On the set of harmonic solutions of periodically perturbed autonomous differential equations on manifolds

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1 Introduction

In this paper, we prove a result which gives information about the structure of the set of harmonic solutions of the parametrized differential equation

$$\dot{x} = g(x) + \lambda f(t, x), \tag{1}$$

where $g : M \to \mathbf{R}^k$ and $f : \mathbf{R} \times M \to \mathbf{R}^k$ are continuous vector fields, tangent to a (not necessarily closed) boundaryless differentiable manifold $M \subset \mathbf{R}^k$, with f T-periodic with respect to the first variable. We investigate the structure of the set of solution pairs of (1); i.e. of those pairs $(\lambda, x) \in [0, \infty) \times C_T(M)$ such that x is a (necessarily T-periodic) solution of (1).

We give conditions ensuring the existence of a non-compact connected component of solution pairs (λ, x) of (1) which emanates from the set of constant solutions of the unperturbed equation

$$\dot{x} = g(x). \tag{2}$$

In the case when M is closed, this component turns out to be unbounded. We point out that the weaker assertion that this connected set emanates from the set of T-periodic solutions of (2) has been previously obtained in [C1] and [C2] under the additional assumption that the set of T-periodic solutions of (2) is compact. For related results regarding continuation principles see [CMZ], [M2] and references therein. The techniques used here are different from those of [C1] and [C2]: we do not use infinite dimensional degree theories.

As an application, we describe the set of forced oscillations of a periodically perturbed pendulum-like equation, providing also a multiplicity result. For a

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nice survey paper regarding forced oscillations for the pendulum equation we suggest the reading of [M1].

2 Preliminaries

We recall first some basic facts and definitions that will turn out to be useful throughout the present paper.

If $M \subset \mathbf{R}^k$ is a differentiable manifold, the metric subspace $C_T(M)$ of $C_T(\mathbf{R}^k)$ of all T-periodic continuous functions $x : \mathbf{R} \to M$ is not necessarily complete, unless M is closed in $\mathbf{R}^k$. Anyway, $C_T(M)$ is always locally complete. The reason is the following: since M is locally compact, given $x \in C_T(M)$, there exists a relatively compact open set $V \subset M$ containing the compact image $x([0,T])$ of x.

A metric space K is said to be precompact, if its completion is compact. Observe that K is precompact if and only if it is totally bounded. In fact, the total boundedness of K is equivalent to the assertion that any sequence in K has a Cauchy subsequence. Consequently K is compact if and only if it is both totally bounded and complete.

The following version of Ascoli's theorem will be useful in the sequel.

Theorem 2.1 *Let X be a subset of $\mathbf{R}^k$ and B a bounded equicontinuous subset of $C([a,b], X)$. Then B is totally bounded. In particular, if X is closed, B is relatively compact.*

Let U be an open subset of the differentiable manifold $M \subset \mathbf{R}^k$, and $v : M \to \mathbf{R}^k$ be a continuous tangent vector field such that the set $v^{-1}(0) \cap U$ is compact. Then, one can associate to the pair (v, U) an integer, often called the Euler characteristic (or index) of v in U, which, roughly speaking, counts (algebraically) the number of zeros of v in U (see e.g. [GP], [H], [Mi], [T], and references therein), and which, for reasons that will became clear in the sequel, we will call degree of the vector field v and denote by $\deg(v, U)$. If $v^{-1}(0) \cap U$ is a finite set, then $\deg(v, U)$ is simply the sum of the indices of the zeros of v. In the general admissible case, i.e. when $v^{-1}(0) \cap U$ is a compact set, $\deg(v, U)$ is defined by taking a convenient smooth approximation of v having finitely many zeros (provided that these zeroes are sufficiently close to $v^{-1}(0) \cap U$).

We stress that no orientability on M is necessary in order to define the degree of a tangent vector field.

The celebrated Poincaré-Hopf theorem says that, if M is a compact manifold (possibly with boundary ∂M), then $\deg(v, M \setminus \partial M) = \chi(M)$ for any tangent vector field v which points outward along ∂M.

In the flat case, namely if U is an open subset of $\mathbf{R}^k$, $\deg(v, U)$ is just the Brouwer degree (with respect to zero) of v in U.

Using the equivalent definition of degree given in [FP2], one can see that all the standard properties of the Brouwer degree on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds.

3 Main results

We deal with the following differential equation

$$\dot{x} = g(x) + \lambda f(t, x). \tag{3}$$

where $f : \mathbf{R} \times M \to \mathbf{R}^k$ and $g : M \to \mathbf{R}^k$ are continuous tangent vector fields defined on a (boundaryless) differentiable manifold $M \subset \mathbf{R}^k$, with f T-periodic in $t \in \mathbf{R}$.

A pair $(\lambda, p) \in [0, \infty) \times M$ is a *starting point* (for T-periodic solutions) if the Cauchy problem

$$\begin{cases} \dot{x} = g(x) + \lambda f(t, x) \\ x(0) = p \end{cases} \tag{4}$$

has a T-periodic solution. A starting point (λ, p) is called *trivial* if $\lambda = 0$ and $p \in g^{-1}(0)$. While the concept of starting point is essentially finite dimensional, strictly related to this, one has the (infinite dimensional) notion of solution pair. We say that $(\lambda, x) \in [0, \infty) \times C_T(M)$ is a *solution pair* if x satisfies (3). If $\lambda = 0$ and x is constant, then (λ, x) is said to be *trivial*.

Denote by X the subset of $[0, \infty) \times C_T(M)$ of all solution pairs and by S the set of all starting points. Notice that X is locally complete, as a closed subset of a locally complete space. The map $h : X \to S$, which associates to $(\lambda, x) \in X$ the starting point $(\lambda, x(0))$, is continuous and onto. Notice that, if (λ, x) is trivial, then the corresponding pair $(\lambda, x(0)) \in S$ is trivial as well. The converse is clearly true if g is C^1.

The following global connectivity result of [FP3] will be crucial in the sequel.

Lemma 3.1 *Let Y be a locally compact Hausdorff space and let Y_0 be a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then $Y \backslash Y_0$ contains a not relatively compact component whose closure (in Y) intersects Y_0.*

For the sake of simplicity, according to [FP4], we make some conventions. We will regard every space as its image in the following diagram of natural inclusions

$$\begin{array}{ccc} [0, \infty) \times M & \longrightarrow & [0, \infty) \times C_T(M) \\ \uparrow & & \uparrow \\ M & \longrightarrow & C_T(M) \end{array} .$$

In particular, we will identify M with its image in $C_T(M)$ under the embedding which associates to any $p \in M$ the map $\hat{p} \in C_T(M)$ constantly equal to p. Moreover we will regard M as the slice $\{0\} \times M \subset [0, \infty) \times M$ and, analogously,

$C_T(M)$ as $\{0\} \times C_T(M)$. We point out that the images of the above inclusions are closed.

According to these identifications, if Ω is an open subset of $[0, \infty) \times C_T(M)$, by $\Omega \cap M$ we mean the open subset of M given by all $p \in M$ such that the pair $(0, \hat{p})$ belongs to Ω. If U is an open subset of $[0, \infty) \times M$, then $U \cap M$ represents the open set $\{p \in M : (0, p) \in U\}$. Moreover, $S \backslash g^{-1}(0)$ stands for the set $S \setminus \left[\{0\} \times g^{-1}(0)\right]$ of nontrivial starting points of (3).

Assume now that (4) has a unique solution for all $p \in M$. By known results on differential equations, the set $D \subset [0, \infty) \times M$ of all the pairs (λ, p) such that the solution of (4) is defined in $[0, T]$ is open, thus locally compact. Obviously the set S of all the starting points of (3) is a closed subset of D, even if it could be not so in $[0, \infty) \times M$. This implies that S is locally compact. If U is an open subset of D, the set $S \cap U$ is open in S, thus it is locally compact as well.

Theorem 3.2 below (see [FS], Theorem 3.1) plays a crucial role in the proof of the main result of this paper.

Theorem 3.2 *Let $f : \mathbf{R} \times M \to \mathbf{R}^k$ and $g : M \to \mathbf{R}^k$ be C^1 tangent vector fields defined on a (boundaryless) differentiable manifold $M \subset \mathbf{R}^k$, with f T-periodic in the first variable. If D and S are as above, U is an open subset of D and $\deg(g, U \cap M)$ is well defined and nonzero, then the set $(S \cap U) \setminus g^{-1}(0)$ of nontrivial starting points (in U) admits a connected subset whose closure in $S \cap U$ meets $g^{-1}(0)$ and is not compact.*

We are now in a position to state our main result. The proof is inspired by [FP4].

Theorem 3.3 *Let $f : \mathbf{R} \times M \to \mathbf{R}^k$ and $g : M \to \mathbf{R}^k$ be continuous tangent vector fields defined on a (boundaryless) differentiable manifold $M \subset \mathbf{R}^k$, with f T-periodic in the first variable. Let Ω be an open subset of $[0, \infty) \times C_T(M)$, and assume that $\deg(g, \Omega \cap M)$ is well defined and nonzero. Then there exists a connected set Γ of nontrivial solution pairs in Ω whose closure in $[0, \infty) \times C_T(M)$ meets $g^{-1}(0) \cap \Omega$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbf{R}^k$ and $\Omega = [0, \infty) \times C_T(M)$, then Γ is unbounded.*

Proof. Let X denote the set of solution pairs of (3). Since X is closed, it is enough to show that there exists a connected set Γ of nontrivial solution pairs in Ω whose closure in $X \cap \Omega$ meets $g^{-1}(0)$ and is not compact.

Assume first that f and g are smooth vector fields. Denote by S the set of all starting points of (3), and take

$$\tilde{S} = \{(\lambda, p) \in S : \text{the solution of (4) is contained in } \Omega\}.$$

Obviously $\tilde{S}$ is an open subset of S, thus we can find an open subset U of D such that $S \cap U = \tilde{S}$ (recall that D is the set of all the pairs (λ, p) such that the solution of (4) is defined in $[0,T]$). We have that

$$g^{-1}(0) \cap \Omega = g^{-1}(0) \cap \tilde{S} = g^{-1}(0) \cap U,$$

thus $\deg(g, U \cap M) = \deg(g, \Omega \cap M) \neq 0$. Applying Theorem 3.2, we get the existence of a connected set $\Sigma \subset (S \cap U) \setminus g^{-1}(0)$ such that its closure in $S \cap U$ is not compact and meets $g^{-1}(0)$. Let $h : X \to S$ be the map which assigns to any solution pair (λ, x) the starting point $(\lambda, x(0))$. Observe that h is continuous, onto and, since f and g are smooth, it is also one to one. Furthermore, by the continuous dependence on initial data, we get the continuity of $h^{-1} : S \to X$. Thus h maps $X \cap \Omega$ homeomorphically onto $S \cap U$, and the trivial solution pairs correspond to the trivial starting points under this homeomorphism. This implies that $\Gamma = h^{-1}(\Sigma)$ satisfies the requirements.

Let us remove the smoothness assumption on g and f. Take $Y_0 = g^{-1}(0) \cap \Omega$ and $Y = X \cap \Omega$. We have only to prove that the pair (Y, Y_0) satisfies the hypothesis of Lemma 3.1. Assume the contrary. We can find a relatively open compact subset C of Y containing Y_0. Thus there exists an open subset W of Ω such that the closure $\overline{W}$ of W in $[0,\infty) \times C_T(M)$ is contained in Ω, $W \cap Y = C$ and $\partial W \cap Y = \emptyset$. Since C is compact and $[0,\infty) \times M$ is locally compact, we can choose W in such a way that the set

$$\{(\lambda\,,\, x(t)) \in [0,\infty) \times M : (\lambda, x) \in W,\; t \in [0,T]\}$$

is contained in a compact subset K of $[0,\infty) \times M$. This implies that W is bounded with complete closure in Ω and $W \cap M$ is a relatively compact subset of $\Omega \cap M$. In particular g is nonzero on the boundary of $W \cap M$ (relative to M). By well known approximation results on manifolds, we can find sequences $\{g_i\}$ and $\{f_i\}$ of smooth tangent vector fields uniformly approximating g and f, with f_i T-periodic in the first variable. For $i \in \mathbf{N}$ large enough, we get

$$\deg(g_i, W \cap M) = \deg(g, W \cap M).$$

Furthermore, by excision,

$$\deg(g, W \cap M) = \deg(g, \Omega \cap M) \neq 0.$$

Therefore, given i large enough, the first part of the proof can be applied to the equation

$$\dot{x} = g_i(x) + \lambda f_i(t, x). \tag{5}$$

Let X_i denote the set of solution pairs of (5). There exists a connected subset Γ_i of $\Omega \cap X_i$ whose closure in Ω meets $g_i^{-1}(0) \cap W$ and is not contained in any compact subset of Ω. Let us prove that, for i large enough, $\Gamma_i \cap \partial W \neq \emptyset$. It is sufficient to show that $X_i \cap \overline{W}$ is compact. In fact, if $(\lambda\,,\, x) \in X_i \cap \overline{W}$ we have, for any $t \in [0,T]$,

$$\|\dot{x}(t)\| \leq \max\left\{\|g(p) + \mu f(\tau, p)\| : (\mu, p) \in K\;,\; \tau \in [0,T]\right\}.$$

Hence, by Ascoli's theorem, $X_i \cap \overline{W}$ is totally bounded and, consequently, compact, since X_i is closed and $\overline{W}$ is complete. Thus, for i large enough, there exists a solution pair $(\lambda_i, x_i) \in \Gamma_i \cap \partial W$ of (5). Again by Ascoli's theorem, we may assume that $x_i \to x_0$ in $C_T(M)$ and $\lambda_i \to \lambda_0$ with $(\lambda_0, x_0) \in \partial W$. Therefore

$$\dot{x}_0(t) = g\,(x_0(t)) + \lambda_0 f\,(t, x_0(t))\ ,\quad t \in \mathbf{R}.$$

Hence (λ_0, x_0) is a solution pair in ∂W. This contradicts the assumption $\partial W \cap Y = \emptyset$.

It remains to prove the last assertion. Let M be closed. There exists a connected set Γ of solution pairs of (3) whose closure is not compact and meets $g^{-1}(0)$. Let Z be the closure in $[0,\infty) \times \mathbf{R}^k$ (or, equivalently, in $[0,\infty) \times M$) of the set

$$\{(\lambda\,,\, x(t)) : (\lambda\,,\, x) \in \Gamma\ ,\ t \in [0,T]\}\,.$$

We need to show that Z is unbounded. Assume the contrary. Hence Z is compact. Thus, by Ascoli's theorem, Γ is totally bounded. Consequently, since $[0,\infty) \times C_T(M)$ is complete, its closure is compact. □

To understand the meaning of Theorem 3.3, consider for example the case when $M = \mathbf{R}^m$. If $g^{-1}(0)$ is compact and $\deg(g, \mathbf{R}^m) \neq 0$, then there exists an unbounded connected set of solution pairs in $[0,\infty) \times C_T(\mathbf{R}^m)$ which meets $g^{-1}(0)$. The existence of this unbounded branch cannot be destroyed by a particular choice of f . However this branch is possibly contained in the slice $\{0\} \times C_T(M)$, as in the following simple two dimensional example:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \lambda \sin t\ . \end{cases}$$

Another direct consequence of Theorem 3.3 can be given when $M \subset \mathbf{R}^k$ is a compact boundaryless manifold with $\chi(M) \neq 0$ and $\Omega = [0,\infty) \times C_T(M)$. In this case, by the Poincaré-Hopf theorem, $\deg(g, M) = \chi(M)$. Thus there exists an unbounded connected set Γ of solution pairs in $[0,\infty) \times C_T(M)$ which meets $g^{-1}(0)$. This extends a result of [FP1] where g is assumed to be identically zero. We point out that, in the case when g and f are locally Lipschitzian in x, the same assertion can be obtained by Theorem 3 in [C2]; where, instead of differentiable manifolds the broader class of ANR's is considered.

4 Applications

In order to enlighten the meaning of Theorem 3.3 we illustrate how the knowledge of the structure of the set of solution pairs can lead to a result about forced oscillations of a pendulum-type equation. Consider the following second order differential equation

$$\ddot{\theta} = g(\theta) + \lambda f(t, \theta), \tag{6}$$

where $g : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ are continuous functions, 2π-periodic with respect to θ. Assume also that f is T-periodic in t. It is convenient to regard g and f as defined on S^1 and $\mathbf{R} \times S^1$ respectively.

Clearly, (6) can be considered as a first order (non-autonomous) differential equation on the tangent bundle $TS^1 = S^1 \times \mathbf{R}$ as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g(x_1) + \lambda f(t, x_1)\,. \end{cases} \tag{7}$$

Assume that g takes both positive and negative values and it has exactly two zeros θ_1 and θ_2 on S^1 (for instance $g(\theta) = -\sin\theta$). Hence $(\theta_1, 0)$ and $(\theta_2, 0)$, which we identify with θ_1 and θ_2, are the unique zeros of the vector field $(x_1, x_2) \mapsto (x_2, g(x_1))$ (which can be regarded as a tangent vector field on $S^1 \times \mathbf{R}$). In what follows, θ_1 and θ_2 will be also identified, respectively, with the trivial solution pairs $(0; \hat{\theta}_1, 0)$, $(0; \hat{\theta}_2, 0)$, where $\hat{\theta}_i$ is the constant map $t \mapsto \theta_i$ for $i \in \{1, 2\}$.

The following theorem holds.

Theorem 4.1 *Let f and g be as above. Denote by C_1 and C_2 the connected components of the set of solution pairs of (7) containing θ_1 and θ_2 respectively. Then C_1 and C_2 are bounded in any subset $[0, \mu] \times C_T(S^1 \times \mathbf{R})$ of $[0, \infty) \times C_T(S^1 \times \mathbf{R})$. Moreover, just one of the following alternatives holds:*

1. *$C_1 = C_2$,*
2. *C_1 and C_2 are disjoint and both unbounded.*

In particular, if the second alternative holds, there exist at least two distinct T-periodic solutions of (6) for each $\lambda \in [0, \infty)$.

Proof. Let $w : [0, \infty) \times C_T(S^1 \times \mathbf{R}) \to \mathbf{Z}$ be the (continuous) function which assigns to any $(\lambda, x) = (\lambda; x_1, x_2)$ the number of turns that $x_1(t)$ makes, in a period, around S^1. More precisely, w associates to $(\lambda, x) \in [0, \infty) \times C_T(S^1 \times \mathbf{R})$ the winding number of the closed curve $t \in [0, T] \mapsto x_1(t) \in S^1$. Regarding θ_1 and θ_2 as solution pairs, we have $w(\theta_1) = w(\theta_2) = 0$. Thus, the continuity of w implies that w must be identically zero on the connected sets C_1 and C_2. This means that, given $(\lambda; x_1, x_2) \in C_1 \cup C_2$, the T-periodic map $x_1 : \mathbf{R} \to S^1$ can be viewed as a T-periodic real function; that is, x_1 is actually a solution of (6).

Let $(\lambda; x_1, x_2)$ be any solution pair of (7) with zero winding number. There exists $t_0 \in [0, T]$ such that $x_2(t_0) = 0$, therefore

$$x_2(t) = x_2(t) - x_2(t_0) = \int_{t_0}^{t} g\left(x_1(s)\right) + \lambda f\left(s, x_1(s)\right)\, ds,$$

hence

$$|x_2(t)| \leq T\left[\max_{\theta\in\mathbf{R}} |g(\theta)| + \lambda \max_{(s,\theta)\in\mathbf{R}\times\mathbf{R}} |f(s, \theta)|\right]. \tag{8}$$

The inequality (8), which holds for any solution pair in $C_1 \cup C_2$, implies that C_1 and C_2 are bounded in any set $[0,\mu] \times C_T(S^1 \times \mathbf{R})$.

Define

$$\begin{aligned} \Omega_1 &= \left\{[0,\infty) \times C_T(S^1 \times \mathbf{R})\right\} \setminus \{\theta_2\}, \\ \Omega_2 &= \left\{[0,\infty) \times C_T(S^1 \times \mathbf{R})\right\} \setminus \{\theta_1\}, \end{aligned}$$

and observe that θ_i, $i = 1, 2$, is the unique zero in $\Omega_i \cap (S^1 \times \mathbf{R})$ of the vector field $(x_1, x_2) \mapsto (x_2, g(x_1))$. Since g changes sign at θ_i, by homotopy arguments one can prove that this vector field has nonzero index at θ_i. Therefore Theorem 3.3 with $M = S^1 \times \mathbf{R}$ implies that $C_i \cap \Omega_i$, $i = 1, 2$, cannot be compact.

We may assume $C_1 \neq C_2$. In this case, C_1 and C_2, being connected components, are disjoint and, consequently, $C_i \subset \Omega_i$ for $i = 1, 2$. Since C_1 and C_2 are closed non-compact subsets of the complete metric space $[0,\infty) \times C_T(S^1 \times \mathbf{R})$, they must be both unbounded. For, since M is closed, Ascoli's theorem implies that any bounded set of solution pairs is actually totally bounded.□

The above theorem leads to the following multiplicity result.

Corollary 4.2 *Let $g : \mathbf{R} \to \mathbf{R}$ be a 2π-periodic continuous function whose image contains 0 in its interior. Assume that g has exactly two zeros $\theta_1, \theta_2 \in [0, 2\pi)$. Then, given a continuous function $(t,\theta) \mapsto f(t,\theta)$, T-periodic in $t \in \mathbf{R}$ and 2π-periodic in θ, there exists $\lambda_f > 0$ such that the equation (6) has at least two T-periodic solutions for each $\lambda \in [0, \lambda_f]$.*

Proof. Let C_1 and C_2 be as in Theorem 4.1. Since the intersections of C_1 and C_2 with the slice $\{0\} \times C_T(S^1 \times \mathbf{R})$ are bounded, hence compact, it is enough to show that there are no connected sets C of solution pairs with $\lambda = 0$ joining θ_1 with θ_2. Suppose such a C exists.

We may assume $\theta_1 < \theta_2$, $g(\theta) < 0$ in (θ_1, θ_2) and $g(\theta) > 0$ in $(\theta_2, \theta_1 + 2\pi)$. The connectedness of C, which is contained in $\{0\} \times C_T(S^1 \times \mathbf{R})$, ensures that in the open neighborhood

$$W = [0,\infty) \times C_T\left((S^1 \backslash \theta_1) \times \mathbf{R}\right)$$

of θ_2 there exists a solution pair

$$(0; \bar{x}_1, \bar{x}_2) \in C \cap W \backslash \{\theta_2\}.$$

The function $\bar{x}_1$ can be regarded as a nonconstant T-periodic solution of $\ddot{\theta} = g(\theta)$ such that $\theta_1 < \bar{x}_1(t) < \theta_1 + 2\pi$ for all $t \in [0, T]$. Let τ_0 and τ_1 be, respectively, a minimum and a maximum point of $\bar{x}_1$ in $[0, T]$. Obviously

$$g\left(\bar{x}_1(\tau_1)\right) = \frac{d^2\bar{x}_1}{dt^2}(\tau_1) \leq 0 \leq \frac{d^2\bar{x}_1}{dt^2}(\tau_0) = g\left(\bar{x}_1(\tau_0)\right).$$

By the properties of g and the fact that $\theta_1 < \bar{x}_1(t) < \theta_1 + 2\pi$, we get $\bar{x}_1(\tau_1) \leq \theta_2 \leq \bar{x}_1(\tau_0)$, which is clearly impossible since $\bar{x}_1$ is nonconstant.□

We point out that, even in the case when g and f are smooth, the above result cannot be proved by simply linearizing (3) about the zeros of g. In fact, one of the two linearized equations could be the following:

$$\ddot{\theta} + \theta = \lambda \sin t.$$

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