is a unit eigenvector of L corresponding to a simple eigenvalue λ_*, then the connected component of Σ containing $(x_*, 0, \lambda_*)$ is either unbounded or includes a trivial solution $(x^*, 0, \lambda^*)$ with $x^* \neq x_*$.*

We point out that, in some particular cases, it is quite evident that the connected component $\mathcal{C}$ of Σ containing a trivial solution $(x_*, 0, \lambda_*)$ satisfies one of the alternatives. For instance, if $N = 0$ or $N = I$ (no matter whether or not λ_* is simple), the component $\mathcal{C}$ is unbounded, since it includes the connected set $\{(x_*, \varepsilon, \lambda_*) : \varepsilon \in \mathbb{R}\}$ or $\{(x_*, \varepsilon, \lambda_* + \varepsilon) : \varepsilon \in \mathbb{R}\}$, respectively. Regarding the second alternative (i.e. $\mathcal{C}$

includes a solution $(x^*, 0, \lambda^*)$ *with* $x^* \neq x_*$), this is trivially satisfied (whatever the map N is) if the sphere of the unit eigenvectors corresponding to λ_* has positive dimension: in this case $\mathcal{C}$ contains infinitely many trivial solutions of the type $(x^*, 0, \lambda^*)$, with $x^* \neq x_*$ and $\lambda^* = \lambda_*$ (see formula (1.3)).

In spite of these considerations, the assertion about the two alternatives is significant when the eigenvalue λ_* is simple: it implies that the connected set $\mathcal{C} \subseteq \Sigma$ is essentially made of nontrivial solutions, consequence of the fact that x_* is *isolated* in the set of the unit eigenvectors.

As previously observed, the equality $\lambda^* = \lambda_*$ is compatible with the assertion of Proposition 1.2. However, λ_* being simple, in this case one necessarily has $x^* = -x_*$. Supported by Theorem 1.1, we believe that Proposition 1.2 could be sharpened as follows:

Conjecture 1.3. *Let* $x_* \in S^{k-1}$ *be a unit eigenvector of* L *corresponding to a simple eigenvalue* λ_**. Then, the connected component of* Σ *containing* $(x_*, 0, \lambda_*)$ *is either unbounded or includes a triple* $(x^*, 0, \lambda^*)$ *with* $\lambda^* \neq \lambda_*$*.*

Up to now we were not able to prove or disprove this conjecture. The difficulty is probably due to the fact that the tools employed to prove Theorem 1.1 are quite different from those used for Theorem 3.8. They are both topological, but the first result is mainly based on Leray–Schauder degree theory while the second one is essentially related to arguments of differential topology, that can be found, for example, in the books [12, 13, 15].

The proof of Theorem 3.8 rests on a result (Lemma 3.3) which seems to have an interest in its own right: it regards the classical eigenvalue problem $Lx = \lambda x$ and states that, *if* $x_* \in S^{k-1}$ *is an eigenvector corresponding to a simple eigenvalue* $\lambda_* \in \mathbb{R}$*, then the map* $(x, \lambda) \in S^{k-1} \times \mathbb{R} \mapsto Lx - \lambda x$ *sends diffeomorphically a neighborhood of* (x_*, λ_*) *in* $S^{k-1} \times \mathbb{R}$ *onto a neighborhood of the origin* $0 \in \mathbb{R}^k$*.*

2. Preliminaries

Given any positive integer k, in the Euclidean space $\mathbb{R}^k$ the standard inner product of two vectors x and y is denoted by $\langle x, y \rangle$. The norm of an element $x \in \mathbb{R}^k$ is the Euclidean one, namely $\|x\| = \sqrt{\langle x, x \rangle}$.

By a *(differentiable) manifold* we shall mean a smooth (i.e. of class C^∞) boundaryless differentiable manifold, embedded in some Euclidean space.

Given a manifold $\mathcal{M}$ and given $p \in \mathcal{M}$, the tangent space of $\mathcal{M}$ at p will be denoted by $T_p(\mathcal{M})$. If $f \colon \mathcal{M} \to \mathcal{N}$ is a C^1 map between two manifolds and $p \in \mathcal{M}$, the derivative of f at p will be written as df_p. This is a linear operator from $T_p(\mathcal{M})$ into $T_{f(p)}(\mathcal{N})$.

If $\mathcal{M}$ is a submanifold of a manifold $\mathcal{N}$ and $p \in \mathcal{M}$, then $T_p(\mathcal{M})$ will be identified with the vector subspace $\operatorname{Img}(dJ_p)$ of $T_p(\mathcal{N})$, where $J \colon \mathcal{M} \hookrightarrow \mathcal{N}$ is the inclusion map. In particular, if $p \in \mathcal{M} \subseteq \mathbb{R}^k$, then $T_p(\mathcal{M})$ is a subspace of $\mathbb{R}^k$.

Remember that, if $f \colon \mathcal{M} \to \mathcal{N}$ is a C^1 map, an element $p \in \mathcal{M}$ is said to be a *regular point* (of f) if df_p is surjective. Non-regular points are called *critical (points)*. The *critical values* of f are those points of the target manifold $\mathcal{N}$ which lie in the image $f(C)$ of the set C of critical points. Any $q \in \mathcal{N}$ which is not in $f(C)$ is a *regular value*. Therefore, in particular, any element of $\mathcal{N}$ which is not in the image of f is a regular value.

The well-known Sard's Lemma implies that *the set of regular values of a smooth map $f\colon \mathcal{M} \to \mathcal{N}$ between two manifolds is dense in $\mathcal{N}$.*

The following famous result (see e.g. [15]) will play an important role in Section 3.

Theorem 2.1 (Regularity of the level set). *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a smooth map between two manifolds of dimensions m and n, respectively.*

If $q \in \mathcal{N}$ is a regular value for f, then $f^{-1}(q)$, if nonempty, is a manifold of dimension $m-n$. Moreover, given $p \in f^{-1}(q)$, one has $T_p(f^{-1}(q)) = \operatorname{Ker} df_p$.

For example, in $\mathbb{R}^k$, the unit sphere S^{k-1} is a 1-codimensional submanifold of $\mathbb{R}^k$, and given $p \in S^{k-1}$ one has

$$T_p(S^{k-1}) = \{\dot p \in \mathbb{R}^k : \langle p, \dot p\rangle = 0\} = (\mathbb{R}p)^\perp.$$

To see this, define $f\colon \mathbb{R}^k \to \mathbb{R}$ by $f(x) = \|x\|^2$ and consider the value $q = 1$, observe that $df_p(\dot p) = 2\langle p, \dot p\rangle$, and notice that $0 \in \mathbb{R}$ is the unique critical value of f.

We recall that a subset X of a metric space is *locally compact* if any point of X admits a compact neighborhood in X. Obviously any compact set, as well as any relatively open subset of a locally compact set, is locally compact. The union of two locally compact sets need not be locally compact (think about an open disk in $\mathbb{C}$ and add a point to the boundary).

3. Results

Consider the following nonlinear eigenvalue problem in $\mathbb{R}^k$:

$$\begin{cases} Lx + \varepsilon N(x) = \lambda x, \\ x \in S^{k-1}, \end{cases} \tag{3.1}$$

where $\varepsilon, \lambda \in \mathbb{R}$, $L\colon \mathbb{R}^k \to \mathbb{R}^k$ is a linear operator and $N\colon S^{k-1} \to \mathbb{R}^k$ is a continuous map.

A *solution* of (3.1) is a *triple* $(x, \varepsilon, \lambda)$ which satisfies the system. The first element x is a *unit eigenvector* of problem (3.1) corresponding to the *eigenpair* (ε, λ).

The solution triples with $\varepsilon = 0$ are called *trivial* and, consequently, the other ones are said to be *nontrivial*. Obviously, if $(x_*, 0, \lambda_*)$ is a trivial solution of (3.1), then x_* is a unit eigenvector (in the usual sense) of L corresponding to the eigenvalue λ_*, and viceversa.

We will denote by Σ the set of the solutions of (3.1) and by $\mathcal{E}$ its projection into the $\varepsilon\lambda$-plane, so that $\mathcal{E}$ is made up of all the eigenpairs of the problem. By Σ_0 we shall mean the subset of Σ of the trivial solutions.

Remark 3.1. *In the particular case in which N is defined on the whole space $\mathbb{R}^k$ and it is linear, the set $\mathcal{E}$ of the eigenpairs is given by*

$$\big\{(\varepsilon, \lambda) \in \mathbb{R}^2 : \det(L + \varepsilon N - \lambda I) = 0\big\}.$$

Clearly, one can find simple examples in which $\mathcal{E}$ is empty (and therefore so is Σ). Obviously, this cannot happen if the dimension k is odd, no matter what is N: in this case $\mathcal{E}$ contains at least one trivial eigenpair $(0, \lambda_*)$, corresponding to a real eigenvalue λ_* of L. However, the following proposition, whose easy proof is left to the reader, gives more information about the structure of the nonempty set $\mathcal{E}$:

Proposition 3.2. *Assume that in problem* (3.1) *the dimension k is odd and that $N\colon \mathbb{R}^k \to \mathbb{R}^k$ is linear.*

Then, if $s > 0$ is bigger than the norm of L, one has

$$\det(L + sI)\det(L - sI) < 0.$$

Consequently, $(0, -s)$ and $(0, s)$ belong to different components of the open set $\mathbb{R}^2 \backslash \mathcal{E}$. In particular, $\mathcal{E}$ is unbounded, and so is Σ, the set $\mathcal{E}$ being its projection.

Our main result regarding (3.1) is Theorem 3.8 below, which implies, in particular, that, *if $x_* \in S^{k-1}$ is an eigenvector of L corresponding to a simple eigenvalue λ_*, then the connected component in Σ containing $(x_*, 0, \lambda_*)$ is either unbounded or meets a trivial solution different from $(x_*, 0, \lambda_*)$.*

Define $\Phi\colon S^{k-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^k$ by $\Phi(x, \varepsilon, \lambda) = Lx + \varepsilon N(x) - \lambda x$, so that $\Sigma = \Phi^{-1}(0)$. Notice that the domain of Φ is a $(k+1)$-dimensional submanifold of the Euclidean space $\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}$.

We denote by $\Psi\colon S^{k-1} \times \mathbb{R} \to \mathbb{R}^k$ the partial map of Φ corresponding to $\varepsilon = 0$. Occasionally, it will be convenient to identify Ψ with the restriction of Φ to the k-dimensional submanifold

$$Z = S^{k-1} \times \{0\} \times \mathbb{R}$$

of $S^{k-1} \times \mathbb{R} \times \mathbb{R}$. Incidentally, we observe that the set Σ_0 of the trivial solutions of (3.1) coincides with $Z \cap \Sigma$.

Before proving Theorem 3.8 we need some preliminary results. One of these is the following one regarding the classical eigenvalue problem:

Lemma 3.3 (On the local diffeomorphism)**.** *Let $L\colon \mathbb{R}^k \to \mathbb{R}^k$ be a linear operator, and assume that $x_* \in S^{k-1}$ is an eigenvector of L corresponding to a simple eigenvalue λ_*.*

Then the function

$$\Psi\colon S^{k-1} \times \mathbb{R} \to \mathbb{R}^k, \quad (x, \lambda) \mapsto Lx - \lambda x$$

maps, diffeomorphically, a neighborhood of (x_*, λ_*) in $S^{k-1} \times \mathbb{R}$ onto a neighborhood of the origin $0 \in \mathbb{R}^k$.

Proof. Let $p_* = (x_*, \lambda_*)$. Because of the (Local) Inverse Function Theorem, it is enough to prove that the derivative $d\Psi_{p_*}\colon T_{p_*}(S^{k-1} \times \mathbb{R}) \to \mathbb{R}^k$ of Ψ at p_* is an isomorphism. Since the manifold $S^{k-1} \times \mathbb{R}$ has the same dimension as $\mathbb{R}^k$, it is sufficient to show that $d\Psi_{p_*}$ is injective.

Observe that the tangent space of $S^{k-1} \times \mathbb{R}$ at p_* is the subspace

$$T_{p_*}(S^{k-1} \times \mathbb{R}) = \big\{\dot{p} = (\dot{x}, \dot{\lambda}) \in \mathbb{R}^k \times \mathbb{R} : \langle \dot{x}, x_* \rangle = 0\big\}$$

of $\mathbb{R}^k \times \mathbb{R}$, and that $\dot{p} = (\dot{x}, \dot{\lambda}) \in T_{p_*}(S^{k-1} \times \mathbb{R})$ is in the kernel of $d\Psi_{p_*}$ if and only if

$$d\Psi_{p_*}(\dot{p}) = (L - \lambda_* I)\dot{x} - \dot{\lambda} x_* = 0. \tag{3.2}$$

Now, the assumption that λ_* is a simple eigenvalue implies the splitting

$$\mathbb{R}^k = \operatorname{Img}(L - \lambda_* I) \oplus \operatorname{Ker}(L - \lambda_* I) = \operatorname{Img}(L - \lambda_* I) \oplus \mathbb{R}x_*.$$

Therefore, from (3.2) we derive $\dot{x} \in \operatorname{Ker}(L - \lambda_* I) = \mathbb{R}x_*$ and $\dot{\lambda} = 0$. Finally, recalling that $\langle \dot{x}, x_* \rangle = 0$, we get $\dot{x} = 0$ and, consequently, $\dot{p} = (\dot{x}, \dot{\lambda}) = 0$. $\square$

We observe that, in the finite-dimensional setting, the local persistence result of Chiappinelli quoted in the Introduction could be deduced from Lemma 3.3. In fact, we obtain the following

Corollary 3.4. *Let $L\colon \mathbb{R}^k \to \mathbb{R}^k$ be a linear operator and $N\colon S^{k-1} \to \mathbb{R}^k$ a Lipschitz continuous map. Assume that $x_* \in S^{k-1}$ is an eigenvector of L corresponding to a simple eigenvalue λ_*.*

Then there exists a neighborhood $(-\delta, \delta)$ of $0 \in \mathbb{R}$ and a Lipschitz curve

$$\varepsilon \in (-\delta, \delta) \mapsto (x(\varepsilon), \lambda(\varepsilon)) \in S^{k-1} \times \mathbb{R}$$

such that $(x(0), \lambda(0)) = (x_*, \lambda_*)$ and

$$Lx(\varepsilon) + \varepsilon N(x(\varepsilon)) = \lambda(\varepsilon)x(\varepsilon), \quad \forall\, \varepsilon \in (-\delta, \delta).$$

Proof. Lemma 3.3 ensures that the function

$$\Psi\colon S^{k-1} \times \mathbb{R} \to \mathbb{R}^k, \quad (x, \lambda) \mapsto Lx - \lambda x$$

maps, diffeomorphically, a neighborhood U of (x_*, λ_*) in $S^{k-1} \times \mathbb{R}$ onto a neighborhood V of the origin $0 \in \mathbb{R}^k$.

Consider the equation $Lx - \lambda x = -\varepsilon N(x)$, with $(x, \lambda) \in U$ and $\varepsilon \in \mathbb{R}$. This can be written in the form $\Psi(x, \lambda) = \varepsilon g(x, \lambda)$, where $g(x, \lambda) = -N(x)$. The diffeomorphism between U and V allows us to transform this equation into an equivalent fixed point problem depending on the real parameter ε.

To this purpose, denote by $\Psi|_U^{-1}$ the inverse of the diffeomorphism between U and V defined by Ψ. Taking U and $V = \Psi(U)$ smaller, if necessary, we may assume that this inverse function is Lipschitz. Thus, putting $q = \Psi(x, \lambda)$, the equation $\Psi(x, \lambda) = \varepsilon g(x, \lambda)$ is equivalent to the fixed point problem $q = \varepsilon f(q)$, where $f\colon V \to \mathbb{R}^k$ is the Lipschitz continuous function, with bounded image, given by $q \mapsto g(\Psi|_U^{-1}(q))$. Therefore, if $|\varepsilon|$ is small, the map $q \mapsto \varepsilon f(q)$ is a contraction whose image is contained in a complete subset of V. Consequently, for any ε in some neighborhood $(-\delta, \delta)$ of $0 \in \mathbb{R}$, this map has a unique fixed point $q(\varepsilon) \in V$. Now, taking into account that f is dominated by some constant M and is Lipschitz with some constant C, by considering δ such that $\delta C < 1$ one gets that $\varepsilon \mapsto q(\varepsilon)$ is a Lipschitz map with constant $M/(1 - \delta C)$, consequence of the inequality

$$\|q(\varepsilon_2) - q(\varepsilon_1)\| \leq \delta C \|q(\varepsilon_2) - q(\varepsilon_1)\| + M|\varepsilon_2 - \varepsilon_1|.$$

Thus, the curve $\varepsilon \in (-\delta, \delta) \mapsto \Psi|_U^{-1}(q(\varepsilon))$ satisfies the assertion. □

The next result will play a fundamental role in the proof of Theorem 3.8.

Lemma 3.5. *Regarding problem* (3.1)*, assume that the map N is smooth and that $x_* \in S^{k-1}$ is an eigenvector of L corresponding to a simple eigenvalue λ_*.*

Given a neighborhood U of $(x_*, 0, \lambda_*)$ in $Z = S^{k-1} \times \{0\} \times \mathbb{R}$, there exists a neighborhood V of the origin $0 \in \mathbb{R}^k$ such that, if $q \in V$ is a regular value for the function

$$\Phi\colon S^{k-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^k, \quad (x, \varepsilon, \lambda) \mapsto Lx + \varepsilon N(x) - \lambda x,$$

then $\Phi^{-1}(q)$ has a connected component that intersects U and is either unbounded or contains at least two points of Z.

Proof. Because of Lemma 3.3, we may assume, without loss of generality, that the restriction of Φ to U is a diffeomorphism onto V. In fact, as already pointed out, the map Ψ may be identified with the restriction of Φ to the submanifold Z of $S^{k-1} \times \mathbb{R} \times \mathbb{R}$.

Now, assume that $q \in V$ is a regular value for Φ. Thus, $\Phi^{-1}(q)$, which is clearly nonempty, is a (boundaryless) manifold, whose dimension is

$$\dim(S^{k-1} \times \mathbb{R} \times \mathbb{R}) - \dim(\mathbb{R}^k) = 1.$$

Any component of this curve is either compact, and in this case diffeomorphic to a circle, or noncompact, and therefore unbounded, as a closed subset of $S^{k-1} \times \mathbb{R} \times \mathbb{R}$.

Let $\mathcal{C}$ denote the component of $\Phi^{-1}(q)$ containing the unique point $p \in U$ such that $\Phi(p) = q$. Since the intersection $\mathcal{C} \cap U$ is nonempty, the first assertion is established.

Assume that $\mathcal{C}$ is bounded. Thus, it is diffeomorphic to a circle. We need to show that this closed curve intersects Z also at some point different from p (so that, necessarily, it is not in U).

To this purpose it is enough to prove that the intersection at p between $\mathcal{C}$ and Z is transversal. In fact, in this case, if $\mathcal{C}$ intersected Z only at p, the intersection number (see e.g. [12]) between $\mathcal{C}$ and Z would be either 1 or -1, according to the orientations of the two manifolds. This would contradict the fact that this number must be zero, since any closed curve in $S^{k-1} \times \mathbb{R} \times \mathbb{R}$ can be homotopically moved away from Z in such a way that, during the homotopy, the intersection set remains confined in a compact subset of Z.

Now, the transversality of the intersection at p between $\mathcal{C}$ and Z is ensured by the fact that Φ maps diffeomorphically U onto V, which implies that q is a regular value not just for Φ but also for the restriction of Φ to U (actually, any value in V is regular for this restriction). In fact, one has

$$T_p(Z) \cap T_p(\mathcal{C}) = T_p(Z) \cap \operatorname{Ker} d\Phi_p = \{0\}.$$

Thus, we have

$$T_p(Z) \oplus T_p(\mathcal{C}) = T_p(S^{k-1} \times \mathbb{R} \times \mathbb{R}),$$

and this concludes the proof. □

The following corollary is an easy consequence of the above two lemmas. Therefore, the proof will be omitted.

Corollary 3.6. *Assume that N is smooth, that $0 \in \mathbb{R}^k$ is a regular value for Φ, and that $x_* \in S^{k-1}$ is an eigenvector of L corresponding to a simple eigenvalue λ_*.*

Then the connected component of $\Phi^{-1}(0)$ containing $(x_*, 0, \lambda_*)$ is a smooth curve which, if bounded, meets a trivial solution of (3.1) different from $(x_*, 0, \lambda_*)$.

The next point-set topological lemma plays an essential role in the proof of Theorem 3.8. It is particularly cut to our purposes and is deduced from general results by C. Kuratowski (see [14], Chapter 5, Vol. 2). We also recommend [1] for a helpful article on connectivity.

Lemma 3.7 ([11])**.** *Let Y_0 be a compact subset of a locally compact metric space Y. Assume that every compact subset of Y containing Y_0 has nonempty boundary.*

Then $Y \setminus Y_0$ contains a connected set whose closure in Y is noncompact and intersects Y_0.

We are now in a position to prove our main result regarding problem (3.1). Recall that Σ is the set of solutions of (3.1) and Σ_0 is the subset of the trivial ones.

Theorem 3.8 (Global continuation of solution triples)**.** *Regarding problem* (3.1)*, assume that $x_* \in S^{k-1}$ is an eigenvector of L corresponding to a simple eigenvalue λ_*.*

Then, the set $\Sigma \setminus \Sigma_0$ of the nontrivial solutions has a connected subset whose closure in Σ contains $(x_*, 0, \lambda_*)$ and is either unbounded or meets a solution $(x^*, 0, \lambda^*)$ with $x^* \neq x_*$.

Proof. We will proceed in two steps: firstly, we shall assume that N is smooth; secondly, we shall suppose that N is merely continuous. In both steps, Lemma 3.7 will play an essential role, and in both steps the metric pair (Y, Y_0) will be the same. So, we shall define it before the first step.

Denote by p_* the "starting" triple $(x_*, 0, \lambda_*) \in \Sigma_0$ and put

$$Y = (\Sigma \setminus \Sigma_0) \cup \{p_*\} \quad \text{and} \quad Y_0 = \{p_*\}.$$

We need to show that Y is locally compact, as required in Lemma 3.7. In fact, Y coincides with $\Sigma \setminus (\Sigma_0 \setminus \{p_*\})$, which is open in Σ, being obtained from this metric space by removing the set $\Sigma_0 \setminus \{p_*\}$, which is closed, because p_* is isolated. Thus, the local compactness of Y follows from the fact that $\Sigma = \Phi^{-1}(0)$ is locally compact, being a closed subset of a finite-dimensional space.

We claim that our proof is complete if the metric pair (Y, Y_0) satisfies the assertion of Lemma 3.7. In fact, assume that $Y \setminus Y_0$, which is the same as $\Sigma \setminus \Sigma_0$, contains a connected set, say Γ, whose closure in Y is noncompact and intersects Y_0. Let $\bar{\Gamma}$ denote the closure of Γ in Σ (or, equivalently, in $S^{k-1} \times \mathbb{R} \times \mathbb{R}$), so that $\bar{\Gamma} \cap Y$ is the closure of Γ in Y. Observe that $\bar{\Gamma}$ includes p_*, this point being in $\bar{\Gamma} \cap Y$.

Assume that $\bar{\Gamma}$ is bounded. Then it must contain a point $p^* = (x^*, 0, \lambda^*) \in \Sigma_0 \setminus \{p_*\}$, since otherwise $\bar{\Gamma}$ would coincide with $\bar{\Gamma} \cap Y$, which is noncompact.

Observe that to any eigenvector of a linear endomorphism of a vector space corresponds a unique eigenvalue (notice that the converse is false). Thus, taking into account that $p^* \neq p_*$, we get $x^* \neq x_*$, and this proves our claim.

Now, it is enough show that the pair (Y, Y_0) satisfies the hypothesis of Lemma 3.7. Assume the contrary. Thus, there exists a compact subset K of Y which contains the point p_* and whose boundary, in Y, is empty. This compact set is relatively open in Y, therefore it is far away from its closed complement $Y \setminus K$. Actually, it is also far away from $\Sigma \setminus K$, since it is disjoint from the closed set $\Sigma_0 \setminus \{p_*\}$. Consequently, there exists an open subset W of $S^{k-1} \times \mathbb{R} \times \mathbb{R}$, that we may assume to be bounded, such that $W \cap \Sigma = K$ and

$$\partial W \cap \Sigma = \emptyset. \tag{3.3}$$

Because of Lemma 3.3, taking W smaller, if necessary, we may also suppose that the intersection $U = Z \cap W$ is mapped by Φ diffeomorphically onto a neighborhood V of the origin $0 \in \mathbb{R}^k$ (recall that $Z = S^{k-1} \times \{0\} \times \mathbb{R}$).

Step 1. Assume that N is smooth. According to Sard's Lemma, the subset Q of V consisting of the regular values for Φ is dense in V. Because of Lemma 3.5, for any $q \in Q$ there exists a connected component of $\Phi^{-1}(q)$ that intersects U and is either unbounded or contains at least two points of Z. Since $\Phi^{-1}(q) \cap U$ consists of a single point, this component must intersect ∂W. Therefore, being $\Sigma = \Phi^{-1}(0)$, the density of Q and the continuity of Φ imply $\partial W \cap \Sigma \neq \emptyset$, contradicting (3.3).

Hence, the pair (Y, Y_0) does satisfy the hypothesis of Lemma 3.7. This proves the assertion in the case when N is smooth. Consequently, recalling that the closure of a connected set is connected, one gets that the component of $\Phi^{-1}(0)$ containing p_* is unbounded or contains at least one point $p^* \in \Sigma_0 \setminus \{p_*\}$.

Step 2. Finally, assume that N is merely continuous. Then, it can be uniformly approximated by a sequence of smooth functions, and for any such a function the conclusion of Step 1 applies. This implies that also in this case we get $\partial W \cap \Sigma \neq \emptyset$, contradicting (3.3). Therefore the metric pair (Y, Y_0) satisfies the hypothesis of Lemma 3.7, and the proof of our main result is completed. □

Remark 3.9. *Under the assumptions of Theorem 3.8, let $\mathcal{C}$ denote the connected component of Σ containing $(x_*, 0, \lambda_*)$. Then, taking into account that Σ is closed in $S^{k-1} \times \mathbb{R} \times \mathbb{R}$ and that the closure of a connected set is connected, we deduce that $\mathcal{C}$ is either unbounded or contains a trivial solution $(x^*, 0, \lambda^*)$ with $x^* \neq x_*$.*

4. Examples

Here we provide some simple examples illustrating the assertion of Theorem 3.8, as well as examples proving that, in this theorem, the assumption that the eigenvalue λ_* is simple cannot be removed. In any considered case the dimension k of the Euclidean space will be "very low" (1, 2 or 3) with the operator L having at least one real eigenvalue. In each example, Σ and $\mathcal{E}$ denote, respectively, the set of solutions and the set of eigenpairs of the given problem. As in the previous sections, Σ_0 is the subset of Σ of the trivial solutions.

The following is a simple example of a linear problem in $\mathbb{R}^2$, in which the operator L has two real eigenvalues (thus, both necessarily simple). The set Σ of solutions is a smooth curve, diffeomorphic to a circle, which contains all the four trivial solutions (two for each eigenvalue). As we shall see, the projection of Σ onto $\mathcal{E}$ is a double covering map.

Example 4.1. In $\mathbb{R}^2$ consider the linear problem

$$\left\{\begin{aligned} x_1 - \varepsilon x_2 &= \lambda x_1, \\ -x_2 + \varepsilon x_1 &= \lambda x_2, \\ x_1^2 + x_2^2 &= 1. \end{aligned}\right. \tag{4.1}$$

Here, the operators are $L\colon (x_1, x_2) \mapsto (x_1, -x_2)$ and $N\colon (x_1, x_2) \mapsto (-x_2, x_1)$. The unperturbed problem has two simple eigenvalues, $\lambda_* = -1$ and $\lambda^* = 1$, with two corresponding pairs of antipodal unit eigenvectors:

$$\pm x_* = \pm(0,1) \quad \text{and} \quad \pm x^* = \pm(1,0).$$

As one can check (see Remark 3.1), the set $\mathcal{E}$ of the eigenpairs of (4.1) is the unit circle $\varepsilon^2 + \lambda^2 = 1$. So, any eigenpair (ε, λ) can be represented as $(\sin t, \cos t)$, with $t \in [0, 2\pi]$. Observe that, given t, the kernel of the linear operator

$$L + (\sin t)N - (\cos t)I$$

is the straight line containing the pair of antipodal unit eigenvectors

$$\pm(\cos(t/2), \sin(t/2)).$$

Therefore, as one can easily verify, Σ can be represented, parametrically, by the simple, regular, closed curve

$$\gamma\colon [0, 4\pi] \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}, \quad \gamma(t) = \Big(\big(\cos(t/2), \sin(t/2)\big), \sin t, \cos t\Big),$$

which (for $t = 0, \pi, 2\pi, 3\pi$) encounters all the four trivial solutions of (4.1).

If from the topological circle Σ we remove the four points of Σ_0, we get four arcs, each of them diffeomorphic to an open real interval and satisfying the assertion of Theorem 3.8. More precisely, any one of the four trivial solutions corresponds to a simple eigenvalue of L and has two arcs satisfying the assertion of Theorem 3.8.

Incidentally, we observe that the projection of Σ onto $\mathcal{E}$ is a double covering map, and the above parametrization γ of Σ is the lifting of the curve

$$\sigma\colon [0,4\pi] \to \mathcal{E}, \quad \sigma(t) = (\sin t, \cos t),$$

with the initial condition $\gamma(0) = \big((1,0),0,1\big)$.

The next system differs from the previous one only for a sign in the first equation and the unperturbed problem is the same. In spite of this, the structure of the set Σ of the solutions is drastically different: it is made up of four unbounded components, each of them containing just one trivial solution.

Example 4.2. In $\mathbb{R}^2$ consider the problem

$$\left\{\begin{array}{rcl} x_1 + \varepsilon x_2 &=& \lambda x_1, \\ -x_2 + \varepsilon x_1 &=& \lambda x_2, \\ x_1^2 + x_2^2 &=& 1. \end{array}\right. \tag{4.2}$$

where L is the same as in Example 4.1, while N maps (x_1, x_2) into (x_2, x_1). As in the first example, our problem has four trivial solutions (two for each eigenvalue):

$$\big((0,\pm 1),0,-1\big) \quad \text{and} \quad \big((\pm 1,0),0,1\big).$$

As one can verify, the set $\mathcal{E}$ of the eigenpairs of (4.2) is the hyperbola $\lambda^2 - \varepsilon^2 = 1$. Therefore, the two branches of $\mathcal{E}$, the lower and the upper in the $\varepsilon\lambda$-plane, can be represented parametrically as

$$(\varepsilon_*(t), \lambda_*(t)) = (\sinh t, -\cosh t) \quad \text{and} \quad (\varepsilon^*(t), \lambda^*(t)) = (\sinh t, \cosh t),$$

with $t \in \mathbb{R}$. Regarding the lower branch, given $t \in \mathbb{R}$, the kernel of the linear operator

$$L + \varepsilon_*(t)N - \lambda_*(t)I$$

is the straight line containing the pair of opposite (not necessarily unit) vectors

$$\pm v_*(t) = \pm(-\sinh t, 1 + \cosh t).$$

Analogously, concerning the upper branch, the kernel of

$$L + \varepsilon^*(t)N - \lambda^*(t)I$$

is the straight line containing

$$\pm v^*(t) = \pm(1 + \cosh t, \sinh t).$$

For example, a parametrization of the component containing the trivial solution $\big((1,0),0,1\big)$ is the following:

$$t \in \mathbb{R} \mapsto \big(v^*(t)/\|v^*(t)\|, \sinh t, \cosh t\big),$$

and the other three components of Σ can be parametrized in a similar manner.

In conclusion, the set Σ has four unbounded connected components, each of them diffeomorphic to $\mathbb{R}$ and containing one and only one trivial solution.

Notice that, here, the set $\Sigma \setminus \Sigma_0$ has eight connected components: two for each trivial solution, and both verifying the assertion of Theorem 3.8.

In the following example the space is again $\mathbb{R}^2$ and N is nonlinear. The operator L has two different real eigenvalues and, consequently, the problem admits four trivial solutions. The set Σ is the union of a topological circle and two straight lines. In spite of the fact that the circle contains only two trivial solutions with the same eigenvalue, Σ connects all the four trivial solutions; compatibly with Conjecture 1.3.

Example 4.3. In $\mathbb{R}^2$, consider the problem

$$\left\{\begin{array}{lcl} x_1+\varepsilon & = & \lambda x_1, \\ 2x_2 & = & \lambda x_2, \\ x_1^2+x_2^2 & = & 1. \end{array}\right. \tag{4.3}$$

Here L and N are $(x_1, x_2) \mapsto (x_1, 2x_2)$ and $(x_1, x_2) \mapsto (1, 0)$, respectively. The operator L has two simple eigenvalues, $\lambda_* = 1$ and $\lambda^* = 2$, with corresponding unit eigenvectors, $\pm x_* = \pm(1, 0)$ and $\pm x^* = \pm(0, 1)$.

Notice that any solution $\big((x_1, x_2), \varepsilon, \lambda\big)$ of (4.3) must have at least one of the following two properties: $x_2 = 0$ (and, consequently, $x_1 = \pm 1$) or $\lambda = 2$ (and, therefore, $x_1 = \varepsilon$ with $|\varepsilon| \leq 1$).

In the first case (with $x_2 = 0$) we get two straight lines in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$:

$$\ell_1 = \big\{\big((-1, 0), \varepsilon, 1-\varepsilon\big) : \varepsilon \in \mathbb{R}\big\} \quad \text{and} \quad \ell_2 = \big\{\big((1, 0), \varepsilon, 1+\varepsilon\big) : \varepsilon \in \mathbb{R}\big\},$$

lying in the two different planes of equations $(x_1, x_2) = (-1, 0)$ and $(x_1, x_2) = (1, 0)$, respectively.

As one can easily check, in the second case (with $\lambda = 2$), the set of solutions can be parametrized as follows:

$$\mathcal{C} = \big\{\big((\sin t, \cos t), \sin t, 2\big) : t \in [0, 2\pi]\big\}, \tag{4.4}$$

which shows that $\mathcal{C}$ is diffeomorphic to a circle and contains two of the four trivial solutions of problem (4.3), both with the eigenvalue $\lambda_* = 2$. Since this connected set is bounded, it seems to furnish a counterexample to the Conjecture 1.3, but this is not the case. In fact, the set Σ of all the solutions of (4.3), which is the union of ℓ_1, ℓ_2 and $\mathcal{C}$ is connected, unbounded, and contains all the four trivial solutions of (4.3). In particular, the connectedness of Σ follows from $\ell_1 \cap \mathcal{C} = \big\{\big((-1, 0), -1, 2\big)\big\}$ and $\ell_2 \cap \mathcal{C} = \big\{\big((1, 0), 1, 2\big)\big\}$.

The simplest example that one can obtain is when the dimension k of the space is one. In this case the unit sphere is $S^0 = \{-1, 1\}$ and, whatever is $N \colon S^0 \to \mathbb{R}$, the set Σ consists of two unbounded connected components.

Example 4.4. Let $\lambda_* \in \mathbb{R}$ be given and, in $\mathbb{R}$, consider the problem

$$\left\{\begin{array}{rcl} \lambda_* x + \varepsilon N(x) & = & \lambda x, \\ x & = & \pm 1, \end{array}\right. \tag{4.5}$$

where $N \colon \{-1, 1\} \to \mathbb{R}$ is arbitrary. Clearly, given any $\varepsilon \in \mathbb{R}$, one gets two solutions:

$$(1, \varepsilon, \lambda_* + \varepsilon N(1)) \quad \text{and} \quad (-1, \varepsilon, \lambda_* - \varepsilon N(-1)).$$

Thus the set Σ of all the solutions of problem (4.5) is composed by two straight lines,

$$\big\{\big(1, \varepsilon, \lambda_* + \varepsilon N(1)\big) : \varepsilon \in \mathbb{R}\big\} \quad \text{and} \quad \big\{\big(-1, \varepsilon, \lambda_* - \varepsilon N(-1)\big) : \varepsilon \in \mathbb{R}\big\},$$

placed in two different planes of $\mathbb{R}^3$, whose equations are $x = 1$ and $x = -1$, respectively.

We close with an example in $\mathbb{R}^3$ showing that, in Conjecture 1.3, the assumption that the eigenvalue λ_* is simple cannot be removed. Probably it could be replaced with the hypotheses that the algebraic and geometric multiplicities of λ_* are the same and odd. Since here the space is 3-dimensional and N is linear, according to Proposition 3.2 the set Σ is necessarily unbounded.

Example 4.5. In $\mathbb{R}^3$, consider the problem

$$\left\{\begin{array}{rcl} \varepsilon x_2 & = & \lambda x_1, \\ 2x_1 - \varepsilon x_1 & = & \lambda x_2, \\ 2x_3 + \varepsilon x_1 & = & \lambda x_3, \\ x_1^2 + x_2^2 + x_3^2 & = & 1. \end{array}\right. \tag{4.6}$$

Here, L and N are, respectively, $(x_1, x_2, x_3) \mapsto (0, 2x_1, 2x_3)$ and $(x_1, x_2, x_3) \mapsto (x_2, -x_1, x_1)$. The operator L has two real eigenvalues: $\lambda_* = 0$, with geometric multiplicity 1 and algebraic multiplicity 2; and $\lambda^* = 2$, which is simple.

One can easily verify that the set $\mathcal{E}$ of the eigenpairs has two connected components: the circle $(\varepsilon - 1)^2 + \lambda^2 = 1$ and the straight line $\lambda = 2$.

Observe that the disconnected set $\mathcal{E}$ is unbounded, as it should be, according to Proposition 3.2. Consequently, Σ is as well unbounded and disconnected.

Any eigenpair of the above circle can be represented as

$$(\varepsilon(t), \lambda(t)) = (1 - \cos t, \sin t), \quad \text{with} \quad t \in [0, 2\pi].$$

Moreover, given any $t \in [0, 2\pi]$, the kernel of the linear operator

$$L + \varepsilon(t)N - \lambda(t)I$$

is spanned by the (nonzero) vector

$$w(t) = \big(\sin(t/2), \cos(t/2), c(t)\big),$$

where $c(t)$ is defined by $2c(t) + \varepsilon(t)\sin(t/2) = \lambda(t)c(t)$, in order to satisfy the third equation of (4.6). Thus, the connected component $\mathcal{C}$ of Σ containing the trivial solution $p_* = \big((0,1,0), 0, 0\big)$ can be parametrized as follows:

$$\sigma\colon [0, 4\pi] \to S^2 \times \mathbb{R} \times \mathbb{R}, \quad \sigma(t) = \big(w(t)/\|w(t)\|, 1 - \cos t, \sin t\big).$$

Consequently, $\mathcal{C}$ is diffeomorphic to S^1 and contains both the trivial solutions corresponding to the eigenvalue $\lambda_* = 0$: $\big((0,1,0), 0, 0\big)$ for $t = 0$ (or, equivalently, for $t = 4\pi$) and $\big((0,-1,0), 0, 0\big)$ for $t = 2\pi$.

Clearly none of the two trivial solutions corresponding to the eigenvalue $\lambda^* = 2$ is in $\mathcal{C}$, and this shows that, in Conjecture 1.3, the assumption that the eigenvalue λ_* is simple cannot be omitted.

We observe that the projection of $\mathcal{C}$ onto the circle $(\varepsilon - 1)^2 + \lambda^2 = 1$ is a double covering map, and the above parametrization σ of $\mathcal{C}$ is the lifting of the curve $t \in [0, 4\pi] \mapsto (1 - \cos t, \sin t)$ with initial condition $\sigma(0) = p_* = \big((0,1,0), 0, 0\big)$.

Associated to the eigenvalue $\lambda^* = 2$ of L, the set Σ has two components: the straight lines

$$\big\{\big((0,0,1), \varepsilon, 2\big) : \varepsilon \in \mathbb{R}\big\} \quad \text{and} \quad \big\{\big((0,0,-1), \varepsilon, 2\big) : \varepsilon \in \mathbb{R}\big\}.$$

We may conclude that Σ has three components: a bounded one, corresponding to $\lambda_* = 0$, and two unbounded, corresponding to $\lambda^* = 2$.

Acknowledgement. We thank the anonymous referees for their useful remarks.

References

[1] Alexander J.C., *A primer on connectivity*, Lecture Notes in Math., Vol. 886, Springer Verlag, Berlin-New York, 1981, 455–483.

[2] Benevieri P. - Calamai A. - Furi M. - Pera M.P., *On the persistence of the eigenvalues of a perturbed Fredholm operator of index zero under nonsmooth perturbations*, Z. Anal. Anwend. **36** (2017), no. 1, 99–128.

[3] Benevieri P. - Calamai A. - Furi M. - Pera M.P., *Global continuation of the eigenvalues of a perturbed linear operator*, Ann. Mat. Pura Appl. **197** (2018), no. 4, 1131–1149.

[4] Chiappinelli R., *Isolated Connected Eigenvalues in Nonlinear Spectral Theory*, Nonlinear Funct. Anal. Appl. **8** (2003), no. 4, 557–579.

[5] Chiappinelli R., *Approximation and convergence rate of nonlinear eigenvalues: Lipschitz perturbations of a bounded self-adjoint operator*, J. Math. Anal. Appl. **455** (2017), no. 2, 1720–1732.

[6] Chiappinelli R., *What do you mean by "nonlinear eigenvalue problems"?*, Axioms **7** (2018), Paper no. 39, 30 pp.

[7] Chiappinelli R. - Furi M. - Pera M.P., *Normalized Eigenvectors of a Perturbed Linear Operator via General Bifurcation*, Glasg. Math. J. **50** (2008), no. 2, 303–318.

[8] Chiappinelli R. - Furi M. - Pera M.P., *Topological persistence of the normalized eigenvectors of a perturbed self-adjoint operator*, Appl. Math. Lett. **23** (2010), no. 2, 193–197.

[9] Chiappinelli R. - Furi M. - Pera M.P., *Persistence of the normalized eigenvectors of a perturbed operator in the variational case*, Glasg. Math. J. **55** (2013), no. 3, 629–638.

[10] Chiappinelli R. - Furi M. - Pera M.P., *Topological Persistence of the Unit Eigenvectors of a Perturbed Fredholm Operator of Index Zero*, Z. Anal. Anwend. **33** (2014), no. 3, 347–367.

[11] Furi M. - Pera M.P., *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory*, Pacific J. Math. **160** (1993), no. 3, 219–244.

[12] Guillemin V. - Pollack A., *Differential Topology*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[13] Hirsch M.W., *Differential Topology*, Graduate Texts in Math., Vol. 33, Springer Verlag, Berlin, 1976.

[14] Kuratowski C., *Topology*, Vol. 2, Academic Press, New York, 1968.

[15] Milnor J.W., *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.

Pierluigi Benevieri - Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, São Paulo - SP - Brasil - CEP 05508-090 - *E-mail address:* pluigi@ime.usp.br

Alessandro Calamai - Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche, Via Brecce Bianche, I-60131 Ancona, Italy - *E-mail address:* calamai@dipmat.univpm.it

Massimo Furi - Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Florence, Italy - *E-mail address:* massimo.furi@unifi.it

Maria Patrizia Pera - Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Florence, Italy - *E-mail address:* mpatrizia.pera@unifi.it