ON THE CHAOTIC BEHAVIOR OF THE SATELLITE HYPERION

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1. Introduction

The behavior of certain satellites of the solar system can be regarded as chaotic. The most striking example is Hyperion, a satellite of the planet Saturn. The irregular oscillations of Hyperion's longest axis with respect to the planet-satellite center line have been studied previously by J. Wisdom, S. Peale, and F. Mignard [12]. The conclusions of the three authors are based on:

1. the images of Hyperion transmitted by Voyager 2 [10];
2. a numerical and theoretical analysis of a differential equation proposed P. Goldreich and S. Peale [4].

The last two authors modified the equation developed by J.M.A. Danby [2] to model the libration in longitude of the moon.

In the first part of their interesting paper, Wisdom, Peale, and Mignard [12] provide numerical evidence that Hyperion's longitudinal librations are chaotic. Using the surface of section method, they plot the pairs $(\phi, \dot{\phi})$, where ϕ is the angle between Hyperion's longest diameter and the planet-satellite center line, when Hyperion crosses the periapsis. A large cloud of points is obtained, which dominates the $p = 1/2$ and $p = 2$ spin-orbit states.

We use a different approach. We first assume, as done in [4, 12], that the spin axis of the satellite remains perpendicular to the orbit plane. Then, the terms of order e^2 or higher, where e is the eccentricity of Hyperion's elliptical orbit, are neglected in the second order nonlinear differential equation proposed in [4, 12]. The equation obtained in this way is transformed, with a change of variable, into an equivalent equation of a forced pendulum with a vertically oscillating pivot. The variable of interest becomes a time dependent angle $x(t)$, measured in radians. The reference position is the one with minimum potential energy, that we set to 0. We regard $x(t)$ as decreasing or increasing according to whether the pendulum travels clockwise or counterclockwise. Moreover, the positions assumed by the pendulum are always given by $x(t) \mod 2\pi$. Consequently, we can write $x(t) = 0$ when the pendulum crosses its lowest position, and we say that the pendulum moves from $-\pi$ to π, if it proceeds counterclockwise from its highest position to its lowest, and then to its highest again.

We consider any non constant and infinite string S of the two symbols $-1, 1$, and we prove that there exists at least one orbit of the pendulum that *represents* (or *is associated to*) S in the following sense. The symbol -1 is used whenever the position $\frac{\pi}{2}$ is crossed with negative (i.e. clockwise) velocity. Likewise, the symbol 1 is used whenever the symmetric position, $-\frac{\pi}{2}$, is crossed with positive (i.e. counterclockwise) velocity. The orbit represents S if it crosses the two positions in the order expressed by S and with velocity of the required sign (see Remark 3.1). We regard the orbit as chaotic if the string S is an irrational number of $[0, 1]$, written in base 2, when -1 is replaced by 0. Finally, using the equivalence mentioned in the previous paragraph, we conclude that the proposed approximate model indicates, although it does not prove, why the planar oscillations of Hyperion are chaotic.

In a future paper we plan to study the full equation (see (2.5) in the next section) governing the motion of Hyperion and other satellites. We would like to mention that problems similar to the one presented here have been investigated by other authors, besides J. Wisdom, S. Peale, and F. Mignard [12]. The following list mentions simply the ones we had the chance to peruse: [1, 5, 6, 7, 8, 9, 11, 13].

2. The model

According to [4, 12], the motion of a tri-axial satellite describing an elliptical orbit around a planet, with spin axis perpendicular to the orbit plane, can be modeled by the second order nonlinear differential equation

$$\ddot{\phi} + \ddot{\theta} = -\tfrac{3(B-A)}{2r^3C} \sin 2\phi. \tag{2.1}$$

The quantities involved in (2.1) are defined as follows. The numbers $A < B < C$ are the principal moments of inertia of the satellite, with C the moment about the spin axis. The orbit is assumed to be a fixed ellipse with semimajor axis a, eccentricity e, instantaneous radius r, and polar angle θ. Recall that θ, measured counterclockwise and expressed in radians, represents the angle between the planet-satellite center line and the major axis of the ellipse, oriented toward the periapsis of the orbit.

Without loss of generality we can assume $a = 1$. Therefore, the relation between r and θ is given by

$$r = \tfrac{1-e^2}{1+e\cos\theta}. \tag{2.2}$$

The orientation of the satellite longest axis relative to the planet-satellite center line is specified by ϕ. Thus $\phi + \theta$ measures the orientation of this axis relative to the longest diameter of the elliptical orbit. The time is rescaled so that the period T of the satellite revolution around the planet is 2π. According to Kepler's second law of planetary motion we have

$$\tfrac{1}{2} r^2 \dot{\theta}(t) = c, \tag{2.3}$$

where c represents the instantaneous area swept by r. Taking into account the equalities $a = 1$ and $T = 2\pi$, we obtain $2c = \sqrt{1-e^2}$. Differentiating (2.2) and (2.3) with respect to t we get

$$\ddot{\theta}(t) = -\tfrac{2e\sin\theta}{r^3}. \tag{2.4}$$

Hence, (2.1) takes the form

$$\ddot{\phi}(t) = \tfrac{2e\sin\theta}{r^3} - \tfrac{3(B-A)}{2r^3C} \sin 2\phi. \tag{2.5}$$

The variable ϕ can be replaced by $\frac{\psi}{2}$ to obtain

$$\ddot{\psi}(t) = \tfrac{4e\sin\theta}{r^3} - \tfrac{3(B-A)}{r^3C} \sin\psi. \tag{2.6}$$

When terms of order e^2 or higher are neglected, (2.6) becomes

$$\ddot{x}(t) = 4e\sin t - k^2(1 + 3e\cos t)\sin x(t) \tag{2.7}$$

where $k^2 = \frac{3(B-A)}{C}$. This is the equation we shall investigate. Since our interest is focused on studying the motion of Hyperion, we shall use $e = 0.11$ and $k^2 = 0.78$, which are the characteristic parameters of this satellite.

3. Preliminary Results

Our readers will notice that the strategy, used in this section and in the next, to study (2.7), is similar to the one used in [3]. There are, however, some significant differences. We mention three.

1. In this paper the variable of interest is the initial velocity of the orbit, while in [3] was the initial position.
2. The solutions of (2.7) are affected by the forcing term $4e \sin t$.
3. Equation (2.7) does not have equilibrium points.

To make the statements and proofs of Sections 3 and 4 more concise and transparent we first introduce some terminology and present three important remarks.

A function $f\colon \mathbb{N} \to \Sigma$, where Σ is a set of symbols, is called a *sequence* or an *infinite string* (in Σ); it is usually identified with its ordered image $\{f(1), f(2), \dots\}$. Likewise, a function $g\colon \{1, 2, \dots, m\} \to \Sigma$ is called a *finite string* or an *m-string* (in Σ); it is identified with its ordered image $\{g(1), g(2), \dots, g(m)\}$. Therefore, a sequence is always infinite, while a string can be finite or infinite. We shall use the term *string*, without adding the qualification *finite* or *infinite*, unless the clarity of the discussion requires otherwise.

Remark 3.1. Let $\Sigma = \{-1, 1, \omega\}$. We consider all finite or infinite strings in Σ of the following form. The last entry of every finite string is ω. The remaining entries of a finite string and all entries of an infinite string are taken from the subset $\Sigma_0 = \{-1, 1\}$.

Let S be an infinite string of the type just described. We say that the solution $x(t)$ of the initial value problem

$$\begin{aligned} &\ddot{x}(t) = 0.44 \sin t - 0.78(1 + 0.33 \cos t) \sin x(t) \\ &x(0) = -\pi, \dot{x}(0) = v \end{aligned} \tag{3.8}$$

represents (or *is associated to*) S, if $x(t)$ crosses the positions $\pm\frac{\pi}{2}$ in the manner described in Section 1, and according to the order in which the entries of S are listed.

Let S be an m-string of the type considered above. We say that the solution $x(t)$ of (3.8) *represents* (or *is associated to*) S, if $x(t)$ crosses $m-1$ times the positions $\pm\frac{\pi}{2}$ as described in Section 1, and according to the order in which the first $m-1$ entries of S are listed, and then oscillates indefinitely around the top position.

Finally, let S be any m-string in Σ_0. We say that the solution $x(t)$ of (3.8) *realizes* S, if the first m crossings of $x(t)$ take place according to the order in which the entries of S are listed.

The reader should notice the difference between *representing* and *realizing*.

Remark 3.2. It is easy to see that by selecting a sufficiently high and positive or negative initial velocity v, the solution of (3.8) will cross as many times as required the positions $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ with positive or negative velocity, respectively.

Remark 3.3. Lemmas and theorems of Sections 3 and 4 are stated and proved for solutions of (2.7) crossing the bottom position while traveling counterclockwise (i.e. with strictly positive velocity). Since (2.7) is equivariant under the symmetry

$x(t) \to -x(-t)$, all statements and proofs remain valid when the bottom position is crossed clockwise.

We are now ready to present the results of this section. The first lemma deals with an interesting property of $\ddot{x}(t)$.

Lemma 3.4. *Let $x(t)$ be a solution of (2.7). Then there exist open arcs A_1 and A_2, centered respectively at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that $x(t) \in A_i$, $i = 1, 2$, implies $x(t)\ddot{x}(t) < 0$.*

Proof. It is sufficient to consider the arcs A_i, $i = 1, 2$, of those angles α such that

$$0.44 \sin t < 0.78(1 + 0.33 \cos t) \sin \alpha \tag{3.9}$$

for every t. An easy computation shows that the choice $A_1 = (-\pi + \beta, -\beta)$ and $A_2 = (\beta, \pi - \beta)$, with $\beta = \arcsin \frac{0.44}{0.78\sqrt{1-0.1089}} \approx 0.640477$, satisfies the condition of the lemma. □

Notice that (2.7) does not have any equilibrium point. However, the role of the top vertical position in the present situation is very similar to its role in the equation without the forcing term $0.44 \sin t$ investigated in [3]. More precisely, we have the following result.

Lemma 3.5. *Let $t_0 > 0$ be given. Then there exists a velocity a such that the solution of*

$$\begin{aligned} &\ddot{x}(t) = 0.44 \sin t - 0.78(1 + 0.33 \cos t) \sin x(t) \\ &x(t_0) = -\pi, \ \dot{x}(t_0) = a \end{aligned} \tag{3.10}$$

will oscillate indefinitely around the top position without entering the arcs A_1 or A_2 for $t > t_0$.

Proof. Given any velocity v, denote by $x_v(t)$ the solution of

$$\begin{aligned} &\ddot{x}(t) = 0.44 \sin t - 0.78(1 + 0.33 \cos t) \sin x(t) \\ &x(t_0) = -\pi, \ \dot{x}(t_0) = v. \end{aligned} \tag{3.11}$$

Define V_1 as the set of those velocities v such that $x_v(t)$ enters A_1 before (or without) entering A_2. The set V_2 is defined in a similar manner. We need to show that $V_1 \cup V_2 \neq \mathbb{R}$.

Since the arcs A_1 and A_2 are open, continuity with respect to initial conditions implies that the sets V_1 and V_2 are open as well. Thus, as these two sets are disjoint, their union cannot coincide with the entire real line. □

Remark 3.6. With the notation of the proof of Lemma 3.5, we observe that, given $v \in \mathbb{R}$, the solution $x_v(t)$ of (3.11) can enter the arcs A_1 or A_2 from the top only with strictly positive or strictly negative velocity, respectively. Then, because of Lemma 3.4, $x_v(t)$ will cross either arc before its velocity changes sign. Therefore, V_1 and V_2 coincide with those values of v such that $x_v(t)$ realizes a string ending with 1 (if $v \in V_1$) or -1, respectively. The solution $x_a(t)$ of Lemma 3.5 represents a string ending with ω. Moreover, numerical estimates based on comparison systems (see [3] for a similar strategy) insure that if $v \in V_1 \cup V_2$, the solution $x_v(t)$ will reach the bottom position, after crossing the arcs A_1 or A_2, with nonzero velocity. Therefore the time $\tau(v)$ of the first arrival at the bottom depends continuously on $v \in V_1 \cup V_2$.

Lemma 3.7. *Let t_0 be given and let $T > t_0$. Then there exists a velocity v_T such that the solution $x(t)$ of (3.11), with $v = v_T$, will reach the bottom position for the first time at the instant T.*

Proof. Let V_1 be as in the proof of Lemma 3.5 and, given $v \in V_1$, denote by $x_v(t)$ the solution of (3.11). Because of Lemmas 3.4 and 3.5, the open set V_1 has an unbounded component $(a, +\infty)$, where a is such that $x_a(t)$ oscillates indefinitely around the top position without ever entering the arcs A_1 or A_2 for $t > t_0$. Given $v \in (a, +\infty)$, let $\tau(v) > t_0$ be the time of first arrival of $x_v(t)$ at the bottom position. As pointed out in Remark 3.6, $\tau(v)$ depends continuously on v. Clearly $\tau(v) \to t_0$ as $v \to +\infty$ and, by continuity with respect to data, $\tau(v) \to +\infty$ as $v \to a$. The assertion now follows from the intermediate value theorem. □

The next two theorems deal with *going over* or *not going over* the top. We establish, in this case, results similar to the ones proved in [3] for a pendulum with variable length but without a forcing term. The times of arrival at the bottom position are different from the ones considered in [3]. In fact, we analyze the behavior of solutions that reach the bottom position at time $t = (n + \frac{1}{2})\pi$ where n is an integer, and study the differences between the cases when n is even (see Theorem 3.8) and the ones when n is odd (see Theorem 3.9).

Theorem 3.8. *Let $x(t)$ be a solution of (2.7) such that:*

1. *$x(t_0) = -\pi$ for some $t_0 \in [0, \infty)$;*
2. *there exists a positive even integer n such that $t_1 := (n + \frac{1}{2})\pi > t_0 + \pi$, $x(t_1) = 0$, and $x(t) \neq x(t_1)$ for every $t \in [t_0, t_1)$;*
3. *$x(t) \notin A_2$ for every $t \in (t_0, t_1)$;*
4. *$x(t)$ enters A_1 from the top during the time interval (t_0, t_1).*

Then $x(t)$ goes over the top in the sense that the positions $-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ and again $-\frac{\pi}{2}$ are crossed successively by $x(t)$ while the velocity $\dot{x}(t)$ remains positive.

Proof. Consider the separatrix

$$y(t) = 2 \arcsin \tanh(\sqrt{0.78}t)$$

of the differential equation

$$\ddot{u}(t) + 0.78 \sin(u(t)) = 0. \tag{3.12}$$

It can be shown that at the bottom position, the non-constant energy

$$E(t) = \tfrac{\dot{x}(t)^2}{2} + 0.78(1 - \cos x(t)) \tag{3.13}$$

of $x(t)$ exceeds the constant energy of $y(t)$ by at least $7e = 0.77$. To establish this result we first multiply both sides of (2.7) by $\dot{x}(t)$, and then integrate in $[t_0, t_1]$. In evaluating the integral we take into account Lemma 3.1 of [3] and the following three properties:

1. $x((n - \frac{1}{2})\pi) \in (-\pi, -\frac{\pi}{2})$ and $\dot{x}(t) \geq 0$ for $t \in ((n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi)$;
2. as long as $x(t)$ oscillates around the top position its energy changes are very small;
3. if the forcing term $0.44 \sin(t)$ is cancelled at any time t when $\dot{x}(t) \geq 0$ and $x(t) \in (-\pi, 0)$, the acceleration will become positive and the solution will never cross again the top position before reaching the bottom position.

The three properties are obtained by means of comparison systems analogous to the ones used in [3].

During the time interval $[(n+\frac{1}{2})\pi, (n+1)\pi]$ we have $\ddot{y}(t-(n+\frac{1}{2})\pi) \leq \ddot{x}(t)$. Hence, $y(\frac{\pi}{2}) \leq x((n+1)\pi)$, and the energy of $x(t)$ still exceeds the energy of the separatrix by at least $7e = 0.77$. It follows that the energy needed by $x(t)$ to reach the top position when $t = (n+1)\pi$ does not exceed $(1 - \cos(2 \arcsin(\tanh \sqrt{0.78}\frac{\pi}{2})))e \approx 0.44213e < e = 0.11$. Thus, $x(t)$ will arrive there with an energy excess of at least $6e = 0.66$. This excess gives the solution a positive velocity, which is sufficiently large to push the pendulum down to the left and through A_1, regardless of the time the top position is crossed. □

Theorem 3.9. *Let $x(t)$ be a solution of (2.7) such that:*

1. $x(t_0) = -\pi$ *for some* $t_0 \in [0, \infty)$*;*
2. *there exists a positive odd integer n such that* $t_1 := (n+\frac{1}{2})\pi > t_0 + \pi$, $x(t_1) = 0$, *and* $x(t) \neq x(t_1)$ *for every* $t \in [t_0, t_1)$*;*
3. $x(t) \notin A_2$ *for every* $t \in (t_0, t_1)$*;*
4. $x(t)$ *enters* A_1 *from the top during the time interval* (t_0, t_1)*.*

Then there exists $t_2 > t_1$ such that

i. $x(t) < \pi$ *for every* $t \in [t_1, t_2]$*;*
ii. $\dot{x}(t) > 0$ *for every* $t \in [t_1, t_2)$*;*
iii. $\dot{x}(t)$ *changes sign at* t_2*.*

Proof. We shall follow the pattern of the previous proof, but with less details. One can show that the non-constant energy of $x(t)$ at the bottom position is less than the constant energy of the separatrix $y(t) = 2\arcsin\tanh(\sqrt{0.78}t)$ by at least $6e = 0.66$. Moreover, during the time interval $[(n+\frac{1}{2})\pi, (n+1)\pi]$ we have $\ddot{y}(t-(n+\frac{1}{2})\pi) \geq \ddot{x}(t)$. Hence, $y(\frac{\pi}{2}) \geq x((n+1)\pi)$, and the energy of the separatrix still exceeds the energy of $x(t)$ by at least $6e = 0.66$. Since, at $t = (n+1)\pi$, the energy needed to reach the top position is larger than $0.44e = 0.0484$ (see Theorem 3.8), the velocity $\dot{x}(t)$ will change sign before the solution can reach it. □

4. CHAOTIC BEHAVIOR

The proof of our main result will now be patterned after [3]. We shall use the lemmas and theorems of the previous section, as well as the following two lemmas. The first one is an easy consequence of the continuity with respect to initial conditions. The statement makes use of the terminology introduced in Remark 3.1. The proof of the lemma is omitted.

Lemma 4.1. *Let S be a string of the symbols $-1, 1$ with exactly m entries. Consider the set V of initial velocities v such that the solution $x_v(t)$ of the initial value problem*

$$\begin{aligned} &\ddot{x}(t) = 0.44\sin t - 0.78(1 + 0.33\cos t)\sin x(t) \\ &x(0) = -\pi,\ \dot{x}(0) = v \end{aligned} \tag{4.14}$$

realizes S. Then V is an open set of the real line. Moreover, if S is not constant, then V is bounded.

Notice that Lemma 4.1 does not imply that the set V is not empty. This property of V will be evident from the proof of Theorem 4.3.

Before discussing our next lemma we need some preliminary definitions. Given a string S with exactly m entries from the symbols -1 and 1, we consider three different types of strings S_1, S_{-1}, and S_ω. The first two have exactly $m+1$ entries and are such that the first m coincide with the entries of S, while the last entry is 1 or -1 respectively. The symbol S_ω denotes a string with exactly m entries. The first $m-1$ coincide with the corresponding entries of S. The last entry is ω. Given an interval $I = [a_0, b_0]$, with $a_0 < b_0$, and a string S as above, we shall say that S is *fully realized* by I if

1. for every $v \in (a_0, b_0)$ the solution $x_v(t)$ of the initial value problem (4.14) realizes S (see Remark 3.1);
2. the solution of the initial value problem

$$\begin{aligned} &\ddot{x}(t) = 0.44 \sin t - 0.78(1 + 0.33 \cos t) \sin x(t) \\ &x(0) = -\pi,\ \dot{x}(0) = a_0 \end{aligned} \tag{4.15}$$

   represents S_ω;
3. the same property holds when a_0 in (4.15) is replaced with b_0.

With this terminology in mind we are ready to state and prove the second lemma of this section.

Lemma 4.2. *Let S be a string with exactly m entries from the symbols $-1, 1$ and let $I = [a_0, b_0]$, $a_0 < b_0$, be such that S if fully realized by I. Then S_1 is fully realized by at least one subinterval $J \subset I$, $J = [c_0, d_0]$, $c_0 < d_0$, and an analogous result holds for S_{-1}.*

Proof. The proof is based on Lemma 3.7 and Theorems 3.8 and 3.9 of the previous section. The idea is straightforward. In fact, close to the point a_0 (or b_0) and inside the interval I, we can find a velocity v_1 such that the corresponding solution $x_1(t)$ of (4.14), with v replaced by v_1, will reach the bottom position at time $t_1 = (n_1 + \frac{1}{2})\pi$, where n_1 is an odd positive integer. Likewise, we can find v_2 such that $x_2(t)$ will reach the bottom position at time $t_2 = (n_2 + \frac{1}{2})\pi$, where n_2 is an even positive integer. □

We are now ready to state and prove the main result of our paper.

Theorem 4.3. *Let S be a sequence of the symbols $-1, 1$ and assume that S is not constant. Then there exists $a \in \mathbb{R}$ such that the solution of the initial value problem*

$$\begin{aligned} &\ddot{x}(t) = 0.44 \sin t - 0.78(1 + 0.33 \cos t) \sin x(t) \\ &x(0) = -\pi,\ \dot{x}(0) = a \end{aligned} \tag{4.16}$$

represents the string S.

Proof. We consider first those cases when the two beginning entries of S are different, and, without loss of generality, we assume that S starts with 1. Let S_m be the string of the first m entries of S. To obtain the desired result we will produce a family of closed nested intervals $I_m = [a_m, b_m]$, $a_m < b_m$, such that S_m is fully realized by I_m. Since $\cap_{m=1}^{\infty} I_m = I_\infty \neq \emptyset$, any solution of (4.16) with $a \in I_\infty$ represents S.

Without loss of generality we may assume that $S = \{1, -1, 1, \dots\}$. Consider a connected component C of V_1 (see Remark 3.6). From the comments made at the beginning of Section 3 (see Remark 3.2) one can easily derive that C is not empty and bounded below. Let a_0 be its greatest lower bound. Then the solution of (4.15) represents the 1-string $\{\omega\}$. With a straightforward argument based on

Lemma 3.7, Theorem 3.8, and Theorem 3.9, we can prove that C contains a velocity v such that the solution $x_v(t)$ of (4.14) realizes the 2-string $\{1, -1\}$. We denote with I_1 the bounded and closed interval contained in C, such that $v \in I_1$, and $S_1 = \{1, -1\}$ is fully realized by I_1. Now let $S_2 = \{1, -1, 1\}$. By Lemma 4.2 the string $S_2 = \{1, -1, 1\}$ is fully realized by at least one closed subinterval of I_1, and the proof can be completed with an induction argument.

Those cases when S starts with $q > 1$ entries of the same sign can be handled in a similar manner with a suitable definition of the set C. □

5. Conclusion

At this point we would like to say one more time that the analysis presented in this paper indicates, although it does not prove, why the librations in longitude of Hyperion are chaotic. The proof will be presented in the study, to appear in a forthcoming paper, of the full nonlinear model (2.6).

However, we frankly admit that our understanding of the many questions raised by both the simplified model investigated here, and the full nonlinear model to be analyzed later, is still very limited. We list below four of these questions.

1. Is there an initial velocity $w > 0$ such that $\dot{x}_w(t) > 0$, for every $t > 0$, where $x_w(t)$ is the solution of the initial value problem

$$\begin{aligned} &\ddot{x}(t) = 0.44 \sin t - 0.78(1 + 0.33 \cos t) \sin x(t) \\ &x(0) = -\pi, \ \dot{x}(0) = w? \end{aligned} \tag{5.17}$$

2. Is the energy of every solution of (2.7) bounded?
3. Let CH be the set of velocities such that the corresponding orbits are chaotic. Does CH have any interior points?
4. Is the chaotic behavior of Hyperion producing an irregular precession of its elliptical orbit?

We sincerely hope that some of our readers will further explore the intricacies of these simple, yet fascinating problems.

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