Second order differential equations on manifolds and forced oscillations

Massimo FURI
Dipartimento di Matematica Applicata "G. Sansone"
Universtità di Firenze
Via di S. Marta 3,
I-50139 Firenze
Italy

Notes by

Marco SPADINI

Abstract

These notes are a brief introductory course to second order differential equations on manifolds and to some problems regarding forced oscillations of motion equations of constrained mechanical systems. The intention is to give a comprehensive exposition to the mathematicians, mainly analysts, that are not particularly familiar with the formalism of differential geometry. The material is divided into five sections. The background needed to understand the subject matter contained in the first three is mainly advanced calculus and linear algebra. The fourth and the fifth sections require some knowledge of degree theory and functional analysis.

We begin with a review of some of the most significant results in advanced calculus, such as the Inverse Function Theorem and the Implicit Function Theorem, and we proceed with the notions of smooth map and diffeomorphism between arbitrary subsets of Euclidean spaces. The second section is entirely devoted to differentiable manifold embedded in Euclidean spaces and tangent bundles. In the third section, dedicated to differential equations on manifolds, a special attempt has been done to introduce the notion of second order differential equation in a very natural way, with a formalism familiar to any analyst. Section four regards the concept of degree of a tangent vector field on a manifold and the Euler-Poincaré characteristic. Finally, in the last section, we deal with forced oscillations for constrained mechanical systems and bifurcation problems. Some recent results and open problems are presented.

1. Notation and preliminaries

We begin this section by briefly reviewing some fundamental notions in calculus of several variables.

Recall that a real function f defined on an open subset U of $\mathbb{R}^k$ is said to be of class C^0 (in U) if it is continuous. Inductively, f is said to be of class C^n, $n \geq 1$, if all its first

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partial derivatives are C^{n-1}. This definition is immediately extended to a vector valued mapping $f : U \to \mathbb{R}^s$ by simply requiring that all its component functions, f_1, f_2,..., f_s, have the same property. Finally, $f : U \to \mathbb{R}^s$ is of class C^∞, or smooth, if it is C^n for all $n \in \mathbb{N}$.

Exercise 1.1. Prove by induction that any C^n map is C^{n-1}.
Hint. The assertion is true for $n = 1$, as a consequence of the Mean Value Theorem.

We point out that this definition of a C^n map, which is clearly equivalent to the classical one, has the advantage that it can be used to give simple inductive proofs of the well-known fact that (whenever it makes sense) the sum, the product, the quotient, and the composition of C^n maps is still a C^n map (defined on an appropriate open domain). The reader is invited to perform the details of these proofs (we suggest to follow the indicated order).

Clearly the notion of C^α map, $\alpha \in \mathbb{N} \cup \{\infty\}$, does not depend on the norms in the source and target spaces. This is due to the fact that in a finite dimensional vector space all the norms are topologically equivalent, and the partial derivative is defined as a limit, a purely topological concept. Therefore, unless otherwise specified, the norm we consider in the spaces $\mathbb{R}^m$, $m \in \mathbb{N}$, is the standard Euclidean one, indicated by $|\cdot|$. The inner product of two vectors $u, v \in \mathbb{R}^m$ will be denoted by $\langle u, v\rangle$; so that $|x|^2 = \langle x, x\rangle$ for all $x \in \mathbb{R}^m$. A *Euclidean set* is just a subset of any Euclidean space $\mathbb{R}^m$.

Exercise 1.2. Let $f : J \to \mathbb{R}$ be a smooth function defined on an open real interval. Assume $f'(x) \neq 0$ for all $x \in J$. Prove that the inverse of f, $f^{-1} : f(J) \to \mathbb{R}$, is smooth.
Hint. Use the inductive definition of C^n function and the well-known formula $(f^{-1})'(y) = 1/f'(f^{-1}(y))$.

Let E, F and G be finite dimensional vector spaces. The vector space of linear operators from E into F will be denoted by $L(E, F)$, or briefly by $L(E)$ in the case where $F = E$. The open subset $GL(E)$ of $L(E)$ stands for the group of non-singular linear endomorphisms of E (the so-called automorphisms of E). The composition of two linear operators $A : E \to F$ and $B : F \to G$ is simply denoted with BA, instead of $B \circ A$, as will be done in the nonlinear case (to avoid confusion with the product of real functions). If the spaces E and F possess standard bases, as in the case when $E = \mathbb{R}^k$ and $F = \mathbb{R}^s$, an element $A \in L(E, F)$ will be identified with its representing matrix. In this way, the notation BA for the composition of two linear operators is consistent with the corresponding row-by-column product of two matrices.

A map $f : U \to \mathbb{R}^s$, defined on an open subset of $\mathbb{R}^k$, is said to be (Fréchet) *differentiable* at $p \in U$ if there exists (and in this case is unique) a linear operator $f'(p) \in L(\mathbb{R}^k, \mathbb{R}^s)$, called the derivative of f at p, in such a way that the map $\varepsilon : (U - p) \setminus \{0\} \to \mathbb{R}^s$, defined by the formula

$$f(p + h) - f(p) = f'(p)h + |h|\varepsilon(h),$$

tends to zero as $h \to 0$. In this case it is convenient to extend the map ε to $U - p$ by putting $\varepsilon(0) = 0$, so that ε turns out to be continuous at the origin of $\mathbb{R}^k$.

We recall that if f is C^1 on U, then it is also Fréchet differentiable on the same domain (i.e. differentiable at every $p \in U$). It is a straightforward consequence of the definition that, if f is differentiable at p, the value that $f'(p)$ takes on a vector $h \in \mathbb{R}^k$ coincides with the directional derivative of f at p along h (and this also shows the uniqueness of the Fréchet derivative). That is, one has

$$f'(p)h = df(p,h) := \lim_{t \to 0^+} \frac{f(p+th) - f(p)}{t}.$$

A generalized version of this formula (Lemma 1.1 below) will play a crucial role in the extension of the notion of derivative for maps which are not (necessarily) defined on open sets.

Since, as pointed out before, $L(\mathbb{R}^k, \mathbb{R}^s)$ is identified with the space of $s \times k$ real matrices, the symbol $f'(p)$ stands also for the matrix representing the derivative of f at p (with respect to the standard bases of $\mathbb{R}^k$ and $\mathbb{R}^s$, respectively): the so called Jacobian matrix of f at p, which, we recall, depends only on the partial derivatives at p of the component functions of f. With this convention, the well-known chain rule for differentiable maps reads as in the one-dimensional case. Namely

$$(g \circ f)'(p) = g'(f(p))f'(p),$$

with the usual row-by-column product between the two matrices $g'(f(p))$ and $f'(p)$.

Notice that $f : U \subseteq \mathbb{R}^k \to \mathbb{R}^s$ is C^n if and only if the map $f' : U \to L(\mathbb{R}^k, \mathbb{R}^s)$, given by $p \mapsto f'(p)$, is C^{n-1} (i.e. all the entries of matrix valued map f' are C^{n-1} real functions). Thus, from the chain rule and the fact that the sum and the product of C^{n-1} real functions is again a C^{n-1} real function, one can immediately deduce that the composition of C^n maps is a C^n map.

As above, let $f : U \to \mathbb{R}^s$ be defined on an open subset U of $\mathbb{R}^k$. Let p be a point in U and E a subspace of $\mathbb{R}^k$. The partial derivative with respect to E of f at p, denoted by $D_E f(p)$, is just the Fréchet derivative at the origin of the composite map $h \mapsto f(p+h)$, regarded as defined on the open subset $(U-p) \cap E$ of E. Obviously, $D_E f(p)$ is a linear operator from E into $\mathbb{R}^s$ and, when f is differentiable at p, as a consequence of the chain rule, coincides with the restriction to E of $f'(p)$.

In the particular case where $\mathbb{R}^k = E_1 \times E_2$, the partial derivatives of f at $p = (p_1, p_2)$ with respect to the subspaces $E_1 \times \{0\}$ and $\{0\} \times E_2$ will be simply denoted by $D_1 f(p)$ and $D_2 f(p)$, respectively. Observe that if f is differentiable at p, one has

$$f'(p)(h_1, h_2) = f'(p)(h_1, 0) + f'(p)(0, h_2).$$

Therefore, since $D_1 f(p)$ and $D_2 f(p)$ coincide, respectively, with the restrictions of $f'(p)$ to $E_1 \times \{0\}$ and $\{0\} \times E_2$, we get

$$f'(p)(h_1, h_2) = D_1 f(p)h_1 + D_2 f(p)h_2.$$

We state without proof two fundamental results in differential calculus which will turn out to be useful later. They can be proved by means of the well-known Banach contraction principle and the mean value theorem for vector valued maps. It is not hard to show, however, that these results can be deduced from each other. Therefore, they may be considered equivalent.

Inverse Function Theorem. *Let $f : U \to \mathbb{R}^k$ be a C^α map on an open subset U of $\mathbb{R}^k$. Assume that for some $p \in U$ the derivative $f'(p)$ is an isomorphism. Then there exists an open neighborhood W of p with the following properties:*

1. *f is one-to-one on W;*
2. *$f(W)$ is open;*
3. *$f^{-1} : f(W) \to \mathbb{R}^k$ is C^α.*

Implicit Function Theorem. *Let $f : U \to \mathbb{R}^s$ be a C^α map on an open subset U of the product space $\mathbb{R}^k \times \mathbb{R}^s$ and let $(p, q) \in U$ be such that $f(p, q) = 0$. Assume that the partial derivative $D_2 f(p, q)$ is an isomorphism. Then, in a convenient neighborhood of (p, q), $f^{-1}(0)$ is the graph of a C^α map, $\varphi : W \to \mathbb{R}^s$, defined in a neighborhood W of p in $\mathbb{R}^k$.*

Exercise 1.3. Prove that if f satisfies the assumptions of the Inverse Function Theorem, then $(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}$ for all $y \in f(W)$.
Hint. Apply the chain rule to the composition $f \circ f^{-1}$.

Exercise 1.4. Prove that the map which assigns to any nonsingular matrix $A \in GL(\mathbb{R}^k)$ its inverse $A^{-1} \in L(\mathbb{R}^k)$ is smooth (thus, it is a diffeomorphism of $GL(\mathbb{R}^k)$ onto itself).

Exercise 1.5. Let $f : U \to \mathbb{R}^k$ be a smooth one-to-one map on an open subset of $\mathbb{R}^k$. Prove that if $f'(x)$ is invertible for all $x \in U$, then $f^{-1} : f(U) \to \mathbb{R}^k$ is smooth.
Hint. Use the inductive definition of C^n map and the formula for the first derivative of f^{-1}.

The second derivative at a point $p \in U$ of a differentiable map $f : U \subseteq \mathbb{R}^k \to \mathbb{R}^s$, denoted by $f''(p)$, is just the derivative at p of $f' : U \to L(\mathbb{R}^k, \mathbb{R}^s)$. Thus, $f''(p)$ belongs to the space $L(\mathbb{R}^k, L(\mathbb{R}^k, \mathbb{R}^s))$ of the linear operators from $\mathbb{R}^k$ into $L(\mathbb{R}^k, \mathbb{R}^s)$. It is easy to check that $L(\mathbb{R}^k, L(\mathbb{R}^k, \mathbb{R}^s))$ is canonically isomorphic to the space $L^2(\mathbb{R}^k; \mathbb{R}^s)$ of the bilinear operators from $\mathbb{R}^k$ into $\mathbb{R}^s$. Throughout this notes it is convenient to regard $f''(p)$ as such a bilinear operator.

To see how $f''(p)$ operates on a pair of vectors $(u, v) \in \mathbb{R}^k \times \mathbb{R}^k$ one may proceed as follows. Fix U and compute, for all x in U, the directional derivative $f'(x)u$. One gets again a map, $x \mapsto f'(x)u$, from U into $\mathbb{R}^s$. Then, take the directional derivative at p, in the direction v, of this map. The result is just $f''(p)(u, v)$. Derivatives of order greater than two may be computed in a similar manner.

If $f : U \subseteq \mathbb{R}^k \to \mathbb{R}^s$ is a C^2 map, then, for any $p \in U$, $f''(p)$ turns out to be a symmetric operator (i.e. $f''(p)(u, v) = f''(p)(v, u)$ for any $(u, v) \in \mathbb{R}^k \times \mathbb{R}^k$). This is a simple consequence of the classical Schwarz theorem on the inversion of partial derivatives. In fact, we have seen before that $f''(p)(u,v)$ can be computed as two successive directional derivatives. Thus, if we consider the C^2 map $\varphi(t,s) = f(p+tu+sv)$, which is defined in a suitable neighborhood of the origin of $\mathbb{R}^2$, we get

$$f''(p)(u,v) = \frac{\partial^2\varphi}{\partial t \partial s}(0,0) = \frac{\partial^2\varphi}{\partial s \partial t}(0,0) = f''(p)(v,u).$$

We are now interested in extending the notion of a C^α map, $\alpha \in \mathbb{N}\cup\{\infty\}$, to the case where $f : X \to \mathbb{R}^s$ is defined on an arbitrary subset X of $\mathbb{R}^k$. In this case, if $p \in X$ is not in the interior of X, the notion of partial derivative of f at p does not make sense any more. Consequently, the above inductive definition of C^n maps on open sets breaks down in this general situation. The needed extension is obtained, loosely speaking, by forcing down the well-known *hereditary property* of C^n maps on open sets. In other words, by requiring that the restriction of a C^n map to any subset of its domain is again a C^n map. We observe that the following definition preserves also the *local property* of C^n maps: that is, any map which is locally C^n is also globally C^n.

Definition 1.1. A map $f : X \to Y$, from a subset of $\mathbb{R}^k$ into a subset of $\mathbb{R}^s$, is said to be C^α, $\alpha \in \mathbb{N}\cup\{\infty\}$, if for any $p \in X$ there exists a C^α map $g : U \to \mathbb{R}^s$, defined on an open neighborhood of p, such that $f(x) = g(x)$ for all $x \in U \cap X$.

In other words, $f : X \subseteq \mathbb{R}^k \to Y \subseteq \mathbb{R}^s$ is C^α if it can be locally extended as a map into $\mathbb{R}^s$ (not merely into Y) to a C^α map defined on an open subset of $\mathbb{R}^k$. To understand why one must seek the extension of f as a map into $\mathbb{R}^s$, observe that the identity $i : [0,1] \to [0,1]$ is not the restriction of any C^1 function $g : U \to [0,1]$ defined on an open neighborhood U of $[0,1]$.

A practical way to assign a C^α map on an arbitrary subset X of $\mathbb{R}^k$ is writing down a C^α map whose "natural domain" is an open set containing X. For example, a real function obtained by the sum, the product, the quotient and the composition of smooth functions defined on open sets is a smooth map on a set which is still open. Thus, if this function is well defined on X, its restriction to X is smooth.

Remark 1.1. Using the well-known fact that any family of open subsets of $\mathbb{R}^k$ admits a subordinate smooth partition of unity, it is easy to show that any C^α map on $X \subseteq \mathbb{R}^k$ is actually the restriction of a C^α map defined on an open neighborhood of X.

As a straightforward consequence of the definition one gets that, given $X \subseteq \mathbb{R}^k$, the identity $i : X \to X$ is a smooth map. Moreover, we observe that the composition of C^α maps between arbitrary Euclidean sets is again a C^α map, since the same is true for maps defined on open sets. Thus, one can view Euclidean sets as objects of a category, whose morphisms are the C^α maps. The study of such a category is the goal of differential topology (in Euclidean spaces).

We recall that in any category one has the concept of *isomorphism*, which by definition is just an invertible morphism. The isomorphisms of some categories have specific names. In the case of topological spaces, for example, they are called homeomorphisms. In the category of Euclidean set with C^α-maps as morphisms, the name of an isomorphism is *diffeomorphism*, or C^α-*diffeomorphism*, to be more specific.

Definition 1.2. A C^α map $f : X \to Y$, from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$, is said to be a C^α-*diffeomorphism* if it is one-one, onto, and f^{-1} is C^α. In this case X and Y are said to be C^α-*diffeomorphic*.

A straightforward consequence of the definition of diffeomorphism (and the hereditary property of C^α-maps) is that

the restriction of a C^α-diffeomorphism is again a C^α-diffeomorphism onto its image.

A very important example of a C^α-diffeomorphism is given by the graph map associated with a C^α map $f : X \to \mathbb{R}^s$ defined on an arbitrary subset of $\mathbb{R}^k$. In fact, let G_f denote the graph of f; that is, the subset of $\mathbb{R}^k \times \mathbb{R}^s$ consisting of the ordered pairs (x, y) given by the equation $y = f(x)$. The graph map of f, $\hat{f} : X \to G_f$, defined by $\hat{f}(x) = (x, f(x))$, is clearly C^α, one-one and onto. Observe now that $\hat{f}^{-1}$ is just the restriction to G_f of the projection $(x, y) \mapsto x$ of $\mathbb{R}^k \times \mathbb{R}^s$ onto the first factor, which is a linear map (and, consequently, smooth). We have therefore proved that

the graph of a C^α map is C^α-diffeomorphic to its domain.

Perhaps, the simplest example of a smooth homeomorphism which is not a diffeomorphism is given by the map $t \mapsto t^3$ from $\mathbb{R}$ onto itself. In fact, observe that if a smooth map $f : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, as a consequence of the chain rule, one has $f'(t) \neq 0$ for all $t \in \mathbb{R}$; and the above map does not have this property.

More generally, if $f : U \to f(U) \subseteq \mathbb{R}^s$ is a C^1-diffeomorphism from an open subset U of $\mathbb{R}^k$ onto its image $f(U)$, then the derivative of f at any point $p \in U$ must be injective. To see this, let g be any C^1-extension of $f^{-1} : f(U) \to \mathbb{R}^k$ to an open subset W of $\mathbb{R}^s$ containing $f(U)$. We have $g(f(x)) = x$ for all $x \in U$. Thus, by the chain rule, $g'(f(p))f'(p)$ is the identity in $L(\mathbb{R}^k)$, and this shows that $f'(p)$ is one-one.

Later we will give a notion of derivative at a point $p \in X$ for any C^1 map $f : X \to \mathbb{R}^s$ defined on an arbitrary subset X of $\mathbb{R}^k$. We shall prove that this extended notion of derivative still has the property that if $f : X \to \mathbb{R}^s$ is a C^1-diffeomorphism onto its image, then the derivative of f at any point $p \in X$ is an injective linear operator (defined on an appropriate subspace $T_p(X)$ of $\mathbb{R}^k$).

Let $X \subseteq \mathbb{R}^2$ denote the graph of the absolute value function. Clearly the restriction to X of the first projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is a smooth homeomorphism, but not a C^1-diffeomorphism, since its inverse $x \mapsto (x, |x|)$ is not C^1. This, however, does not prove that X is not C^1-diffeomorphic to $\mathbb{R}$. We will see later, with the aid of the extended notion of derivative, that no C^1-diffeomorphism is possible between X and $\mathbb{R}$ (as our intuition suggests).

Let X be a subset of $\mathbb{R}^k$ and let $p \in X$. A unit vector $v \in S^{k-1} = \{x \in \mathbb{R}^k : |x| = 1\}$ is said to be tangent to X at p if there exists a sequence $\{p_n\}$ in $X \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$. Observe that, because of the compactness of the unit sphere S^{k-1}, if p is an accumulation point of X, there exists at least a unit vector tangent to X at p. The following definition of tangent cone is based on the above notion of unit tangent vector. It is fairly easy to check (see Exercise 1.6 below) that it is equivalent to the classical one introduced by Bouligand in [Bo] (see also [Se], p. 149, for a precursor of this notion). However, we find the definition below more convenient to prove some useful properties of C^α maps defined on arbitrary subsets of Euclidean spaces.

Definition 1.3. Let X be a subset of $\mathbb{R}^k$ and let $p \in X$. If p is an isolated point of X, then the *tangent cone* of X at p, $C_p(X)$, is just the trivial subspace $\{0\}$ of $\mathbb{R}^k$. If p is an accumulation point of X, then $C_p(X)$ is the cone generated by the set of unit tangent vectors. That is, $C_p(X) = \{\lambda v : \lambda \geq 0, v \in S^{k-1}$ is tangent to X at $p\}$. The *tangent space* of X at p, $T_p(X)$, is the vector space spanned by $C_p(X)$.

Observe that the notion of tangent cone is local. That is, if two sets X and Y coincide in a neighborhood of a common point p, they have the same tangent cone. Another important property is *translation invariance*: $T_p(X) = T_{x+p}(x + X)$, for all $x \in \mathbb{R}^k$.

Exercise 1.6. Prove that the tangent cone defined above coincides with the Bouligand cone. That is, given $X \subseteq \mathbb{R}^k$ and $p \in X$, a vector $v \in \mathbb{R}^k$ is in $C_p(X)$ if and only if

$$\liminf_{t \to 0+} \frac{\operatorname{dist}(p + tv, X)}{t} = 0,$$

where $\operatorname{dist}(p + tv, X)$ denotes the distance between the point $p + tv$ and the set X.

Exercise 1.7. Let X be a subset of $\mathbb{R}$ and $p \in X$. Prove that for $C_p(X)$ we have only four possibilities: $\{0\}$, $\mathbb{R}$, $(-\infty, 0]$, $[0, +\infty)$.

Exercise 1.8. Let $X \subseteq \mathbb{R}^k$ and $p \in X$. Prove that $C_p(X)$ is closed in $\mathbb{R}^k$.

Exercise 1.9. Let $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^s$. Given $(p, q) \in X \times Y$, prove that

$$T_{(p,q)}(X \times Y) = T_p(X) \times T_q(Y).$$

Exercise 1.10. Prove that if X is convex, then $C_p(X)$ coincides with the closure of the set $\{\lambda(x - p) : \lambda \geq 0, x \in X\}$.

Exercise 1.11. Prove that a locally compact subset X of $\mathbb{R}^m$ is open if and only if $C_x(X) = \mathbb{R}^m$ for all $x \in X$. Find an example to show that this assertion is not true if the assumption that X is locally compact is removed.

Exercise 1.12. Let $f : U \to \mathbb{R}^s$ be a continuous map defined on an open subset of $\mathbb{R}^k$. Prove that f is differentiable at $p \in U$ if and only if the tangent space at $(p, f(p))$ to the graph G_f of f is a graph (in this case $T_{(p,f(p))}(G_f)$ is just the graph of $f'(p)$). Use this fact to give a possible definition of derivative for maps defined on arbitrary subsets of $\mathbb{R}^k$.

The following result is helpful for the computation of the tangent cone of a set defined by inequalities. Its proof, based on the Inverse Function Theorem, is left to the experienced reader.

Theorem 1.1. *Let $f : U \to \mathbb{R}^s$ be a C^1 map defined on an open subset of $\mathbb{R}^k$. Let $Y \subseteq \mathbb{R}^s$ and $p \in f^{-1}(Y)$. Assume that p is a regular point of f, i.e. the derivative $f'(p) : \mathbb{R}^k \to \mathbb{R}^s$ of f at p is surjective. Then*

$$C_p(f^{-1}(Y)) = \{v \in \mathbb{R}^k : f'(p)v \in C_{f(p)}(Y)\} = f'(p)^{-1}(C_{f(p)}(Y)).$$

To understand the meaning of Theorem 1.1, consider for example in $\mathbb{R}^2$ a set X defined by three inequalities of the type $f_1(x_1, x_2) \le 0$, $f_2(x_1, x_2) \le 0$, $f_3(x_1, x_2) \le 0$, where f_1, f_2, f_3 are C^1 real functions on $\mathbb{R}^2$. So, depending on the three functions, one can think of X as some kind of triangular patch. We assume that X is locally defined by at most two of the three above inequalities. In other words if, for example, $p = (p_1, p_2)$ satisfies the conditions $f_1(p) = 0$ and $f_2(p) = 0$, then $f_3(p)$ must be negative; so that the function f_3 does not contribute to the local definition of X in a neighborhood of p (in this case a convenient neighborhood of p is given by $\{(x_1, x_2) \in \mathbb{R}^2 : f_3(x_1, x_2) < 0\}$). According to Theorem 1.1, to compute $C_p(X)$ for such a (vertex) point p we proceed as follows. We define a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by considering the two functions f_1 and f_2 as components of f. The assumption that p is a regular point for f means that the two linear functionals $f_1'(p)$ and $f_2'(p)$ are linearly independent; therefore the intersection of the two closed half planes $f_1'(p)v \le 0$, $f_2'(p)v \le 0$ is a nontrivial non-flat convex angle (as in the vertex of a triangle). By Theorem 1.1, this angle is just the tangent cone to X at p.

The case $f_1(p) = 0$, $f_2(p) < 0$ and $f_3(p) < 0$ may be treated analogously. In this case, in fact, only the function f_1 contributes to the local definition of X. Therefore the regularity assumption of Theorem 1.1 means that the gradient of f_1 at p, $\nabla f_1(p)$, does not vanish. Hence, for such a point, again as a consequence of Theorem 1.1, $C_p(X)$ is the half plane $\{v \in \mathbb{R}^2 : f_1'(p)v \le 0\}$.

A slightly more general situation may be considered. We could define $X \subseteq \mathbb{R}^2$ by a finite number of inequalities of the form

$$f_1(x_1, x_2) \le 0,\ f_2(x_1, x_2) \le 0, \dots,\ f_n(x_1, x_2) \le 0,$$

obtaining, in this case, some kind of polygonal patch in $\mathbb{R}^2$. As before, in order that the regularity assumption be satisfied, X must be locally defined by at most two of these n inequalities (there are no surjective linear operators from $\mathbb{R}^2$ into $\mathbb{R}^s$ if $s > 2$). Now, the analysis of this apparently more complicated example proceeds as before, and one gets three possibilities for the tangent cone: $\mathbb{R}^2$, a half plane, a convex nontrivial angle. Therefore, in any of these cases the tangent space is $\mathbb{R}^2$.

Another interesting simple example is given by considering in $\mathbb{R}^3$ the hemisphere

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.$$

Applying Theorem 1.1 to a point $p = (p_1, p_2, p_3)$ which does not belong to the equator (i.e. with $p_3 > 0$) we obtain

$$C_p(X) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : p_1 v_1 + p_2 v_2 + p_3 v_3 = 0\},$$

and, consequently, $T_p(X) = C_p(X)$. On the other hand, if p is on the equator, one has

$$C_p(X) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : p_1 v_1 + p_2 v_2 = 0, v_3 \geq 0\}.$$

Therefore $T_p(X)$ is the plane

$$\{(v_1, v_2, v_3) \in \mathbb{R}^3 : p_1 v_1 + p_2 v_2 = 0\},$$

which can be obtained, as in the case of p not on the equator, by simply linearizing (at p) the equation $x_1^2 + x_2^2 + x_3^2 = 1$ (the inequality $x_3 \geq 0$ does not contribute in defining $T_p(X)$).

Roughly speaking, what we may deduce from the analysis of the previous examples can be summarized and generalized as follows. The tangent cone at $p \in X$ of a subset X of $\mathbb{R}^k$ "regularly" and "essentially" defined (in a neighborhood of p) by a system consisting of a finite number of inequalities, $f_1(x) \leq 0$, $f_2(x) \leq 0, \ldots f_r(x) \leq 0$, and a finite number of equations, $g_1(x) = 0$, $g_2(x) = 0, \ldots,$ $g_s(x) = 0$, is obtained by "linearizing" at p the given system. The tangent space is defined just by the equations of the linearized system. More precisely, if p is a regular point for the map $h : \mathbb{R}^k \to \mathbb{R}^{r+s}$ obtained by considering (as components) the real functions $f_1, \ldots,$ f_r, $g_1, \ldots g_s$, and if the condition $h(p) = 0 \in \mathbb{R}^{r+s}$ is satisfied, then as a consequence of Theorem 1.1, we get

$$C_p(X) = \{v \in \mathbb{R}^k : f_1'(p)v \leq 0, \ldots,\ f_r'(p)v \leq 0;\ g_1'(p)v = 0, \ldots,\ g_s'(p)v = 0\},$$

and

$$T_p(X) = \{v \in \mathbb{R}^k : g_1'(p)v = 0, \ldots,\ g_s'(p)v = 0\}.$$

Given a C^1 map $f : X \to Y$ from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$ and a point $p \in X$, we shall define a linear operator $f'(p)$ from $T_p(X)$ into $T)f(p)(Y)$, called the derivative of f at p, which maps the tangent cone of X at p into the tangent cone of Y at $f(p)$. This derivative will turn out to satisfy the two well-known functorial properties of the Fréchet derivative. That is, "the derivative of the identity map $i : X \to X$ is the identity on $T_p(X)$", and "the derivative at $p \in X$ of the composition of two C^1 maps, $f : X \to Y$ and $g : Y \to Z$, is the composition $g'(f(p))f'(p)$ of the two derivatives". To achieve this, we need the following three lemmas. The first one extends the well-known fact that the Fréchet derivative can be computed as a directional derivative. Its elementary proof is left to the reader.

Lemma 1.1. *Let $f : U \to \mathbb{R}^s$ be defined on an open subset of $\mathbb{R}^k$ and differentiable at $p \in U$. If $v \in S^{k-1}$ is a unit vector, then*

$$f'(p)v = \lim_{n\to\infty} \frac{f(p_n) - f(p)}{|p_n - p|},$$

where $\{p_n\}$ is any sequence in $U \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$.

Lemma 1.2. *Let $f : U \to \mathbb{R}^s$ be defined on an open subset of $\mathbb{R}^k$ and differentiable at $p \in U$. If f maps a subset X of U containing p into a subset Y of $\mathbb{R}^s$, then $f'(p)$ maps $C_p(X)$ into $C_{f(p)}(Y)$. Consequently, because of the linearity of $f'(p)$, it also maps $T_p(X)$ into $T_{f(p)}(Y)$.*

Proof. It is sufficient to show that if $v \in S^{k-1}$ is tangent to X at p, then $f'(p)v$ is tangent to Y at $f(p)$. For this, let $\{p_n\}$ be a sequence in $X \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$. By Lemma 1.1, we have $(f(p_n) - f(p))/|p_n - p| \to f'(p)v$. If $f'(p)v = 0$ there is nothing to prove, since $0 \in C_{f(p)}(Y)$ by the definition of tangent cone. On the other hand, if $f'(p)v \neq 0$, we have $f(p_n) \neq f(p)$, provided that n is large enough. Thus, for such n's, we can write

$$\frac{f(p_n) - f(p)}{|f(p_n) - f(p)|} = \frac{|p_n - p|}{|f(p_n) - f(p)|}\frac{f(p_n) - f(p)}{|p_n - p|}.$$

Therefore,

$$\lim_{n\to\infty} \frac{f(p_n) - f(p)}{|f(p_n) - f(p)|} = \frac{f'(p)v}{|f'(p)v|},$$

and this shows that $f'(p)v = \lambda w$, where $\lambda > 0$ and $w \in S^{k-1}$ is tangent to Y at $f(p)$. □

Lemma 1.3. *Let $f, g : U \to \mathbb{R}^s$ be defined on an open subset of $\mathbb{R}^k$ and differentiable at $p \in U$. Assume that f and g coincide on some subset of X containing p. Then $f'(p)$ and $g'(p)$ coincide on $C_p(X)$ and, consequently, on $T_p(X)$.*

Proof. Let $\varphi : U \to \mathbb{R}^s$ be defined by $\varphi(x) = f(x) - g(x)$; so that φ maps X into the trivial subspace $Y = \{0\}$ of $\mathbb{R}^s$. Thus, by Lemma 1.2 we obtain

$$\varphi'(p)v = f'(p)v - g'(p)v = 0 \quad \text{for any } v \in C_p(X).$$

□

Lemma 1.3 ensures that if $f : X \to \mathbb{R}^s$ is a C^1 map on a subset X of $\mathbb{R}^s$ and p is a point in X, then the restriction to $T_p(X)$ of the derivative at p of any C^1 local extension of f to a neighborhood of p in $\mathbb{R}^k$ does not depend on the chosen extension. In other words, all the C^1 extensions of f to an open neighborhood of p have the same derivative with respect to the subspace $T_p(X)$. Moreover, Lemma 1.2 implies that if g is such an extension and f maps X into Y, then $g'(p)$ maps $T_p(X)$ into $T_{f(p)}(Y)$. These two facts justify the following definition.

Definition 1.4. Let $f : X \to Y$ be a C^1 map from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$. The derivative of f at p, $f'(p) : T_p(X) \to T_{f(p)}(Y)$, is the restriction to $T_p(X)$ of the derivative at p of any C^1 extension of f to a neighborhood of p in $\mathbb{R}^k$.

We point out that this extended derivative inherits the two functorial properties of the classical derivative (the easy proof of this fact is left to the reader). As a consequence of this we get the following result.

Theorem 1.2. *Let $f : X \subseteq \mathbb{R}^k \to Y \subseteq \mathbb{R}^s$ be a C^1-diffeomorphism. Then for any $p \in X$, $f'(p) : T_p(X) \to T_{f(p)}(Y)$ is an isomorphism mapping $C_p(X)$ onto $C_{f(p)}(Y)$.*

Proof. To simplify the notation, put $q = f(p)$. By the definition of diffeomorphism we have $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$, where i_X and i_Y denote the identity on X and Y, respectively. Therefore, by the functorial properties of the extended derivative, the two compositions $(f^{-1})'(q)f'(p)$ and $f'(p)(f^{-1})'(q)$ coincide, respectively, with the identity on $T_p(X)$ and $T_q(Y)$. This means that $f'(p)$ is invertible and $f'(p)^{-1} = (f^{-1})'(q)$. The fact that $C_p(X)$ and $C_q(Y)$ correspond to each other under $f'(p)$ is a direct consequence of Lemma 1.2. □

As an application of Theorem 1.2, consider the graph $X \subseteq \mathbb{R}^2$ of the absolute value function in $\mathbb{R}$. Clearly $T_{(0,0)}(X) = \mathbb{R}^2$. Therefore X cannot be diffeomorphic to $\mathbb{R}$, since $T_p(X) = \mathbb{R}$ for any p in $\mathbb{R}$.

Exercise 1.13. Prove that the set $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$ is not diffeomorphic to $Y = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ or } y \geq 0\}$.
Hint. Use the fact $C_{(0,0)}(X)$ is convex and $C_{(0,0)}(Y)$ is not.

Problem. Let $f : X \subseteq \mathbb{R}^k \to Y \subseteq \mathbb{R}^s$ be a C^1-diffeomorphism. Is it true that if f is smooth, then f is actually a C^∞-diffeomorphism?

Problem. Assume $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^s$ are C^n-diffeomorphic, for all $n \in \mathbb{N}$. Are they C^∞-diffeomorphic?

Let X be a subset of $\mathbb{R}^k$. We say that a point $p \in X$ is *regular* for X if $T_p(X) = C_p(X)$. In other words, since $T_p(X)$ is the space spanned by $C_p(X)$, saying that p is a regular point for X means that $C_p(X)$ is a vector space. A point which is not regular will be said *singular*. The set of singular points of X will be denoted by δX.

For example, if X is an n-simplex in $\mathbb{R}^k$, δX is just the union of all the $(n-1)$-faces of X, $\delta\delta X$, denoted by $\delta^2 X$, is the union of all the $(n-2)$-faces of X, and so on.

If $X \subseteq \mathbb{R}^2$ is the graph of the absolute value function, we obviously have $\delta X = \{(0, 0)\}$ and $\delta^2 X = \emptyset$.

Observe also that if X is an open subset of $\mathbb{R}^k$, then $\delta X = \emptyset$.

The following straightforward consequence of Theorem 1.2 shows that the concept of singular point is invariant under diffeomorphisms.

Theorem 1.3. *If $f : X \to Y$ is a C^α-diffeomorphism, then it maps δX onto δY. Consequently, for any $n \in \mathbb{N}$, $\delta^n X$ and $\delta^n Y$ are C^α-diffeomorphic.*

Let us see now how the notion of a tangent cone may turn out to be useful in studying minimum problems for real C^1-functions in arbitrary Euclidean sets.

Theorem 1.4 (First order necessary condition for a minimum point). *Let $p \in X$ be a relative minimum point for a C^1 map $f : X \to \mathbb{R}$. Then $f'(p)v \geq 0$ for all $v \in C_p(X)$. Consequently, if p is a regular point of X, then $f'(p) : T_p(X) \to \mathbb{R}$ is the null map.*

Proof. Since both the notions of tangent cone and of minimum point are local ones, one may assume, replacing X with a neighborhood of p in X if necessary, that $f(X)$ is contained in the half line $Y = [f(p), \infty)$. Thus $f'(p)$ maps $C_p(X)$ into $C_{f(p)}([f(p), \infty)) = [0, \infty)$, and the assertion is proved. □

Obviously, if $p \in X$ is a maximum point for f, applying Theorem 1.4 to $-f$, we get $f'(p)v \leq 0$ for all $v \in C_p(X)$. From this we shall deduce the following sufficient condition for a (relative) minimum point. Thus, in some sense, we may regard the sufficient condition as a consequence of the necessary one.

Theorem 1.5 (First order sufficient condition for a minimum point). *Assume that $f : X \to \mathbb{R}$ is C^1. If $f'(p)v > 0$ for all $v \in C_p(X) \setminus \{0\}$, then p is a (strict) relative minimum point for f.*

Proof. Assume that the assertion is false and define $A = \{x \in X : f(x) \leq f(p)\}$. Clearly p is an accumulation point for A and, consequently, there exists a nonzero vector $v \in C_p(A)$. By the definition of A, p is a maximum point for f in A; therefore, by the necessary condition (for a maximum point), we get $f'(p)v \leq 0$. This contradicts our assumption, since the inclusion $A \subseteq X$ implies $C_p(A) \subseteq C_p(X)$. □

We will go further with the analysis of the relationship between the notion of tangent cone and the study of minimum problems for a “nice” function f restricted to a “not necessarily nice” subset X of $\mathbb{R}^k$. We will analyze what can be deduced from the knowledge of the second order Taylor formula of the function f at a point $p \in X$. In this case, however, we shall assume f to be defined in a neighborhood U of X. The reason for this, as we shall see later, is due to the fact that the second order conditions for $p \in X$ to be a relative minimum point for the restriction of f to X do depend on the behavior of f in a complete neighborhood of p in $\mathbb{R}^k$. This is in contrast with the first order case.

Theorem 1.6 (Second order necessary condition for a minimum point). *Let X be a subset of $\mathbb{R}^k$, U an open set containing X and $f : U \to \mathbb{R}$ a C^2 real function. Assume that $p \in X$ is a relative minimum point for f in X. If $f'(p)v = 0$, for all $v \in \mathbb{R}^k$, then $f''(p)(v, v) \geq 0$, for all $v \in C_p(X)$.*

Proof. It is enough to prove that if $v \in C_p(X)$ is a unit vector, then $f''(p)(v,v) \geq 0$. So, let $\{p_n\}$ be a sequence in $X \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$. Since p is a relative minimum point for f in X, we have for n sufficiently large

$$0 \leq f(p_n) - f(p) = f'(p)(p_n - p) + \frac{1}{2} f''(p)(p_n - p, p_n - p) + o(|p_n - p|^2).$$

Thus

$$0 \leq f''(p)(p_n - p, p_n - p) + |p_n - p|^2 \omega(p_n - p),$$

where $\omega(x) \to 0$ as $x \to 0$.

Dividing by $|p_n - p|^2$ and passing to the limit we get $f''(p)(v,v) \geq 0$, as claimed. □

As for the first order case, the second order sufficient condition can be deduced directly from the necessary condition. The following result requires that the gradient of f at $p \in X$ is zero. We shall later remove this assumption.

Theorem 1.7 (Second order sufficient condition for a minimum point). *Let X be a subset of $\mathbb{R}^k$, U an open set containing X and $f : U \to \mathbb{R}$ a C^2 real function. Let $p \in X$ be such that $f'(p)v = 0$ for all $v \in \mathbb{R}^k$, and $f''(p)(v,v) > 0$ for all $v \in C_p(X) \setminus \{0\}$. Then p is a (strict) relative minimum point for f in X.*

Proof. Assume that the assertion is false. As in the proof of Theorem 1.5 define $A = \{x \in X : f(x) \leq f(p)\}$ and let $v \in C_p(A) \setminus \{0\}$. Since p is a minimum point for $-f$ in A, by the necessary condition we get $f''(p)(v,v) \leq 0$. And this is a contradiction, since $A \subseteq X$ implies $v \in C_p(X) \setminus \{0\}$. □

We observe that in Theorem 1.7, the assumption that the gradient of f vanishes at $p \in X$ is not restrictive when p is an interior point of X. In fact, in this case, this condition is necessary for p to be a relative minimum point. The assumption is not restrictive even when $X \subseteq \mathbb{R}^k$ is C^1-diffeomorphic to an open subset of $\mathbb{R}^m$ (as in the case of the Lagrange multipliers). In fact, Theorem 1.4 implies that if p is a (relative) minimum point for f in X, then the gradient of f at p, $\nabla f(p)$, must be orthogonal to $T_p(X)$; and in this case one may show that it is possible to (locally) modify f, exclusively outside X, by adding a new function (vanishing in X) in such a way that the gradient of the modified function turns out to be zero at p. This is actually what one does dealing with Lagrange multipliers, where the modified map is of the type $F_\lambda(x) = f(x) - \langle \lambda, g(x) \rangle$, with $\lambda \in \mathbb{R}^{k-m}$ a suitable "multiplier", and $g : W \to \mathbb{R}^{k-m}$ a convenient map (according to Theorem 2.2 below).

The example below shows that the condition $\nabla f(p) = 0$ of Theorem 1.7 may be too restrictive in some cases. We shall present therefore an extension of this theorem, where the above condition is replaced by the following weaker one:

$$\langle \nabla f(p), x - p \rangle \geq 0 \textit{ for all } x \textit{ in a convenient neighborhood of } p \textit{ in } X.$$

Let X be the subset $[0, \infty) \times [0, \infty)$ of $\mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = y \cos x + x^2 - y^2 - xy^2 + x \sin(y^3 - x^2).$$

Thus, one can write

$$f(x,y) = y + x^2 - y^2 + o(|v|^2),$$

where $v = (x,y)$. The necessary condition for $p = (0,0)$ to be a (relative) minimum point (for f in X) is satisfied, since the first derivative of f at p applied to a vector (x,y) is just y (i.e. the homogeneous polynomial of degree one in the Taylor expansion of f at p). On the other hand, we observe that neither Theorem 1.5 nor Theorem 1.7 can be applied to check whether p is actually a minimum point. Moreover, since $T_p(X) = \mathbb{R}^2$, there is no way to modify f outside X in order to get a map with zero gradient at p. However, Theorem 1.8 below shows immediately that p is a (relative) minimum point for f in X (the conditions ensuring this can be easily checked in the above second order Taylor formula). As far as we know the following simple result seems to be unknown.

Theorem 1.8 (Mixed order sufficient condition for a minimum point). *Let X be a subset of $\mathbb{R}^k$, U an open set containing X and $f : U \to \mathbb{R}$ a C^2 real function. Assume that $p \in X$ is a relative minimum point for the restriction to X of the linear functional $f'(p) : \mathbb{R}^k \to \mathbb{R}$. If*

$$f''(p)(v,v) > 0 \quad \textit{for all } v \in (\operatorname{Ker} f'(p) \cap C_p(X)) \setminus \{0\},$$

then p is a (strict) relative minimum point for f in X.

Proof. By the Taylor formula we have

$$f(x) - f(p) = f'(p)(x-p) + \frac{1}{2}f''(p)(x-p, x-p) + |x-p|^2\omega(x-p),$$

where $\omega(z) \to 0$ as $z \to 0$. Therefore, assuming that the assertion is false, one can find a sequence $\{p_n\}$ in $X \setminus \{p\}$ such that $p_n \to p$, $(p_n - p)/|p_n - p| \to v \in C_p(X)$ and

$$0 \geq f'(p)(p_n - p) + \frac{1}{2}f''(p)(p_n - p, p_n - p) + |p_n - p|^2\omega(p_n - p).$$

Dividing by $|p_n - p|$ and passing to the limit we get $f'(p)v \leq 0$. Since p is a relative minimum point for $f'(p)$ in X, we have $f'(p)(p_n - p) = f'(p)p_n - f'(p)p \geq 0$, for n sufficiently large. Thus $f'(p)v \geq 0$ and consequently $v \in \operatorname{Ker} f'(p) \cap C_p(X)$. Since $f'(p)(p_n - p) \geq 0$, a fortiori one has

$$0 \geq \frac{1}{2}f''(p)(p_n - p, p_n - p) + |p_n - p|^2\omega(p_n - p).$$

Therefore, dividing by $|p_n - p|^2$ and passing to the limit we obtain $f''(p)(v,v) \leq 0$, contradicting the assumption that $f''(p)$ is positive definite on $\operatorname{Ker} f'(p) \cap C_p(X)$. □

We point out that in the case where X is convex (at least in a neighborhood of $p \in X$) the condition of Theorem 1.8 that p is a relative minimum point for the linear functional $f'(p)$ is necessary for p to be a (relative) minimum point for f in X. To see this, observe that when p is an interior point, this condition is equivalent to requiring that $f'(p)$ is the zero functional. If, on the other hand, p is on the boundary of X and $f'(p)$ is nonzero, the condition means that X lies (locally) in the halfspace

$$\{x \in \mathbb{R}^k : f'(p)(x-p) \geq 0\}.$$

Thus, since (for a convex set X) $C_p(X)$ coincides with the closure of

$$\{\lambda(x-p) : \lambda \geq 0, x \in X\},$$

the assumption that p is a relative minimum point for the linear functional $f'(p)$ is equivalent to the first order necessary condition $f'(p)v \geq 0$ for all $v \in C_p(X)$.

2. Differentiable manifolds in Euclidean spaces

A subset M of $\mathbb{R}^k$ is called a (boundaryless) *m-dimensional (differentiable) manifold of class C^α*, $\alpha \in \mathbb{N} \cup \{\infty\}$, if it is locally C^α-diffeomorphic to $\mathbb{R}^m$; meaning that any point p of M admits a neighborhood (in M) which is C^α-diffeomorphic to an open subset of $\mathbb{R}^m$. A C^α-diffeomorphism $\varphi : W \to V \subseteq M$ from an open subset W of $\mathbb{R}^m$ onto an open subset V of M is called a *parametrization* (of class C^α of V). The inverse of a parametrization $\varphi^{-1} : V \to W$ is called a *chart* or a *coordinate system* on V, and its component functions, x_1, x_2,..., x_m, are the *coordinate functions* of φ^{-1} on V.

As an example observe that the graph of any C^α-map $f : W \to \mathbb{R}^s$ defined on an open subset of $\mathbb{R}^m$ is a C^α-manifold of dimension m. In fact, as observed before, the graph of any C^α map is C^α-diffeomorphic to its domain. Therefore, in particular, the m-dimensional sphere $S^m = \{x \in \mathbb{R}^{m+1} : |x|^2 = 1\}$ is a smooth m-dimensional manifold, being locally the graph of a C^∞ real function defined on the open unit ball of an m-dimensional subspace of $\mathbb{R}^{m+1}$. Observe also that any open subset of a differentiable manifold is again a differentiable manifold.

As a straightforward consequence of the definition, any point p of an m-dimensional C^1-manifold M is non-singular (i.e. $C_p(M) = T_p(M)$). Moreover, $\dim T_p(M) = m$. In fact, since this property is true for open subsets of $\mathbb{R}^m$, according to Theorem 1.2, it holds true for m-dimensional C^1-manifolds. Incidentally, observe that Theorem 1.2 provides a practical method for computing $T_p(M)$. That is, if $\varphi : W \to V$ is a any C^1-parametrization of a neighborhood V of p in M, then $T_p(M) = \operatorname{Im} \varphi'(w)$, where $\varphi(w) = p$.

The following direct consequence of the Implicit Function Theorem can be used to produce a large variety of examples of differentiable manifolds. It gives also a useful tool to compute the tangent space at any given point of a manifold. We recall first that if $f : U \to \mathbb{R}^s$ is a C^1 map on an open subset U of $\mathbb{R}^k$, an element $p \in U$ is said to be a *regular point* (of f) if the derivative $f'(p)$ of f at p is surjective. Non-regular points are called *critical (points)*. The *critical values* of f are those points of the target space $\mathbb{R}^s$ which lie in the image $f(C)$ of the set C of critical points. Any $y \in \mathbb{R}^s$ which is not in $f(C)$ is a *regular value*. Therefore, in particular, any element of $\mathbb{R}^s$ which is not in the image of f is a regular value. Notice that, in this terminology, the words "point" and "value" refer to the source and target spaces, respectively.

Theorem 2.1 (Regularity of the level set). *Let $f : U \to \mathbb{R}^s$ be a C^α mapping of an open subset of $\mathbb{R}^k$ into $\mathbb{R}^s$. If $0 \in \mathbb{R}^s$ is a regular value for f, then $f^{-1}(0)$ is a C^α-manifold of dimension $m = k - s$. Moreover, given $p \in f^{-1}(0)$, we have*

$$T_p(f^{-1}(0)) = \operatorname{Ker} f'(p).$$

Proof. Choose a point $p \in f^{-1}(0)$ and split $\mathbb{R}^k$ into the direct sum $\operatorname{Ker} f'(p) \oplus \operatorname{Ker} f'(p)^\perp$. Since by assumption $f'(p) : \mathbb{R}^k \to \mathbb{R}^s$ is onto, the restriction of $f'(p)$ to $\operatorname{Ker} f'(p)^\perp$ is an isomorphism. Observe that this restriction is just the second partial derivative, $D_2 f(p)$, of f at p with respect to the given decomposition. It follows by the Implicit Function Theorem that in a neighborhood of p, $f^{-1}(0)$ is the graph of a C^α map $\varphi : W \to \operatorname{Ker} f'(p)^\perp$ defined on an open subset W of $\operatorname{Ker} f'(p)$. Recalling that the graph of a C^α map is C^α diffeomorphic to its domain, we get that in a neighborhood of p, $f^{-1}(0)$ is a C^α-differentiable manifold whose dimension is $\dim \operatorname{Ker} f'(p) = k - s$.

To prove that $T_p(f^{-1}(0)) = \operatorname{Ker} f'(p)$ observe first that $T_p(f^{-1}(0)) \subseteq \operatorname{Ker} f'(p)$. In fact, f maps $f^{-1}(0)$ into $\{0\}$ and, consequently, $f'(p)$ maps $T_p(f^{-1}(0))$ into $T_0(\{0\}) = \{0\}$. The equality follows by computing the dimensions of the two spaces. □

To see how Theorem 2.1 can easily be applied to produce examples of differentiable manifolds, consider the map $f : \mathbb{R}^k \to \mathbb{R}$, given by $f(x) = |x|^2$. Differentiating f we get $f'(x)h = 2\langle x, h\rangle$. Therefore $x = 0$ is the only critical point of f and, consequently, $f(0) = 0 \in \mathbb{R}$ is the only critical value. Thus, for any $a > 0$, the set

$$S_a^{k-1} = \{x \in \mathbb{R}^k : |x| = a\}$$

is a $(k-1)$-dimensional smooth manifold in $\mathbb{R}^k$, called the $(k-1)$-dimensional sphere of radius a. Given $x \in S_a^{k-1}$, the tangent space at x is the hyperplane orthogonal to x, i.e.

$$T_x(S_a^{k-1}) = \{h \in \mathbb{R}^k : \langle x, h\rangle = 0\}.$$

A different interesting example is the configuration space of a rigid body in $\mathbb{R}^3$. Choosing a triangle of vertices p_1, p_2, p_3 in a rigid body, the position in $\mathbb{R}^3$ of these points gives complete information about the location of the body in the space. Therefore, the configuration space can be regarded as the set

$$M = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : |p_1 - p_2| = a_3, |p_2 - p_3| = a_1, |p_3 - p_1| = a_2\},$$

where the edges a_1, a_2, a_3 of the chosen triangle are fixed. To see that M is a smooth 6-dimensional manifold, consider the map $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$f(p_1, p_2, p_3) = \left(|p_1 - p_2|^2, |p_2 - p_3|^2, |p_3 - p_1|^2\right).$$

Differentiating f at a given position $(p_1, p_2, p_3) \in M$, we get

$$f'(p_1, p_2, p_3)(\dot{p}_1, \dot{p}_2, \dot{p}_3) = 2\left(\langle p_1 - p_2, \dot{p}_1 - \dot{p}_2\rangle, \langle p_2 - p_3, \dot{p}_2 - \dot{p}_3\rangle, \langle p_3 - p_1, \dot{p}_3 - \dot{p}_1\rangle\right).$$

In order to apply Theorem 2.1 we have to check that the linear map

$$f'(p_1, p_2, p_3) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

is onto. To see this observe first that the vector $e_1 = (1, 0, 0)$ belongs to its image. In fact, to solve the system

$$\begin{cases} \langle p_1 - p_2, \dot{p}_1 - \dot{p}_2\rangle = 1, \\ \langle p_2 - p_3, \dot{p}_2 - \dot{p}_3\rangle = 0, \\ \langle p_3 - p_1, \dot{p}_3 - \dot{p}_1\rangle = 0 \end{cases}$$

it is enough to choose $\dot{p}_2 = \dot{p}_3 = 0$ and $\dot{p}_1$ orthogonal to $p_3 - p_1$ in such a way that $\langle p_1 - p_2, \dot{p}_1\rangle = 1$. This is clearly possible, since we have assumed the two vectors $p_1 - p_2$ and $p_3 - p_1$ to be linearly independent (recall that the three points p_1, p_2, p_3 are geometrically independent). The same method shows that $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ are in the image of $f'(p_1, p_2, p_3)$. Thus, Theorem 2.1 applies to show that M is a smooth submanifold of $\mathbb{R}^9$. The tangent space of M at (p_1, p_2, p_3) is the subspace of $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ given by

$$\left\{(\dot{p}_1, \dot{p}_2, \dot{p}_3) : \langle p_1 - p_2, \dot{p}_1 - \dot{p}_2\rangle = 0, \langle p_2 - p_3, \dot{p}_2 - \dot{p}_3\rangle = 0, \langle p_3 - p_1, \dot{p}_3 - \dot{p}_1\rangle = 0\right\}.$$

Exercise 2.1. The configuration space of a double pendulum is given by

$$M = \{(p_1, p_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |p_1| = a_1, |p_1 - p_2| = a_2\},$$

where a_1, a_2 are two fixed positive numbers. Show that M is a smooth manifold (a two dimensional torus).

Theorem 2.1 can be partially inverted, in the sense that any C^α differentiable manifold in $\mathbb{R}^k$ can be locally regarded as a *regular level set* (i.e. as the inverse image of a regular value of a C^α map on an open subset of $\mathbb{R}^k$). In fact, the following theorem holds.

Theorem 2.2. *Let M be an m-dimensional manifold of class C^α in $\mathbb{R}^k$. Then, given $p \in M$, there exists a map $f : U \to \mathbb{R}^{k-m}$, C^α on a neighborhood U of p in $\mathbb{R}^k$, which defines $M \cap U$ as a regular level set.*

Proof. Let $\varphi : W \to \mathbb{R}^k$ be a C^α-parametrization of M around p and let $w = \varphi^{-1}(p)$. Consider any linear map $L : \mathbb{R}^{k-m} \to \mathbb{R}^k$ such that $\operatorname{Im} L \oplus T_p(M) = \mathbb{R}^k$ (this is clearly possible since $\dim T_p(M) = m$), and define $g : W \times \mathbb{R}^{k-m} \to \mathbb{R}^k$ by putting $g(x, y) = \varphi(x) + Ly$. The derivative of g at $(w, 0) \in W \times \mathbb{R}^{k-m}$ is given by

$$g'(w, 0)(h, k) = \varphi'(w)h + Lk,$$

which is clearly surjective (therefore an isomorphism), since $\operatorname{Im}\varphi'(0) = T_p(M)$. By the Inverse Function Theorem, g is a C^α-diffeomorphism of a neighborhood of $(w, 0)$ in $W \times \mathbb{R}^{k-m}$ onto a neighborhood U of p in $\mathbb{R}^k$. Let ψ be the inverse of such a diffeomorphism and define $f : U \to \mathbb{R}^{k-m}$ as the composition $\pi_2 \circ \psi$ of ψ with the projection $\pi_2 : W \times \mathbb{R}^{k-m} \to \mathbb{R}^{k-m}$ of $W \times \mathbb{R}^{k-m}$ onto the second factor. Clearly, f satisfies the assertion. □

We point out that there are differentiable manifolds in $\mathbb{R}^k$ which cannot be globally defined as regular level sets. One can prove, in fact, that when this happens the manifold must be orientable (the definition of orientability and the proof of this assertion would carry us too far away from the goal of this course). As an intuitive example consider a Möbius strip M embedded in $\mathbb{R}^3$ and assume $M = f^{-1}(0)$, where $f : U \to \mathbb{R}$ is a C^1 map on an open subset of $\mathbb{R}^3$. If $0 \in \mathbb{R}$ were a regular value for f, the gradient of f at any point $p \in f^{-1}(0)$, $\nabla f(p)$, would be nonzero. Therefore, the map $\nu : M \to \mathbb{R}^3$, given by $\nu(p) = \nabla f(p)/|\nabla f(p)|$, would be a continuous normal unit vector field on M, and this is well-known to be impossible on the Möbius strip (a one-sided surface).

We want to define now "a concrete" notion of tangent bundle TM associated with a C^α manifold M in $\mathbb{R}^k$. We will prove that if $\alpha \geq 2$, TM is a $C^{\alpha-1}$ differentiable manifold in $\mathbb{R}^k \times \mathbb{R}^k$. In order to do this, it is convenient to define the concept of tangent bundle for any subset of $\mathbb{R}^k$, and to prove that when two sets X and Y are C^α-diffeomorphic, the corresponding tangent bundles are $C^{\alpha-1}$-diffeomorphic.

Definition 2.1. Given $X \subseteq \mathbb{R}^k$, the subset

$$TX = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^k : x \in X, y \in T_x(X)\}$$

of $\mathbb{R}^k \times \mathbb{R}^k$ is called the *tangent bundle* of X. The canonical projection $\pi : TX \to X$ is the restriction to TX of the projection of $\mathbb{R}^k \times \mathbb{R}^k$ onto the first factor (thus, π is always a smooth map).

Definition 2.2. Let $f : X \to Y$ be a C^α-map from a subset X of $\mathbb{R}^k$ into a subset Y of $\mathbb{R}^s$ and assume $1 \leq \alpha \leq \infty$. The *tangent map* of f, $Tf : TX \to TY$, is given by

$$Tf(x, y) = (f(x), f'(x)y).$$

As pointed out in Remark 1.1, one may regard a C^α map $f : X \to Y$ as the restriction of a C^α map $g : U \to \mathbb{R}^s$ defined on an open neighborhood U of X. Consequently, if $\alpha \geq 1$, $Tg : TU \to T\mathbb{R}^s$, given by $(x, y) \mapsto (g(x), g'(x)y)$, is a $C^{\alpha-1}$ map from the open neighborhood $TU = U \times \mathbb{R}^k$ of TX into $T\mathbb{R}^s = \mathbb{R}^s \times \mathbb{R}^s$. This proves that Tf, which is just the restriction to TX of Tg, is a $C^{\alpha-1}$ map.

Clearly, if $f : X \to Y$ and $g : Y \to Z$ are C^α maps, one has $T(g \circ f) = Tg \circ Tf$. Moreover, if $i : X \to X$ is the identity on X, then $Ti : TX \to TX$ is the identity on TX. Therefore, one may regard T as a (covariant) functor from the category of Euclidean sets with C^α maps into the category of Euclidean sets with $C^{\alpha-1}$ maps. This implies that if $f : X \to Y$ is a C^α-diffeomorphism, then $Tf : TX \to TY$ is a $C^{\alpha-1}$-diffeomorphism. Therefore, if M is a C^α manifold of dimension m, since it is locally C^α-diffeomorphic to the open subsets of $\mathbb{R}^m$, its tangent bundle TM is a $C^{\alpha-1}$ manifold of dimension $2m$. Moreover, if $\varphi : W \to V \subseteq M$ is a parametrization of an open set V in M, $T\varphi : W \times \mathbb{R}^m \to TV \subseteq TM$ is a parametrization of the open set $TV = \pi^{-1}(V)$ of TM.

Exercise 2.2. Let $M \subseteq \mathbb{R}^k$ be an m-dimensional manifold regularly defined (as a zero level set) by a C^α map $f : U \to \mathbb{R}^s$ on an open set U of $\mathbb{R}^k$. Prove that if $\alpha \geq 2$, TM is regularly defined by Tf. That is

$$TM = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^k : f(x) = 0, f'(x)y = 0\}$$

where $(0, 0) \in \mathbb{R}^s \times \mathbb{R}^s$ is a regular value for Tf.

Definition 2.3. Let X be a subset of $\mathbb{R}^k$. A C^α *tangent vector field* on X is a C^α map $g : X \to \mathbb{R}^k$ with the property that $g(x) \in T_x(X)$ for all $x \in X$. The tangent vector field g on X is said to be *inward* if $g(x) \in C_x(X)$ for all $x \in X$.

In many textbooks in differential geometry a tangent vector field on a differentiable manifold M is defined as a cross section of the tangent bundle TM. That is, a map $w : M \to TM$ with the property that the composition $\pi \circ w : M \to M$ of w with the bundle projection π is the identity on M. However, in our "concrete" situation (i.e. M in $\mathbb{R}^k$) this "abstract" definition turns out to be redundant. In fact, observe that, for M embedded in $\mathbb{R}^k$, a map $w : M \to \mathbb{R}^k \times \mathbb{R}^k$ is a cross section of TM if and only if for all $x \in M$ one has $w(x) = (x, g(x))$, with $g(x) \in T_x(M)$. Therefore, forgetting x in the pair $(x, g(x))$, one may accept the simpler definition given above.

To any C^1 function $f : M \to \mathbb{R}$ on a differentiable manifold $M \subseteq \mathbb{R}^k$ one can assign a tangent vector field on M, called the *gradient* of f and denoted by ∇f, in the following way:

$$\langle \nabla f(x), v \rangle = f'(x)v \quad \text{for any } v \in T_x(M).$$

In other words, given $x \in M$, the gradient of f at x is the vector of $T_x(M)$ corresponding to $f'(x) \in T_x(M)^*$ under the isomorphism $j : T_x(M) \to T_x(M)^*$ defined by $j(u)v = \langle u, v \rangle$.

Exercise 2.3. Let M be a differentiable manifold in $\mathbb{R}^k$ and let $f : M \to \mathbb{R}$ be the restriction to M of a C^1 function $\hat{f}$ defined on an open set U containing M. Prove that, given $x \in M$, $\nabla f(x)$ is just the "tangential component" of $\nabla \hat{f}(x)$; i.e. the image of $\nabla \hat{f}(x)$ under the orthogonal projection of $\mathbb{R}^k$ onto $T_x(M)$.

3. Ordinary differential equations on manifolds

An autonomous first order differential equation on a manifold $M \subseteq \mathbb{R}^k$ (or, more generally, on a subset of $\mathbb{R}^k$) is given by assigning a (continuous) tangent vector field

$g : M \to \mathbb{R}^k$ on M. The first order (autonomous) differential equation associated with g will be written in the form

$$\dot{x} = g(x), \qquad x \in M. \tag{3.1}$$

However, the important fact about a differential equation (or any equation, in general) is not the way this is written: what counts is the exact definition of what we mean by a solution (and this implicitly defines the notion of equation). By a solution of (3.1) we mean a C^1 curve $x : J \to \mathbb{R}^k$, defined on a (nontrivial) interval $J \subseteq \mathbb{R}$, which satisfies the conditions $x(t) \in M$ and $\dot{x}(t) = g(x(t))$, identically on J. Thus, even if, according to Remark 1.1, the map g may be thought as defined on an open set U containing M, a solution $x : J \to \mathbb{R}^k$ of

$$\dot{x} = g(x), \qquad x \in U \tag{3.2}$$

is a solution of (3.1) if and only if its image lies in M. However, if M is closed in U, under the uniqueness assumption of the Cauchy problem for (3.2), any solution of (3.2) starting from a point of M must lie entirely in M (see Exercise 3.1 below).

If $\varphi : M \to N$ is a C^1-diffeomorphism between two differentiable manifolds and g is a tangent vector field on M, one gets a tangent vector field h on N by putting $h(z) = \varphi'(\varphi^{-1}(z))g(\varphi^{-1}(z))$. In this way, if $x \in M$ and $z \in N$ correspond under φ, the two vectors $h(z)$ and $g(x)$ correspond under the isomorphism $\varphi'(x) : T_x(M) \to T_z(N)$. For this reason we say that the two vector fields g and h *correspond* under φ (or they are *φ-related*). We observe that in this case, as an easy consequence of the chain rule for the derivative (and the definition of solution of a differential equation), equation (3.1) is equivalent to

$$\dot{z} = h(z), \qquad z \in N, \tag{3.3}$$

in the sense that $x : J \to M$ is a solution of (3.1) if and only if the composition $z = \varphi \circ x$ is a solution of (3.3). That is, the solutions of (3.1) and (3.3) correspond under the diffeomorphism φ.

A non-autonomous first order differential equation on a manifold $M \subseteq \mathbb{R}^k$ is given by assigning, on an open subset V of $\mathbb{R} \times M$, a non-autonomous (continuous) vector field $g : V \to \mathbb{R}^k$, which is tangent to M for all $t \in \mathbb{R}$. That is, for any $t \in \mathbb{R}$, the map $g_t : V_t \to \mathbb{R}^k$, given by $g_t(x) = g(t, x)$, is a tangent vector field on the (possibly empty) open subset $V_t = \{x \in M : (t, x) \in V\}$ of M. In other words, $g(t, x) \in T_x(M)$ for each $(t, x) \in V$.

The first order differential equation associated with g is denoted as follows:

$$\dot{x} = g(t, x), \qquad (t, x) \in V. \tag{3.4}$$

A solution of (3.4) is a C^1 map $x : J \to M$, on an interval $J \subseteq \mathbb{R}$, such that, for all $t \in J$, $(t, x(t)) \in V$ and $\dot{x}(t) = g(t, x(t))$.

We point out that (3.4) can be thought as a special autonomous equation on the open submanifold V of $\mathbb{R} \times M \subseteq \mathbb{R}^{k+1}$. In fact (3.4) is clearly equivalent to the system

$$\begin{cases} \dot{t} = 1, \\ \dot{x} = g(t, x), \qquad (t, x) \in V \end{cases} \tag{3.5}$$

and the vector field $(t, x) \mapsto (1, g(t, x))$ is tangent to V. By "equivalent" we mean that the solutions (3.4) and (3.5) are in a one-to-one correspondence.

As pointed out before, any differential equation on a manifold M is transformed into an equivalent one by a diffeomorphism $\varphi : M \to N$. Thus, since differentiable manifolds are locally diffeomorphic to open subsets of Euclidean spaces, the classical results about local existence and uniqueness for differential equations in $\mathbb{R}^m$ apply immediately to this more general context. Therefore, given $(t_0, x_0) \in V$, the continuity of the vector field $g : V \to \mathbb{R}^k$ is sufficient to ensure the existence, on an open interval J, of a solution $x : J \to M$ of (3.4) satisfying the Cauchy condition $x(t_0) = x_0$. If g is C^1 (or, more generally, locally Lipschitz), two solutions satisfying the same Cauchy condition coincide in their common domain. Moreover, by considering the partial ordering associated with the graph inclusion, one gets that any solution of (3.4) can be extended to a maximal one (i.e. to a solution which is not the restriction of any different solution).

With the same method used to deal with differential equations in $\mathbb{R}^m$, one gets that the domain of any maximal solution $x(\cdot)$ of (3.4) is an open interval (α, β), with $-\infty \leq \alpha < \beta \leq +\infty$. Moreover, given any $t_0 \in (\alpha, \beta)$ and any compact set K in the domain V of $g : V \to \mathbb{R}^k$, both the graphs of the restrictions of $x(\cdot)$ to $(\alpha, t_0]$ and $[t_0, \beta)$ are not contained in K. This is referred as the *Kamke property* of the maximal solution (in a differentiable manifold). In particular, if M is a compact manifold and $V = \mathbb{R} \times M$, any maximal solution of (3.4) is defined in the whole real axis.

Exercise 3.1. Let $g : U \to \mathbb{R}^k$ be a continuous vector field defined on an open subset U of $\mathbb{R}^k$. Assume that for each $p \in U$ the equation (3.2) admits a unique maximal solution $x(\cdot)$ satisfying the Cauchy condition $x(0) = p$. Prove that if g is tangent to a C^1 manifold $M \subseteq U$, which is relatively closed in U, then any maximal solution of (3.2) which meets M must lie entirely in M.

Hint. Use the Kamke property of the maximal solutions of (3.1).

As regards the continuous dependence on data, with the same method used for differential equations in $\mathbb{R}^m$, one has the following result.

Theorem 3.1. *Let $M \subseteq \mathbb{R}^k$ be a differentiable manifold and $g : V \to \mathbb{R}^k$ a locally Lipschitz non-autonomous vector field (tangent to M) defined on an open subset V of $\mathbb{R} \times M$. Given $(\tau, p) \in V$ denote (when defined) by $x(t, \tau, p)$ the value at t of the maximal solution through (τ, p). Let $\{(\tau_n, p_n)\}$ be a sequence in V converging to $(\tau_0, p_0) \in V$ and $[a, b]$ a compact interval contained in the domain of $x(\cdot, \tau_0, p_0)$. Then, for n sufficiently large, $x(\cdot, \tau_n, p_n)$ is defined in $[a, b]$ and*

$$x(t, \tau_n, p_n) \to x(t, \tau_0, p_0)$$

uniformly in $[a, b]$ as $n \to \infty$. In particular, the set of all (t, τ, p) such that $x(t, \tau, p)$ is well defined is an open subset of $\mathbb{R} \times V$ (obviously containing any (τ, τ, p) with $(\tau, p) \in V$).

As pointed out before, the tangent bundle TM of an m-dimensional C^α-manifold $M \subseteq \mathbb{R}^k$ is a $2m$-dimensional manifold of class $C^{\alpha-1}$ in $\mathbb{R}^k \times \mathbb{R}^k$. Therefore, an autonomous differential equation on TM will be written in the form

$$\begin{cases} \dot{x} = g(x, y), \\ \dot{y} = h(x, y), \qquad (x, y) \in TM \end{cases} \tag{3.6}$$

where the pair of $\mathbb{R}^k$-vectors $\big(g(x, y), h(x, y)\big)$ belongs to the tangent space $T_{(x,y)}TM$ for any $(x, y) \in TM$. A solution of (3.6) is a C^1 map $t \mapsto (x(t), y(t))$ from an interval $J \subseteq \mathbb{R}$ into TM such that $\dot{x}(t) = g(x(t), y(t))$, $\dot{y}(t) = h(x(t), y(t))$ for all $t \in J$.

As regards the non-autonomous case, we shall, for the sake of simplicity, from now on consider only the following situation:

$$\begin{cases} \dot{x} = g(t, x, y), \\ \dot{y} = h(t, x, y), \qquad (t, x, y) \in \mathbb{R} \times TM, \end{cases}$$

where $f : \mathbb{R} \times TM \to \mathbb{R}^k$ and $g : \mathbb{R} \times TM \to \mathbb{R}^k$ are continuous maps such that

$$\big(g(t, x, y), h(t, x, y)\big) \in T_{(x,y)}TM,$$

for all $(t, x, y) \in \mathbb{R} \times TM$. In other words, we shall assume that the domain V of the vector field $(t, x, y) \mapsto (g(t, x, y), h(t, x, y))$ coincides with the whole differentiable manifold $\mathbb{R} \times TM$.

To understand the meaning of a differential equation on TM, it is important to write down a necessary and sufficient condition for a pair of vectors $(\dot{x}, \dot{y}) \in \mathbb{R}^k \times \mathbb{R}^k$ to be tangent to TM at some point (x, y). To compute a generic tangent vector of $T_{(x,y)}(TM)$ we proceed as follows.

Because of Theorem 2.2, we may assume without loss of generality that M is regularly defined as the zero level set of a C^α map $f : U \to \mathbb{R}^s$ on an open subset of $\mathbb{R}^k$. Here, for simplicity, we will assume $\alpha = \infty$. Since in this case the tangent space to M at $x \in M$ is given by

$$T_x(M) = \{y \in \mathbb{R}^k : f'(x)y = 0\},$$

we get

$$TM = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^k : f(x) = 0,\ f'(x)y = 0\}.$$

According to Exercise 2.2, TM is regularly defined in $U \times \mathbb{R}^k$ by the two equations

$$\begin{cases} f(x) = 0, \\ f'(x)y = 0. \end{cases}$$

Thus, we can iterate the procedure, in order to compute TTM as a regularly defined (smooth) submanifold of $(\mathbb{R}^k \times \mathbb{R}^k) \times (\mathbb{R}^k \times \mathbb{R}^k)$. Differentiating these two equations, we obtain

$$TTM = \{(x, y; \dot{x}, \dot{y}) : f(x) = 0, f'(x)y = 0; f'(x)\dot{x} = 0, f''(x)(\dot{x}, y) + f'(x)\dot{y} = 0\},$$

and this implies that $(\dot{x}, \dot{y}) \in T_{(x,y)}(TM)$ if and only if

$$\begin{cases} f'(x)\dot{x} = 0, \\ f''(x)(\dot{x}, y) + f'(x)\dot{y} = 0. \end{cases}$$

In other words, given $(x, y) \in TM$, $(\dot{x}, \dot{y}) \in \mathbb{R}^k \times \mathbb{R}^k$ is in $T_{(x,y)}(TM)$ if and only if $\dot{x}$ is an arbitrary vector in $T_x(M)$ and $\dot{y} \in \mathbb{R}^k$ satisfies the condition

$$f'(x)\dot{y} = \psi(x; y, \dot{x}), \tag{3.7}$$

where $\psi(x, y, \dot{x}) = -f''(x)(\dot{x}, y)$. Observe that ψ is a smooth mapping from the vector bundle

$$T^2M = \{(x; u, v) \in \mathbb{R}^k \times (\mathbb{R}^k \times \mathbb{R}^k) : x \in M; u, v \in T_x(M)\},$$

into $\mathbb{R}^k$, and is bilinear and symmetric with respect to the last two variables.

Since M is regularly defined by the equation $f(x) = 0$, given $x \in M$, the linear map $f'(x)$ is onto. This means that once the two vectors $y, \dot{x} \in T_x(M)$ have been assigned, the equation (3.7) can be solved with respect to $\dot{y}$. Let us compute the set of solutions $\dot{y}$ of the equation (3.7). Given $z \in \mathbb{R}^s$, denote by Az the unique solution of $f'(x)\dot{y} = z$ which lies in the space $\operatorname{Ker} f'(x)^\perp = T_x(M)^\perp$. That is, $A : \mathbb{R}^k \to T_x(M)^\perp$ is the inverse of the restriction to $T_x(M)^\perp$ of $f'(x)$. Thus, any solution of the linear equation (3.7) can be expressed in the form

$$\dot{y} = \nu(x; y, \dot{x}) + w,$$

where $\nu(x; y, \dot{x}) = A\psi(x; y, \dot{x})$ is the unique solution of (3.7) in the space $T_x(M)^\perp$ and w is an arbitrary vector in $T_x(M) = \operatorname{Ker} f'(x)$.

From the above argument one can deduce that, given (x, y) in TM, to assign an arbitrary vector $(\dot{x}, \dot{y})$ in $T_{(x,y)}(TM)$ it is sufficient to choose v and w arbitrarily in $T_x(M)$ and to define $(\dot{x}, \dot{y}) = (v, \nu(x; y, v) + w)$. Observe that this is in accord with the fact that $T_{(x,y)}(TM)$ must be a $2m$-dimensional subspace of $\mathbb{R}^k \times \mathbb{R}^k$.

The above definition of the mapping $\nu : T^2M \to \mathbb{R}^k$ seems to depend on the function $f : U \to \mathbb{R}^s$ we have used to cut M (as a regular level set). This is a false impression. Roughly speaking, ν depends only on how M is twisted in $\mathbb{R}^k$. To convince oneself, observe that the concepts of tangent space and tangent bundle have been given for any subset of $\mathbb{R}^k$. If the set M happens to be a smooth manifold (actually, C^2 is sufficient), then given $x \in M$ and $y, \dot{x} \in T_x(M)$, the vector $\nu(x; y, \dot{x})$ turns out to be the common normal component (with respect to the decomposition $\mathbb{R}^k = T_x(M) \times T_x(M)^\perp$) of all the vectors $\dot{y}$ with the property that $(\dot{x}, \dot{y}) \in T_{(x,y)}(TM)$. The fact that M can be thought as a regular level set of a smooth map has been used to prove that ν is well defined and satisfies some properties that we summarize in the following result.

Theorem 3.2. *Let M be a smooth manifold in $\mathbb{R}^k$ and define*

$$T^2M = \{(x; u, v) \in \mathbb{R}^k \times (\mathbb{R}^k \times \mathbb{R}^k) : x \in M; u, v \in T_x(M)\}.$$

Then there exists a unique smooth map $\nu : T^2M \to \mathbb{R}^k$ such that

1. *$\nu(x; u, v) \in T_x(M)^\perp$ for all $(x, u, v) \in T^2M$;*
2. *for any $x \in M$, the map $\nu(x; \cdot, \cdot) : T_x(M) \times T_x(M) \to T_x(M)^\perp$ is bilinear and symmetric;*

3. $(\dot{x}, \dot{y}) \in T_{(x,y)}(TM)$ *if and only if* $\dot{x} \in T_x(M)$ *and the orthogonal projection of* $\dot{y}$ *onto* $T_x(M)^\perp$ *coincides with* $\nu(x; y, \dot{x})$.

Exercise 3.2. Find the explicit expression of the map ν for the following cases;

$$S^2 = \{x \in \mathbb{R}^3 : |x|^2 = 1\}; \qquad \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}.$$

According to what we have shown above, any (non-autonomous) first order differential equation on TM can be written in the form

$$\begin{cases} \dot{x} = g(t, x, y), \\ \dot{y} = \nu(x; y, g(t, x, y)) + f(t, x, y), \qquad (t, x, y) \in \mathbb{R} \times TM, \end{cases} \tag{3.8}$$

where $g, f : \mathbb{R} \times TM \to \mathbb{R}^k$ are continuous and such that $g(t, x, y), f(t, x, y) \in T_x(M)$ for all $(t, x, y) \in \mathbb{R} \times TM$.

We are now ready to introduce the concept of second order differential equation on a smooth m-dimensional manifold $M \subseteq \mathbb{R}^k$. We will see how this equation can be written exactly in the same way as if M were an open subset of $\mathbb{R}^k$.

When M is an open subset U of $\mathbb{R}^k$ one has $TM = U \times \mathbb{R}^k$, and a second order differential equation in M is written in the form

$$\ddot{x} = h(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM. \tag{3.9}$$

where $h : \mathbb{R} \times TM \to \mathbb{R}^k$ is a continuous map. As pointed out before, the important fact about an equation is what we mean by a solution. In this case a solution is a C^2 map $x(\cdot) : J \to M$, defined in a nontrivial interval $J \subseteq \mathbb{R}$, such that $\ddot{x}(t) = h(t, x(t), \dot{x}(t))$, identically in J. Therefore if $t \mapsto x(t)$ is a solution of (3.9), the associated curve $t \mapsto (x(t), y(t))$, where $y(t) = \dot{x}(t)$, lies in TM and satisfies the following first order equation in TM:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = h(t, x, y), \qquad (t, x, y) \in \mathbb{R} \times TM.)) \end{cases} \tag{3.10}$$

Writing (3.9) in this form allows us to adapt the well-known existence and uniqueness results of first order differential equations (as well as the Kamke property of the maximal solutions) to the context of second order equations on open subsets of Euclidean spaces. For example, as regards the existence property, one obtains that given $(\tau, p, v) \in \mathbb{R} \times TM$, (3.9) has a solution $x(\cdot)$, which is defined in an open interval J containing τ and satisfies the initial conditions $x(\tau) = p$ and $\dot{x}(\tau) = v$.

We point out that in the above considered case, i.e. when M is an open subset U of $\mathbb{R}^k$, no matter how the continuous map h of (3.9) is given, the (time dependent) vector field $G : \mathbb{R} \times TM \to \mathbb{R}^k \times \mathbb{R}^k$, given by $G(t, x, y) = (y, h(t, x, y))$, is always tangent to TM (which, in this case, is just the open subset $U \times \mathbb{R}^k$ of $\mathbb{R}^k \times \mathbb{R}^k$). Of course, this is not so when M is a general submanifold of $\mathbb{R}^k$. Consequently, if we want to apply the classical results to the associated system (3.10), the mapping $h : \mathbb{R} \times TM \to \mathbb{R}^k$ must be given in such a way that the map G defined above turns out to be a tangent vector field on TM. This suggests the following definition of second order differential equation on M.

Definition 3.1. Let M be a smooth differentiable submanifold of $\mathbb{R}^k$ and let $h : \mathbb{R} \times TM \to \mathbb{R}^k$ be a continuous map. An expression of the type

$$\ddot{x} = h(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM \tag{3.11}$$

is called a *(time dependent) second order differential equation* on M, provided that the associated vector field $G : \mathbb{R} \times TM \to \mathbb{R}^k \times \mathbb{R}^k$, given by $G(t, x, y) = (y, h(t, x, y))$, is tangent to TM (i.e. $(y, h(t, x, y)) \in T_{(x,y)}(TM)$ for all $(t, x, y) \in \mathbb{R} \times TM$). A solution of (3.11) is a C^2 curve $x : J \to \mathbb{R}^k$, defined on a (nontrivial) interval $J \subseteq \mathbb{R}$, in such a way that $x(t) \in M$ and $\ddot{x}(t) = h(t, x(t), \dot{x}(t))$, identically on J.

With this definition of second order differential equation, one could consider a more general situation than the case where M is a differentiable manifold. In fact, the above definition makes sense even if M is an arbitrary subset of $\mathbb{R}^k$. However, in order to obtain a meaningful situation, the subset M of $\mathbb{R}^k$ and the map $h : \mathbb{R} \times TM \to \mathbb{R}^k$ should be given in such a way that for any $(\tau, p) \in \mathbb{R} \times M$ and $v \in C_p(M)$ one gets the existence of (at least) a solution $x : [\tau, \beta) \to M$, defined on a right neighborhood of τ, satisfying the Cauchy conditions $x(\tau) = p$ and $\dot{x}(\tau) = v$. For interesting sufficient conditions which ensure the existence and uniqueness of a solution of a Cauchy problem associated with a first order differential equation on a set, we refer to the work of Nagumo [Na]. As far as we know, general conditions on M and h which ensure that the associated system (3.10) satisfies Nagumo's assumptions have not been given so far.

Going back to the case when M is a smooth submanifold of $\mathbb{R}^k$, it is important to see how one can practically write down a second order differential equation on M. Roughly speaking, we shall see how one can decompose the mapping $h : \mathbb{R} \times TM \to \mathbb{R}^k$ into a normal part, which depends only on the geometry of M, and a tangential part, which can be arbitrarily assigned. As a physical interpretation, (3.11) represents the motion equation of a constrained system, the manifold being the constraint. The normal component of h is the *reactive force* and the tangential component is the *active force*. When the tangential part of h is zero, the equation (3.11) is called *inertial*, and its solutions are the *geodesics* of M.

The general form (3.8) of any (non-autonomous) first order differential equation on TM helps us to write down any second order differential equation on M. In fact, let $\nu : T^2M \to \mathbb{R}^k$ be as in Theorem 3.2 and define $r : TM \to \mathbb{R}^k$ by $r(x, y) = \nu(x; y, y)$. Clearly, r is smooth and, given $x \in M$, $y \mapsto r(x, y)$ is a quadratic map from $T_x(M)$ into $T_x(M)^\perp$. Therefore, from Theorem 3.2 we get that

$$\ddot{x} = h(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM$$

is a second order differential equation on M if and only if, for any $(t, x, \dot{x}) \in \mathbb{R} \times TM$, the orthogonal projection of $h(t, x, \dot{x})$ onto $T_x(M)^\perp$ coincides with $r(x, \dot{x})$. Thus, any second order differential equation has the form

$$\ddot{x} = r(x, \dot{x}) + f(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM, \tag{3.12}$$

where $f : \mathbb{R} \times TM \to \mathbb{R}^k$ is an arbitrary continuous (time dependent, velocity dependent) tangent vector field on M. That is, the active force f must satisfy the condition $f(t, x, \dot{x}) \in T_x(M)$ for all $(t, x, \dot{x}) \in \mathbb{R} \times TM$. When f is zero, the equation (3.12) is called inertial and its solutions are the geodesics of M. When the active force has the special form

$$f(t, x, \dot{x}) = -\alpha \dot{x} + a(t, x),$$

where $\alpha > 0$ is given, the expression $-\alpha \dot{x}$ is the *frictional force* (α is the coefficient of friction) and $a(t, x)$ represents the *applied force*.

Exercise 3.3. Prove that if $x : J \to M$ is a geodesic of a smooth (or, more generally C^2) manifold $M \subseteq \mathbb{R}^k$, then the map $t \mapsto |\dot{x}(t)|^2$ is constant.

Exercise 3.4. Prove that if M is an open subset of an affine subspace of $\mathbb{R}^k$, then the map $r : TM \to \mathbb{R}^k$ is trivial.

Exercise 3.5. Find the explicit expression of $r : TM \to \mathbb{R}^k$ for

$$M = S_a^{k-1} = \{x \in \mathbb{R}^k : |x| = a\}.$$

As in the case where M is an open subset of $\mathbb{R}^k$, a second order differential equation on M can be equivalently written as a first order equation in TM. Actually, by definition, $\dot{x} = h(t, x, \dot{x})$ is a second order equation on M if and only if the equivalent system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = h(t, x, y) \end{cases}$$

is a first order equation on TM. Writing the continuous map $h : \mathbb{R} \times TM \to \mathbb{R}^k$ in the form $h(t, x, y) = r(x, y) + f(t, x, y)$, we get the general expression of a system on TM which corresponds to a second order differential equation on M:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = r(x, y) + f(t, x, y), \end{cases}$$

where $f : \mathbb{R} \times TM \to \mathbb{R}^k$ is an arbitrary continuous vector field such that $f(t, x, y) \in T_x(M)$ for all $(t, x, y) \in \mathbb{R} \times TM$.

There is a different and more concise way of writing a second order differential equation on a smooth manifold. By definition, any solution $x : J \to M$ of (3.12) satisfies the condition

$$\ddot{x}(t) = r(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t))$$

for all $t \in J$. Given any $t \in J$, the second derivative $\ddot{x}(t)$ can be uniquely decomposed into the sum of two components: a normal part $\ddot{x}_\nu(t) \in T_{x(t)}(M)^\perp$ and a parallel (or tangential) part $\ddot{x}_\pi(t) \in T_{x(t)}(M)$. It is not difficult to prove (see Exercise 3.6 below) that any C^2 curve $x : J \to M$ satisfies the condition

$$\ddot{x}_\nu(t) = r(x(t), \dot{x}(t)),$$

identically on J, no matter if it is a solution of (3.12) or not. Therefore, $x : J \to M$ is a solution of (3.12) if and only if one has

$$\ddot{x}_\pi(t) = f(t, x(t), \dot{x}(t))$$

for all $t \in J$. That is, the tangential acceleration (called the *covariant derivative* of the velocity) of any solution of (3.12) must equal the active force. Thus, the equation (3.12) can be written in the form

$$\ddot{x}_\pi = f(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM,$$

and the geodesics of M are the solutions of

$$\ddot{x}_\pi = 0. \tag{3.13}$$

Exercise 3.6. Prove that any C^2 curve $x : J \to M$ on a smooth manifold $M \subseteq \mathbb{R}^k$ satisfies the condition $\ddot{x}_\nu(t) = r(x(t), \dot{x}(t))$ for all $t \in J$, where $\ddot{x}_\nu(t)$ stands for the orthogonal projection of the acceleration $\ddot{x}(t)$ onto the space $T_{x(t)}(M)^\perp$.

Exercise 3.7. Prove that given $(p, v) \in TM$, the norm $|r(p, v)|$ is the curvature (in $\mathbb{R}^k$) at the point $p \in M$ of the geodesic $x(\cdot)$ satisfying the Cauchy conditions $x(0) = p$, $\dot{x}(0) = v$. Prove that the center of curvature of such a geodesic lies in the half-line with endpoint p and direction $r(p, v)$.

Exercise 3.8. Prove that if (3.13) admits a nontrivial closed geodesic (i.e. a non-constant periodic solution), then given $T > 0$, it admits T-periodic solutions $x : \mathbb{R} \to M$ with arbitrarily large speed $|\dot{x}(t)|$.

4. The degree of a tangent vector field

As in the previous section, for the sake of simplicity, all the differentiable manifolds we shall consider are supposed to be smooth, unless otherwise specified. Assume $M \subseteq \mathbb{R}^k$ is such a manifold and let $g : M \to \mathbb{R}^k$ be a continuous tangent vector field on M. If g is admissible on M, i.e. if the set of its zeros is compact, then (see e.g. [GP], [Hi], [Mi], [Tr] and references therein) one can assign to g an integer, $\deg(g, M)$, called the degree (or index, or Euler characteristic, or rotation) of the tangent vector field g on M. All the standard properties of the Brouwer degree of vector fields on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds.

To avoid any possible confusion, we point out that in the literature there exists a different extension of the Brouwer degree to the context of differentiable manifolds (see e.g. [Mi] and references therein), called the Brouwer degree for maps on manifolds. This second extension, roughly speaking, counts the (algebraic) number of solutions of an equation of the form $h(x) = y$, where $h : M \to N$ is a map between oriented manifolds of the same dimension and $y \in N$ is such that $h^{-1}(y)$ is compact. This dichotomy of notions in the context of manifolds arises from the fact that counting the solutions of an equation of the form $h(x) = y$ cannot be reduced to the problem of counting the zeros of a vector field, as one can do in $\mathbb{R}^k$ by defining $g(x) = h(x) - y$. Therefore, from the point of view of global analysis, the degree of a vector field and the degree of a map are necessarily two separated notions. The first one, which we are interested in, does not require any orientability and is particularly important for differential equations, since, we recall, a tangent vector field on a manifold can be regarded as a differential equation.

We give here a brief idea of how this degree can be defined (for an equivalent definition based on fixed point index theory see [FP1]). We need first the following result (see e.g. [Mi]).

Theorem 4.1. *Let $g : M \to \mathbb{R}^k$ be a C^1 tangent vector field on a differentiable manifold $M \subseteq \mathbb{R}^k$. If g is zero at some point $p \in M$, then the derivative $g'(p) : T_p(M) \to \mathbb{R}^k$ maps $T_p(M)$ into itself. Therefore, $g'(p)$ can be regarded as an endomorphism of $T_p(M)$ and consequently its determinant* $\det(g'(p))$ *is well defined.*

Proof. It suffices to show that $g'(p)v \in T_p(M)$ for any $v \in C_p(M)$ such that $|v| = 1$. Given such a vector v, consider a sequence in $M \setminus \{p\}$ such that $p_n \to p$ and $(p_n - p)/|p_n - p| \to v$. By Lemma 1.1 we have

$$g'(p)v = \lim_{n\to\infty} \frac{g(p_n) - g(p)}{|p_n - p|} = \lim_{n\to\infty} \frac{g(p_n)}{|p_n - p|}.$$

Observe that for all $n \in \mathbb{N}$, the vector $w_n = g(p_n)/|p_n - p|$ is tangent to M at p_n. Let us show that this implies that the limit $w = g'(p)v$ of $\{w_n\}$ is in $T_p(M)$. In fact, because of Theorem 2.2, we may assume that M (around p) is a regular level set of a smooth map $f : U \to \mathbb{R}^s$ defined on some open subset U of $\mathbb{R}^k$. Thus, by Theorem 2.1, $f'(p_n)w_n = 0$, and passing to the limit we get $f'(p)w = 0$, which means $w \in T_p(M)$, as claimed. □

Exercise 4.1. Let X be a subset of $\mathbb{R}^k$ with the property that the multivalued map which assigns to any $x \in X$ the compact set $T_x(X) \cap S^{k-1}$ is upper semicontinuous (see e.g. [AC]). Prove that if $g : X \to \mathbb{R}^k$ is a C^1 tangent vector field on X and $p \in X$ is a zero of g, then, as in Theorem 4.1, $g'(p)$ maps $T_p(X)$ into itself.

Let $g : M \to \mathbb{R}^k$ be a C^1 tangent vector field on a differentiable manifold $M \subseteq \mathbb{R}^k$. A zero $p \in M$ of g is said to be *nondegenerate* if $g'(p)$, as a map from $T_p(M)$ into itself, is an isomorphism. In this case, its index, $i(g, p)$, is defined to be 1 or -1 according to the sign of the determinant $\det(g'(p))$.

Exercise 4.2. Let $f : X \to \mathbb{R}^s$ be a C^1 map on a subset X of $\mathbb{R}^k$ and let $p \in X$ be such that $f(p) = 0$. Prove that if $f'(p) : T_p(X) \to \mathbb{R}^s$ is one to one, then p is an isolated zero of f. Use this fact to show that nondegenerate zeros of a tangent vector field on a manifold are isolated.

Hint. Use Lemma 1.1 (or Lemma 1.2, taking the restriction of f to $f^{-1}(0)$).

In the particular case when an admissible tangent vector field $g : M \to \mathbb{R}^k$ is *nondegenerate* (i.e. smooth, with only nondegenerate zeros), its degree, $\deg(g, M)$, is defined just summing up the indices at its zeros. This makes sense, since $g^{-1}(0)$ is compact (g being admissible) and discrete (as pointed out in Exercise 4.2). Therefore, the sum if finite. Using transversality arguments (see e.g. [Hi]) one can show that if two such tangent vector fields can be joined by a smooth homotopy, then they have the same degree, provided that this homotopy is admissible (i.e. the set of zeros remains in a compact subset of M). Moreover, it is clear that given g as above, if V is any open subset of M containing $g^{-1}(0)$, then $\deg(g, M) = \deg(g, V)$.

The above "homotopy invariance property" gives an idea of how to proceed in the general case. If $g : M \to \mathbb{R}^k$ is any continuous admissible tangent vector field on M, consider any relatively compact open subset V of M containing the zeros of g and observe that, since the boundary ∂V of V (in M) is compact, we have $\min\{|g(x)| : x \in \partial V\} = \delta > 0$. Let g_1 be any nondegenerate tangent vector field on $\bar{V}$ such that

$$\max\{|g(x) - g_1(x)| : x \in \partial V\} < \delta.$$

Then $\deg(g, M)$ is defined as $\deg(g1, V)$. To see that this definition does not depend on the approximating map, observe that if g_2 is a different nondegenerate vector field satisfying the same inequality, we get $(1 - s)g_1(x) + sg_2(x) \neq 0$ for all $s \in [0, 1]$ and $x \in \partial V$. Therefore $(x, s) \mapsto (1 - s)g_1(x) + sg_2(x)$ is an admissible homotopy of tangent vector fields on V. This proves that $\deg(g_1, V) = \deg(g_2, V)$. The fact that this definition does not depend on the open set V containing $g^{-1}(0)$ is very easy to check and left to the reader.

The following are the main properties of the degree for tangent vector fields on manifolds.

Solution. If $\deg(g, M) \neq 0$ then g has a zero on M.

Additivity. If V_1 and V_2 are open in M, $V_1 \cap g^{-1}(0)$ and $V_2 \cap g^{-1}(0)$ are compact, and $V_1 \cap V_2 \cap g^{-1}(0)$ is empty, then $\deg(g, V_1 \cup V_2) = \deg(g, V_1) + \deg(g, V_2)$.

Homotopy invariance. If $h : M \times [0, 1] \to \mathbb{R}^k$ is a continuous admissible homotopy of tangent vector fields, that is $h(x, s) \in T_x(M)$ for all $(x, s) \in M \times [0, 1]$ and $h^{-1}(0)$ is compact, then $\deg(h(\cdot, s), M)$ does not depend on $s \in [0, 1]$.

The above definition of degree implies immediately that if two vector fields $g_1 : M \to \mathbb{R}^k$ and $g_2 : N \to \mathbb{R}^s$ correspond under a diffeomorphism $\varphi : M \to N$, then, if one is admissible, so is the other one, and they have the same degree (on M and N respectively). Moreover, if M is an open subset of $\mathbb{R}^k$ and $g : M \to \mathbb{R}^k$ is admissible, then $\deg(g, M)$ is just the classical Brouwer degree at zero, $\deg(g, V, 0)$, of the restriction of g to any bounded open subset V of M containing $g^{-1}(0)$ and such that $\bar{V} \subseteq M$ (see e.g. [Ll]).

Exercise 4.3. Regarding the complex plane $\mathbb{C}$ as a real two dimensional manifold, prove that the (topological) degree, $\deg(P, \mathbb{C})$, of any polynomial map $P : \mathbb{C} \to \mathbb{C}$ coincides with its algebraic degree. This gives a proof of the fundamental theorem of algebra.

We observe that, as a consequence of the homotopy invariance property, the degree of a tangent vector field on a compact manifold $M \subseteq \mathbb{R}^k$ does not depend on the vector field. In fact, if g_1 and g_2 are two tangent vector fields on M, then $h : M \times [0, 1] \to \mathbb{R}^k$, given by $h(x, s) = (1 - s)g_1(x) + sg_2(x)$, is an admissible homotopy. This implies that to any compact manifold M one can assign an integer, $\chi(M)$, called the *Euler-Poincaré characteristic* of M, by putting

$$\chi(M) := \deg(g, M),$$

where $g : M \to \mathbb{R}^k$ is any tangent vector field on M. Clearly, if two compact manifolds M and N are diffeomorphic, then $\chi(M) = \chi(N)$. Moreover, if M is compact with $\chi(M) \neq 0$, then any tangent vector field on M must vanish at some point.

Exercise 4.4. Prove that $\chi(S^2) = 2$.
Hint. Consider the gradient of the map $h : S^2 \to \mathbb{R}$, given by $h(x, y, z) = z$.

Exercise 4.5. Prove that $\chi(S^1) = 0$ by defining a nonvanishing tangent vector field on S^1.

Exercise 4.6. Compute $\chi(S^m)$ by considering the gradient of the real function $h : S^m \to \mathbb{R}$, given by $h(x_1, x_2, ...x_{m+1}) = x_{m+1}$.

Exercise 4.7. Give an example of a (necessarily noncompact) manifold M with two admissible vector fields g_1 and g_2 for which $\deg(g_1, M) \neq \deg(g_2, M)$.

Exercise 4.8. Prove that if a tangent vector field g on an m-dimensional manifold M is admissible, then $-g$ is admissible and $\deg(-g, M) = (-1)^m \deg(g, M)$.

Exercise 4.9. Deduce from the previous exercise that if M is a compact odd dimensional manifold, then $\chi(M) = 0$.

So far we have considered only manifolds without boundary. An m-dimensional differentiable manifold with boundary is just a subset X of $\mathbb{R}^k$ which is locally diffeomorphic to the open subsets of a closed half subspace of $\mathbb{R}^m$. The boundary ∂X of X coincides, by definition, with the set δX of singular points of X. The interior of X is $X \setminus \partial X$. Observe that as a consequence of Theorem 1.3, ∂X is a boundaryless manifold of dimension $m - 1$ and the tangent cone to X at a boundary point p is a closed half subspace of the m-dimensional space $T_p(X) \subseteq \mathbb{R}^k$. Moreover, ∂X is (relatively) closed in X and the open subset $X \setminus \partial X$ of X is a boundaryless m-dimensional manifold. Observe also that any boundaryless manifold M (as defined in section 2) can be regarded as a particular manifold with boundary (with the property $\partial M = \emptyset$).

Assume now that $X \subseteq \mathbb{R}^k$ is a compact manifold with boundary. If $g : X \to \mathbb{R}^k$ is a tangent vector field on X which does not vanish on ∂X, then g is admissible on $M = X \setminus \partial X$. In fact, $g^{-1}(0)$ is a compact subset of X contained in M. This means that $\deg(g, M)$ is well defined. In this case we say that g is admissible on X and we put $\deg(g, X) := \deg(g, M)$. If, in particular, g is strictly outward along the boundary, that is, $-g(x)$ is in the (relative) interior of the half subspace $C_x(X)$ of $T_x(X)$ for all $x \in \partial X$, then g is admissible and $\deg(g, X)$ is well defined. If g_1 and g_2 are two tangent vector fields on X, both strictly outward along the boundary, then the homotopy $h(x, s) = (1 - s)g_1(x) + sg_2(x)$ does not vanish on ∂X. Therefore, $h^{-1}(0)$ is a compact subset of $(X \setminus \partial X) \times [0, 1]$, and this implies $\deg(g_1, X) = \deg(g_2, X)$. Hence, it makes sense to define the Euler-Poincaré characteristic $\chi(X)$ of a manifold X with boundary, by considering the common degree of all tangent vector fields on X pointing outward along ∂X. This extends the previous definition for boundaryless manifolds.

Actually, there are other equivalent (and better) ways to define the Euler-Poincaré characteristic of a compact manifold with boundary. One of these is via homology theory (see for example [Sp]). The powerful homological method has the advantage that can be applied to a large class of topological spaces, which includes those of the same homotopy type as compact polyhedra (such as compact manifolds with boundary). The celebrated Poincaré-Hopf theorem asserts that

"the definition of the Euler-Poincaré characteristic considered above coincides with the homological one".

Exercise 4.10. Prove that the Euler-Poincaré characteristic of the k-dimensional disk $D^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$ is one.

5. Forced oscillations on manifolds and bifurcation

This last section is devoted to the problem concerning the existence of forced oscillations of a periodically excited constrained mechanical system. The system is represented by a second order differential equation on a manifold M (the constraint) which throughout this section will be assumed to be smooth, boundaryless, m-dimensional and embedded in $\mathbb{R}^k$. The proofs of most of the results are too long to be included here.

Consider the following second order, time dependent differential equation on M:

$$\ddot{x} = r(x, \dot{x}) + f(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM, \tag{5.1}$$

where the forcing term $f : \mathbb{R} \times TM \to \mathbb{R}^k$ is a continuous T-periodic tangent vector field on M. That is, f satisfies the condition

$$f(t + T, x, \dot{x}) = f(t, x, \dot{x}) \in T_x(M),$$

for all $(t, x, \dot{x}) \in \mathbb{R} \times TM$. This represents the motion equation of a constrained mechanical system acted on by a T-periodic "generalized" force f, which, without loss of generality, can be assumed to be tangent to the constraint M (any normal component of a force is neutralized by the constraint). We are interested in conditions on the constraint M and on the force f which ensure the existence of forced (or harmonic) oscillations of (5.1); i.e. periodic solutions of the same period as that of the forcing term f.

To study this problem, it is convenient to embed (5.1) in a one parameter family of second order differential equations in the following way:

$$\ddot{x} = r(x, \dot{x}) + \lambda f(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM, \ \lambda \geq 0. \tag{5.2}$$

Thus, (5.2) becomes (5.1) when $\lambda = 1$ and reduces to the inertial equation for $\lambda = 0$. An appropriate space to look for solutions of (5.2) is the Cartesian product $[0, \infty) \times C^1_T(M)$, consisting of all the pairs $(\lambda, x(\cdot))$, with $\lambda \geq 0$ and $x : \mathbb{R} \to M$ a T-periodic C^1 map. This is obviously a metric space, since $C^1_T(M)$ is a subset of the Banach space $C^1_T(\mathbb{R}^k)$ of the C^1 T-periodic maps $x : \mathbb{R} \to \mathbb{R}^k$, with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in \mathbb{R}\} + \sup\{|\dot{x}(t)| : t \in \mathbb{R}\}.$$

However, $[0, \infty) \times C^1_T(M)$ need not be complete. To see this, think of the most significant and simple example of differentiable manifold: an open subset of $\mathbb{R}^k$ (different from $\mathbb{R}^k$).

Exercise 5.1. Prove that $C^1_T(M)$ is complete if and only if M is closed in $\mathbb{R}^k$.

Exercise 5.2. Use the preceding exercise (and the fact that any differentiable manifold is locally compact) to show that $C^1_T(M)$ is locally complete.

An element $(\lambda, x) \in [0, \infty) \times C^1_T(M)$ will be called a *solution pair* of (5.2) provided that $x(\cdot)$ is a (clearly T-periodic) solution of the differential equation (5.2). Denote by X the subset of $[0, \infty) \times C^1_T(M)$ of all the solution pairs of (5.2), and observe that the points of M are in a one-to-one correspondence with the solution pairs of the type $(0, x)$, where $x : \mathbb{R} \to M$ is a constant map. Therefore, in the sequel, the manifold M will be identified with these elements, called the *trivial solution pairs* of (5.2). This simplifies some notation. For example if W is a subset of $[0, \infty) \times C^1_T(M)$, $M \cap W$ stands for the subset of M consisting of those points $p \in M$ such that $(0, \bar{p}) \in W$, where $\bar{p}$ is the constant map $t \mapsto p$. A neighborhood of a point $p \in M$ in the space $[0, \infty) \times C^1_T(M)$ is actually a neighborhood of $(0, \bar{p})$. According to this identification and terminology, the elements of the subset $X \setminus M$ of $[0, \infty) \times C^1_T(M)$ are regarded as the *nontrivial solution pairs* of (5.2). We observe that there may exist nontrivial solution pairs even for $\lambda = 0$. This happens when (and only when) the inertial equation admits nontrivial closed geodesics, as in the case of the inertial motion of a mass point constrained in a circle or in a sphere.

Exercise 5.3. Prove that the set X of solution pairs of (5.2) is closed in $[0, \infty) \times C^1_T(M)$.

Exercise 5.4. Prove that M may be regarded as a closed subset of $[0, \infty) \times C^1_T(M)$.

Exercise 5.5. Using Ascoli's theorem prove that any bounded subset of X is precompact (i.e. it has compact completion or, equivalently, is totally bounded).

Exercise 5.6. Use the above exercise and the fact that X is closed in the locally complete space $[0,\infty) \times C^1_T(M)$ to show that X is locally compact.

An element p of M is called a *bifurcation point* (of forced oscillations) for (5.2) if any neighborhood of p in $[0,\infty) \times C^1_T(M)$ contains a nontrivial solution (pair). In spite of the fact that one may have nontrivial solutions for $\lambda = 0$, any bifurcation point must be an accumulation point of solution pairs having $\lambda > 0$. The reason of this is a well-known result in Riemannian geometry: *there are no (nontrivial) closed geodesics in a convenient neighborhood of a point.*

The following is a necessary condition for a point $p \in M$ to be of bifurcation (see [FP4]).

Theorem 5.1. *Let M be a boundaryless m-dimensional smooth manifold in $\mathbb{R}^k$ and let $f : \mathbb{R} \times TM \to \mathbb{R}^k$ be a T-periodic continuous active force on M. If $p \in M$ is a bifurcation point (of forced oscillations) for the parametrized second order equation*

$$\ddot{x} = r(x,\dot{x}) + \lambda f(t,x,\dot{x}), \qquad (t,x,\dot{x}) \in \mathbb{R} \times TM, \quad \lambda \geq 0.$$

then the average force vanishes at p. That is

$$\bar{f}(p) = \frac{1}{T}\int_0^T f(t,p,0)dt = 0.$$

Observe that Theorem 5.1 is trivial when the reactive force $r : TM \to \mathbb{R}^k$ is identically zero (the flat case). In fact, let $\{(\lambda_n, x_n)\}$ be a sequence of nontrivial solution pairs such that $\lambda_n \to 0$, $x_n(t) \to p$ uniformly and $\dot{x}_n(t) \to 0$ uniformly. Integrating from 0 to T both members of the equalities

$$\ddot{x}_n(t) = \lambda_n f(t, x_n(t), \dot{x}_n(t)), \quad n \in \mathbb{N}, \quad t \in \mathbb{R},$$

we get

$$\int_0^T f(t, x_n(t), \dot{x}_n(t))dt = 0,$$

and the assertion in the flat case is obtained passing to the limit.

The following global result of [FP2] provides a sufficient condition for bifurcation.

Theorem 5.2. *Assume that the constraint M is compact with nonzero Euler-Poincaré characteristic. Then (5.2) admits an unbounded connected set Σ of nontrivial solution pairs whose closure in $[0,\infty) \times C^1_T(M)$ contains a bifurcation point.*

As a consequence of the above theorem, we get the following perturbation result:

If M is compact and $\chi(M) \neq 0$, then (5.2) admits T-periodic solutions for $\lambda > 0$ sufficiently small.

In fact, the unbounded branch Σ, being connected, cannot be entirely contained in the "inertial" slice $\{0\} \times C^1_T(M)$; otherwise one would have an accumulation point of nontrivial closed geodesics (and this, as pointed out before, is impossible). As far as we know, this perturbation result was proved for the first time by Benci and Degiovanni in [BD], with completely different methods.

Observe that the condition $\chi(M) \neq 0$ is necessary for the existence of a bifurcation point for any T-periodic forcing term. It is known, in fact, that, if $\chi(M) = 0$, there exists an autonomous, nonvanishing, tangent vector field f on M. Thus, f can be interpreted as a T-periodic force coinciding with its average, and the necessary condition for bifurcation of Theorem 5.1 is not satisfied.

The question is if, under the assumption that M is compact with $\chi(M) \neq 0$, (5.1) has a forced oscillation, at least in the case when f is independent of the velocity (or, more generally, bounded). We do not yet know the answer to this problem, even though we are incline to believe that it is affirmative.

An interesting result related to this conjecture has been obtained by Benci in [Be], where he proved the existence of infinitely many forced oscillations for a system whose constraint M is a smooth manifold with finite fundamental group (as in the case of S^m), provided that the force admits a time periodic Lagrangian satisfying certain physically reasonable assumptions.

Positive partial answers to the above conjecture have been obtained in [FP3] for $M = S^2$ (the spherical pendulum) and in [FP5] for $M = S^{2n}$. In both cases, the applied force is assumed to be independent of the velocity. The crucial tool to get these results is Theorem 5.2 above, which ensures the existence of an unbounded connected set Σ of nontrivial solution pairs (recall that $\chi(S^{2n}) = 2$). Therefore, if (5.1) does not admit forced oscillations, then Σ must be contained in $[0,1) \times C^1_T(S^{2n})$, and this was shown to be impossible if the applied force f does not depend on the velocity. Actually (see [FP3] and [FP5]), with this assumption on f, in spite of the fact that the set of closed geodesics of S^{2n} is unbounded (with the C^1 norm), any connected set of solution pairs contained in $[0,1) \times C^1_T(S^{2n})$ must be bounded. The technique to get these estimates is based upon the existence of a convenient, continuous, integer valued function (the rotation index with respect to the origin) defined on a suitable class of T-periodic C^1 curves on S^{2n}. The idea is similar to the one previously used by Capietto, Mawhin and Zanolin in [CMZ1] to get solutions of a superlinear periodic boundary value problem in $\mathbb{R}^k$.

Theorem 5.2 above gives a sufficient condition for the existence of bifurcation points of the equation (5.2). However, because of the compactness assumption on M, this condition cannot be applied to the most common situation: the flat case. Moreover, since (according to Theorem 5.1) a bifurcation point of (5.2) is a zero of the average force $\bar{f}$, it is natural to ask when a zero $p \in M$ of $\bar{f}$ is actually a bifurcation point. On the other hand, the above sufficient condition does not answer this question: it cannot be applied to a small neighborhood of a point p in M. This justifies the interest of the following extension of Theorem 5.2 (see [FP4]), which includes the cases when M

is an open subset of $\mathbb{R}^k$ or an open neighborhood of a point p in M. We recall first that an autonomous tangent vector field $\bar{f}$ on a differentiable manifold M is said to be admissible on an open subset V of M if $\bar{f}^{-1}(0) \cap V$ is compact (see section 4). In this case the degree, $\deg(\bar{f}, V)$, is well defined.

Theorem 5.3. *Let M be a boundaryless smooth manifold in $\mathbb{R}^k$ and let $f : \mathbb{R} \times TM \to \mathbb{R}^k$ be a T-periodic continuous active force on M. Denote by $\bar{f} : M \to \mathbb{R}^k$ the autonomous tangent vector field*

$$\bar{f}(p) = \frac{1}{T} \int_0^T f(t, p, 0)dt.$$

Let V be an open subset of M and assume that the degree $\deg(\bar{f}, V)$ of $\bar{f}$ on V is defined and nonzero. Then the parametrized second order differential equation

$$\ddot{x} = r(x, \dot{x}) + \lambda f(t, x, \dot{x}), \qquad (t, x, \dot{x}) \in \mathbb{R} \times TM, \quad \lambda \geq 0,$$

admits a connected set Σ of nontrivial solution pairs, whose closure (in $[0, \infty) \times C^1_T(M)$) meets V (at some bifurcation point) and has (at least) one of the following three properties:

1. *it is unbounded;*
2. *it is not complete;*
3. *it contains a bifurcation point in $M \setminus V$.*

Observe that in the above result, if the manifold M is a closed subset of $\mathbb{R}^k$, then the second alternative cannot occur, since, in this case, $[0, \infty) \times C^1_T(M)$ is a complete metric space. Thus, choosing $V = M$, if $\deg(\bar{f}, M)$ is defined and nonzero, then the branch Σ must be unbounded. This is, in fact, the situation of Theorem 5.2, where M is compact and $\deg(\bar{f}, M) = \chi(M) \neq 0$.

Assume now that $p \in M$ is an isolated zero for the average force $\bar{f}$ and let $V \subseteq M$ be an open neighborhood of p. By the excision property of the degree, $\deg(\bar{f}, V)$ does not depend on V, provided that $V \cap \bar{f}^{-1}(0) = \{p\}$. This shared integer is called the index of $\bar{f}$ at the isolated zero p and denoted by $i(\bar{f}, p)$. This clearly extends to the continuous case the notion of index of a C^1 tangent vector field at a nondegenerate zero (introduced in section 4). Observe, in fact, that if the force f is C^1 and $p \in M$ is a nondegenerate zero of $\bar{f}$, then p is an isolated zero of $\bar{f}$ and $i(\bar{f}, p) = \pm 1$ according to whether the derivative $\bar{f}'(p) : T_p(M) \to T_p(M)$ preserves or inverts the orientation of $T_p(M)$.

Using the notion of index at an isolated zero, we have as a direct consequence of Theorem 5.3 the following sufficient condition for a given $p \in M$ to be a bifurcation point of (5.2).

Corollary 5.1. *Let M and f be as in Theorem 5.3. Assume that $p \in M$ is an isolated zero of the average force $\bar{f}$. If $i(\bar{f}, p) \neq 0$, then p is a bifurcation point of* *forced oscillations for (5.3). In particular this holds when f is C^1 and the derivative $\bar{f}'(p) : T_p(M) \to T_p(M)$ is one-to-one.*

As pointed out in section 3 any second order differential equation on a differentiable manifold $M \subseteq \mathbb{R}^k$ can be regarded as a first order differential equation on the tangent bundle $TM \subseteq \mathbb{R}^k \times \mathbb{R}^k$. On the other hand, any differential equation on TM can be extended to a differential equation on an open subset U of $\mathbb{R}^k \times \mathbb{R}^k$ containing TM. Thus, the equation (5.2) can be written in the form

$$\dot{z} = g(z) + \lambda h(t, z), \qquad (t, z) \in \mathbb{R} \times U, \tag{5.3}$$

where $g(z) + \lambda h(t, z) \in T_z(TM)$, whenever $z \in TM$. Since $h(t, z) \equiv h(t + T, z)$, this equation can be regarded as a T-periodic perturbation of a first order autonomous differential equation in $\mathbb{R}^{2k}$.

A very interesting continuation principle for equations of the above form (and not necessarily related to second order equations) is given in [CMZ2], where the existence of a bifurcating branch of solution pairs (λ, z) is ensured, provided that the Brouwer topological degree of g is well defined and nonzero (no assumptions on the perturbation h are needed for the existence of such a branch).

What seems peculiar to us, and interesting for further investigations, is the fact that in Theorem 5.3 is just the role of the periodic perturbation h (or, equivalently, of the applied force f) which is important for the existence of a bifurcating branch. The map g does not satisfy any assumptions, except that the vector field $g : U \to \mathbb{R}^k \times \mathbb{R}^k$ must be tangent to TM for any z in the subset TM of U. Actually, in the situation of Theorem 5.3, the vector field g need not be admissible (from the point of view of the degree theory). In fact, in this case, $g^{-1}(0)$ coincides with the trivial section $M \times \{0\}$ of TM, which need not be compact (observe, in fact, that M can be an open subset of $\mathbb{R}^k$).

Partial results regarding the periodically perturbed equation (5.3) have been recently obtained by P. Morassi in [Mo], where the subset $g^{-1}(0)$ of W is assumed to be a differentiable manifold; even though, in his case, (5.3) is not necessarily associated with a second order differential equation.

References

[AC] Aubin J.P. - Cellina A., *Differential Inclusions: Set valued Maps and Viability Theory*, A Series of Comprehensive Studies in Math. 264, Springer-Verlag, Berlin, 1984.

[BC] Brickell F. - Clark R.S., *Differentiable manifolds*, Van Nostrand Reinhold Company, London, 1979.

[BD] Benci V. - Degiovanni M., *Periodic solutions of dissipative dynamical systems*, Proc. Conf. on "Variational Problems", Berestycki, Coron and Ekeland editors, Birkhäser, Basel, New York, 1990.

[Be] Benci V., *Periodic solutions of Lagrangian systems on a compact manifold*, J. Differential Equations, 63 (1986), 135–161.

[Bo] Bouligand G., *Introduction à la géométrie infinitesimale directe*, Gauthier-Villard, Paris, 1932.

[CMZ1] Capietto A. - Mawhin J. - Zanolin F., *A continuation approach to superlinear periodic boundary value problems*, J. Differential Equations, 88 (1990), 347–395.

[CMZ2] Capietto A. - Mawhin J. - Zanolin F., *Continuation theorems for periodic perturbations of autonomous systems*, Trans. Amer. Math. Soc., 329 (1992), 41–72.

[FP1] Furi M. - Pera M.P., *A continuation principle for forced oscillations on differentiable manifolds*, Pacific J. of Math., 121 (1986), 321-338.

[FP2] Furi M. - Pera M.P., *A continuation principle for the forced spherical pendulum*, in "Fixed Point Theory and Applications", Thera M.A. and Baillon J.-B. Editors, Pitman Research Notes in Math. 252, Longman Scientific & Technical, 1991, 141–154.

[FP3] Furi M. - Pera M.P., *The forced spherical pendulum does have forced oscillations*, in "Delay Differential Equations and Dynamical Systems", Busenberg S. and Martelli M. Editors, Lecture Notes in Math. 1475, Springer Verlag, Berlin, 1991, 176–183.

[FP4] Furi M. - Pera M.P., *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory*, Pacific J. of Math., 160 (1993), 219–244.

[FP5] Furi M. - Pera M.P., *On the notion of winding number for closed curves and applications to forced oscillations on even dimensional spheres*, Boll. Un. Mat. Ital., (7-A) 3 (1993), 397–407.

[GP] Guillemin V. - Pollack A., *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.

[Hi] Hirsh M.W., *Differential Topology*, Graduate Texts in Math., Vol. 33, Springer Verlag, Berlin, 1976.

[La] Lang S., *Introduction to differentiable manifolds*, John Wiley & Sons, Inc., New York, 1966.

[Ll] Lloyd N.G., *Degree Theory*, Cambridge Tracts in Math., Vol. 73, Cambridge Univ. Press, London, 1978.

[Mi] Milnor J.W., *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.

[Mo] Morassi P., *Principi di continuazione per equazioni funzionali dipendenti da un parametro e applicazioni alla biforcazione di soluzioni periodiche di equazioni differenziali ordinarie*, Ph.D. Thesis, Firenze, 1994.

[Na] Nagumo M., *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Phys. Math. Soc. Japan, (3) 24 (1942), 551–559.

[Se] Severi F., *Conferenze di geometria algebrica*, Zanichelli, Bologna, 1927.

[Sp] Spanier E.H., *Algebraic Topology*, McGraw-Hill Series in Higher Math., McGraw-Hill, New York, 1966.

[Tr] Tromba A.J., *The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree*, Advances in Math., 28 (1978), 148–173.