GLOBAL STABILITY OF EQUILIBRIA

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1. Introduction

Let $F\colon \mathbb{R}^p \to \mathbb{R}^p$ be of class C^1 (i.e. differentiable in the sense of Fréchet with continuous derivative). In a paper published in 1960 Markus L. and Yamabe H. [20] conjectured that the origin is a global attractor for the autonomous system of differential equations

$$\mathbf{x}'(t) = F(\mathbf{x}(t)) \tag{1.1}$$

provided that

1. $F(\mathbf{0}) = \mathbf{0}$, and
2. $$\sigma(F'(\mathbf{x})) \subset \mathbf{C}^-$$
   for every $\mathbf{x} \in \mathbb{R}^p$.

The symbol $\sigma(F'(\mathbf{x}))$ stands for the spectrum of the linear operator $F'(\mathbf{x})$, while $\mathbf{C}^-$ denotes the set of complex numbers with real part strictly less than 0. We know today that the Markus-Yamabe conjecture is true in $\mathbb{R}^2$ [9, 12], and false in $\mathbb{R}^p\colon p \geq 3$ [6].

In 1976 La Salle J. P. [19] proposed a similar result for discrete dynamical systems by conjecturing that every orbit of

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n) \tag{1.2}$$

converges to $\mathbf{0}$ provided that

1. $F(\mathbf{0}) = \mathbf{0}$, and
2. $$\rho(F'(\mathbf{x})) < 1$$
   for every $\mathbf{x} \in \mathbb{R}^p$.

It is assumed that F is of class C^1 and the symbol $\rho(F'(\mathbf{x})) = \max\{|\lambda|\colon \lambda \in \sigma(F'(\mathbf{x}))\}$ denotes the spectral radius of the linear operator $F'(\mathbf{x})$.

The conjecture of La Salle is false even in $\mathbb{R}^2$ [23]. However, it is true in $\mathbb{R}^2$ for polynomial vector fields [8].

It is not hard to show that when F is a gradient every orbit of (1.1) and of (1.2) converges to $\mathbf{0}$ [22]. It has been shown that the same conclusion can be reached when F is upper or lower triangular. Moreover, in the case of the discrete dynamical system (1.2), the assumption $\rho(F'(\mathbf{x})) < 1$ can be relaxed [1, 16].

Hartman [14] proved the global convergence conjectured by Markus-Yamabe for (1.1) when the spectrum of the symmetric part $H(\mathbf{x})$ of $F'(\mathbf{x})$ is contained in $\mathbf{C}^-$.

The present paper has two main purposes. First we would like to show that Hartman's condition [14] can be considerably relaxed (see Theorem 2.5) using an idea developed in [10]. Second, we would like to establish a companion result (see Theorem 2.7) for discrete dynamical systems.

2. Results

The theorems presented in this section regard the global behavior of continuous and discrete dynamical systems when the time dependence is not contained explicitly in the equation, i.e. the systems are *autonomous*. We shall assume that the function $F\colon \mathbb{R}^p \to \mathbb{R}^p$ is continuous and $F(\mathbf{0}) = \mathbf{0}$. Moreover we shall also assume that F is Gateaux differentiable [27] except possibly on a linearly countable set S, i.e. such that $\mathbf{0} \in S$ and $S \cap [\mathbf{0}, \mathbf{x}]$ is at most countable, where $[\mathbf{0}, \mathbf{x}]$ denotes the line segment joining $\mathbf{0}$ with $\mathbf{x}$, and $\mathbf{x}$ is any point of $\mathbb{R}^p$.

Our theorems are based on extensions of previous results established in [10, 11, 21] (see also [4]). We shall now state and prove the results and their generalizations with the adjustments that make them applicable to the situations envisioned in this paper.

Theorem 2.1. *Let $a < b$ be two real numbers and let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$. Assume that f is differentiable in (a, b) except possibly at finitely many points of (a, b). Then both sets $S^+ = \{c \in (a, b)\colon f(b) - f(a) \le f'(c)(b - a)\}$, and $S^- = \{d \in (a, b)\colon f(b) - f(a) \ge f'(d)(b - a)\}$ are not empty.*

Proof. Assume first that f is constant. Then S^+ and S^- are obviously not empty. Assume now that f is not constant. In the case when f is differentiable in (a, b) the stated result is a consequence of the Mean Value Theorem. In the case when f is not differentiable at n points of (a, b) we can apply Corollary 4 of [10] to obtain

$$f(b) - f(a) = \sum_{i=1}^{n+1} \alpha_i f'(c_i)(b - a), \tag{2.3}$$

where $\alpha_1, \ldots, \alpha_{n+1}$ are positive real numbers such that $\sum_{i=1}^{n+1} \alpha_i = 1$. Let c_j, c_k be such that $f'(c_j) \le f'(c_i) \le f'(c_k)$ for every $i = 1, \ldots, n + 1$. Then

$$f'(c_j)(b - a) \le f(b) - f(a) \le f'(c_k)(b - a). \tag{2.4}$$

(see also Theorem 2 of [21]). □

We shall now establish a result that is a more general than Theorem 2.1. We first need the following lemma. The symbol E^c denotes the complement of the set E.

Lemma 2.2. *Let $a < b$ and $g\colon [a, b] \to \mathbb{R}$ be continuous. Assume that $E \subset (a, b)$ is a countable subset of (a, b) such that g is differentiable on $E^c \cap (a, b)$, $g'(x) \ge 0$, and g is not constant. Then the set $N = \{x \in (a, b)\colon g'(x) > 0\}$ is non-negligible. Moreover, we have $g(b) > g(a)$.*

Proof. The image of g is the union of three sets

$$\mathrm{Im}(g) = g(E) \cup g(F) \cup g(N)$$

where $F = \{x \in (a, b)\colon\ g'(x) = 0\}$. Since g is not constant, $\mathrm{Im}(g)$ is non-negligible. We know that $g(F)$ is negligible [21] and $g(E)$ is countable. Hence, N cannot be negligible, for, otherwise, $g(N)$ would be negligible [21]. The inequality $g(b) > g(a)$ is now a consequence of Theorem 2 of [24]. □

From Lemma 2.2 we derive the following important consequence.

Theorem 2.3. *Let $a < b$ and $f\colon [a,b] \to \mathbb{R}$ be continuous. Assume that f is differentiable except possibly at the points of a countable set $E \subset (a,b)$. Then there are points $c_1, c_2 \in E^c \cap (a,b)$ such that*

$$f'(c_1)(b-a) \le f(b) - f(a) \le f'(c_2)(b-a) \tag{2.5}$$

Proof. The result is obviously true if f is constant. Hence, assume that f is not constant. We shall establish the existence of c_2 such that

$$f(b) - f(a) \le f'(c_2)(b-a).$$

The existence of c_1 can be obtained in a similar manner.

In the case when

$$\sup\{f'(x)\colon x \in E^c \cap (a,b)\} = +\infty \tag{2.6}$$

the existence of c_2 is obvious.

Hence, assume that

$$\sup\{f'(x)\colon x \in E^c \cap (a,b)\} = M < \infty$$

and define $k(x) = \mathrm{M}(x-b) - f(x)$. Then k is continuous in $[a,b]$ and it is differentiable in $E^c \cap (a,b)$. Moreover, we have $k'(x) = \mathrm{M} - f'(x) \ge 0$. Assume first, that k is constant. Then $k(a) = M(a-b) - f(a) = k(b) = -f(b)$ and $M = f'(x)$ for every $x \in E^c \cap (a,b)$. Thus, we obtain

$$\frac{f(b) - f(a)}{b-a} = M = f'(c_2) \tag{2.7}$$

for some $c_2 \in (a,b)$. Hence,the right-hand-side inequality stated in the theorem is clearly true.

Let us now assume that k is not constant. Then $k'(x) > 0$ for $x \in N$ where $N \subset (a,b)$ is non-negligible. Moreover, $k(b) > k(a)$ (see Lemma 2.2). From

$$k(b) = -f(b)$$

and

$$k(a) = \mathrm{M}(a-b) - f(a)$$

we derive

$$\mathrm{M} > \frac{f(b) - f(a)}{b-a}, \tag{2.8}$$

and this inequality establishes the right hand side of (2.5). □

We are now in a position to prove the following result that has important applications to global asymptotic stability.

Theorem 2.4. *Let $F\colon \mathbb{R}^p \to \mathbb{R}^p$ be continuous. Assume that $F(\mathbf{0}) = \mathbf{0}$, and F is Gateaux differentiable except possibly on a linearly countable subset S of $\mathbb{R}^p$. Let $\mathbf{x} \in \mathbb{R}^p$ and $\mathbf{v} \neq \mathbf{0}$ be any vector of $\mathbb{R}^p$. Then there exist $\mathbf{c}_1, \mathbf{c}_2$ in the open line segment joining $\mathbf{0}$ with $\mathbf{x}$ such that*

$$\mathbf{v} \cdot F_G^{'}(\mathbf{c}_1)\ \mathbf{x} \le \mathbf{v} \cdot F(\mathbf{x}) \le \mathbf{v} \cdot F_G^{'}(\mathbf{c}_2)\ \mathbf{x}. \tag{2.9}$$

where $\cdot$ denotes the standard inner product.

Proof. Let $\mathbf{x}$ be any element of $\mathbb{R}^p$ different from $\mathbf{0}$ and define $g\colon [0,1] \to \mathbb{R}$ by $g(s) = \mathbf{v} \cdot F(\alpha(s))$ where $\alpha\colon [0,1] \to \mathbb{R}^p$ is given by $\alpha(s) = s\mathbf{x}$. The stated result is now a consequence of Theorem 2.3 (see also [27]). □

Our first result on global asymptotic stability considers the behavior of all orbits of continuous and autonomous dynamical systems.

Theorem 2.5. *Let $F\colon \mathbb{R}^p \to \mathbb{R}^p$ be a map that is locally Lipschitz at every $\mathbf{x} \in \mathbb{R}^p$, and Gateaux differentiable except possibly on a linearly countable set $S \subset \mathbb{R}^p$. Assume that $F(\mathbf{0}) = \mathbf{0}$ and the eigenvalues of the symmetric part $H_G(\mathbf{x})$ of $F'_G(\mathbf{x})$ are strictly negative at every point where $F'_G(\mathbf{x})$ exists. Then every solution of the autonomous system*

$$\mathbf{x}'(t) = F(\mathbf{x}(t)) \tag{2.10}$$

goes to $\mathbf{0}$ as t goes to ∞.

Proof. Let $\mathbf{x}_0$ be an initial condition for equation (2.10) and let $\mathbf{x}(t)$ be the unique solution of (2.10) such that $\mathbf{x}(0) = \mathbf{x}_0$. The assumption that F is locally Lipschitz at every $\mathbf{x} \in \mathbb{R}^p$ insures that the solution of (2.10) is uniquely determined by its initial value $\mathbf{x}_0$ and varies continuously with it [29]. According to Theorem 2.4 there exists $\mathbf{c}_2$ in the open line segment joining $\mathbf{x}$ with $\mathbf{0}$ such that

$$\frac{d}{dt}\|\mathbf{x}(t)\|^2 = 2\mathbf{x}(t)\cdot\mathbf{x}'(t) = 2\mathbf{x}(t)\cdot F(\mathbf{x}(t)) \le 2\mathbf{x}(t)\cdot F'_G(\mathbf{c}_2)\mathbf{x}(t). \tag{2.11}$$

Note that

$$2\mathbf{x}(t)\cdot F'_G(\mathbf{c}_2)\mathbf{x}(t) = \mathbf{x}(t)\cdot H_G(\mathbf{c}_2)\mathbf{x}(t).$$

Since the eigenvalues of $H_G(\mathbf{c}_2)$ are strictly negative we obtain that $\|\mathbf{x}(t)\|$ is decreasing. Hence, the solution exists for all $t > 0$ and we have

$$\lim_{t\to+\infty} \|\mathbf{x}(t)\| = a \ge 0. \tag{2.12}$$

Assume that $a > 0$. Then the Ω limit set of the solution is contained in the sphere $\{\mathbf{z} \in \mathbb{R}^p : \|\mathbf{z}\| = a\}$. This, however, is impossible, since it would imply the existence of a solution $\mathbf{w}$ of (2.10) defined in $[0, +\infty)$ and with the property $\|\mathbf{w}(t)\| = a$ for all $t \ge 0$ (see [2], pg. 227). □

We now provide an example showing that the conditions listed in Theorem 2.5 appear to be optimal, in the sense that the intersection between S and a segment of the form $[\mathbf{0}, \mathbf{x}]$ cannot be uncountable.

Example 2.6. Let $T\colon [0,1] \to [0,1]$ be the Cantor ternary function. Extend T to the entire real line in the following manner. For $x \in (1,2]$ define $F(x) = 1+T(x-1)$, and for $x \in (2,3]$ set $F(x) = 2 + T(x-2)$. For $x > 2$ the function F is defined in a similar manner. We then extend F to the entire real line so that the extension is odd. Shift F to the left by $\frac{1}{2}$ and lower it by $\frac{1}{2}$. Then, define $G\colon \mathbb{R} \to \mathbb{R}$ by $G(x) = F(x + \frac{1}{2}) - \frac{1}{2} - \epsilon x$ where $0 < \epsilon < 0.1$. Consider the differential equation

$$x'(t) = G(x(t)) \tag{2.13}$$

Clearly $G(0) = 0$ and G is continuous. Moreover $G'(x) < 0$ for every x where the derivative exists. In particular, when $x \in (-\frac{1}{6}, \frac{1}{6})$ we have $G(x) = -\epsilon x$. Hence, $x = 0$ is a local attractor. However, notice that the solution of the initial value problem

$$x'(t) = G(x(t)),\ x(0) = 1$$

does not go to 0. It should be pointed out that the function $G(x)$ is not locally Lipschitz at every point of an uncountable set of measure 0.

In the following theorem we establish a criterion similar to the one of the previous result and regarding discrete dynamical systems.

Theorem 2.7. *Let $F\colon \mathbb{R}^p \to \mathbb{R}^p$ be continuous. Assume that F is Gateaux differentiable except possibly on a linearly countable set S, $F(\mathbf{0}) = \mathbf{0}$, and the spectral radius $\rho(P_G(\mathbf{x}))$, of $P_G(\mathbf{x}) = F'_G(\mathbf{x})TF'_G(\mathbf{x})$, where $TF'_G(\mathbf{x})$ is the transpose of $F'_G(\mathbf{x})$, is less than 1 at every point where $F'_G(\mathbf{x})$ exists. Then every orbit of the system*

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n) \tag{2.14}$$

goes to $\mathbf{0}$ as n goes to $+\infty$.

Proof. Set $\mathbf{v} = F(\mathbf{x})$ in (2.9) to obtain

$$\|F(\mathbf{x})\|^2 \leq F(\mathbf{x}) \cdot F'_G(\mathbf{c}_2)\ \mathbf{x}.$$

By a well known inequality we have

$$F(\mathbf{x}) \cdot F'_G(\mathbf{c}_2)\ \mathbf{x} \leq \|F(\mathbf{x})\|\ \|F'_G(\mathbf{c}_2)\ \mathbf{x}\|.$$

It follows that

$$\|F(\mathbf{x})\| \leq \|F'_G(\mathbf{c}_2)\ \mathbf{x}\|,$$

and

$$\|F(\mathbf{x})\|^2 \leq \mathbf{x} \cdot TF'_G(\mathbf{c}_2)F'_G(\mathbf{c}_2)\mathbf{x}, \tag{2.15}$$

where $TF'_G(\mathbf{c}_2)$ is the transpose of $F'_G(\mathbf{c}_2)$. Let $\mathbf{x}_0$ be the starting point of an orbit. Equation (2.15) and the assumption $\rho(P_G(\mathbf{x})) < 1$ imply that

$$\|\mathbf{x}_{n+1}\| < \|\mathbf{x}_n\|, \tag{2.16}$$

for every $n = 0, 1, \ldots$. Hence, the sequence $\{\|\mathbf{x}_n\|\colon\ n = 0, 1, \ldots\}$ is decreasing. Let $L(\mathbf{x}_0)$ be the set of limit points of the orbit $O(\mathbf{x}_0)$. Since F is continuous we have (see [22]) $F(L(\mathbf{x}_0)) = L(\mathbf{x}_0)$. Therefore, we conclude that $\mathbf{0}$ is the only limit point of the orbit $O(\mathbf{x}_0)$, i.e. $L(\mathbf{x}_0) = \{\mathbf{0}\}$. □

The following example is an application of Theorem 2.7 to a discrete dynamical system in $\mathbb{R}^2$.

Example 2.8. Consider a discrete dynamical system $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that F is continuous and the equation

$$F(\mathbf{x}) = \mathbf{x}$$

is solved if and only if $\mathbf{x} = \mathbf{0}$. The function F fails to be differentiable on a linearly conuntable set S (see below). At every other point the entries of the 2×2 matrix $F'(\mathbf{x})$ are the following

$$\begin{aligned} a_{11} &= 0.4(1 - \cos 2\pi x) \\ a_{12} &= g(y) \\ a_{21} &= g(x) \\ a_{22} &= 0.4(1 - \cos 2\pi y). \end{aligned}$$

Consequently, the elements of the product of the derivative of F and of its transpose are

$$\begin{aligned} b_{11} &= \tfrac{(1-\cos 2\pi x)^2}{6.25} + (g(y))^2 \\ b_{12} &= 0.4(g(x)(1 - \cos 2\pi x) + g(y)(1 - \cos 2\pi y)) \\ b_{21} &= b_{12} \\ b_{22} &= \tfrac{(1-\cos 2\pi y)^2}{6.25} + (g(x))^2. \end{aligned}$$

The eigenvalues are found to be

$$\lambda_i = \frac{b_{11} + b_{22} + (-1)^i \sqrt{(b_{11} - b_{22})^2 + 4(b_{12})^2}}{2}, \; i = 1, 2. \tag{2.17}$$

Select $g(x) = \frac{(-1)^{[2x+0.5]}+1.5}{8}$, where the symbol $[x]$ denotes the greatest integer contained in x. Hence, the function $g(x)$ is discontinuous on the family of lines

$$L = \{x = \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \dots\}.$$

Analogously, $g(y) = \frac{(-1)^{[2y+0.5]}+1.5}{8}$ and the function is discontinuous on the family of lines

$$M = \{y = \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \dots\}.$$

It can be easily shown that $\lambda_i \in (-1, 1)$ at every point where F is differentiable. Moreover, we can select a function $G(x)$ so that G is continuous, $G'(x) = g(x)$ at every point where g is continuous, and $G(0) = 0$. A similar choice can be made for $G(y)$.

Hence, on the basis of Theorem 2.7, we conclude that every orbit of the discrete dynamical system

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n)$$

converges to $\mathbf{0}$.

The following example shows that the set S cannot be selected so that its intersection with a segment of the form $[\mathbf{0}, \mathbf{x}]$ is uncountable.

Example 2.9. Let F be the function defined in Example 2.6 and consider the discrete dynamical system

$$x_{n+1} = F(x_n + \frac{1}{2}) - \frac{1}{2} + \epsilon x_n = G(x_n) \tag{2.18}$$

Assume that $0 < \epsilon < 0.1$. Notice that $G(0) = 0$. Hence 0 is a stationary state of our system. Moreover, for every $x_0 \in (-\frac{1}{6}, \frac{1}{6})$ we have $G'(x) = \epsilon < 1$. Hence, the orbit starting at x_0 will converge to 0, which is a local attractor for the dynamical system (2.18).

Moreover, at every point where the derivative of G is defined we have $G'(x) = \epsilon < 1$. However, let us consider the orbit starting from $x_0 = 1$. Then

$$x_1 = G(x_0) = 1 + \epsilon x_0 = 1 + \epsilon$$

Hence, $x_0 < x_1$. Continue the process to show that the sequence of iterates is monotone and it does not converge to 0. Thus, 0 is not a global attractor.

The following result does not imply global asymptotic stability, but it is included in this paper because it appears to be the natural extension of Theorem 2.5 to discrete dynamical systems of the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + hF(\mathbf{x}_n). \tag{2.19}$$

Notice that (2.19) can be obtained with the application of the one-step Euler method to (2.10). The parameter h represents the time step used in the scheme.

Theorem 2.10. *Let $K\colon \mathbb{R}^p \to \mathbb{R}^p$ be defined by $K(\mathbf{x}) = \mathbf{x} + hF(\mathbf{x})$, where F is continuous in $\mathbb{R}^p$ and such that $F(\mathbf{0}) = \mathbf{0}$. Assume that F is Gateaux differentiable except possibly on a linearly countable set S, and the symmetric part of $F'_G(\mathbf{x})$ is negative definite. Then, given an initial position $\mathbf{x}_0$ there exists $h_0 = h(\mathbf{x}_0) > 0$ such that for any $k \in [0, h_0)$ the orbit $O(\mathbf{x}_0)$ of the dynamical system*

$$\mathbf{x}_{n+1} = \mathbf{x}_n + kF(\mathbf{x}_n) \tag{2.20}$$

converges to $\mathbf{0}$.

Proof. From (2.19) we easily obtain

$$\mathbf{x}_{n+1} \cdot \mathbf{x}_{n+1} = \mathbf{x}_n \cdot \mathbf{x}_n + 2h\mathbf{x}_n \cdot F(\mathbf{x}_n) + h^2 F(\mathbf{x}_n) \cdot F(\mathbf{x}_n).$$

Using Theorem 2.4 we can write

$$2\mathbf{x} \cdot F(\mathbf{x}) + hF(\mathbf{x}) \cdot F(\mathbf{x}) \leq 2\mathbf{x} \cdot F'_G(\mathbf{c}_2)\mathbf{x} + hF(\mathbf{x}) \cdot F(\mathbf{x})$$

where $\mathbf{c}_2$ is a suitable point in the line segment joining $\mathbf{x}$ with $\mathbf{0}$. Since

$$2\mathbf{x} \cdot F'_G(\mathbf{c}_2)\mathbf{x} = \mathbf{x} \cdot H_G(\mathbf{c}_2)\mathbf{x},$$

our assumptions on F imply that

$$\mathbf{x}_0 \cdot H_G(\mathbf{c}_{20})\mathbf{x}_0 < 0.$$

where $H_G(\mathbf{c}_{20})$ is the symmetric part of $F'_G(\mathbf{c}_{20})$, and $\mathbf{c}_{20}$ is a suitable point in the line segment joining $\mathbf{0}$ with $\mathbf{x}_0$. Let $Q = \max\{\|F(\mathbf{x})\|^2,\ \|\mathbf{x}\| \leq \|\mathbf{x}_0\|\}$ and choose h_0 so that

$$\mathbf{x}_0 \cdot H_G(\mathbf{c}_{20})\mathbf{x}_0 + h_0 Q = 0.$$

Then for every $k < h_0$ we have $\|\mathbf{x}_1\| < \|\mathbf{x}_0\|$. The procedure can now be repeated and the reasoning of Theorem 2.7 can be applied to conclude that $L(\mathbf{x}_0) = \{\mathbf{0}\}$. □

Recall that a result less general than Theorem 2.5 was established by Hartman (see [14]). Hartman's proof requires that F is (differentiable in the sense of Fréchet and) of class C^1 in $\mathbb{R}^p$ and the symmetric part of the Fréchet derivative $F'(\mathbf{x})$ is negative definite except possibly at $\mathbf{0}$. To the best of our knowledge, no results similar to Theorems 2.7 and 2.10 have been established for discrete dynamical systems.

Hartman and Olech ([15], see Theorem 2.1) proved that the $\mathbf{0}$ solution of (1.1) is globally asymptotically stable provided that

1. $F(\mathbf{x}) = \mathbf{0}$ if an only if $\mathbf{x} = \mathbf{0}$;
2. $\alpha(\mathbf{x}) \leq 0$ with
$$\alpha(\mathbf{x}) = \max\{\lambda_i(\mathbf{x}) + \lambda_j(\mathbf{x}) :\ 1 \leq i < j \leq p\},$$
where $\{\lambda_k(\mathbf{x}), k = 1, \ldots, p\}$ are the eigenvalues of the symmetric part of $F'(\mathbf{x})$;
3. $$\int_0^\infty p(u)\mathrm{d}u = \infty,$$
with $p(u) = \min\{\|F(\mathbf{x})\| \colon \|\mathbf{x}\| = u\}$.

Hartman's condition (see [14]) was weakened by Smith (see [30], Theorem 7) who proved that the bounded semi-orbits of (2.10) converge to $\mathbf{0}$ provided that

$$\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) < 0$$

where $\lambda_1(\mathbf{x})$ and $\lambda_2(\mathbf{x})$ are the two largest eigenvalues of the symmetric part of $F'(\mathbf{x})$.

We do not know if the result of Hartman-Olech or the one of Smith can be obtained or improved with the method outlined in Theorem 2.5.

Recall that when $F(\mathbf{0}) = \mathbf{0}$, $F'(\mathbf{x})$, the Fréchet derivative of F at $\mathbf{x}$, is continuous at $\mathbf{0}$ and the spectral radius of $F'(\mathbf{0})$ is smaller than 1, then $\mathbf{0}$ is a sink. In other words, there exists $r > 0$ such that every orbit starting at a point $\mathbf{x}_0$ closer than r to the origin will converge to $\mathbf{0}$ in an exponential manner. The proof can be based on the equivalence of all norms in $\mathbb{R}^p$, on the Mean Value Inequality, and on the fundamental result

$$\lim_{n\to\infty} \|A^n\|_0^{\frac{1}{n}} = \rho(A), \tag{2.21}$$

where A is a $p \times p$ matrix with real entries, $\|\cdot\|_0$ is any operator norm, and $\rho(A)$ is the spectral radius of A (see [22]). A proof of this result without using the differentiability of the function in a neighborhood of the origin, but still anchored on (2.21), was proposed in [18]. There are advantages and disadvantages in the approach of [22] with respect to the one of [18]. The interested reader can easily compare the two methods and their respective proofs.

3. Open Questions

We would like to mention that when $p = 1$, $f(0) = 0, |f^{'}(0)| = 1$ and $|f^{'}(x)| < 1$ for $0 < |x| < r$ one can easily prove that 0 is a sink (see [22]). As a matter of fact, the differentiability of f at 0 is not needed.

For $p > 1$ one could try to obtain a similar result by assuming that $F(\mathbf{0}) = \mathbf{0}$, $\rho(F'(\mathbf{0})) = 1$, and there exists $r > 0$ such that $\rho(F'(\mathbf{x})) < 1$ whenever $\|\mathbf{x}\| \in (0, r)$. However, such a result is still in the works if no other assumption is made about $F'(\mathbf{x})$. We suspect that the result is true when the eigenvalues of $F'(\mathbf{0})$ with modulus 1 are semisimple, but does not hold when at least one of these eigenvalues is not semisimple.

Another interesting question is the following. Suppose that $F(\mathbf{0}) = \mathbf{0}$, $\rho(F'(\mathbf{x})) < 1$ for all $\|\mathbf{x}\| \le 1$, and F maps the unit disk $D(\mathbf{0}, 1)$ into itself. Is it true that every orbit of F with initial point $\mathbf{x}_0 \in D(\mathbf{0}, 1)$ converges to $\mathbf{0}$?

A problem strictly related to the well-known Jacobian Conjecture (JC) regards the existence of non-zero fixed points for a C^1 map $F : \mathbb{R}^p \to \mathbb{R}^p$ such that $F(\mathbf{0}) = \mathbf{0}$ and $\rho(F'(\mathbf{x})) < 1$ for every $\mathbf{x} \in \mathbb{R}^p$. It has been shown [7] that when F is a polynomial vector field the uniqueness of the fixed point (Fixed Point Conjecture or FPC, for short) is equivalent to the well-known Jacobian Conjecture (JC) [17]. The proof of this equivalence (see [7]) is anchored to the very clever work of Bass, Connell and Wright [3]. Nothing is known when F is not a polynomial map, but it satisfies the conditions listed above.

There are others unsolved problems regarding convergence of continuous or discrete dynamical systems. The ones just mentioned provide interesting open questions about this area of research.

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