ATYPICAL BIFURCATION WITHOUT COMPACTNESS

P. BENEVIERI, M. FURI, M. MARTELLI, AND M.P. PERA

1. Introduction

We consider the semilinear operator equation

$$Lx + \lambda h(\lambda, x) = 0 \tag{1.1}$$

where $L : E \to F$ is a linear Fredholm operator of index zero between real Banach spaces and $h : \Omega \to F$ is a continuous map defined in a simply connected open subset Ω of $\mathbb{R} \times E$. We assume that, for any $\lambda \in \mathbb{R}$, the partial map $x \mapsto Lx + \lambda h(\lambda, x)$ is a nonlinear Fredholm map of index 0 on the (possibly empty) section

$$\Omega_\lambda = \{x \in E : (\lambda, x) \in \Omega\}.$$

In addition we require the continuity of the map

$$(\lambda, x) \mapsto \partial_2 h(\lambda, x),$$

where the symbol $\partial_2 h(\lambda, x)$ denotes the partial Fréchet derivative of h with respect to the second variable at the point (λ, x).

The set of *trivial solutions* of (1.1) is obtained when $\lambda = 0$. It coincides with $(\{0\} \times \operatorname{Ker} L) \cap \Omega$, assumed to be not empty. One of the problems related to equation (1.1) is to establish under what conditions the set of nontrivial solutions is not empty and to determine topological properties of this set. One of them is the existence of a *bifurcation point*; that is, a point p in $\operatorname{Ker} L \cap \Omega_0$ such that $(0, p)$ lies in the closure of the set of nontrivial solutions. The related bifurcation theory is sometimes called *cobifurcation* [6] or *atypical bifurcation* [8].

Some authors [6, 7] have studied the case when h is compact and proved the existence of a connected bifurcating branch of nontrivial solutions that is either unbounded or its closure contains at least two bifurcation points. In this paper we obtain an analogous result removing the compactness assumption on h. The proof is based on a degree theory developed by Benevieri and Furi [1] for the class of Fredholm maps of index zero.

2. Terminology and Preliminary Results

We start this section with a brief summary of the degree theory presented by Benevieri and Furi in [1] (see also [2] and [3]). It is based on a notion of orientation for nonlinear Fredholm maps of index zero between Banach spaces.

The starting point is a definition of orientation for linear Fredholm operators of index zero between real vector spaces. Let E and F be real vector spaces. We recall that a linear operator $L : E \to F$ is called *algebraic Fredholm* if $\operatorname{Ker} L$ and $F/\operatorname{Im} L$ have finite dimension. The *index* of L is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim F/\operatorname{Im} L.$$

Given an algebraic Fredholm operator of index zero $L : E \to F$, a linear operator $A : E \to F$, having finite dimensional image, is a *corrector* of L if $L + A$ is an isomorphism. The set $\mathcal{C}(L)$ of correctors of L is nonempty, since it contains any linear operator $A : E \to F$ such that $\operatorname{Ker} A \oplus \operatorname{Ker} L = E$ and $\operatorname{Im} A \oplus \operatorname{Im} L = F$.

We introduce in $\mathcal{C}(L)$ the following equivalence relation. Let $A, B \in \mathcal{C}(L)$ and consider the automorphism

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)$$

of E. Observe that $K = I - T$ has finite dimensional image. Given any nontrivial finite dimensional subspace E_0 of E containing $\operatorname{Im} K$, the restriction of T to E_0 is clearly an automorphism of this subspace. Then, its determinant is well defined and nonzero. One can check that this determinant does not depend on the space E_0 containing $\operatorname{Im} K$. Thus this value can be defined as the *determinant* of $(L + B)^{-1}(L + A)$, in symbols $\det(L + B)^{-1}(L + A)$.

We say that A is *equivalent* to B or, more precisely, A is L *-equivalent* to B, if $\det(L + B)^{-1}(L + A) > 0$. This is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes.

Definition 2.1. An *orientation* of an algebraic Fredholm operator of index zero L is the choice of one of the two equivalence classes of $\mathcal{C}(L)$. We say that L is *oriented* when an orientation is chosen.

Given an oriented operator L, it will be useful to call *positive correctors* of L the elements of its orientation.

According to Definition 2.1, an oriented operator L is actually a pair (L, ω), where ω is one of the two equivalence classes of $\mathcal{C}(L)$. However, to simplify the notation, we shall not use different symbols to distinguish between oriented and nonoriented operators.

Definition 2.2. An oriented isomorphism L is said to have the *natural orientation* if the trivial operator 0 is a positive corrector of L. In addition we define $\operatorname{sign} L = 1$ if L is naturally oriented and $\operatorname{sign} L = -1$ otherwise.

Notice that, if we choose a positive corrector A of an oriented isomorphism L, then $\operatorname{sign} L$ is just the sign of

$$\det(L + A)^{-1}L = \det(I - (L + A)^{-1}A),$$

which is well defined, as seen above, since the operator $(L + A)^{-1}A$ has finite dimensional image.

We embed now the notion of orientation in the framework of Banach spaces. From now on E and F are real Banach spaces and $\mathrm{L}(E, F)$ denotes the Banach space of bounded linear operators from E into F. In what follows all linear operators will assumed to be bounded. For the sake of simplicity, the set of bounded correctors of a Fredholm operator of index zero $L : E \to F$ is still denoted by $\mathcal{C}(L)$, as in the algebraic case. It is clear that an orientation of L can be regarded as an equivalence class of bounded correctors of L.

Let us recall that the set $\Phi(E,F)$ of Fredholm operators from E into F is open in $\mathrm{L}(E,F)$. In fact, for any $n \in \mathbb{Z}$, the set $\Phi_n(E,F)$ of Fredholm operators of index n is open in $\mathrm{L}(E,F)$.

As one can notice, Definition 2.1 does not require any topological setting, being referred to algebraic Fredholm operators. However, in the less general context of Banach spaces, an orientation of a Fredholm operator of index zero induces an orientation of any sufficiently close linear operator. Indeed, given $L \in \Phi_0(E,F)$ and a corrector A of L, $L' + A$ is still an isomorphism for every L' in a suitable neighborhood U of L. If, in addition, L is oriented and A is a positive corrector of L, then any $L' \in U$ can be oriented choosing A as a positive corrector of L'. This elementary consideration brings us to the following notion of orientation for a continuous map with values in $\Phi_0(E,F)$.

Definition 2.3. Let X be a topological space and $\psi : X \to \Phi_0(E,F)$ be continuous. An *orientation* of ψ is a continuous choice of an orientation $\alpha(x)$ of $\psi(x)$ for each $x \in X$, where "continuous" means that for any $x \in X$ there exists $A_x \in \alpha(x)$ which is a positive corrector of $\psi(x')$ for any x' in a neighborhood of x. A map is *orientable* when it admits an orientation and *oriented* when an orientation is chosen.

In [2] it is shown that if X is simply connected and locally path connected, then any continuous map from X into $\Phi_0(E,F)$ is orientable.

From Definition 2.3 we obtain the following notion of orientation for (nonlinear) Fredholm maps of index zero between Banach spaces. We recall that, given an open subset V of E, a map $f : V \to F$ is called *Fredholm* if it is C^1 and its Fréchet derivative, $Df(x)$, is a Fredholm operator for all $x \in V$. The *index* of f at x is the index of $Df(x)$ and f is said to be of index n if it is of index n at any point of its domain.

Definition 2.4. An *orientation* of a Fredholm map of index zero $f : V \to F$ is an orientation of the continuous map $Df : x \mapsto Df(x)$, and f is *orientable*, or *oriented*, if so is Df according to Definition 2.3.

Let us now sketch the construction of the degree and recall its main properties.

Definition 2.5. Let $f : V \to F$ be an oriented (Fredholm) map (of index zero) and $y \in F$. Given an open subset U of V, we say that the triple (f,U,y) is *admissible* for the degree if $f^{-1}(y) \cap U$ is compact.

The degree of an admissible triple (f,U,y) is preliminarily defined when y is a regular value for f in U. In this case we put

$$\deg(f,U,y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} Df(x).$$

To define the degree of an admissible triple (f,U,y) in the case when y is a critical value, we consider an open subset U' of U, containing $f^{-1}(y) \cap U$ and such that f is proper on the closure $\overline{U}'$ of U' (U' exists since Fredholm maps are locally proper - see [9]). Then, the degree of (f, U, y) is defined as

$$\deg(f, U, y) = \deg(f, U', z),$$

where z is any regular value for the restriction of f to U', sufficiently close to y. This definition is well posed as proven in [1].

The properties of this degree are analogous to the classical properties of the Leray-Schauder degree (see [1] for details). Here we mention the following ones.

i) (*Normalization*) Let the identity I of E be naturally oriented. For any open subset U of E and any $y \in U$, one has

$$\deg(I, U, y) = 1.$$

ii) (*Additivity*) If (f, U, y) is an admissible triple and U_1, U_2 are two disjoint open subsets of U such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then

$$\deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y).$$

iii) (*Existence*) Let (f, U, y) be admissible. If

$$\deg(f, U, y) \neq 0,$$

then the equation $f(x) = y$ has a solution in U.

iv) (*Excision*) If (f, U, y) is an admissible triple and U_1 is an open subset of U containing $f^{-1}(y) \cap U$ then

$$\deg(f, U_1, y) = \deg(f, U, y).$$

Another important property satisfied by this notion of degree, and which is crucial for the results obtained in this work, is an extended version of the Homotopy Invariance Property.

Let W be an open subset of $\mathbb{R} \times E$ and $H : W \to F$ be continuous. We say that H is a *continuous family of Fredholm maps of index zero* if the following conditions are satisfied:

a) for any $\lambda \in \mathbb{R}$ the partial map $H_\lambda = H(\lambda, \cdot)$ is Fredholm of index zero on the section W_λ;
b) the partial derivative $\partial_2 H : W \to \Phi_0(E, F)$ is continuous.

In [3] it was proved that a continuous family of Fredholm maps is locally proper, extending a well-known result of Smale for the C^1 case (see [9]). This fact will be crucial in the sequel.

We say that H is *orientable*, or *oriented*, if so is the map $\partial_2 H : W \to \Phi_0(E, F)$ according to Definition 2.3. Let us now state the following version of the Homotopy Invariance Property of the degree (see [3, Theorem 3.4]).

Theorem 2.6 (*General Homotopy Invariance*). *Let W be open in $[0, 1] \times E$ and $H : W \to F$ an oriented family of Fredholm maps of index zero. Let $y : [0, 1] \to F$ be a (continuous) path. If the set $\{(\lambda, x) \in W : H(\lambda, x) = y(\lambda),\ \lambda \in [0, 1]\}$ is compact, then $\deg(H_\lambda, W_\lambda, y(\lambda))$ does not depend on λ.*

We conclude the brief summary of this degree theory with another property needed in the next section.

Let $f : U \to F$ be an oriented map and let F_0 be an oriented finite dimensional subspace of F. Assume that F_0 is transverse to f; that is, $\operatorname{Im} Df(x) + F_0 = F$ for every $x \in f^{-1}(F_0)$. In this case $f^{-1}(F_0)$ is a differentiable manifold of the same dimension as F_0. In [1] it is shown how the orientations of f and F_0 give an orientation on $f^{-1}(F_0)$, that we shall refer to as *induced orientation*.

Theorem 2.7 (*Reduction*)**.** *Let $(f, U, 0)$ be an admissible triple and let F_0 be an oriented finite dimensional subspace of F, transverse to f. Then*

$$\deg(f, U, 0) = \deg(f_0, f^{-1}(F_0) \cap U, 0),$$

where $f_0 : f^{-1}(F_0) \to F_0$ is the restriction of f to the manifold $f^{-1}(F_0)$ endowed with the induced orientation, and the right hand side is the Brouwer degree of the triple $(f_0, f^{-1}(F_0) \cap U, 0)$.

We end this section with the following result (see Lemma 1.4 of [5]) which plays a crucial role in the proof of the main theorem of this paper.

Lemma 2.8. *Let K be a compact subset of a locally compact metric space X. Assume that any compact subset of X containing K has nonempty boundary. Then $X \backslash K$ contains a not relatively compact component whose closure in X intersects K.*

3. Results

We come back to the study of equation (1.1). Denote by $\pi : F \to F/ \operatorname{Im} L$ the canonical projection, and by $R : F \to \operatorname{Im} L$ a bounded linear retraction, namely an operator from F onto $\operatorname{Im} L$ such that $Ry = y$ for every $y \in \operatorname{Im} L$. Equation (1.1) is equivalent to the system

$$\begin{cases} Lx + \lambda Rh(\lambda, x) & = & 0 \\ \lambda \pi h(\lambda, x) & = & 0 \end{cases} \tag{3.1}$$

Notice that all points in Ω of the form $(0, x)$, with $x \in \operatorname{Ker} L$, are solutions of (3.1). They are called *trivial solutions*. We say that $p \in \operatorname{Ker} L$ is a *bifurcation point* for (1.1) if $(0, p)$ is in the closure of the set of nontrivial solutions.

In order to investigate the set of nontrivial solutions it is convenient to consider the system

$$\begin{cases} Lx + \lambda Rh(\lambda, x) & = & 0 \\ \pi h(\lambda, x) & = & 0 \end{cases} \tag{3.2}$$

which is equivalent to (3.1) for $\lambda \neq 0$.

Theorem 3.1 (necessary condition)**.** *Assume that p is a bifurcation point for the equation* (1.1). *Then $h(0, p) \in \operatorname{Im} L$ or, equivalently, $\pi h(0, p) = 0$.*

Proof. Since p is a bifurcation point, there exists a sequence $\{(\lambda_n, x_n)\}$ of solutions of (3.2) converging to $(0, p)$. The result follows from the continuity of πh. □

The following result provides a sufficient condition for the existence of a bifurcation point. The statement involves the degree of a map between $\operatorname{Ker} L$ and $F/ \operatorname{Im} L$.

Therefore these spaces should be considered oriented. However, the result is independent of the chosen orientations. As before, given $W \subseteq \Omega \subseteq \mathbb{R} \times E$, W_0 and Ω_0 denote the sections of W and Ω at $\lambda = 0$.

Theorem 3.2 (sufficient condition)**.** *Let* $v : \operatorname{Ker} L \cap \Omega_0 \to F/\operatorname{Im} L$ *be defined by* $v(p) = \pi(h(0,p))$. *Given* $W \subseteq \Omega$ *open, assume that* $\deg(v, W_0 \cap \operatorname{Ker} L, 0)$ *is defined and different from 0. Then there exists a connected set of nontrivial solutions of* (1.1) *whose closure in* W *is not compact and intersects* $\{0\} \times \operatorname{Ker} L$.

Proof. Let $H : \Omega \to \operatorname{Im} L \times (F/\operatorname{Im} L)$ denote the map

$$(\lambda, x) \mapsto (Lx + \lambda R h(\lambda, x), \pi h(\lambda, x)),$$

which is clearly a continuous family of Fredholm maps of index 0. Since Ω is simply connected, the map

$$\partial_2 H : \Omega \to \Phi_0(E, \operatorname{Im} L \times (F/\operatorname{Im} L))$$

defined by

$$\partial_2 H(\lambda, x) = (Lx + \lambda R \partial_2 h(\lambda, x), \pi \partial_2 h(\lambda, x))$$

is orientable (see the previous section). Choose an orientation of $\partial_2 H$. By definition, this gives an orientation of the family H and of any partial map H_λ.

The set

$$Y = \big\{(\lambda, x) \in W : H(\lambda, x) = 0\big\}$$

is locally compact since, as observed before, H is locally proper. Moreover, $Y_0 = v^{-1}(0) \cap W_0$ is compact, because we assumed that $\deg(v, W_0 \cap \operatorname{Ker} L, 0)$ is defined.

We apply Lemma 2.8 to the pair $(Y, \{0\} \times Y_0)$. Assume, by contradiction, that there exists a compact set $C \subseteq Y$ containing $\{0\} \times Y_0$ and with empty boundary in Y. This implies the existence of an open subset $\widetilde{W}$ of W such that $\widetilde{W} \cap Y = C$. Since C is compact, the General Homotopy Invariance property implies that $\deg(H_\lambda, \widetilde{W}_\lambda, 0)$ does not depend on $\lambda \in \mathbb{R}$. Moreover, $\widetilde{W}_\lambda \cap C_\lambda$ is empty for some λ's. Hence, we obtain $\deg(H_0, \widetilde{W}_0, 0) = 0$. The inclusions $v^{-1}(0) \cap W_0 \subseteq \widetilde{W}_0 \subseteq W_0$ imply, using the Excision property of the degree, $\deg(H_0, W_0, 0) = 0$.

Now, observe that the subspace $F_0 = \{0\} \times (F/\operatorname{Im} L)$ is transverse to H_0 and $H_0^{-1}(F_0) = \operatorname{Ker} L \cap \Omega_0$. Thus, from the Reduction property of the degree, we obtain

$$\deg(H_0, W_0, 0) = \deg(v, W_0 \cap \operatorname{Ker} L, 0) \neq 0,$$

which is a contradiction. □

Corollary 3.3. *Let the assumptions of Theorem 3.2 be satisfied. Assume, moreover, that the map* $(\lambda, x) \mapsto Lx + \lambda h(\lambda, x)$ *is proper on bounded and closed subsets of* W. *Then* (1.1) *admits a connected set* Σ *of nontrivial solutions such that its closure in* $\mathbb{R} \times E$ *intersects* $\{0\} \times \operatorname{Ker} L$ *and is either unbounded or reaches the boundary of* W. *In particular, if* $W = \mathbb{R} \times E$, *then* Σ *is unbounded.*

Proof. Let $\overline{\Sigma}$ denote the closure in $\mathbb{R} \times E$ of a connected branch Σ as in Theorem 3.2. Suppose $\overline{\Sigma} \cap \partial W = \emptyset$. Thus, the closure of Σ in W coincides with $\overline{\Sigma}$. Thus $\overline{\Sigma}$ cannot be bounded, since the properness of $(\lambda, x) \mapsto Lx + \lambda h(\lambda, x)$ on bounded closed subsets of W implies that H has the same property. □

The following Corollary can be regarded as an extension of a result proved in [7].

Corollary 3.4. *Let W and v be as in Theorem 3.2. Assume that the map $(\lambda, x) \mapsto Lx + \lambda h(\lambda, x)$ is proper on bounded and closed subsets of W. Let $p \in \operatorname{Ker} L \cap W_0$ be such that $v(p) = 0$ and let $v'(p) : \operatorname{Ker} L \to F/\operatorname{Im} L$ be invertible. Then (1.1) admits a connected set of nontrivial solutions such that its closure contains p and satisfies at least one of the following three conditions:*

i. *is unbounded;*
ii. *contains a point $q \in \operatorname{Ker} L \cap W_0$, $q \neq p$;*
iii. *intersects ∂W.*

Proof. The assumptions $v(p) = 0$ and $v'(p)$ invertible imply the existence of an open neighborhood $\widetilde{W}_0$ of p in W_0 such that $v^{-1}(0) \cap \widetilde{W}_0 = \{p\}$ and $\deg(v, \widetilde{W}_0, 0) = \pm 1$. Now apply Corollary 3.3 replacing W with the set

$$\widetilde{W} = (\{0\} \times \widetilde{W}_0) \cup \{(\lambda, x) \in W : \lambda \neq 0\},$$

which is open, being obtained from W by removing the closed subset

$$\{(0, x) \in W : x \notin \widetilde{W}_0\}.$$

□

Remark 3.5. Using a result of the first two authors (see [4]) it can be shown, essentially with the same proof of Theorem 3.2, that the assertion is still valid when

$$h(\lambda, x) = h_1(\lambda, x) + h_2(\lambda, x),$$

with h_1 satisfying the conditions previously required and h_2 locally compact. Under this more general formulation the result extends the previous theorems of [5, 7]. To better understand how the cited results are extended we should mention that the simple connectivity of Ω is required only when $h_1 \neq 0$. In fact, when $h(x, \lambda) = h_2(x, \lambda)$ the orientability of the Fredholm map used in the proof of Theorem 3.2 reduces to the orientability of L. Since L is Fredholm, its orientability does not require the simple connectivity of Ω. Therefore, the result is a clear extension of the theorems proved in [5, 7].

We end this paper with an example that illustrates how Theorem 3.2 can be applied. Before presenting our example we would like to show that the open connected subset V of the Banach space $C^1 := C^1([0, T], \mathbb{R}^n)$, $n \geq 3$, defined by

$$V = \{x : x(t) \neq 0, \forall t \in [0, T]\}$$

is simply connected. In fact, let $k : [\alpha, \beta] \to V$ be a closed curve. Define the homotopy $H : [\alpha, \beta] \times [0, 1] \to V$ by the action $H(\tau, s)(t) = k(\tau)((1 - s)t)$. Then $H(\tau, 0) = k(\tau)$ while $H(\tau, 1)$ is a closed curve of constant functions that can be identified with the closed curve $\tau \mapsto k(\tau)(0)$ in $\mathbb{R}^n \backslash \{0\}$. Since $\mathbb{R}^n \backslash \{0\}$ is simply connected, the assertion follows.

Example 3.6. Consider the following boundary value problem depending on a real parameter λ:

$$\begin{cases} \dot{x}(t) + \lambda\phi(t, x(t), \dot{x}(t)) = 0 \\ x(0) = x(T)\,. \end{cases} \tag{3.3}$$

We assume that $\phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and T-periodic with respect to the first variable. Our boundary value problem can be rewritten in the form

$$\begin{cases} Dx + \lambda h(x) = 0 \\ Bx = 0 \end{cases} \tag{3.4}$$

where $L, h : C^1 \to F := C([0,T], \mathbb{R}^n)$, and $B : C^1 \to \mathbb{R}^n$ are defined by $(Lx)(t) = \dot{x}(t)$, $h(x)(t) = \phi(t, x(t), \dot{x}(t))$, and $Bx = x(T) - x(0)$.

Set $E = \operatorname{Ker} B$ and define $f : \mathbb{R} \times E \to F$ by $f(\lambda, x) = Lx + \lambda h(x)$. Clearly a solution of (3.3) is a pair (λ, x) such that $f(\lambda, x) = 0$. To apply Theorem 3.2 to this problem it is enough to find an open, simply connected subset Ω of $\mathbb{R} \times E$ such that for any $(\lambda, x) \in \Omega$ the partial Fréchet derivative $\partial_2 f(\lambda, x) : E \to F$, which is given by

$$\partial_2 f(\lambda, x) = L + \lambda h'(x),$$

is a Fredholm operator of index 0. Observe that

$$(\partial_2 f(\lambda, x)q)(t) = \dot{q}(t) + \lambda\partial_2 h(t, x(t), \dot{x}(t))q(t) + \lambda\partial_3 h(t, x(t), \dot{q}(t))\dot{q}(t).$$

Therefore, the above operator can be rewritten in the following way:

$$(\partial_2 f(\lambda, x)q)(t) = (I + \lambda A_x(t))\dot{q}(t) + \lambda B_x(t)q(t)$$

where I is the $n \times n$ real identity matrix and, given $x \in E$, A_x and B_x are $n \times n$ matrices of real functions defined in $[0,T]$. Observe that $\partial_2 f(0, x) : E \to F$ is a Fredholm operator of index 0 since it can be regarded as the composition of the inclusion $E \hookrightarrow C^1$ that is Fredholm of index $-n$, with a differential operator from C^1 to F that is onto with a n-dimensional kernel. It is easily seen that $\partial_2 f(\lambda, x) : E \to F$ is Fredholm of index 0 in an open set Ω if for every $(\lambda, x) \in \Omega$ one has

$$\det(I + \lambda A_x(t)) \neq 0, \quad \forall t \in [0,T].$$

This is the case if, for example, for every $x \in E$ and $t \in [0,T]$, the eigenvalues of $A_x(t)$ are never negative, a property that is certainly verified when $A_x(t)$ is positive semidefinite. In this case we can choose

$$\Omega = \{(\lambda, x) \in \mathbb{R} \times E : 1 + \lambda M_x > 0\},$$

where

$$M_x = \max_{t \in [0,T]} \|A_x(t)\|.$$

The set Ω is contractible, and therefore simply connected, since it can be deformed into the contractible subset $\{0\} \times E$ via the homotopy $H : \Omega \times [0,1] \to \Omega$ defined by $H((\lambda, x), s) = (s\lambda, x)$.

The application of Theorem 3.2 to our example requires now the existence of an open set $W \subseteq \Omega$ such that the vector field $v : \operatorname{Ker} L \cap W_0 \to F/\mathrm{Im}L \cong \operatorname{Ker} L$ defined by

$$v(p) = \frac{1}{T}\int_0^T \phi(t, p, 0)dt$$

has degree different from 0 on $W_0 \cap \operatorname{Ker} L$.

References

[1] Benevieri P. - Furi M., *A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory*, Ann. Sci. Math. Québec **22** (1998), 131–148.
[2] Benevieri P. - Furi M., *On the concept of orientability for Fredholm maps between real Banach manifolds*, Topol. Methods Nonlinear Anal. **16**, 2(2000), 279–306.
[3] Benevieri P. - Furi M., *Bifurcation Results for Families of Fredholm Maps of Index Zero between Banach Spaces*, Nonlinear Analysis Forum **6**, 1(2001), 35–47.
[4] Benevieri P. - Furi M., *A Degree theory for locally compact perturbations of Fredholm maps in Banach spaces*, preprint.
[5] Furi M. - Pera M. P., *A Continuation Principle for Periodic Solutions of Forced Motion Equations on Manifolds and Applications to Bifurcation Theory*, Pac. J. of Math. **160** (1993)2, 219–243.
[6] Furi M. - Pera M. P., *Co-Bifurcating Branches of Solutions for Nonlinear Eigenvalue Problems in Banach Spaces*, Annali di Matem. Pura e Applicata **85** (1983)4, 119–132.
[7] Martelli M., *Large Oscillations of Forced Nonlinear Differential Equations*, AMS Contemporary Mathematics **21** (1983), 151–159.
[8] Prodi G. - Ambrosetti A., *Analisi non lineare*, Quad. 1, Scuola Norm. Sup. Pisa, 1973.
[9] Smale S., *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

Pierluigi Benevieri, Massimo Furi, and Maria Patrizia Pera,
Dipartimento di Matematica Applicata ''Giovanni Sansone''
Università degli Studi di Firenze
Via S. Marta 3
50139 Firenze, Italy

Mario Martelli
Department of Mathematics
Claremont McKenna College
Claremont, CA, 91711, USA

e-mail addresses:
benevieri@dma.unifi.it
furi@dma.unifi.it
mmartelli@mckenna.edu
pera@dma.unifi.it