ON THE LONGITUDINAL LIBRATIONS OF HYPERION

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Abstract. We study a nonlinear, second order ordinary differential equation that models the longitudinal librations of the longest axis L of a satellite with respect to the planet-satellite center line C. Combining theoretical arguments and numerical evidence we prove that, in the case of Hyperion, a satellite of Saturn, the angle A between L and C can change in a chaotic manner at the moments when the distance from Hyperion to Saturn reaches its minimum value. More precisely, given an arbitrary sequence of zeros and ones, we show that there is at least one initial velocity of A such that its successive positions reproduce the given sequence.

1. Introduction

Hyperion is a satellite of Saturn. The irregular oscillations of its longest axis with respect to the planet-satellite center line have been studied previously by J. Wisdom, S. Peale, and F. Mignard [15]. Their conclusions were derived from:

1. Hyperion's images transmitted by Voyager 2 [13];
2. a numerical and theoretical analysis of a differential equation modeling a planetary motion and proposed by P. Goldreich and S. Peale [5].

The last two authors modified the equation developed by J. M. A. Danby [2] to model the longitudinal librations of the moon.

In the first part of their interesting paper Wisdom, Peale, and Mignard [15] provide numerical evidence that Hyperion's longitudinal librations are chaotic. They denote by φ the angle between Hyperion's longest diameter and the planet-satellite center line. Using the surface of section method, they plot the pairs $(\varphi, \dot{\varphi})$ when Hyperion crosses the *periapsis*, namely the point when the satellite is closest to Saturn. A large cloud of points is obtained, which dominates the 1/2 and 2 spin-orbit states.

We use a different approach, since our goal is to provide a mathematical explanation of the unpredictable behavior of the longitudinal librations of Hyperion. We assume that the spin axis of the satellite remains perpendicular to the orbit plane, and set $x = 2\varphi$ in the equation of motion proposed in [5, 15]. In this framework we identify x with a point mass moving without friction on S^1 with its lowest position (called *South Pole*) used as a reference and set equal to 0. The point mass is acted on by two forces. One, a sort of oscillating *gravitational force*, pushes it vertically toward the South Pole. The other gives the point mass a clockwise or counterclockwise rotational acceleration.

We consider x as decreasing or increasing according to whether the point mass travels clockwise or counterclockwise. Moreover, the positions on S^1 are given by

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$x \mod 2\pi$. Consequently, $x \equiv 0$ when the point mass crosses the South Pole. Moreover, we say that x moves from $-\pi$ to π (the *North Pole*), if it proceeds counterclockwise from its highest position to its lowest, and then to its highest again. All results of this paper are stated with these conditions.

Two properties of the motion of the point mass on S^1 are used to prove the unpredictability of the longitudinal librations of Hyperion. The first one, which is established theoretically, states that the South Pole can be reached at any selected time (see Theorem 3.7). The second one states that the time of arrival at the South Pole is crucial, since it dictates if the point mass will (see Theorem 4.2) or will not (see Theorem 4.3) gain enough energy to complete a full revolution around S^1. Accurate numerical estimates play a determinant role in the proof of this second property.

We include in this paper the terms neglected in [4] (powers of the eccentricity e with exponent 2 or higher), complicating considerably the differential equation governing the motion of the point mass. Despite this, we are able to show that the longitudinal librations of Hyperion can be chaotic.

Besides J. Wisdom, S. Peale, and F. Mignard [15], other authors, such as [1, 4, 7, 8, 9, 11, 12, 14, 17], have investigated problems similar to the one presented here.

2. The model

According to [5, 15], the motion of a tri-axial satellite describing an elliptical orbit around a planet, with spin axis perpendicular to the orbit plane, can be modeled by the second order nonlinear differential equation

$$\ddot{\varphi} + \ddot{\theta} = -\tfrac{3(B-A)}{2C}\left(\tfrac{a}{r}\right)^3 \sin 2\varphi. \tag{2.1}$$

The quantities involved in (2.1) are defined as follows (see [16]). The numbers $A < B < C$ are the principal moments of inertia of the satellite, with C the moment about the spin axis. The orbit is assumed to be a fixed ellipse with semimajor axis a, eccentricity e, instantaneous radius r, and polar angle θ. Recall that θ, measured counterclockwise and expressed in radians, is the angle between the planet-satellite center line and the major axis of the ellipse, oriented toward the periapsis of the orbit. The orientation of the satellite longest axis relative to the planet-satellite center line is specified by φ. Thus $\varphi + \theta$ is the angle between this axis and the longest diameter of the elliptical orbit. We make here the standard approximation that the center of mass of the satellite describes an elliptical orbit around the planet. Consequently, the distance r between the satellite and the planet is expressed by the following function of the angle θ:

$$r = \frac{p}{1 + e\cos\theta}, \tag{2.2}$$

where $p = a(1 - e^2)$ is the *parameter* of the elliptical orbit. The angle θ and the time t are related by Kepler's second law of planetary motion and by the standard initial condition $\theta(0) = 0$. Hence, we obtain the initial value problem:

$$\tfrac{1}{2} r^2 \dot{\theta} = c, \quad \theta(0) = 0, \tag{2.3}$$

where the constant c represents the instantaneous area swept by r.

Let us introduce the variable ψ such that (see [6])

$$r = a(1 - e\cos\psi).$$

The relation between θ and ψ is easily obtained. We find

$$\cos\theta = \frac{\cos\psi - e}{1 - e\cos\psi}.$$

Customarily, we set

$$\theta = \psi = 0$$

when the distance between the satellite and the planet is

$$r = \frac{p}{1+e},$$

namely, at the *periapsis* of the orbit. Moreover,

$$\theta = \psi = \pi$$

when the distance is

$$r = \frac{p}{1-e},$$

i.e. the position of the satellite is symmetric to the periapsis with respect to the origin. The relation between ψ and t is found to be (see [6])

$$\omega t = \psi - e\sin\psi,$$

where ω is the frequency of the elliptical motion.

Differentiating (2.2) and (2.3) with respect to t yields

$$\ddot{\theta} = -\tfrac{8c^2}{r^3}\tfrac{e\sin\theta}{a(1-e^2)}.$$

Hence, (2.1) takes the form

$$\ddot{\varphi} = (\tfrac{a}{r})^3(\tfrac{8c^2 e\sin\theta}{a^4(1-e^2)} - \tfrac{3(B-A)}{2C}\sin 2\varphi). \tag{2.4}$$

Define $x = 2\varphi$. Then (2.4) becomes

$$\ddot{x} = (\tfrac{a}{r})^3(\tfrac{16c^2 e\sin\theta}{a^4(1-e^2)} - \tfrac{3(B-A)}{C}\sin x).$$

Since we are interested in the motion of Hyperion, we use $e = 0.11$ ([10]) and $\frac{3(B-A)}{C} = 0.78$ ([15], pg. 138), which are good *approximations* of the characteristic parameters of this satellite. Moreover, we set $a = 1$ and we normalize the time so that the period of revolution is 2π. As an immediate consequence, we obtain

$$4c^2 = 1 - e^2 = 0.9879.$$

In this case the three variables θ, ψ and t coincide at $n\pi$, $n \in \mathbb{Z}$. In fact, geometric considerations show that the C^∞ functions $\eta(t) := \theta(t) - t$ and $\gamma(t) := \theta(t) - \psi(t)$ are 2π-periodic, odd, and vanish at $t = \pi$.

With the above settings we obtain the autonomous system of differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= 1.0372(1 + .11\cos\theta)^3(.44\sin\theta - .78\sin x), \\ \dot{\theta} &= 1.01843(1 + .11\cos\theta)^2, \end{aligned} \tag{2.5}$$

where the numbers 1.0372 and 1.01843 have been rounded up.

The initial value problem (2.3) can be written as

$$\dot{\theta} = 1.01843(1 + .11\cos\theta)^2, \quad \theta(0) = 0.$$

Thus, considering the unique solution $\theta(t)$ of this problem, the autonomous system (2.5) is transformed into the non-autonomous differential equation

$$\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x). \tag{2.6}$$

Proposition 2.2 below shows that this equation can be regarded as governing the motion of a point mass constrained on a vertical circle and acted on by two periodic forces of the same period: an oscillating gravitational force, and a forcing term acting clockwise and counterclockwise, alternatively. The proof of Proposition 2.2 is based on the following simple result.

Lemma 2.1. *Let $f(t)$ be a C^∞ odd function, periodic with period 2π and such that $f(-\pi) = f(\pi) = 0$. Then there exists an even C^∞ function $a(t)$ such that*

$$f(t) = a(t)\sin t.$$

Proof. It is known that $\sin t = k(t)t$, where $k(t)$ is a C^∞ even function such that $k(0) = 1$. Additionally, the C^∞ odd function $f(t)$ can be written in the form $f(t) = g(t)t$, where $g(t)$ is C^∞ and even. Hence, $f(t) = a_1(t)\sin t$, where $a_1(t) = \frac{g(t)}{k(t)}$ in the interval $(-\pi, \pi)$.

Similarly, one can write $\sin t = h(t)(t-\pi)$ with $h(\pi) = -1$, and $f(t) = d(t)(t-\pi)$. Define $a_2(t) = \frac{d(t)}{h(t)}$ in the interval $(0, 2\pi)$. Notice that $a_1(t) = a_2(t)$ in the interval $(0, \pi)$. Hence, we can define the C^∞ function

$$\delta(t) = \begin{cases} a_1(t), & t \in [0, \pi), \\ a_2(t), & t \in [\pi, 2\pi). \end{cases}$$

The C^∞ function $\delta(t)$ can now be extended in a periodic manner (of period 2π) to the entire real line. Its C^∞ extension is the function $a(t)$. □

As a consequence of Lemma 2.1 we obtain the following

Proposition 2.2. *The differential equation (2.6) can be rewritten in the form*

$$\ddot{x} = \alpha(t)\sin t - \beta(t)\sin x$$

where the functions $\alpha(t)$ and $\beta(t)$ are positive, even, C^∞, and 2π-periodic.

Proof. Recall that $\theta(t) = t + \eta(t)$, where $\eta(t)$ is 2π-periodic, odd, and vanishes at $t = \pi$. Thus, the existence of $\alpha(t)$ is established using Lemma 2.1. The function $\beta(t)$ is given by $.809016(1 + .11\cos\theta(t))^3$. □

3. Preliminary Results

Our goal is to investigate the behavior of the solutions of the non-autonomous differential equation (2.6) which, as pointed out in the previous section, is 2π-periodic.

We already mentioned that the dependent variable $x(t)$ denotes the location (measured in radians) of a point mass on S^1, with its lowest position set equal to 0 and referred to as the South Pole.

The methods and strategy we use to study (2.6) have similarities with the ones of [4]. There is, however, a significant difference since we do not approximate the eccentricity of Hyperion's orbit. To make the statements and proofs of the following sections more concise and transparent we now present some terminology, lemmata, propositions, theorems, and appropriate remarks.

The first lemma contains an important property of the product $x\ddot{x}$.

Lemma 3.1. *There exist open arcs A_W and A_E, centered respectively at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that $x \in A_W \cup A_E$ implies $x\ddot{x} < 0$.*

Proof. The readers can easily verify that

$$A_W = (-\pi + \arcsin \tfrac{22}{39}, -\arcsin \tfrac{22}{39}).$$

In fact, $\ddot{x}$ is positive as long as $-.44 - .78 \sin x > 0$. Notice that, when $-\pi < x < 0$ we have

$$-.78 \sin x > 0.$$

In a similar manner we obtain

$$A_E = (\arcsin \tfrac{22}{39}, \pi - \arcsin \tfrac{22}{39}),$$

and this completes the proof. □

In the sequel we will call A_W and A_E the *west arc* and the *east arc* respectively. We write $A_W = (\alpha_W, \beta_W)$ with $\alpha_W = -\pi + \arcsin \frac{22}{39}$, $\beta_W = -\arcsin \frac{22}{39}$. Likewise, we have $A_E = (\beta_E, \alpha_E)$ with $\alpha_E = \pi - \arcsin \frac{22}{39}$, $\beta_E = \arcsin \frac{22}{39}$. Moreover, the arc (α_W, α_E) containing π will be called the *north arc* and denoted by A_N, while the arc (β_W, β_E) that contains 0 will be called the *south arc* and denoted by A_S. Accordingly, the center points of the four arcs A_W, A_E, A_N and A_S will be called *West, East, North Pole* and *South Pole*, respectively.

We will need the following physically meaningful result. For completeness' sake we include its elementary proof. Given $x_0, x_1 \in \mathbb{R}$, $x_0 \neq x_1$, by $\overline{x_0 x_1}$ we shall denote the closed interval $[\min\{x_0, x_1\}, \max\{x_0, x_1\}]$.

Lemma 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f_- : \mathbb{R} \to \mathbb{R}$ and $f_+ : \mathbb{R} \to \mathbb{R}$ be continuous and such that*

$$f_-(x) \leq f(t,x) \leq f_+(x)$$

for all $(t,x) \in \mathbb{R}^2$. Let $x(t)$, $t \geq t_0$, be a solution of the differential equation $\ddot{x} = f(t,x)$ satisfying the initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$. Define the real functions k_- and k_+ by

$$k_\pm(x) = \frac{v_0^2}{2} + \int_{x_0}^{x} f_\pm(s)\, ds,$$

and, given $x_1 \neq x_0$, denote by u_- and u_+ the numbers

$$u_\pm = \min\left\{k_\pm(x) : x \in \overline{x_0 x_1}\right\}.$$

If $x_1 > x_0$, $v_0 > 0$, and $u_- > 0$, then $\dot{x}(t) \geq \sqrt{2u_-}$ for all $t \geq t_0$ such that $x_0 \leq x(t) \leq x_1$. In particular $x(t)$ reaches x_1 at some $t_1 \leq t_0 + (x_1 - x_0)/\sqrt{2u_-}$.

If $x_1 < x_0$, $v_0 < 0$, and $u_+ > 0$, then $\dot{x}(t) \leq -\sqrt{2u_+}$ for all $t \geq t_0$ such that $x_0 \leq x(t) \leq x_1$. In particular $x(t)$ reaches x_1 at some $t_1 \leq t_0 + (x_0 - x_1)/\sqrt{2u_+}$.

If $x_1 > x_0$, $v_0 > 0$, and $u_+ < 0$, then $x(t)$ does not reach x_1 before it stops.

If $x_1 < x_0$, $v_0 < 0$, and $u_- < 0$, then $x(t)$ does not reach x_1 before it stops.

Proof. Observe that

$$\frac{\dot{x}(t)^2}{2} = \frac{v_0^2}{2} + \int_{t_0}^{t} f(\tau, x(\tau))\dot{x}(\tau)\, d\tau, \quad \forall\, t \geq t_0, \tag{3.7}$$

as can be easily verified by differentiating both members of the equation and noticing that they coincide when $t = t_0$.

We have to examine four cases, and in any of them the initial velocity $v_0 = \dot{x}(t_0)$ is assumed to be nonzero. Therefore, it make sense to consider the maximal interval containing t_0 and contained in the relatively open subset $\{t \geq t_0 : \dot{x}(t) \neq 0\}$ of $[t_0, +\infty)$. This nonempty interval will be denoted by $[t_0, t_*)$, with $t_0 < t_* \leq +\infty$.

Consider the first case. That is, assume $x_1 > x_0$, $v_0 > 0$, and $u_- > 0$. From the equation (3.7) and the inequality $f_-(x) \leq f(t,x)$, given any $t \in [t_0, t_*)$, we obtain

$$\frac{\dot{x}(t)^2}{2} \geq \frac{v_0^2}{2} + \int_{t_0}^{t} f_-(x(\tau))\dot{x}(\tau)\, d\tau = \frac{v_0^2}{2} + \int_{x_0}^{x(t)} f_-(s)\, ds = k_-(x(t)).$$

Since $\dot{x}(t_0) > 0$, we get $\dot{x}(t) \geq \sqrt{2u_-}$ for all $t \in [t_0, t_*)$ such that $x(t) \leq x_1$, and the assertion follows in the first case.

The second case can be treated as the previous one, but taking into account that now $\dot{x}(\tau)$ is negative for all $\tau \in [t_0, t] \subset [t_0, t_*)$.

Consider now the third case. Namely, suppose $x_1 > x_0$, $v_0 > 0$, and $u_+ < 0$. Then, from the equation (3.7) and the inequality $f(t,x) \leq f_+(x)$, we get

$$\frac{\dot{x}(t)^2}{2} \leq \frac{v_0^2}{2} + \int_{x_0}^{x(t)} f_+(s)\, ds = k_+(x(t)), \quad \forall\, t \in [t_0, t_*).$$

Since $u_+ < 0$, there exists a point $\bar{x}$ in the open interval (x_0, x_1) such that $k_+(\bar{x}) < 0$. Thus, because of the above inequality, $x(t)$ cannot cross the point $\bar{x}$ before it stops.

The last case can be treated as the previous one, but using the fact that now $\dot{x}(\tau) < 0$, $\forall\, \tau \in [t_0, t]$. □

Remark 3.3. The equation (2.6) is of the type $\ddot{x} = f(t,x)$, with

$$f_-(x) \leq f(t,x) \leq f_+(x)$$

for all $(t,x) \in \mathbb{R}^2$, where

$$\begin{aligned} f_-(x) &= 1.0372(1 - \mathrm{sign}(-.44 - 0.78\sin(x)))^3(-.44 - 0.78\sin(x)),\\ f_+(x) &= 1.0372(1 + \mathrm{sign}(+.44 - 0.78\sin(x)))^3(+.44 - 0.78\sin(x)). \end{aligned}$$

Our goal is to show that the velocity $\dot{x}$ of the point mass at the South Pole never vanishes when the point mass arrives there after crossing with positive (negative) velocity the arc A_W (A_E). This important result is stated as a proposition.

Proposition 3.4. *Assume that t_0 is such that the point mass $x(t)$ is at the position α_W (or α_E) when $t = t_0$, and it travels counterclockwise (clockwise). Let $t_1 > t_0$ be the first time after t_0 such that the point mass reaches the South Pole. Then $\dot{x}(t) \neq 0$, $\forall\, t \in [t_0, t_1]$.*

Proof. We first give the proof when the South Pole is reached by the point mass while it travels with positive velocity.

With the notation of Lemma 3.2, let $x_0 = \alpha_W$ be the highest point of the west arc and let $x_1 = 0$ be the South Pole. Notice that the function

$$g(x) = \int_{x_0}^{x} f_-(s)\, ds$$

is increasing in the west arc (x_0, β_W), $f_-(s)$ being positive in this arc, and decreasing in (β_W, x_1). Computing $g(x_1)$ we get $g(x_1) = g(0) = .13355\ldots$. Since the initial velocity v_0 (i.e. the velocity of $x(t)$ when the point mass enters the west arc) cannot be negative, Lemma 3.2 shows that $x(t)$ reaches the South Pole with positive velocity without stopping. In fact, a numerical computation shows that $\dot{x}(t_1) > .5205$.

In the case when the point mass travels clockwise (i.e. with negative velocity) the computation is similar. With a procedure parallel to the one just used for the previous case, we arrive at the conclusion that the velocity of the point mass at the South Pole satisfies the inequality $\dot{x}(t_1) < -0.5205$.

Hence, in both cases, the velocity at the South Pole cannot be 0. $\square$

Notice that the differential equation (2.6) does not have any equilibrium points. However, the role of the North Pole for $x(t)$ is very similar to its role in the equation investigated in [4]. More precisely, we have the following result.

Proposition 3.5. *Let $t_0 \in \mathbb{R}$ be given. Then there exists v_0 such that the unique solution $x(t)$ of the initial value problem*

$$\begin{aligned}&\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x),\\ &x(t_0) = -\pi,\ \dot{x}(t_0) = v_0,\end{aligned}$$

will oscillate around the North Pole without ever entering A_W or A_E for $t > t_0$.

Proof. Given any v, denote by $x_v(t)$ the unique solution of the initial value problem

$$\begin{aligned}&\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x),\\ &x(t_0) = -\pi,\ \dot{x}(t_0) = v.\end{aligned} \tag{3.8}$$

Define I_W as the set of those velocities v such that $x_v(t)$ enters A_W before (or without) entering A_E. Likewise, we define I_E as the set of those values v such that $x_v(t)$ enters A_E before (or without) entering A_W. We need to show that $I_W \cup I_E \neq \mathbb{R}$.

The set I_W is nonempty since, as a consequence of Lemma 3.2 and Remark 3.3, $v \in I_W$ when $v > 0$ is large enough. Analogously $I_E \neq \emptyset$. Now observe that, given $v \in \mathbb{R}$, the solution $x_v(t)$ can enter the arcs A_W or A_E from the top only with strictly positive or strictly negative velocity respectively. Then, because of Proposition 3.4, after entering one of the two arcs, $x_v(t)$ will cross the South Pole before $\dot{x}$ changes sign. Therefore, continuity with respect to initial conditions implies that I_W and I_E are open. Thus, as these two sets are disjoint, their union cannot coincide with the entire real line. $\square$

Remark 3.6. With the notation of the proof of Proposition 3.5, given $v \in I_W \cup I_E$, according to Proposition 3.4 the point mass $x_v(t)$ will reach the South Pole (for the first time after t_0) with nonzero velocity.

Our next result is a consequence of the above remark.

Theorem 3.7. *Let t_0 be given and let $\tau > t_0$. Then there exists a velocity v_W^τ (v_E^τ) such that the solution $x(t)$ of the initial value problem*

$$\begin{aligned}&\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x),\\ &x(t_0) = -\pi,\ \dot{x}(t_0) = v_W^\tau\ (\dot{x}(t_0) = v_E^\tau)\end{aligned}$$

will reach with positive (negative) velocity the South Pole for the first time at the instant τ.

Proof. Let I_W and I_E be as in the proof of Proposition 3.5. Notice that $v \in I_W \cup I_E$ when $|v| \neq 0$ is large enough. Therefore, because of Proposition 3.5, the open set I_W has an unbounded component $(v_0, +\infty)$. Moreover v_0 does not belong to I_E, since this set is open. Consequently, v_0 satisfies the assertion of Proposition 3.5.

Given $v \in (v_0, +\infty)$, let $\tau(v) > t_0$ be the time of first arrival of $x_v(t)$ at the South Pole. From Remark 3.6 we derive that $\tau(v)$ depends continuously on v. Clearly $\tau(v) \to t_0$ as $v \to +\infty$ and, by continuity with respect to initial conditions, $\tau(v) \to +\infty$ as $v \to v_0$. A similar reasoning applies to I_E.

In both cases the assertion follows from the Intermediate Value Theorem. □

4. OVER OR NOT

The two results of this section deal with *going over* (see Theorem 4.2) or *not going over* (see Theorem 4.3) the top. We establish results similar to the ones proved in [4]. The procedure, however, is different.

The following definition will be used repeatedly.

Definition 4.1. We say that the point mass crosses or enters one of the arcs A_S, A_W, A_N, A_E or crosses one of the corners S, W, N, E *immediately after* a time t_0 when the event occurs for the first time after t_0, and between t_0 and the event its velocity either does not change sign, or it changes sign only inside A_N. In this case we require the point mass to enter and exit A_N from different border points.

The following result deals with going over the top.

Theorem 4.2. *Consider a solution $x(t)$ of the differential equation (2.6) which, for some instant $t_0 \in \mathbb{R}$, satisfies the following two conditions:*

- *$x(t_0)$ belongs to the north arc A_N;*
- *immediately after t_0 the point mass $x(t)$ crosses either A_W or A_E and, for some $n \in \mathbb{Z}$, arrives at the South Pole at time $t_1 = \frac{\pi}{2} + 2n\pi$ in the first case, or at time $t_1 = -\frac{\pi}{2} + 2n\pi$ in the second one.*

Then the point mass crosses the South Pole immediately after t_1. Hence, it crosses the South Pole a second time immediately after t_0.

Proof. First of all notice that the equation (2.6) is periodic of period 2π. Thus, if $x(t)$ is a solution, so is $y(t) := x(t + 2n\pi)$ for any $n \in \mathbb{Z}$, and we can suppose that t_1 is $\frac{\pi}{2}$ in the first case and $-\frac{\pi}{2}$ in the second.

Consider the first case. That is, assume that immediately after $t_0 < \frac{\pi}{2}$ the point mass crosses A_W and arrives at the South Pole at time $t_1 = \frac{\pi}{2}$. We shall establish that in this case the point mass arrives there with velocity greater than 2.271. Moreover, we will show that if the point mass arrives at the South Pole at time $t_1 = \frac{\pi}{2}$ with velocity greater that 2, then it will reach the west arc immediately after t_1. Consequently, because of Proposition 3.4, it will cross again the South Pole immediately after t_1.

The computations can be made by solving numerically the differential equation (2.6) with the following conditions: $x(\frac{\pi}{2}) = 0$, $\dot{x}(\frac{\pi}{2}) = v$, $v \in \mathbb{R}$. Given a *shooting velocity* v, the problem will be solved backward to show that the point mass arrives at the bottom with velocity greater than 2.271, and will be solved forward to show that with $v > 2$ the point mass will reach the west arc.

Let us start by shooting with $v = 2$ and solving the problem forward. Numerical computations show that in this case the highest point α_W of the west arc is reached (without stopping) at time $\bar{t} = 3.5956...$ with velocity $\bar{v} = 1.7054...$.

If we let the initial velocity v vary between 2 and 3, we observe that the point mass reaches the upper point α_W of the west arc with a velocity $\bar{v}$ which increases almost linearly with v. Here are some values obtained numerically (many other

values have been checked and all of them agrees with interpolation):
$v = 2.0 \mapsto \bar{v} = 1.7054...$,
$v = 2.1 \mapsto \bar{v} = 1.8449...$,
$v = 2.2 \mapsto \bar{v} = 1.9771...$,
$v = 2.3 \mapsto \bar{v} = 2.1039...$,
$v = 2.4 \mapsto \bar{v} = 2.2263...$,
$v = 2.5 \mapsto \bar{v} = 2.3454...$,
$v = 2.6 \mapsto \bar{v} = 2.4618...$,
$v = 2.7 \mapsto \bar{v} = 2.5761...$,
$v = 2.8 \mapsto \bar{v} = 2.6884...$,
$v = 2.9 \mapsto \bar{v} = 2.7994...$,
$v = 3.0 \mapsto \bar{v} = 2.9089...$.

Moreover, as a consequence of Lemma 3.2 and Remark 3.3, we obtain that if the point mass starts from the South Pole **at any time** with velocity greater than 3, then it will reach, without stopping, the west arc with velocity greater than 0.5386. Therefore, we may conclude that the solution of any initial value problem

$$\begin{aligned} &\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x),\\ &x(\tfrac{\pi}{2}) = 0, \quad \dot{x}(\tfrac{\pi}{2}) > 2 \end{aligned}$$

reaches α_W with positive velocity immediately after $t_1 = \frac{\pi}{2}$. Consequently, because of Proposition 3.4, it will continue until it crosses the South Pole immediately after t_1.

Now we want to show that if $x(t)$ enters the west arc with positive velocity and arrives at the South Pole at time $t_1 = \frac{\pi}{2}$, then $\dot{x}(t_1) > 2.271$. Therefore, as a consequence of the previous argument, $x(t)$ will continue without stopping until it crosses the South Pole a second time.

We shoot now with the following final conditions $x(\frac{\pi}{2}) = 0$, $\dot{x}(\frac{\pi}{2}) = v$, $v \in \mathbb{R}$, and we follow the solution backward. The entering velocity in the west arc is denoted by $\hat{v}$. We write below some values obtained numerically (many other values have been checked and all of them agrees with interpolation). We point out that exactly the same values of $\hat{v}$ can be obtained by solving (this time forward) the equation (2.6) with initial conditions $x(-\frac{\pi}{2}) = 0$, $\dot{x}(-\frac{\pi}{2}) = v$, and computing the velocity $\hat{v}$ of the solution when it reaches the highest point α_E of the east arc. In fact, if $x(t)$ is a solution of (2.6), so is $y(t) := -x(-t)$. It is not difficult to check, numerically, that if $x(-\frac{\pi}{2}) = 0$ and $\dot{x}(-\frac{\pi}{2}) < 2.2713$, then the initial energy is not sufficient to reach the point α_E immediately after $t_1 = -\frac{\pi}{2}$. For this reason we compute $\hat{v}$ for $v \geq 2.2714$.
$v = 2.2714 \mapsto \hat{v} = 0.0045...$,
$v = 2.3714 \mapsto \hat{v} = 0.4668...$,
$v = 2.4714 \mapsto \hat{v} = 0.7763...$,
$v = 2.5714 \mapsto \hat{v} = 1.0288...$,
$v = 2.6714 \mapsto \hat{v} = 1.2485...$,
$v = 2.7714 \mapsto \hat{v} = 1.4467...$,
$v = 2.8714 \mapsto \hat{v} = 1.6297...$,
$v = 2.9714 \mapsto \hat{v} = 1.8008...$,
$v = 3.0714 \mapsto \hat{v} = 1.9633...$,
$v = 3.1714 \mapsto \hat{v} = 2.1187...$,
$v = 3.2714 \mapsto \hat{v} = 2.2680...$.

We do not need to continue the computation for final velocities v greater than 3.2714. In fact, as a consequence of Lemma 3.2 and Remark 3.3, if the point mass enters the west arc with any positive velocity $\hat{v}$, it will arrive at the South Pole with velocity v greater than $\hat{v}$ (whatever is the time of arrival). Actually, as in the proof of Proposition 3.4, one has

$$\frac{v^2}{2} \geq \frac{\hat{v}^2}{2} + 0.13355\ldots .$$

Therefore, we conclude that if the point mass enters A_W with positive velocity and, for some $n \in \mathbb{Z}$, arrives at the South Pole at time $t_1 = \frac{\pi}{2} + 2n\pi$, then it will cross again the South Pole immediately after t_1.

Consider now the case in which immediately after $t_0 < -\frac{\pi}{2}$ the point mass crosses A_E and arrives at the South Pole at time $t_1 = -\frac{\pi}{2}$.

With a numerical procedure similar to that of the previous case, one can check that any solution $x(t)$ of (2.6) that reaches the South Pole at time $t_1 = -\frac{\pi}{2}$ immediately after crossing A_E will arrive there with velocity less than -1.698 (i.e. the speed is higher than 1.698 and clockwise). The computations can be made by solving backward the differential equation (2.6) with initial conditions $x(-\frac{\pi}{2}) = 0$, $\dot{x}(-\frac{\pi}{2}) = v$, $v \in \mathbb{R}$.

Again numerically, we can prove that the solution of any initial value problem

$$\begin{aligned} &\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x), \\ &x(-\tfrac{\pi}{2}) = 0, \quad \dot{x}(-\tfrac{\pi}{2}) < -1.65 \end{aligned}$$

crosses the South Pole immediately after $t_1 = -\frac{\pi}{2}$. Therefore, also in this last case the point mass will cross the South Pole a *second time* immediately after t_0. □

Our next result deals with not reaching the north arc A_N.

Theorem 4.3. *Consider a solution $x(t)$ of the differential equation (2.6) which, for some instant $t_0 \in \mathbb{R}$, satisfies the following three conditions:*

- *$x(t_0)$ belongs to the north arc A_N;*
- *$\dot{x}(t_0) = 0$;*
- *immediately after t_0 the point mass $x(t)$ crosses either A_W or A_E and, for some $n \in \mathbb{Z}$, arrives at the South Pole at time $t_1 = -\frac{\pi}{2} + 2n\pi$ in the first case, or at time $t_1 = \frac{\pi}{2} + 2n\pi$ in the second one.*

Then the point mass will not reach the north arc A_N immediately after t_1.

Proof. As in the proof of Theorem 4.2 we can set $t_1 = -\frac{\pi}{2}$ in the first case and $t_1 = \frac{\pi}{2}$ in the second one.

Assume first that the point mass starts at time $t_0 < -\frac{\pi}{2}$ from a point in A_N with zero velocity, and immediately after t_0 it crosses A_W and arrives at the South Pole at time $t_1 = -\frac{\pi}{2}$. We can show (numerically) that in this case the point mass will reach the South Pole with positive velocity less than 1.412.

Still numerically we can prove that the solution of any problem

$$\begin{aligned} &\ddot{x} = 1.0372(1 + .11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x), \\ &x(-\tfrac{\pi}{2}) = 0, \quad 0 < \dot{x}(-\tfrac{\pi}{2}) < 2.271 \end{aligned}$$

does not reach A_N immediately after t_1. Thus the assertion is proved in the first case.

Consider now the second case. We can show that the point mass will cross the South Pole with clockwise velocity less negative than -1.65 (i.e. with speed less than 1.65).

Moreover, we can show that the solution of any problem

$$\begin{aligned}&\ddot{x} = 1.0372(1+.11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x),\\ &x(\tfrac{\pi}{2}) = 0, \quad -1.698 < \dot{x}(\tfrac{\pi}{2}) < 0\end{aligned}$$

does not reach A_N immediately after $t_1 = \frac{\pi}{2}$. Thus the assertion is proved also in this last case. □

5. CHAOTIC BEHAVIOR

We are now ready to show that the librations in longitude of Hyperion can be chaotic. We first need some definitions.

Given $v \in \mathbb{R}$, let $x(t)$ be the solution of the initial value problem

$$\begin{aligned}&\ddot{x} = 1.0372(1+.11\cos\theta(t))^3(.44\sin\theta(t) - .78\sin x),\\ &x(0) = -\pi, \quad \dot{x}(0) = v.\end{aligned} \tag{5.9}$$

Definition 5.1. Whenever $x(t)$ crosses A_W with positive velocity or A_E with negative velocity we say that the crossing is a *significant event*. When the velocity is positive we identify the crossing with 1, and when it is negative with -1.

Denote by Σ all infinite *strings* with entries from $\{-1, 1\}$.

Definition 5.2. Given $\sigma \in \Sigma$ and $n \in \mathbb{N}$, we say that $x(t)$ is *n-compatible with σ* if it starts with at least n-significant events and the list of symbols associated to them coincides with the first n entries of σ, with their order preserved.

Definition 5.3. We say that $x(t)$ *realizes* $\sigma \in \Sigma$ if it is n-compatible with σ for every $n \in \mathbb{N}$.

Let $\sigma \in \Sigma$ and $n \in \mathbb{N}$ be given.

1. We denote by $I_n^\sigma \subset \mathbb{R}$ the set of initial velocities $v \in \mathbb{R}$ such that the corresponding solution of (5.9) is n-compatible with σ. Notice that continuity with respect to initial conditions implies that I_n^σ is open.
2. Given $v \in I_n^\sigma$, denote by $t_n^\sigma(v)$ the time when the solution of (5.9) with initial velocity v crosses the South Pole immediately after n-significant events. Proposition 3.4 and the continuous dependence on initial conditions imply that the function $t_n^\sigma \colon I_n^\sigma \to \mathbb{R}$ is continuous.

We are now ready to state and prove the main result of our paper.

Theorem 5.4. *Let $\sigma \in \Sigma$. Then there exists a solution of (5.9) that realizes σ.*

Proof. We shall determine a sequence $\{J_n\}$ of nonempty, bounded, open intervals such that for all $n \in \mathbb{N}$:

(a_n) $J_n \subset I_n^\sigma$;
(b_n) the closure of J_{n+1} is contained in J_n.

The two properties just mentioned imply that

$$J_\infty = \bigcap_{n \geq 1} J_n \neq \emptyset, \tag{5.10}$$

and a solution of (5.9) with initial velocity $v \in J_\infty$ realizes σ.

We now describe how to define $J_1,\ J_2$ and J_3 for a *specific* $\sigma \in \Sigma$. An induction procedure can be used to define J_n for all $n \in N$ so that (5.10) holds.

Without loss of generality we can assume that the first element of σ is 1. Notice that a point mass with a large and counterclockwise initial speed will first cross A_W. Analogously, if the initial speed is large enough and clockwise, the point mass will first cross A_E. Thus the open set I_1^σ is nonempty and bounded below. Therefore it contains a bounded interval $J_1 = (\omega_1, v_1)$ such that $\omega_1 \notin I_1^\sigma$. Continuity with respect to initial conditions implies that the unique solution of (5.9) with initial velocity ω_1 cannot have 1 or -1 as the first significant event. Thus this solution will always remain in A_N oscillating indefinitely. Hence, continuity with respect to initial conditions implies

$$\lim_{v \to \omega_{1+}} t_1^\sigma(v) = +\infty. \tag{5.11}$$

Suppose now that the second element of σ is different from the previous one, i.e. it is -1. The continuity of t_1^σ, equation (5.11) and Theorem 4.2 imply the existence of an initial velocity $v_2 \in J_1$ such that the corresponding solution of (5.9) crosses A_W a second time with no change in the sign of its velocity. Now observe that continuity with respect to initial conditions implies that any solution of (5.9) with initial velocity $v \in J_1$ close to ω_1 will stop inside A_N at some instant $t_0 \geq 0$ before the first significant event (recall that the solution of (5.9) with $v = \omega_1$ will oscillate indefinitely inside A_N). Thus, equation (5.11) and Theorem 4.3 imply the existence of an initial velocity $w_2 \in (\omega_1, v_2)$ such that the corresponding solution of (5.9) does not reach the north arc A_N immediately after the first significant event. Therefore, continuity with respect to initial conditions implies the existence of a solution of (5.9) with initial velocity $u_2 \in (w_2, v_2)$ which enters A_N (immediately after the first significant event) and then goes back crossing A_E with negative velocity. Consequently its second significant event is -1. This shows that the open set $I_2^\sigma \cap (u_2, v_2)$ is nonempty. Moreover, since this set is clearly strictly contained in (u_2, v_2), it contains an interval $J_2 = (u_2, \omega_2)$ with $\omega_2 \notin I_2^\sigma$. Continuity with respect to initial conditions implies that the solution of (5.9) with initial velocity ω_2 cannot have 1 or -1 as the second significant event. Thus, this solution, after the first significant event, will enter A_N and remain there oscillating indefinitely.

At this point, the situation is as follows:

(a_2) the interval $J_2 = (u_2, \omega_2)$ is contained in I_2^σ;
(b_2) the closure of J_2 is contained in J_1;
(c_2) ω_2 is the initial velocity of a solution of (5.9) that after the first significant event enters A_N and remains there oscillating indefinitely.

Because of continuity with respect to initial conditions, we have

$$\lim_{v \to \omega_{2-}} t_2^\sigma(v) = +\infty. \tag{5.12}$$

Let us assume that the third entry of σ is the same as the previous one, i.e. it is again -1. Observe that, because of (5.12) and condition (c_2), in J_2 we can find an initial velocity of a solution of (5.9) that satisfies the two assumptions of Theorem 4.2 for some instant of time t_0 after the first significant event and before the second (which, we point out, is labeled -1). Similarly, in J_2 we can find an initial velocity of a solution that satisfies the three assumptions of Theorem 4.3 for some instant of time between the first and the second significant events. Therefore, with a procedure analogous to the one described for the construction of J_2, we can select an open interval J_3 with the following properties:

(a_3) all velocities of the interval J_3 are 3-compatible with the string σ;
(b_3) the closure of J_3 is contained in J_2;
(c_3) one of the extremes of J_3 is the initial velocity of a solution of (5.9) that after the second significant event enters A_N and remains there oscillating indefinitely.

An induction argument can now be used to complete the proof. □

6. Open Problems

The analysis presented in Sections 3, 4 and 5 proves that the librations in longitude of Hyperion can be chaotic. However, we must admit that our understanding of the many questions raised by the motion of this satellite is still very limited. We list below two of these questions.

1. Let CH be the set of velocities such that the corresponding orbits are chaotic. Does CH have any interior points?
2. We have made the standard approximation that the center of mass of Hyperion describes an elliptical orbit and the spin axis of the satellite remains perpendicular to the orbit's plane. An interesting question can now be raised: is the chaotic behavior of Hyperion producing an irregular precession of its elliptical orbit?

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