BIFURCATION OF FIXED POINTS FROM A MANIFOLD OF TRIVIAL FIXED POINTS

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ABSTRACT. We consider a parametrized fixed point equation (or, more generally, a coincidence equation) in a finite dimensional manifold and we give necessary as well as sufficient conditions for bifurcation from a manifold of trivial fixed points. The abstract results are then applied to forced oscillations of second order differential equations on manifolds, providing a necessary condition and a sufficient condition for an equilibrium point to be a bifurcation point of periodic orbits.

1. Introduction

Let Z be a finite dimensional differentiable manifold and consider the parametrized fixed point equation

$$f(\lambda, z) = z, \tag{1.1}$$

where $f : \mathbb{R} \times Z \to Z$ is a C^1 map. In [7], assuming that $f(0, z) = z$, for all $z \in Z$, we obtained conditions for bifurcation from the manifold $\{0\} \times Z$, actually regarded as the set of trivial solutions to (1.1). In the same paper, we gave an application of such results to the one parameter family of first order periodic problems

$$\begin{cases} \dot{x} = \lambda F(t, x), \quad \lambda \in \mathbb{R} \\ x(0) = x(T)\,, \end{cases} \tag{1.2}$$

with $F : \mathbb{R} \times N \to \mathbb{R}^s$ a time dependent T-periodic C^1 tangent vector field on a differentiable manifold $N \subseteq \mathbb{R}^s$. A pair $(\lambda, q) \in \mathbb{R} \times N$, where q is the value at time $t = 0$ of a T-periodic solution of $\dot{x} = \lambda F(t, x)$, has been called a starting point. Clearly, for $\lambda = 0$, any pair $(0, q)$, $q \in N$ is a starting point of the constant solution $x(t) \equiv q$. The abstract bifurcation results of [7] apply to (1.2) by taking $Z = N$ and f to be the Poincaré T-translation operator $P^T : \mathbb{R} \times N \to N$ associated with (1.2). Since the starting points of the form $(0, q)$ satisfy $P^T(0, q) = q$, it is natural to regard $\{0\} \times N$ as the manifold of trivial solutions. We proved that a sufficient condition for a trivial starting point $(0, q_0)$ to be a bifurcation point is that q_0 is a nondegenerate zero of the autonomous tangent vector field

$$w(q) = \frac{1}{T} \int_0^T F(t, q)\, dt,$$

called in [7] the "average wind".

The idea of extending a similar result to periodic problems for second order differential equations on manifolds led us to study the one parameter motion problem associated with a force λF, where now $F : \mathbb{R} \times TN \to \mathbb{R}^s$ is a T-periodic C^2 vector field defined on the tangent bundle TN of N and is assumed to be tangent to N, that means $F(t, q, v)$ tangent to N at q for all $(t, q, v) \in \mathbb{R} \times TN$. Clearly, as for first order equations, any $q \in N$ is still a rest point of the motion problem with $\lambda = 0$, the so-called inertial problem. However, as well-known, in the non flat case the inertial problem may also have nonconstant T-periodic solutions, as, for instance, in the case of an inertial motion on a sphere. In fact, closed geodesics may be T-periodic if they have appropriate speed. Consequently, when dealing with the parametrized fixed point equation involving the Poincaré T-translation operator $P^T : \mathbb{R} \times TN \to TN$ associated with the T-periodic second order problem, namely $P^T(\lambda, q, v) = (q, v)$, even if it is natural to suppose the triples $(0, q, 0)$, $q \in N$, to be the trivial solutions (observe that, now, the constraint is $N \times \{0\} \subseteq TN$), one should keep in mind that, for $\lambda = 0$, the equation $P^T(0, q, v) = (q, v)$ may also have fixed points (q, v) with $v \neq 0$.

The situation arising in the constrained periodic motion problem, and already described, is the main motivation of this paper. In particular, in Section 4 where we are concerned with the equation (1.1), we assume the existence of a manifold $M_0 \subseteq Z$ such that $f(0, z) = z$ for all $z \in M_0$. We emphasize the fact that M_0 may be strictly contained in the set of fixed points of f for $\lambda = 0$ and we will refer to $\{0\} \times M_0$ as to the set of trivial solutions of (1.1). Our aim is to get bifurcation from $\{0\} \times M_0$. We give necessary conditions (Theorem 4.1 and Corollary 4.5) and sufficient conditions (Theorem 4.2 and Corollary 4.7) for a point $p \in M_0$ to be a bifurcation point of (1.1). Such results are deduced from quite general bifurcation theorems obtained in Section 3 for a coincidence equation of the form $f(x) = h(x)$, with f and h maps between two finite dimensional manifolds (extensions to the infinite dimensional context will appear elsewhere). We would like to point out that, as one may expect, our sufficient conditions for bifurcation are, in some sense, second order conditions. Actually, they are given in terms of the Hessian of a C^2 map between manifolds (see Section 2), since, as well-known and easy to check, the second derivative is not intrinsically defined for maps acting between two differentiable manifolds.

As observed just few lines above, in the present context the nontrivial solutions (λ, z) of (1.1) may have $\lambda = 0$. However, an extra condition yielding that nontrivial pairs sufficiently close to $\{0\} \times M_0$ have $\lambda \neq 0$ can be assumed (see (H) of Section 4). It seems interesting to observe that this condition is satisfied by the Poincaré operator $P^T : \mathbb{R} \times TN \to TN$ associated with the second order periodic problem we are interested in (see Theorem 5.2 below). As a consequence, the nontrivial triples (λ, q, v) which are close to $(0, q, 0)$ and such that $P^T(\lambda, q, v) = (q, v)$ have necessarily $\lambda \neq 0$. This corresponds to the well-known physical fact that in a Riemaniann manifold there are no nonconstant closed geodesics too close to a given point.

Finally, from the abstract results of Section 4, we are able to deduce for the constrained T-periodic second order problem, the analogue of the bifurcation result obtained for the first order problem (1.2). Namely, we prove that a trivial starting point $(0, q_0, 0)$ of the motion equation is a bifurcation point provided that q_0 is a nondegenerate zero of the "average force" vector field

$$\bar{F}(q) = \frac{1}{T} \int_0^T F(t, q, 0)\, dt.$$

2. Notation and Preliminaries

In this paper all the manifolds are assumed to be real and smooth. Thus, for simplicity, the term smooth will be omitted. Clearly, most of the statements make sense even assuming less regularity of the involved manifolds. However, we are not interested here in this more general situation.

Given two manifolds X and Y and a C^1 map $f : X \to Y$, the *(first) derivative* of f at $x \in X$ will be denoted by $Df(x)$ or, also, by $f'(x)$. As well-known, $Df(x)$ is a linear operator sending the tangent space T_xX of X at x into the tangent space $T_{f(x)}Y$ of Y at $f(x)$.

When $X = X_1 \times X_2$, the *partial derivative* with respect to the first (respectively, the second) variable at (x_1, x_2) will be indicated with $\partial_1 f(x_1, x_2)$ (respectively, $\partial_2 f(x_1, x_2)$). For any pair of tangent vectors $(u_1, u_2) \in T_{x_1}X_1 \times T_{x_2}X_2$, one has

$$Df(x_1, x_2)(u_1, u_2) = \partial_1 f(x_1, x_2)u_1 + \partial_2 f(x_1, x_2)u_2.$$

In particular, if $X_1 = \mathbb{R}$, the partial derivative $\partial_1 f(x_1, x_2)$, which is actually a linear operator from $\mathbb{R}$ to the tangent space $T_{f(x_1,x_2)}Y$, will be identified with the tangent vector $\partial_1 f(x_1, x_2)(1) \in T_{f(x_1,x_2)}Y$. With this notation, for the (total) derivative $Df(x_1, x_2)$ one has the equality

$$Df(x_1, x_2)(u_1, u_2) = u_1\partial_1 f(x_1, x_2) + \partial_2 f(x_1, x_2)u_2,$$

where $(u_1, u_2) \in \mathbb{R} \times T_{x_2}X_2$.

When X and Y are Euclidean (or, more generally, Banach) spaces, the *second derivative* of a C^2 map $f : X \to Y$ at $x \in X$ is a symmetric bilinear operator from X to Y, i.e. an element of the space $L^2_s(X, Y)$, and will be denoted by $D^2f(x)$. A practical method for its computation is the following: given $u, v \in X$, consider the function of two real variables $\sigma(r, s) = f(x + ru + sv)$; then,

$$D^2f(x)(u, v) = \frac{\partial^2\sigma}{\partial r\partial s}(0, 0).$$

However, when $f : X \to Y$ acts between two differentiable manifolds, then the second derivative of f at $x \in X$ is not intrinsically defined, since only a part of this derivative is independent of coordinates, as can be easily seen by a simple computation. More precisely, one can define (see e.g. [1]) an intrinsic symmetric bilinear operator $Hf(x)$, called the *Hessian* of f at x, acting from $\operatorname{Ker} Df(x)$ to $\operatorname{coKer} Df(x) = T_{f(x)}Y/\operatorname{Im} Df(x)$, i.e. an element of $L^2_s(\operatorname{Ker} Df(x), \operatorname{coKer} Df(x))$. For example, if f is a real function on X and $x \in X$ is a critical point of f, then $\operatorname{Ker} Df(x) = T_xX$ and $\operatorname{coKer} Df(x) = \mathbb{R}$. Thus, in this case, $Hf(x)$ is the classical Hessian, which can be regarded either as a symmetric bilinear form or as a quadratic form on the tangent space T_xX.

By taking charts $\varphi : U \subseteq X \to \mathbb{R}^k$ and $\psi : V \subseteq Y \to \mathbb{R}^l$ about x and $y = f(x)$ respectively, one can define $Hf(x)$ as follows

$$Hf(x)(u, v) = \pi\big(D\psi^{-1}(\psi(y))D^2\hat{f}(\varphi(x))(D\varphi(x)u, D\varphi(x)v)\big), \tag{2.1}$$

where $u, v \in \operatorname{Ker} Df(x)$, $\hat{f} = \psi \circ f \circ \varphi^{-1}$, and $\pi : T_yY \to T_yY/\operatorname{Im} Df(x)$ is the canonical projection. We will show below that $Hf(x)$ is a well-defined element of $L^2_s(\operatorname{Ker} Df(x), \operatorname{coKer} Df(x))$, i.e. that the above definition does not depend on the particular choice of the charts φ and ψ. More precisely, if $\varphi_1 : U_1 \subseteq X \to \mathbb{R}^k$ and

$\varphi_2 : U_2 \subseteq X \to \mathbb{R}^k$ are charts about x, $\psi_1 : V_1 \subseteq Y \to \mathbb{R}^l$ and $\psi_2 : V_2 \subseteq Y \to \mathbb{R}^l$ are charts about $y = f(x)$, we will show that

$$\begin{aligned} &\pi\big(D\psi_1^{-1}(\psi_1(y))D^2\hat{f}_1(\varphi_1(x))(D\varphi_1(x)u, D\varphi_1(x)v)\big) \\ &= \pi\big(D\psi_2^{-1}(\psi_2(y))D^2\hat{f}_2(\varphi_2(x))(D\varphi_2(x)u, D\varphi_2(x)v)\big), \end{aligned} \tag{2.2}$$

where $\hat{f}_i = \psi_i \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \subseteq \mathbb{R}^k \to \mathbb{R}^l$, $i = 1, 2$. Clearly, one has $\hat{f}_2 = \beta \circ \hat{f}_1 \circ \alpha^{-1}$, where we have set $\alpha = \varphi_2 \circ \varphi_1^{-1}$ and $\beta = \psi_2 \circ \psi_1^{-1}$. In other words, we are reduced to consider the diagram

$$\begin{array}{ccc} \varphi_1(U_1) & \xrightarrow{\hat{f}_1} & \psi_1(V_1) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \varphi_2(U_2) & \xrightarrow{\hat{f}_2} & \psi_2(V_2) \end{array} \tag{2.3}$$

and to investigate the relationship between the second derivatives $D^2\hat{f}_1$ and $D^2\hat{f}_2$. This will be carried out in the following two steps.

Lemma 2.1. *Let W_1, W_2 be open subsets of $\mathbb{R}^k$, $f_1 : W_1 \to \mathbb{R}^l$ be a C^2 map, $\alpha : W_1 \to W_2$ be a C^2 diffeomorphism. Then, if $f_2 : W_2 \to \mathbb{R}^l$ denotes the composition $f_1 \circ \alpha^{-1}$ and $x_2 = \alpha(x_1)$, we have*

(a) *$Df_1(x_1)u_1 = Df_2(x_2)u_2$, where $u_1, u_2 \in \mathbb{R}^k$ are such that $u_2 = D\alpha(x_1)u_1$;*
(b) *$w_1 - w_2 \in \operatorname{Im} Df_1(x_1) = \operatorname{Im} Df_2(x_2)$, where $w_i = D^2f_i(x_i)(u_i, v_i)$ with $u_i, v_i \in \mathbb{R}^k$ $(i = 1, 2)$ such that $u_2 = D\alpha(x_1)u_1$, $v_2 = D\alpha(x_1)v_1$.*

Proof. (a) The assertion follows immediately from the chain rule of the derivative.

(b) Let us compute the second derivative $w_1 = D^2f_1(x_1)(u_1, v_1)$ and compare it with $w_2 = D^2f_2(x_2)(u_2, v_2)$. As observed above, it is enough to compute the second derivative at the origin of the function of two real variables $\sigma_1(r, s) = f_1(x_1 + ru_1 + sv_1) = f_2(\alpha(x_1 + ru_1 + sv_1))$. One has

$$\frac{\partial \sigma_1}{\partial r}(0, s) = Df_2(\alpha(x_1 + sv_1))D\alpha(x_1 + sv_1)u_1$$

and

$$\begin{aligned} \frac{\partial^2 \sigma_1}{\partial s \partial r}(0, 0) = D^2f_2(\alpha(x_1))(D\alpha(x_1)u_1, D\alpha(x_1)v_1) + \\ Df_2(\alpha(x_1))(D^2\alpha(x_1)(u_1, v_1)). \end{aligned}$$

Therefore,

$$D^2f_1(x_1)(u_1, v_1) = D^2f_2(x_2)(u_2, v_2) + Df_2(x_2)(D^2\alpha(x_1)(u_1, v_1)).$$

This means that w_1 coincides with w_2 up to an element belonging to $\operatorname{Im} Df_2(x_2)$, as claimed. □

Lemma 2.2. *Let W be an open subset of $\mathbb{R}^k$, Ω_1 and Ω_2 open subsets of $\mathbb{R}^l$, $f_1 : W \to \Omega_1$ a C^2 map, $\beta : \Omega_1 \to \Omega_2$ a C^2 diffeomorphism. Then, if $f_2 : W \to \Omega_2$ denotes the composition $\beta \circ f_1$ and $y_1 = f_1(x)$, we have*

(a) *$Df_2(x)u = D\beta(y_1)(Df_1(x)u)$, for any $u \in \mathbb{R}^k$;*
(b) *$D\beta(y_1)w_1 = w_2$, where $w_i = D^2f_i(x)(u, v), i = 1, 2$, with $u, v \in \operatorname{Ker} Df_1(x)$.*

Proof. (a) The assertion follows immediately from the chain rule of the derivative.

(b) Given $u, v \in \mathbb{R}^k$, let us compute $D^2 f_2(x)(u, v)$. As in Lemma 2.1, consider the function $\sigma_2(r, s) = f_2(x + ru + sv) = \beta(f_1(x + ru + sv))$. One has

$$\frac{\partial \sigma_2}{\partial r}(0, s) = D\beta(f_1(x + sv))(Df_1(x + sv)u)$$

and

$$\frac{\partial^2 \sigma_2}{\partial s \partial r}(0, 0) = D^2\beta(f_1(x))(Df_1(x)u, Df_1(x)v) + D\beta(f_1(x))(D^2 f_1(x)(u, v)).$$

Now, by taking $u, v \in \operatorname{Ker} Df_1(x)$, the first term in the above sum is zero. Thus $D^2 f_2(x)(u, v) = D\beta(y_1)(D^2 f_1(x)(u, v))$, and the assertion is proved. □

Observe now that the equality (2.2) follows by directly applying Lemmas 2.1 and 2.2, with $\alpha = \varphi_2 \circ \varphi_1^{-1}$, $\beta = \psi_2 \circ \psi_1^{-1}, W_1 = \varphi_1(U_1), W_2 = \varphi_2(U_2), \Omega_1 = \psi_1(V_1), \Omega_2 = \psi_2(V_2)$, and by recalling that $\pi(w) = 0$ if and only if $w \in \operatorname{Im} Df(x)$.

Remark 2.3. Let us compute in coordinates the derivative and the Hessian of a smooth map $f : X \to Y$ between two manifolds X and Y. Given $x \in X$ and $y = f(x) \in Y$, let $\{x_i\}_{i=1,\ldots,k}$ and $\{y_h\}_{h=1,\ldots,l}$ be coordinate systems about x and y, respectively. Thus, if u is a vector tangent to X at x, the derivative $Df(x)u$ in coordinates is given by

$$\sum_h \Big(\sum_i \alpha_i \frac{\partial f_h}{\partial x_i}(x) \Big) \Big(\frac{\partial}{\partial y_h} \Big)_y,$$

where $u = \sum_i \alpha_i \Big(\frac{\partial}{\partial x_i} \Big)_x$ and $f_h = y_h \circ f$. Moreover, if u and v are tangent vectors belonging to $\operatorname{Ker} Df(x)$, it is not hard to check that the Hessian $Hf(x)(u, v)$ can be represented in coordinates, up to elements belonging to the image of $Df(x)$, as follows

$$\sum_h \Big(\sum_{i,j} \alpha_i \beta_j \frac{\partial^2 f_h}{\partial x_i \partial x_j}(x) \Big) \Big(\frac{\partial}{\partial y_h} \Big)_y,$$

where $v = \sum_j \beta_j \Big(\frac{\partial}{\partial x_j} \Big)_x$.

The following property of $Hf(x)$ will be used in the sequel.

Lemma 2.4. *Let $f : X \to Y$ be a C^2 map between two finite dimensional manifolds and assume that f is constant on a submanifold M of X. Then, given $x \in M$ and $u, v \in T_x M$ one has $Hf(x)(u, v) = 0$.*

Proof. Since f is constant in the submanifold M of X, then, according to definition introduced in (2.1), given $x \in M$, the map $\hat{f} : \varphi(U) \to \psi(V)$ is constant in $\varphi(U \cap M)$, where we may assume that $\varphi : U \to W$ is a chart about x transforming $U \cap M$ in $W \cap E$, where E is a subspace of $\mathbb{R}^k$. Therefore, given $u, v \in T_x M$, the corresponding vectors $D\varphi(x)u$, $D\varphi(x)v$ belong to E. Hence, the map $\hat{\sigma}(r, s) = \hat{f}(\varphi(x) + rD\varphi(x)u + sD\varphi(x)v)$ is constant, so that

$$D^2 \hat{f}(\varphi(x))(D\varphi(x)u, D\varphi(x)v) = \frac{\partial^2 \hat{\sigma}}{\partial r \partial s}(0, 0) = 0.$$

This clearly implies

$$Hf(x)(u, v) = \pi\big(D\psi^{-1}(\psi(y)) D^2 \hat{f}(\varphi(x))(D\varphi(x)u, D\varphi(x)v)\big) = 0,$$

which is our assertion. □

3. General bifurcation

Let $f, h : X \to Y$ be maps between two finite dimensional manifolds and consider the coincidence equation

$$f(x) = h(x). \tag{3.1}$$

Let us denote by S the solution set to the above equation and suppose that one is interested in regarding a distinguished subset M of S as the set of *trivial solutions* of (3.1). Consequently, $S \setminus M$ will be the set of nontrivial solutions. According to this terminology, a trivial solution $p \in M$ will be called a *bifurcation point* for the equation (3.1) if any neighborhood of p in X contains elements of $S \setminus M$. Actually, some structure is required on the trivial set M; for instance, assume that

- the set M of trivial solutions of the equation (3.1) is an m-dimensional manifold.

Our purpose now is to prove a necessary condition (Theorem 3.4 below) and a sufficient condition (Theorem 3.6) for the coincidence equation (3.1) to possess bifurcation from M. To this end, we will make use of finite dimensional versions (Lemmas 3.1 and 3.2 below) of two results obtained in [5], by means of the Implicit Function Theorem, in the more general context of Fredholm maps between Banach spaces (a forthcoming joint paper with M. Martelli will deal with coincidence problems for maps between Banach manifolds). In particular, given a map $g : \mathbb{R}^k \to \mathbb{R}^l$ and a set $M \subseteq g^{-1}(0)$, an element $p \in M$ is a bifurcation point for the equation $g(x) = 0$ if any neighborhood of p in $\mathbb{R}^k$ contains elements of $g^{-1}(0) \setminus M$. To understand the meaning of the following lemma, observe that, if g is C^1, the condition $M \subseteq g^{-1}(0)$ implies $T_x M \subseteq \operatorname{Ker} Dg(x)$ for all $x \in M$.

Lemma 3.1. *Let $g : \mathbb{R}^k \to \mathbb{R}^l$ be a C^1 map and let M be an m-dimensional manifold contained in $g^{-1}(0)$. A necessary condition for $p \in M$ to be a bifurcation point (from M) for the equation $g(x) = 0$ is that $\dim \operatorname{Ker} Dg(p) > m$.*

Lemma 3.2. *Let $g : \mathbb{R}^k \to \mathbb{R}^l$, with $k - l = 1$, be a C^2 map and let M be an m-dimensional manifold contained in $g^{-1}(0)$. Assume that for some $p \in M$ there exists $u \in \operatorname{Ker} Dg(p) \setminus T_p M$ such that the linear operator*

$$v \in T_p M \mapsto \pi D^2 g(p)(u, v)\,,$$

where $\pi : \mathbb{R}^l \to \mathbb{R}^l / \operatorname{Im} Dg(p)$ denotes the canonical projection, is onto. Then p is a bifurcation point (from M) for the equation $g(x) = 0$.

Remark 3.3. Since in Lemma 3.2 we have assumed $k - l = 1$, we have

$$\dim \operatorname{Ker} Dg(p) = 1 + \dim(\mathbb{R}^l / \operatorname{Im} Dg(p)).$$

Moreover, the existence of $u \in \operatorname{Ker} Dg(p) \setminus T_p M$ implies

$$\dim \operatorname{Ker} Dg(p) > m.$$

On the other hand, from the assumption that the map

$$v \in T_p M \mapsto \pi D^2 g(p)(u, v) \in \mathbb{R}^l / \operatorname{Im} Dg(p)$$

is onto, we get

$$\dim(\mathbb{R}^l / \operatorname{Im} Dg(p)) \leq \dim T_p M = m.$$

Therefore,

$$\dim \operatorname{Ker} Dg(p) = m + 1$$

and, thus, $\dim(\mathbb{R}^l / \operatorname{Im} Dg(p)) = m$. Consequently, the map

$$v \in T_pM \mapsto \pi D^2 g(p)(u,v) \in \mathbb{R}^l / \operatorname{Im} Dg(p)$$

is also one-to-one. This shows that the surjectivity assumption of Lemma 3.2 can be replaced (as in [5]) with the following equivalent condition

$$\begin{cases} \dim \operatorname{Ker} Dg(p) = m+1, \\ v \in T_pM \text{ and } D^2 g(p)(u,v) \in \operatorname{Im} Dg(p) \implies v = 0\,. \end{cases} \tag{3.2}$$

Let us now go back to the coincidence equation (3.1). We can prove the following results.

Theorem 3.4. *Let $f, h : X \to Y$ and M be as above, and let $p \in M$ be a bifurcation point for the equation* (3.1). *If f and h are C^1 in a neighborhood of p in X, then*

$$\dim \operatorname{Ker}(Df(p) - Dh(p)) > m.$$

Proof. Observe first that, as one can easily check, the notion of bifurcation and the statement of the theorem are invariant under diffeomorphisms. Therefore, recalling that a manifold is locally diffeomorphic to a whole Euclidean space, one can think of f and h as maps between Euclidean spaces, say $\mathbb{R}^k$ and $\mathbb{R}^l$. Hence, the assertion follows by a straightforward application of Lemma 3.1 to the map $g = f - h$. □

Remark 3.5. Observe that, for any $x \in M$, the following inclusion holds

$$T_xM \subseteq \operatorname{Ker}(Df(x) - Dh(x)). \tag{3.3}$$

To see this, it suffices to reduce, as in the proof of Theorem 3.4, to the map $g : \mathbb{R}^k \to \mathbb{R}^l$, $g = f - h$, and to observe that the fact that g is constant on M implies, as already observed, that $T_xM \subseteq \operatorname{Ker} Dg(x)$, for all $x \in M$.

As a consequence of (3.3) and recalling that M is m-dimensional, one has

$$\dim \operatorname{Ker}(Df(p) - Dh(p)) > m$$

if and only if T_pM is strictly contained in $\operatorname{Ker}(Df(p) - Dh(p))$. In other words, the necessary condition of Theorem 3.4 is equivalent to the following:

there exists $u \notin T_pM$ such that $Df(p)u = Dh(p)u$.

Theorem 3.6. *Let $f, h : X \to Y$ and M be as above, and suppose* $\dim X - \dim Y = 1$. *Given $p \in M$, assume that f and h are C^2 in a neighborhood of p in X. If there exists $u \in \operatorname{Ker}(Df(p) - Dh(p)) \setminus T_pM$ such that the linear operator*

$$L_u : T_pM \to T_{f(p)}Y / \operatorname{Im}(Df(p) - Dh(p))$$

given by

$$L_u v = Hf(p)(u,v) - Hh(p)(u,v)$$

is onto, then p is a bifurcation point (from M) for the equation (3.1).

Proof. As in the proof of Theorem 3.4, one can reduce to the case of a map $g = f - h$ acting between Euclidean spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ and to an m-dimensional manifold (still denoted by M) contained in $g^{-1}(0)$. In this context, our assumption is transformed in the existence of $p \in M$ and $u \in \operatorname{Ker} Dg(p) \setminus T_pM$ such that the map $v \in T_pM \mapsto Hg(p)(u,v) \in \mathbb{R}^k / \operatorname{Im} Dg(p)$ is onto. Now, observe that the map $Hg(p)(u,v)$ is nothing else but the composition $\pi D^2 g(p)(u,v)$, where $D^2 g(p)$ is the second derivative of g at p and $\pi : \mathbb{R}^l \to \mathbb{R}^l / \operatorname{Im} Dg(p)$ denotes the canonical projection. Consequently, the condition that the map $v \in T_pM \mapsto \pi D^2 g(p)(u,v)$ is onto implies, by Lemma 3.2, that p is a bifurcation point for the equation $g(x) = 0$. Thus, the same is true also for the coincidence equation $f(x) = h(x)$, as claimed. $\square$

Remark 3.7. From the assumption $\dim X - \dim Y = 1$ it follows

$$\dim \operatorname{Ker}(Df(p) - Dh(p)) = 1 + T_{f(p)}Y/\operatorname{Im}(Df(p) - Dh(p)).$$

Therefore, as in Remark 3.3, it is easy to verify that the following conditions in Theorem 3.6 are equivalent:

(a) L_u is onto;
(b) L_u is an isomorphism;
(c) $\dim \operatorname{Ker}(Df(p) - Dh(p)) = m + 1$ and L_u is one-to-one.

Theorem 3.8. *Let L_u be the linear operator defined in Theorem 3.6. Then, the property of L_u of being onto does not depend on $u \in \operatorname{Ker}(Df(p) - Dh(p)) \setminus T_pM$. More precisely, given $u_1 \in \operatorname{Ker}(Df(p) - Dh(p)) \setminus T_pM$, there exists $\alpha \neq 0$ such that $L_{u_1} = \alpha L_u$.*

Proof. Let $u_1 \in \operatorname{Ker}(Df(p) - Dh(p)) \setminus T_pM$. Since, in view of Remark 3.7, one has $\dim \operatorname{Ker}(Df(p) - Dh(p)) = 1 + \dim T_pM$, there exists $\alpha \neq 0$ and $w \in T_pM$ such that $u_1 = \alpha u + w$. Hence, recalling that (by Lemma 2.4) the bilinear operator $Hf(p)$ vanishes for pair of vectors in T_pM, given any $v \in T_pM$, we obtain

$$Hf(p)(u_1, v) = Hf(p)(\alpha u + w, v) = \alpha Hf(p)(u, v) + Hf(p)(w, v) = \alpha Hf(p)(u, v).$$

Analogously, $Hh(p)(u_1, \cdot) = \alpha Hh(p)(u, \cdot)$. Thus $L_{u_1} = \alpha L_u$, as claimed. $\square$

4. Bifurcation of fixed points

In this section we are concerned with bifurcation for the parametrized fixed point equation

$$f(\lambda, z) = z, \tag{4.1}$$

where z belongs to a finite dimensional manifold Z and f is a Z-valued map defined in $\mathbb{R} \times Z$ or, more generally, in an open subset U of $\mathbb{R} \times Z$ containing $\{0\} \times Z$. For any $\lambda \in \mathbb{R}$ we denote by $f_\lambda : Z \to Z$ the partial map $f_\lambda(\cdot) = f(\lambda, \cdot)$. We use the notation $\operatorname{Fix} f_\lambda$ to indicate the subset of Z of the fixed points of f_λ. Moreover, we set

$$S = \{(\lambda, z) \in \mathbb{R} \times Z : f(\lambda, z) = z\}$$

and we assume that

- there exists an m-dimensional submanifold M_0 of Z such that $f(0, z) = z$ for all $z \in M_0$.

In other words, we assume the existence of a distinguished subset M_0 of $\operatorname{Fix} f_0$ in such a way that we can think of $\{0\} \times M_0 \subseteq \mathbb{R} \times Z$ as the set of *trivial solutions* to (4.1). Let us point out that M_0 may be strictly contained in $\operatorname{Fix} f_0$ and, in fact, this is precisely the situation we have in mind in view of the applications to second order differential equations on manifolds that we are going to present in the next section.

We will say that an element $p \in M_0$ is a *bifurcation point* of the equation (4.1) if in any neighborhood of $(0, p)$ there exists a nontrivial solution of (4.1), i.e. a pair $(\lambda, z) \in S \setminus (\{0\} \times M_0)$.

Clearly, the equation (4.1) is a particular case of the coincidence equation (3.1) with $X = \mathbb{R} \times Z$, $Y = Z$, $M = \{0\} \times M_0$ and $h = P_2$, where $P_2 : \mathbb{R} \times Z \to Z$ is the projection onto the second component Z.

By using the terminology introduced in Section 3, we have here that $p \in M_0$ is a bifurcation point of (4.1) if and only if $(0, p) \in \{0\} \times M_0$ is a bifurcation point for the coincidence equation $f(\lambda, z) = P_2(\lambda, z)$. We emphasize the fact that, in the present context, a pair of the form $(0, z)$, with $z \in \operatorname{Fix} f_0 \setminus M_0$, must be considered as a nontrivial solution.

In this section, we are interested in obtaining, for the equation (4.1), necessary conditions and sufficient conditions providing bifurcation from M_0.

To this end, let $z \in M_0$ and assume that f is C^1 in a neighborhood of $(0, z)$ in $\mathbb{R} \times Z$. Denote by I_z the identity map on the tangent space T_zZ. Since z is a fixed point of the partial map f_0, the partial derivative $\partial_2 f(0, z)$ of f at $(0, z)$, which coincides with the derivative $Df_0(z)$ of f_0 at z, maps T_zZ into itself. Consequently, the linear operator $\partial_2 f(0, z) - I_z$ maps T_zZ into itself as well. Also observe that, since the derivative $DP_2(0, z) : \mathbb{R} \times T_zZ \to T_zZ$ of P_2 at $(0, z)$ is the projection $(\mu, w) \mapsto w$, then the partial derivative $\partial_2 P_2(0, z) : T_zZ \to T_zZ$ coincides with I_z.

Straightforward consequences of Theorems 3.4 and 3.6 are the following conditions for bifurcation.

Theorem 4.1 (Necessary condition)**.** *Let $f : \mathbb{R} \times Z \to Z$ and M_0 be as above and let $p \in M_0$ be a bifurcation point of* (4.1)*. If f is C^1 in a neighborhood of* $(0, p)$ *in $\mathbb{R} \times Z$, then there exists $(\mu, w) \in (\mathbb{R} \times T_pZ) \setminus (\{0\} \times T_pM_0)$ such that*

$$-\mu \partial_1 f(0, p) = \partial_2 f(0, p)w - w\,.$$

Proof. The assumption that $(0, p)$ is a bifurcation point for the coincidence equation $f(\lambda, z) = P_2(\lambda, z)$ implies, by Theorem 3.4,

$$\dim \operatorname{Ker}(Df(0, p) - DP_2(0, p)) > m\,.$$

Therefore, as already observed in Remark 3.5, the tangent space of $M = \{0\} \times M_0$ at $(0, p)$ is strictly contained in the kernel of $Df(0, p) - DP_2(0, p)$. Thus, there exists $(\mu, w) \notin T_{(0,p)}M = \{0\} \times T_pM_0$ such that

$$\mu \partial_1 f(0, p) + \partial_2 f(0, p)w - w = 0,$$

which is our assertion. □

Theorem 4.2 (Sufficient condition)**.** *Let $f : \mathbb{R} \times Z \to Z$ and M_0 be as above and let $p \in M_0$. Assume that f is C^2 in a neighborhood of* $(0, p)$ *in $\mathbb{R} \times Z$, and that there exists $(\mu, w) \in (\mathbb{R} \times T_pZ) \setminus (\{0\} \times T_pM_0)$ such that*

$$-\mu \partial_1 f(0, p) = \partial_2 f(0, p)w - w\,.$$

If the linear operator

$$v \in T_pM_0 \mapsto Hf((0, p))((\mu, w), (0, v)) \in T_pZ/\operatorname{Im}(Df(0, p) - DP_2(0, p))$$

is onto, then p is a bifurcation point of (4.1) *from M_0.*

Proof. The assertion follows immediately by applying Theorem 3.6 to the coincidence equation $f(\lambda, z) = P_2(\lambda, z)$, noting that the equality $\mu \partial_1 f(0, p) + \partial_2 f(0, p)w - w = 0$ is equivalent to $(\mu, w) \in \operatorname{Ker}(Df(0, p) - DP_2(0, p))$, and that, obviously, one has $HP_2(0, p) = 0$. □

Remark 4.3. In the case of the bifurcation equation (4.1), one clearly has $\dim(\mathbb{R}\times Z)-\dim Z = 1$. Therefore, similar arguments to those in Remarks 3.3 and 3.7 prove that the following conditions in Theorem 4.2 are equivalent

(a) the map $v \in T_pM_0 \mapsto Hf((0,p))((\mu,w),(0,v))$ is onto;
(b) the map $v \in T_pM_0 \mapsto Hf((0,p))((\mu,w),(0,v))$ is an isomorphism;
(c) $\dim \operatorname{Ker}(Df(0,p) - DP_2(0,p)) = m+1$ and the map
$v \in T_pM_0 \mapsto Hf((0,p))((\mu,w),(0,v))$ is one-to-one.

As already pointed out, the manifold M_0 may be strictly contained in the set $\operatorname{Fix} f_0$. Thus, the nontrivial pairs (λ, z) involved in equation (4.1) may have $\lambda = 0$. However, as we shall see later, an extra condition yielding that any nontrivial pair sufficiently close to $\{0\}\times M_0$ has $\lambda \neq 0$ turns out to be satisfied in many applications to differential equations. In the abstract setting, such a condition can be interpreted by assuming that f is C^1 in a neighborhood of $\{0\} \times M_0$ and that

$$(H) \qquad T_zM_0 = \operatorname{Ker}(\partial_2 f(0,z) - I_z) \quad \text{for all } z \in M_0\,.$$

Lemma 4.4 below shows that M_0 is isolated in the set $\operatorname{Fix} f_0$, provided that (H) is satisfied.

Lemma 4.4. *Assume that f is C^1 in a neighborhood of a given $(0,p) \in \{0\} \times M_0$ and that condition*

$$(H_p) \qquad T_pM_0 = \operatorname{Ker}(\partial_2 f(0,p) - I_p)$$

is satisfied. Then, there exists a neighborhood V of $(0,p)$ in $\mathbb{R} \times Z$ such that if $(\lambda, z) \in V$ is a nontrivial solution of (4.1), then $\lambda \neq 0$. Consequently, if condition (H) is satisfied, there exists an isolating neighborhood W of M_0 in Z, i.e. $M_0 = \operatorname{Fix} f_0 \cap W$.

Proof. Assume by contradiction that in any neighborhood of $(0,p)$ in $\mathbb{R} \times Z$ there exists a solution $(0,z)$ of (4.1) with $z \in \operatorname{Fix} f_0 \setminus M_0$. This means that p is a bifurcation point, relatively to the manifold Z, for the coincidence equation $f(0,z) = z$. Therefore, by applying Theorem 3.4 to $f_0 = f(0,\cdot)$, to the identity of Z and to M_0, one gets $\dim \operatorname{Ker}(Df_0(p) - I_p) > m = \dim T_pM_0$. This contradicts condition (H_p), and the first assertion is proved. The last statement is a trivial consequence. □

Corollary 4.5 below is a direct consequence of Theorem 4.1 and assumption (H_p).

Corollary 4.5. *Let $p \in M_0$ and $f : \mathbb{R} \times Z \to Z$ be as in Theorem 4.1. Assume that (H_p) is satisfied. Then, a necessary condition for p to be a bifurcation point of (4.1) is that there exists $w \in T_pZ$ such that*

$$\partial_1 f(0,p) = \partial_2 f(0,p)w - w\,.$$

Proof. By Theorem 4.1, there exists $(\mu, \hat{w}) \in (\mathbb{R} \times T_pZ) \setminus (\{0\} \times T_pM_0)$ such that

$$-\mu\partial_1 f(0,p) = \partial_2 f(0,p)\hat{w} - \hat{w}\,.$$

Now, if $\mu \neq 0$, then, by setting $w = -\hat{w}/\mu$, we obtain $\partial_1 f(0,p) = \partial_2 f(0,p)w - w$, as claimed. Let us show that μ cannot be equal to zero. In fact, otherwise, $\partial_2 f(0,p)\hat{w} - \hat{w} = 0$; that is, $\hat{w} \in \operatorname{Ker}(\partial_2 f(0,p) - I_p)$. By assumption (H_p), this implies $\hat{w} \in T_pM_0$ and, thus, $(0,\hat{w}) \in T_{(0,p)}M = \{0\} \times T_pM_0$, a contradiction. □

Remark 4.6. The above necessary condition can be interpreted as the fact that, if (H_p) is satisfied, then one can find $w \in T_pZ$ such that the vector $(1, -w)$, which is tangent to $\mathbb{R} \times Z$ at $(0, p)$, belongs to the kernel of $Df(0, p) - DP_2(0, p)$.

In this case, an equivalent manner of writing the necessary condition is also

$$\partial_1 f(0, p) \in \operatorname{Im}(\partial_2 f(0, p) - I_p)\,.$$

The following particular case of Theorem 4.2 will be used in the applications to differential equations presented in the next section.

Corollary 4.7. *Let $p \in M_0$ and $f : \mathbb{R} \times Z \to Z$ be as in Theorem 4.2. Assume that, there exists $w \in T_pZ$ such that*

$$\partial_1 f(0, p) = \partial_2 f(0, p)w - w\,.$$

If the linear operator

$$v \in T_pM_0 \mapsto Hf((0, p))((1, -w), (0, v)) \in T_pZ/\operatorname{Im}(\partial_2 f(0, p) - I_p)$$

is onto, then p is a bifurcation point of (4.1) *from M_0. In addition, any nontrivial solution (λ, z) of* (4.1) *close to $(0, p)$ has $\lambda \neq 0$.*

Proof. Apply Theorem 4.2 with the pair $(1, -w)$ and observe that the assumption $\partial_1 f(0, p) \in \operatorname{Im}(\partial_2 f(0, p) - I_p)$ implies

$$\operatorname{Im}(\partial_2 f(0, p) - I_p) = \operatorname{Im}(Df(0, p) - DP_2(0, p)).$$

Thus, p is a bifurcation point of (4.1).

In order to prove the last assertion, according to Lemma 4.4 it is enough to show that the assumptions in the corollary guarantee the validity of condition (H_p). To this end, observe first that, from Remark 4.3, we get

$$\dim \operatorname{Ker}(Df(0, p) - DP_2(0, p)) = m + 1.$$

Moreover, the existence of a vector $w \in T_pZ$ such that

$$(1, -w) \in \operatorname{Ker}(Df(0, p) - DP_2(0, p))$$

clearly implies

$$\dim\Big(\operatorname{Ker}(Df(0, p) - DP_2(0, p)) \cap (\{0\} \times T_pZ)\Big) < m + 1\,.$$

On the other hand, since

$$\operatorname{Ker}(Df(0, p) - DP_2(0, p)) \cap (\{0\} \times T_pZ)$$

contains $\{0\} \times T_pM_0$, one has

$$\dim\Big(\operatorname{Ker}(Df(0, p) - DP_2(0, p)) \cap (\{0\} \times T_pZ)\Big) \geq \dim(\{0\} \times T_pM_0) = m.$$

Therefore,

$$\dim\Big(\operatorname{Ker}(Df(0, p) - DP_2(0, p)) \cap (\{0\} \times T_pZ)\Big) = m$$

and (H_p) follows noting that

$$\operatorname{Ker}(Df(0, p) - DP_2(0, p)) \cap (\{0\} \times T_pZ) = \{0\} \times \operatorname{Ker}(\partial_2 f(0, p) - I_p).$$

Consequently, by Lemma 4.4, any nontrivial solution (λ, z) of (4.1) close to $(0, p)$ has $\lambda \neq 0$. $\square$

Remark 4.8. In the case when $\dim M_0 = 1$, the linear operator

$$v \in T_pM_0 \mapsto Hf((0,p))((1,-w),(0,v)) \in T_pZ/\operatorname{Im}(\partial_2 f(0,p) - I_p)$$

introduced in Corollary 4.7 is a map between 1-dimensional spaces. Thus, our assumption that such an operator is onto can be interpreted as a Crandall-Rabinowitz type condition ([3]).

In order to illustrate how the assumptions in Corollary 4.5 and 4.7 above can be explicitly computed in local coordinates, we give below two examples of parametrized fixed point equations in the projective space $\mathbb{P}^2$.

Example 4.9. Let $\mathbb{P}^2$ be the 2-dimensional projective space. We can think of $\mathbb{P}^2$ as the Grassmannian $G_1(\mathbb{R}^3)$; that is, the smooth manifold of all straight lines in $\mathbb{R}^3$ through the origin. Consider the map $\sigma : [-\pi/2, \pi/2] \times \mathbb{R} \to \mathbb{P}^2$ that associates to any $(\theta, \varphi) \in [-\pi/2, \pi/2] \times \mathbb{R}$ the straight line of $\mathbb{P}^2$ containing the point $(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$. It is easy to check that σ is a quotient map, i.e. that a set $A \subseteq \mathbb{P}^2$ is open if and only if $\sigma^{-1}(A)$ is open in $[-\pi/2, \pi/2] \times \mathbb{R}$. As in the terminology used for the coordinates of the Earth, θ will be called the *latitude* and φ the *longitude*. Moreover, the Equator is the image of $\{0\} \times \mathbb{R}$ (under σ) and the North-South Pole (that is, the vertical line), denoted NS, is the element $\sigma(\{\pi/2\} \times \mathbb{R})$ or, equivalently, $\sigma(\{-\pi/2\} \times \mathbb{R})$. Consider the map

$$\hat{f} : \mathbb{R} \times [-\pi/2, \pi/2] \times \mathbb{R} \to [-\pi/2, \pi/2] \times \mathbb{R}$$

given by $\hat{f}(\lambda, \theta, \varphi) = (-\theta + \sin\lambda\,\sin\theta, \varphi + \sin\lambda\,\sin\varphi)$. Since $\hat{f}$ sends fibers of the quotient map $I \times \sigma$ (here I denotes the identity of $\mathbb{R}$) into fibers of σ, then it induces a well-defined map $f : \mathbb{R} \times \mathbb{P}^2 \to \mathbb{P}^2$, which is continuous since σ is a quotient map. One can also check that f is smooth on $\mathbb{R} \times (\mathbb{P}^2 \setminus NS)$. By taking $Z = \mathbb{P}^2$ and f as above, we are reduced in $\mathbb{R} \times \mathbb{P}^2$ to a parametrized fixed point equation as (4.1). For $\lambda = 0$, the fixed points of $f_0 = f(0, \cdot)$ are the elements of the Equator and the North-South Pole. It is quite natural to think of the Equator, which obviously is a 1-dimensional submanifold of $\mathbb{P}^2$, as the set M_0 of trivial fixed points of f_0. Let us consider in the Equator the element $p = \sigma(0,0)$; that is, the line through the origin and $(1,0,0)$. It is immediately seen that p is a bifurcation point from the Equator, since any pair $(\lambda, p) \in \mathbb{R} \times \mathbb{P}^2$ is a solution to our equation (and is nontrivial if $\lambda \neq 0$). Our aim here is to compute in coordinates for such a bifurcation point p and for the map f defined above the necessary condition as well as the sufficient condition given in Corollaries 4.5 and 4.7. To this end observe first that the restriction of σ to the open subset $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ is clearly a smooth parametrization of a neighborhood U of p in $\mathbb{P}^2$. Thus, its inverse $z \in U \mapsto (\theta(z), \varphi(z))$ is a system of coordinates about p sending p in $(0,0)$. Let us take the same local coordinates (θ, φ) about $p \in \mathbb{P}^2$ both for p in the domain of f_0 and in its image. By recalling Remark 2.3, it is easy to verify that, if w is a vector tangent to $\mathbb{P}^2$ at p, $w = \alpha_1\left(\frac{\partial}{\partial\theta}\right)_p + \alpha_2\left(\frac{\partial}{\partial\varphi}\right)_p$, then the vector $(\partial_2 f(0,p) - I_p)w$ is given by

$$-\alpha_1\left(\frac{\partial}{\partial\theta}\right)_p + \alpha_2\left(\frac{\partial}{\partial\varphi}\right)_p - \alpha_1\left(\frac{\partial}{\partial\theta}\right)_p - \alpha_2\left(\frac{\partial}{\partial\varphi}\right)_p = -2\alpha_1\left(\frac{\partial}{\partial\theta}\right)_p.$$

In other words, the image of the linear map $\partial_2 f(0,p) - I_p$ consists of the vectors which are tangent at p to the meridian passing through p and its kernel consists of the vectors which are tangent to the Equator at p. Therefore, assumption (H_p)

holds and, since the derivative $\partial_1 f(0,p)$ of f with respect to λ at $(0,p)$ is represented by $\left(\frac{\partial}{\partial\theta}\right)_p$, the necessary condition "$\partial_1 f(0,p) = \partial_2 f(0,p)w - w$, for some $w \in T_p\mathbb{P}^2$", stated in Corollary 4.5, is clearly satisfied with $\alpha_1 = -1/2$. As regards the sufficient condition of Corollary 4.7, if w is as above and v is a vector tangent to the Equator at p, say $v = \beta\left(\frac{\partial}{\partial\varphi}\right)_p$, then (again recall Remark 2.3) an easy computation shows that the Hessian $Hf(0,p)((1,-w)(0,v))$ can be represented as $2\beta\left(\frac{\partial}{\partial\varphi}\right)_p$, up to elements belonging to the image of $\partial_2 f(0,p) - I_p$; that is, up to vectors of the form $\gamma\left(\frac{\partial}{\partial\theta}\right)_p$. Thus, since if $\beta \neq 0$ such an element does not belong to $\mathrm{Im}(\partial_2 f(0,p) - I_p)$, this proves that the linear operator $v \mapsto Hf((0,p))((1,-w),(0,v))$ is onto, and the sufficient condition stated in Corollary 4.7 is verified.

In Example 4.10 below, the dimension of the manifold M_0 of trivial fixed points is strictly greater than 1; that is, grater than the difference between the dimension of $X = \mathbb{R} \times \mathbb{P}^2$ and that of $Y = \mathbb{P}^2$. Consequently, since this difference is an integer representing the Fredholm index of the map f, our example cannot be interpreted, as the previous one, in the context of the Crandall-Rabinowitz bifurcation result.

Example 4.10. Given the quotient map σ as in Example 4.9, let us consider the map $f : \mathbb{R} \times \mathbb{P}^2 \to \mathbb{P}^2$ induced by $\hat{f}(\lambda,\theta,\varphi) = (\theta + \sin\lambda \sin\theta, \varphi + \sin\lambda \sin\varphi)$. For $\lambda = 0$, any element of $\mathbb{P}^2$ is clearly a fixed point of f_0 and will be assumed to be a trivial fixed point. Consequently, M_0 coincides with the whole space $Z = \mathbb{P}^2$. As previously, let us take the element $p = \sigma(0,0)$ of the Equator to be the bifurcation point at which computing the conditions of Corollaries 4.5 and 4.7. It is immediately seen that, in this case, $\partial_2 f(0,p) - I_p$ is the null operator. Thus,

$$\mathrm{Ker}(\partial_2 f(0,p) - I_p) = T_p\mathbb{P}^2 \quad \text{and} \quad \mathrm{Im}(\partial_2 f(0,p) - I_p) = \{0\}.$$

Moreover, since $\partial_1 f(0,p) = 0$, the necessary condition of Corollary 4.5 is satisfied with any $w \in T_p\mathbb{P}^2$. Let us take, for simplicity, $w = 0$ and observe that if v is any vector in T_pM_0, then $Hf((0,p))((1,0),(0,v))$ belongs to T_pM_0 as well (recall that here $T_pZ = T_pM_0$ and $\mathrm{Im}(\partial_2 f(0,p) - I_p) = \{0\}$). Hence, if $v = \beta_1\left(\frac{\partial}{\partial\theta}\right)_p + \beta_2\left(\frac{\partial}{\partial\varphi}\right)_p$, by computing $Hf((0,p))((1,0),(0,v))$, one gets again $\beta_1\left(\frac{\partial}{\partial\theta}\right)_p + \beta_2\left(\frac{\partial}{\partial\varphi}\right)_p$. This shows that the linear operator $v \mapsto Hf((0,p))((1,0),(0,v))$ is onto, as required by our sufficient condition.

5. Applications to differential equations

In this section we give an application to second order differential equations on manifolds of the obtained bifurcation results. Similar results have been obtained in [6] by means of topological tools, as the fixed point index and its relationship with the degree of a tangent vector field on a differentiable manifold.

Let N be an m-dimensional manifold in $\mathbb{R}^s$. As previously, given $q \in N$, let $T_qN \subseteq \mathbb{R}^s$ and $(T_qN)^\perp \subseteq \mathbb{R}^s$ denote, respectively, the tangent space and the normal space of N at q. Given any $q \in N$ and any $u \in \mathbb{R}^s$, the vector u can be uniquely decomposed into a parallel component $u_\pi \in T_qN$ and a normal component $u_\nu \in (T_qN)^\perp$. Obviously, the decomposition of u depends on the chosen element q. By

$$TN = \{(q,v) \in \mathbb{R}^s \times \mathbb{R}^s : q \in N \ , \ v \in T_qN\}$$

we indicate the tangent bundle of N. Clearly, TN contains a natural copy of N via the embedding $q \mapsto (q, 0)$.

Let $F : \mathbb{R} \times TN \to \mathbb{R}^s$ be a continuous map such that $F(t, q, v) \in T_q N$ for all $(t, q, v) \in \mathbb{R} \times TN$. For brevity, we will say that F is *tangent to* N although it is not a tangent vector field on N.

We will consider in N the parametrized motion equation

$$\ddot{x}_\pi = \lambda F(t, x, \dot{x}), \quad \lambda \in \mathbb{R}\,. \tag{5.1}$$

A solution of (5.1), corresponding to a given $\lambda \in \mathbb{R}$, is a C^2 map $x : J \to N$, defined on a nontrivial interval J, such that $\ddot{x}_\pi(t) = \lambda F\,(t, x(t), \dot{x}(t))$ for all $t \in J$, where $\ddot{x}_\pi(t)$ is the parallel (or tangential) part of the acceleration $\ddot{x}(t) \in \mathbb{R}^s$ and is obtained by taking the orthogonal projection of $\ddot{x}(t)$ onto $T_{x(t)}N$.

It is known that there exists a unique smooth map $r : TN \to \mathbb{R}^s$, called the *reactive force* (or *inertial reaction*) with the following properties:

(a) $r(q, v) \in (T_q N)^\perp$ for any $(q, v) \in TN$;
(b) r is quadratic in the second variable;
(c) any C^2 curve $x : J \to N$ is a solution of the differential equation
$$\ddot{x}_\nu = r(x, \dot{x}),$$
i.e., for any $t \in J$, the normal component $\ddot{x}_\nu(t)$ of the acceleration $\ddot{x}(t)$ equals $r(x(t), \dot{x}(t))$.

The map r is strictly related to the second fundamental form on N and may be interpreted as the reactive force due to the constraint N. Actually, given $(q, v) \in TN$, $r(q, v)$ is the unique vector of $\mathbb{R}^s$ which makes $(v, r(q, v))$ tangent to TN at (q, v).

Due to condition (c) above, equation (5.1) can be equivalently written as

$$\ddot{x} = r(x, \dot{x}) + \lambda F(t, x, \dot{x}). \tag{5.2}$$

For $\lambda = 0$, it reduces to the so-called *inertial equation*

$$\ddot{x} = r(x, \dot{x})\,,$$

whose solutions are the geodesics of N.

Clearly, (5.2) can be written, in an equivalent way, as a first order order differential equation on TN as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = r(x, y) + \lambda F(t, x, y)\,, \end{cases} \tag{5.3}$$

where, as it is not hard to verify, the map

$$(\lambda, t, q, v) \in \mathbb{R} \times \mathbb{R} \times TN \mapsto (v, r(q, v) + \lambda F(t, q, v)) \in \mathbb{R}^s \times \mathbb{R}^s$$

is a tangent vector field on TN.

For a more extensive treatment on the subject of second-order ODEs on manifolds from this embedded viewpoint see e.g. [4].

Assume from now on that F is T-periodic with respect to t and (at least) C^1. In what follows, we will be concerned with T-periodic solutions of the motion equation (5.1). More precisely, a pair (λ, x), with $\lambda \in \mathbb{R}$ and $x : \mathbb{R} \to N$ a T-periodic solution of (5.1) corresponding to λ, will be called a *T-pair* of (5.1).

Let

$$D = \{(\lambda, q, v) \in \mathbb{R} \times TN : \text{the maximal solution } (x(\cdot), y(\cdot)) \text{ of (5.3)}$$
$$\text{satisfying } x(0) = q, y(0) = v \text{ is defined in } [0, T]\},$$

and let $P^T : D \to TN$ be the *Poincaré T-translation operator* which associates to any $(\lambda, q, v) \in D$ the value $(x(T), y(T))$ at time T of the solution of (5.3) with initial conditions (q, v). It can be shown that D is an open set (clearly containing $\{0\} \times N \times \{0\}$) and that P^T is C^k provided that so is F. Since we will deal with a local problem, for the sake of simplicity we assume $D = \mathbb{R} \times TN$. However, all the statements below remain true also in the case when D is a proper subset of $\mathbb{R} \times TN$, but their proofs require cumbersome notation.

Consider the parametrized fixed point equation

$$P^T(\lambda, q, v) = (q, v). \tag{5.4}$$

The equation (5.4) is strictly related to the T-periodic problem associated with equation (5.1). More precisely, a triple (λ, q, v) is such that $P^T(\lambda, q, v) = (q, v)$ if and only if (λ, x) with $x(\cdot)$ a solution of (5.1) corresponding to λ and satisfying $x(0) = q$, $\dot{x}(0) = v$, is a T-pair of (5.1). Such a triple (λ, q, v) is also called a *starting point* of the T-pair (λ, x).

When $\lambda = 0$, the fixed points (q, v) of $P_0^T = P^T(0, \cdot, \cdot)$ are initial conditions of closed (T-periodic) geodesics on N. Among these pairs, those of the form $(q, 0)$ correspond to the constant solutions $x(t) \equiv q$. Therefore, as far as we are concerned with equation (5.4), it turns out to be quite natural to think of the starting points $(0, q, 0)$, $q \in N$, as the *trivial* ones. We will be interested in detecting those elements $(q, 0) \in N \times \{0\}$ such that in any neighborhood of $(0, q, 0)$ in $\mathbb{R} \times TN$ there exists a nontrivial starting point. More precisely, we will apply the results of Section 4, with $Z = TN$, $M_0 = N \times \{0\}$ and $f = P^T$, in order to obtain a necessary condition and a sufficient condition for a pair $(q, 0) \in N \times \{0\}$ to be a bifurcation point for the equation (5.4). As already pointed out (see Section 4), in the second order ODEs context that we are investigating here, the set $N \times \{0\}$ may be strictly contained in $\operatorname{Fix} P_0^T$. In other words, there may exist triples $(0, q, v)$, with (q, v) belonging to $\operatorname{Fix} P_0^T \setminus (N \times \{0\})$ that are (nontrivial) starting points of nonconstant T-periodic geodesics on N. For example, this occurs for the inertial motion on a sphere.

In what follows, we will say that a constant solution $x(t) \equiv q_0 \in N$ is a *bifurcation point of (the second order equation)* (5.1) if $(q_0, 0) \in TN$ is a bifurcation point of $P^T(\lambda, q, v) = (q, v)$. Due to the continuous dependence of the solutions of differential equations on the initial conditions, given a bifurcation point $t \mapsto q_0$ of (5.1), if (λ, x) is a T-pair associated with a nontrivial starting point (λ, q, v) close in $\mathbb{R} \times TN$ to the triple $(0, q_0, 0)$, then λ is close to 0 in $\mathbb{R}$ and $t \mapsto x(t)$ is close to $t \mapsto q_0$ in the usual C^1 norm. Thus, a bifurcation point of the motion equation (5.1) has, in some sense, also an infinite dimensional meaning since it can be interpreted as a bifurcation point of T-pairs as well.

We recall below a well-known result on ODEs that we will use several times in the sequel (see e.g. [2]).

Theorem 5.1. *Consider the following initial value problem*

$$\begin{cases} \dot{x} = g(t, x) \\ x(0) = a\,, \end{cases} \tag{5.5}$$

where $g : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function and $a \in \mathbb{R}^n$. Assume that, for any $a \in \mathbb{R}^n$, the solutions of (5.5) are continuable at least to T. Let $\Phi^T : \mathbb{R}^n \to \mathbb{R}^n$ be the translation operator which associates to $a \in \mathbb{R}^n$ the value $x(T)$ of the solution of $\dot{x} = g(t,x)$ such that $x(0) = a$. Then, Φ^T is C^1 and, for any $a \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, the derivative $D\Phi^T(a)h$ of Φ^T at the point a along the vector h coincides with the value at the time T of the solution $\xi(\cdot)$ of the linear problem

$$\begin{cases} \dot{\xi} = \partial_2 g(t, x_0(t))\, \xi \\ \xi(0) = h\,, \end{cases}$$

where $x_0(\cdot)$ is the solution of $\dot{x} = g(t,x)$ with initial value a.

It is well-known that in a Riemannian manifold there are no nonconstant closed geodesics too close to a given point. Roughly speaking, this fact, if interpreted in our context, means that the manifold $N \times \{0\} \subseteq TN$ is isolated in $\operatorname{Fix} P_0^T$. As already observed in Section 4 (see Lemma 4.4), this is a consequence of condition (H). Actually, we prove below that the T-translation operator P^T in (5.4) satisfies assumption (H). As a by-product, according to Lemma 4.4, we will obtain that if $(q_0, 0)$, $q_0 \in N$, is any bifurcation point of (5.4), then there exists a neighborhood of $(0, q_0, 0)$ in $\mathbb{R} \times TN$ in which any nontrivial solution (λ, q, v) of (5.4) has necessarily $\lambda \neq 0$.

Theorem 5.2. *The T-translation operator P^T satisfies assumption (H) of Section 4 with $M_0 = N \times \{0\}$, i.e., for any $q \in N$, one has*

$$T_{(q,0)}(N \times \{0\}) = \operatorname{Ker}(DP_0^T(q,0) - I_{(q,0)}) \tag{5.6}$$

(here $I_{(q,0)}$ denotes the identity map on the tangent space $T_{(q,0)}TN$).

Proof. In order to prove that P^T satisfies (5.6), we need first to compute the derivative of P_0^T at any point $(q,0), q \in N$. To this end observe that, since N is an m-dimensional manifold in $\mathbb{R}^s$, there exists a diffeomorphism of a neighborhood of q in N onto $\mathbb{R}^m$. Clearly, this diffeomorphism can be extended to a C^∞ map φ defined on an open neighborhood of q in $\mathbb{R}^s$. Since we are dealing with local problems, without loss of generality we may therefore assume that φ is a map from an open neighborhood U of N in $\mathbb{R}^s$ and that the restriction of φ to N is a diffeomorphism onto $\mathbb{R}^m$. The map φ induces on the fiber bundle $TU = U \times \mathbb{R}^s$ the tangent map

$$T\varphi : TU \to \mathbb{R}^m \times \mathbb{R}^m,\ (x,y) \mapsto (\varphi(x), D\varphi(x)y).$$

Since the restriction $\varphi|_N : N \to \mathbb{R}^m$ of φ to N is a diffeomorphism, so is the tangent map $T(\varphi|_N) : TN \to \mathbb{R}^m \times \mathbb{R}^m$. By means of the change of variables

$$\begin{cases} x_1 = \varphi(x) \\ y_1 = D\varphi(x)y\,, \end{cases}$$

system (5.3) is transformed, after some calculations, in a first order system in $\mathbb{R}^m \times \mathbb{R}^m$ of the form

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = s(x_1, y_1) + \lambda G(t, x_1, y_1)\,, \end{cases} \tag{5.7}$$

where s is quadratic in y_1 for any fixed x_1 and G is the vector field which corresponds to F under $D\varphi$. More precisely, we have

$$s(x_1, y_1) = D^2\varphi(\varphi^{-1}(x_1))\Big(\big(D(\varphi^{-1}(x_1))\big)^{-1} y_1, \big(D(\varphi^{-1}(x_1))\big)^{-1} y_1\Big)$$

$$+D\varphi\big(\varphi^{-1}(x_1)\big)r\Big(\varphi^{-1}(x_1),\big(D(\varphi^{-1}(x_1))\big)^{-1}y_1\Big)$$

and

$$G(t,x_1,y_1)=D\varphi\big(\varphi^{-1}(x_1)\big)F\Big(t,\varphi^{-1}(x_1),\big(D(\varphi^{-1}(x_1))\big)^{-1}y_1\Big)$$

Moreover, it is easy to see that the Poincaré T-translation operator Q^T which associates to any $(\lambda,\hat{q},\hat{v})\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m$ the value $(x_1(T),y_1(T))$ at time T of the solution of (5.7) with initial condition $(\hat{q},\hat{v})$ corresponds to P^T under $T(\varphi|_N)$.

Our aim is to apply Theorem 5.1 to system (5.7) with $\lambda=0$. We need to linearize (5.7) about the constant solution $(x_1(t),y_1(t))\equiv(\hat{q},0)\in\mathbb{R}^m\times\{0\}$. Since $s(x_1,0)=0$ for all $x_1\in\mathbb{R}^m$, one has $\partial_1 s(\hat{q},0)=0$. Moreover, recalling that s is quadratic in the second variable, one also gets $\partial_2 s(\hat{q},0)=0$. Thus $Ds(\hat{q},o)=0$, and the required linearization is given by the initial value problem

$$\begin{cases}\dot{\xi}=\eta\\ \dot{\eta}=0\\ \xi(0)=h\\ \eta(0)=k,\end{cases}\tag{5.8}$$

with $(h,k)\in\mathbb{R}^m\times\mathbb{R}^m$. By Theorem 5.1, the derivative $DQ_0^T(\hat{q},0)(h,k)$ of $Q_0^T:=Q^T(0,\cdot,\cdot)$ at $(\hat{q},0)$ in (h,k) coincides with the value $(\xi(T),\eta(T))$ of the solution of (5.8). By computing $(\xi(T),\eta(T))$ one gets

$$\begin{cases}\xi(T)=h+kT\\ \eta(T)=k\end{cases}$$

Consequently,

$$\big(DQ_0^T(\hat{q},0)-I\big)(h,k)=(h+kT-h,k-k)=(kT,0),\tag{5.9}$$

where I denotes the identity of $\mathbb{R}^m\times\mathbb{R}^m$. Thus, the kernel of $DQ_0^T(\hat{q},0)-I$ is the subset of $\mathbb{R}^m\times\mathbb{R}^m$ given by

$$\{(h,k)\in\mathbb{R}^m\times\mathbb{R}^m : k=0\}=\mathbb{R}^m\times\{0\}.$$

Therefore, going back to the manifold N, we have

$$\operatorname{Ker}\big(DP_0^T(q,0)-I_{(q,0)}\big)=T_qN\times\{0\}$$

and noting that

$$T_{(q,0)}(N\times\{0\})=T_qN\times\{0\}\,,$$

the assertion follows. □

Let us now consider the *average force* $\bar{F}:N\to\mathbb{R}^s$ given by

$$\bar{F}(q)=\frac{1}{T}\int_0^T F(t,q,0)dt\,.$$

Clearly, $\bar{F}$ is an autonomous tangent vector field on N. The zeros of the average force play an important role in obtaining bifurcation conditions for the second order equation (5.1). Results similar to Theorems 5.3 and 5.4 below can be deduced also from the global bifurcation context discussed in [6] by means of topological degree methods.

Theorem 5.3 (Necessary condition)**.** *Assume that the constant solution $t\mapsto q_0\in N$ is a bifurcation point of the motion equation* (5.1)*. Then* $\bar{F}(q_0)=0$.

Proof. By definition, the constant solution $x(t) \equiv q_0 \in N$ is a bifurcation point of (5.1) if and only if $(q_0, 0) \in TN$ is a bifurcation point of $P^T(\lambda, q, v) = (q, v)$. Our aim is to apply Corollary 4.5 to the Poincaré T-translation operator P^T and to the bifurcation point $(q_0, 0)$. To this end, observe first that, as proved in Theorem 5.2, assumption (H) is satisfied. Therefore, by Corollary 4.5 (see also Remark 4.6), it follows that a necessary condition for $(q_0, 0)$ to be a bifurcation point of (5.4) is that there exists $w = (w_1, w_2) \in T_{(q_0,0)}TN = T_{q_0}N \times T_{q_0}N$ such that

$$(1, -w_1, -w_2) \in \operatorname{Ker}(DP^T(0, q_0, 0) - DP_2(0, q_0, 0)), \tag{5.10}$$

where $P_2 : \mathbb{R} \times TN \to TN$ denotes the projection onto the second component TN.

As in the proof of Theorem 5.2, in order to compute $DP^T(0, q_0, 0)$ we can reduce to $\mathbb{R}^m$ and apply Theorem 5.1 to the following initial value problem in $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$

$$\begin{cases} \dot{\lambda} = 0 \\ \dot{x}_1 = y_1 \\ \dot{y}_1 = s(x_1, y_1) + \lambda G(t, x_1, y_1) \\ \lambda(0) = \lambda \\ x_1(0) = \hat{q} \\ y_1(0) = \hat{v} \end{cases} \tag{5.11}$$

with $(\hat{q}, \hat{v}) \in \mathbb{R}^m \times \mathbb{R}^m$.

By linearizing (5.11) about the constant solution $(\lambda(t), x_1(t), y_1(t)) \equiv (0, \hat{q_0}, 0) \in \{0\} \times \mathbb{R}^m \times \{0\}$ (here $\hat{q_0}$ is the point of $\mathbb{R}^m$ that corresponds to $q_0 \in N$ under the diffeomorphism φ), we get

$$\begin{cases} \dot{\mu} = 0 \\ \dot{\xi} = \eta \\ \dot{\eta} = \mu\, G(t, \hat{q_0}, 0) \\ \mu(0) = \mu \\ \xi(0) = h \\ \eta(0) = k \end{cases}$$

whose solution at time T is

$$\begin{cases} \mu(T) = \mu \\ \xi(T) = h + kT + \mu \int_0^T (T - t) G(t, \hat{q_0}, 0) dt \\ \eta(T) = k + \mu \int_0^T G(t, \hat{q_0}, 0) dt \end{cases}$$

By Theorem 5.1, the triple $(\mu(T), \xi(T), \eta(T))$ coincides with the value along the vector (μ, h, k) of the derivative at $(0, \hat{q_0}, 0)$ of the translation operator associated with (5.11). In particular, the derivative of the Poincaré operator Q^T at $(0, \hat{q_0}, 0)$ along (μ, h, k) is given by the last two components $(\xi(T), \eta(T))$, i.e.

$$\begin{gathered} DQ^T(0, \hat{q_0}, 0)(\mu, h, k) = \\ (h + kT + \mu \int_0^T (T - t) G(t, \hat{q_0}, 0) dt, k + \mu \int_0^T G(t, \hat{q_0}, 0) dt). \end{gathered} \tag{5.12}$$

Consequently, by again interpreting the above operator in $\mathbb{R} \times TN$, we obtain

$$\begin{gathered} DP^T(0, q_0, 0)(\mu, u_1, u_2) - DP_2(0, q_0, 0)(\mu, u_1, u_2) = \\ \Big(u_2 T + \mu \int_0^T (T - t) F(t, q_0, 0) dt, \mu \int_0^T F(t, q_0, 0) dt\Big), \end{gathered}$$

where $(u_1, u_2) \in T_{(q_0,0)}TN$.

Thus, the triple $(1, -w_1, -w_2)$ of (5.10) must satisfy

$$\Big(- w_2 T + \int_0^T (T-t)F(t,q_0,0)dt, \int_0^T F(t,q_0,0)dt \Big) = (0,0). \tag{5.13}$$

This implies

$$\bar{F}(q_0) = \frac{1}{T}\int_0^T F(t,q_0,0)dt = 0,$$

which is our assertion. □

Theorem 5.4 (Sufficient condition)**.** *Let F be C^2 and assume that $q_0 \in N$ is a zero of the average force $\bar{F}$ such that $D\bar{F}(q_0) : T_{q_0}N \to \mathbb{R}^s$ is one-to-one. Then, the constant solution $t \mapsto q_0$ is a bifurcation point of* (5.1)*.*

Proof. According to our definition, $q_0 \in N$ is a bifurcation point of (5.1) if $(q_0, 0)$ is a bifurcation point of (5.4) from $N \times \{0\}$. We will apply Corollary 4.7 with $Z = TN$, $M_0 = N \times \{0\}$, $f = P^T$, $p = (q_0, 0)$. Since $\bar{F}(q_0) = 0$, by taking $w = (0, w_2)$ with

$$w_2 = -\frac{1}{T}\int_0^T tF(t,q_0,0)dt,$$

from (5.13) one has $(1, 0, -w_2) \in \operatorname{Ker}(DP^T(0,q_0,0) - DP_2(0,q_0,0))$ or, equivalently,

$$\partial_1 P^T(0,q_0,0) = (DP_0^T(q_0,0) - I_{(q_0,0)})w,$$

as required in Corollary 4.7.

Therefore, in order to prove that $(q_0, 0)$ is a bifurcation point, it remains to show that the linear operator

$$\dot{q} \in T_{q_0}N \mapsto HP^T(0,q_0,0)((1,0,-w_2),(0,\dot{q},0))$$
$$\in T_{(q_0,0)}TN/\operatorname{Im}(DP_0^T(q_0,0) - I_{(q_0,0)})$$

is onto or, equivalently, that it is one-to-one (recall Remark 4.3 and observe that the dimension of $\operatorname{Ker}(DP^T(0,q_0,0) - DP_2(0,q_0,0))$ is $m+1$). To this end, as in Theorem 5.2, we may reduce to system (5.7) in $\mathbb{R}^m \times \mathbb{R}^m$ and consider the linear operator $L : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ given by

$$L(h) = D^2Q^T(0,\hat{q_0},0)\big((1,0,-\hat{w_2}),(0,h,0)\big),$$

where $\hat{q_0}$ and $\hat{w_2}$ are the elements in $\mathbb{R}^m$ that correspond to q_0 and w_2 respectively, and, as previously, Q^T is the Poincaré T-translation operator that corresponds to P^T under $T(\varphi|_N)$. Since our statements are invariant under diffeomorphisms, it is enough to prove that if $L(h)$ belongs to $\operatorname{Im}\big(DQ_0^T(\hat{q_0},0) - I\big)$, then $h = 0$.

As well-known, $L(h)$ can be calculated by introducing the map

$$\gamma : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m, \quad x_1 \mapsto DQ^T\big((0,x_1,0)\big)(1,0,-\hat{w_2}),$$

and by taking the derivative $D\gamma(\hat{q_0})(h)$. Thus, recalling (5.12), a standard computation gives

$$L(h) = \Big(\int_0^T (T-t)\partial_2 G(t,\hat{q_0},0)h\,dt, \int_0^T \partial_2 G(t,\hat{q_0},0)h\,dt\Big).$$

On the other hand, from (5.9), we get immediately

$$\operatorname{Im}\big(DQ_0^T(\hat{q_0},0) - I\big) = \operatorname{Ker}\big(DQ_0^T(\hat{q_0},0) - I\big) = \mathbb{R}^m \times \{0\}.$$

Consequently, if $L(h)$ belongs to $\operatorname{Im}\big(DQ_0^T(\hat{q}_0,0)-I\big)$, then

$$\int_0^T \partial_2 G(t,\hat{q}_0,0)h\,dt = 0. \tag{5.14}$$

To achieve the proof, we will show that (5.14) implies $h=0$. To see this, observe that the term $\int_0^T \partial_2 G(t,\hat{q}_0,0)h\,dt$ coincides with the derivative of the map $\hat{q}\mapsto \int_0^T G(t,\hat{q},0)\,dt$ at $\hat{q}_0$ along h. Therefore, our claim will follow from the injectivity of the above derivative. Clearly, up to diffeomorphisms, this operator coincides with the derivative of the average force $\bar{F}$ at q_0. Thus, it turns out to be one-to-one, since we have assumed that so is $D\bar{F}(q_0)$. □

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