BIFURCATION RESULTS FOR FAMILIES OF FREDHOLM MAPS OF INDEX ZERO BETWEEN BANACH SPACES

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1. Introduction

In [1] we defined an oriented degree, with values in $\mathbb{Z}$, for a class of nonlinear Fredholm maps of index zero between Banach spaces (and, more generally, Banach manifolds). This notion extends and simplifies the theory given by Elworthy and Tromba in [3] and [4], avoiding any additional structure on the manifolds (such as the Fredholm structure). Our theory is, in some sense, parallel to that of Fitzpatrick, Pejsachowicz and Rabier (see [7] and [8]), but essentially based on a simple algebraic notion of orientation for a Fredholm linear operator of index zero (instead of their notion of parity). According to this concept, any Fredholm operator of index zero between vector spaces can be oriented and admits exactly two orientations. Moreover, when an oriented operator acts between Banach spaces and, in addition, is bounded, its orientation induces, by a sort of stability, an orientation on any sufficiently close bounded operator. This property is crucial, because it permits to define in a simple manner a notion of orientation for a nonlinear Fredholm map of index zero g between Banach spaces (or, more generally, Banach manifolds). Roughly speaking, an orientation of g is a "continuous" assignment of an orientation to the Fréchet derivative $Dg(x)$ of g at every x. The orientable maps are, by definition, those which admit an orientation, and the oriented maps are those with a chosen orientation. The degree of an oriented map g at a point y is defined whenever $g^{-1}(y)$ is compact.

In this paper we apply our notions of orientation and degree to obtain some local and global bifurcation results for a class of maps between Banach spaces (and, more generally, Banach manifolds). The maps we take into account, called *(continuous) families of Fredholm maps of index zero*, are continuous maps of the form $f\colon \Omega \to F$, where Ω is an open subset of $\mathbb{R} \times E$, E and F are real Banach spaces, and f is continuously differentiable with respect to the second variable with Fredholm of index zero partial derivative $D_2 f(\lambda, x)$. With the assumption that $f(\lambda, 0) = 0$ for all $(\lambda, 0) \in \Omega$, we study the existence and global behavior of "branches" of nontrivial solutions of the equation $f(\lambda, x) = 0$ emanating from the set $(\mathbb{R} \times \{0\}) \cap \Omega$ of the trivial solutions. This is done by the classical change of degree argument, which goes back to [10] and [12]. To check the change of degree we introduce the simple concept of *sign jump* at a point $\lambda_0 \in \mathbb{R}$ of the path $\lambda \mapsto D_2 f(\lambda, 0)$ of Fredholm operators of index zero. As we shall see, the existence of a sign jump at a point λ_0 ensures that this is a bifurcation point, in the same way as the change of sign of the jacobian $\det(D_2 f(\lambda, 0))$ does in the finite dimensional case.

Some results are stated in elementary terms, without any notion of degree or orientation. In their proofs, however, these notions are needed.

In the context of Fredholm maps between Banach spaces, important bifurcation results have been obtained by Fitzpatrick, Pejsachowicz and Rabier. In [5], Fitzpatrick and Pejsachowicz proved a local bifurcation theorem for C^1 Fredholm maps. Afterwards, Fitzpatrick, Pejsachowicz and Rabier provided a global bifurcation theorem for C^2 Fredholm maps (see [8]), recently extended to the C^1 case by Pejsachowicz and Rabier in [11]. Their results (in Banach spaces) are strictly correlated with ours because of the similar approach, based on the same idea of change of degree. Indeed, starting from the very interesting notion of parity, introduced by Fitzpatrick and Pejsachowicz in [6], Fitzpatrick, Pejsachowicz and Rabier defined a notion of orientability and degree for Fredholm maps of index zero, which they applied to obtain global bifurcation results. However, there are two important differences between our results and those mentioned above: firstly, thanks to our notion of orientation and degree, we are able to provide bifurcation results in the not necessarily C^1 case; secondly, our global results hold for maps between arbitrary real Banach manifolds (without any need of additional structures). In Section 3 the reader will find a comparison between our results and those of Fitzpatrick, Pejsachowicz and Rabier.

2. Orientation and degree for Fredholm maps

This section is devoted to a brief summary of the notions of orientation and degree for nonlinear Fredholm maps of index zero between Banach spaces introduced in [1] and [2]. We give here just the basic definitions and properties of these concepts.

The starting point is a notion of orientation for a linear Fredholm operator of index zero between real vector spaces. We recall that, given two real vector spaces E and F, a linear operator $L\colon E \to F$ is called (*algebraic*) *Fredholm* if $\mathrm{Ker}L$ and $\mathrm{coKer}L$ have finite dimension. The integer

$$\mathrm{ind}L = \dim \mathrm{Ker}L - \dim \mathrm{coKer}L.$$

is the *index* of L.

A linear operator $A\colon E \to F$, having finite dimensional range, is a *corrector* of a Fredholm operator of index zero L if $L + A$ is an isomorphism. It is not difficult to see that the set of correctors of L, indicated by $\mathcal{C}(L)$, is nonempty. In fact, $\mathcal{C}(L)$ contains any (possibly trivial) linear operator $A\colon E \to F$ such that $\mathrm{Ker}A \oplus \mathrm{Ker}L = E$ and $\mathrm{Range}A \oplus \mathrm{Range}L = F$.

We introduce in $\mathcal{C}(L)$ the following equivalence relation. Let $A, B \in \mathcal{C}(L)$ and consider the automorphism

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)$$

of E. Clearly, the operator $K = I - T$ has finite dimensional range. Consequently, given any finite dimensional subspace E_0 of E containing the range of K, the restriction of T to E_0 is an automorphism of E_0. Then, its determinant, $\det T|_{E_0}$, is well defined and nonzero (we set such a determinant equal to 1 when E_0 is trivial, and this can happen only if $A = B$). It is easy to check that this determinant does not depend on E_0 (see [1]). Consequently, the number $\det(L + B)^{-1}(L + A)$ is well defined.

We say that A is *equivalent* to B or, more precisely, A is *L-equivalent* to B, if $\det(L + B)^{-1}(L + A) > 0$. This is actually an equivalence relation on $\mathcal{C}(L)$ with just two equivalence classes.

Definition 2.1. An *orientation* of a Fredholm operator of index zero L is one of the two equivalence classes of $\mathcal{C}(L)$. We say that L is *oriented* when an orientation is chosen.

In the sequel, given an oriented operator L, the elements of its orientation will be called the *positive correctors* of L.

According to Definition 2.1, an oriented operator L is a pair (L, ω), where ω is one of the two equivalence classes of $\mathcal{C}(L)$. However, to simplify the notation, we shall not use different symbols to distinguish between oriented and nonoriented operators (unless it is necessary).

Definition 2.2. Let L be an isomorphism. Since the trivial operator 0 is a corrector of L, it defines an equivalence class of $\mathcal{C}(L)$, called the *natural orientation* of L. If L has already an orientation, we define its sign as follows: $\mathrm{sign} L = 1$ if the trivial operator 0 is a positive corrector of L and $\mathrm{sign} L = -1$ otherwise. If a Fredholm operator of index zero L is not invertible, we put $\mathrm{sign} L = 0$.

Notice that if we choose a positive corrector A of an oriented operator L, then $\mathrm{sign} L$ is just the sign of the following (possibly zero) real number:

$$\det(L + A)^{-1} L = \det(I - (L + A)^{-1} A),$$

which is well defined, since the operator $(L + A)^{-1} A$ has finite dimensional range.

We consider now the notion of orientation in the framework of bounded Fredholm operators between Banach spaces. From now on, unless otherwise stated, E and F will denote two real Banach spaces, and $L(E, F)$ will be the Banach space of bounded linear operators from E into F. For the sake of simplicity, the set of continuous correctors of a bounded Fredholm operator of index zero $L\colon E \to F$ will be still denoted by $\mathcal{C}(L)$, as in the algebraic case, instead of $\mathcal{C}(L) \cap L(E, F)$. It is clear that an orientation of L can be regarded as an equivalence class of continuous correctors of L.

Let us recall that the set $\Phi(E, F)$ of Fredholm operators from E into F is open in $L(E, F)$ and the integer valued map $\mathrm{ind}\colon \Phi(E, F) \to \mathbb{Z}$ is continuous. Consequently, given $n \in \mathbb{Z}$, the set $\Phi_n(E, F)$ of Fredholm operators of index n is an open subset of $L(E, F)$.

As one can notice, Definition 2.1 is referred to algebraic Fredholm operators between vector spaces and does not require any topological structure. On the other hand, in the context of Banach spaces, an orientation of a continuous Fredholm operator of index zero induces, by a sort of stability, an orientation to any sufficiently close bounded operator. Precisely, consider an operator $L \in \Phi_0(E, F)$ and a corrector A of L. Since the set of the isomorphisms from E into F is open in $L(E, F)$, $L' + A$ is still an isomorphism for every L' in a suitable neighborhood U of L. If, in addition, L is oriented and A is a positive corrector of L, then any $L' \in U$ can be oriented by considering A as a positive corrector for L'. This elementary consideration brings us to the following notion of orientation for a continuous map with values in $\Phi_0(E, F)$.

Definition 2.3. Let X be a topological space and $h\colon X \to \Phi_0(E, F)$ be continuous. An *orientation* of h is a continuous choice of an orientation $\alpha(x)$ of $h(x)$ for each $x \in X$; where "continuous" means that for any $x \in X$ there exists $A_x \in \alpha(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is *orientable* when it admits an orientation and *oriented* when an orientation has been chosen.

As pointed out in [2], if X is simply connected, then any continuous map $h\colon X \to \Phi_0(E,F)$ is orientable (with just two possible orientations).

From Definition 2.3 we derive the following notion of orientation for Fredholm maps of index zero between Banach spaces. We recall that, given an open subset W of E, a map $g\colon W \to F$ is called *Fredholm* if it is C^1 and its Fréchet derivative, $Dg(x)$, is a Fredholm operator for all $x \in W$. The *index* of g at x is the index of $Dg(x)$ and g is said to be of index n if it is of index n at any point of its domain.

Definition 2.4. An *orientation* of a Fredholm map of index zero $g\colon W \to F$ is an orientation of the continuous map $Dg\colon x \mapsto Dg(x)$, and g is *orientable*, or *oriented*, if so is Dg according to Definition 2.3.

Let W be an open subset of E and $H\colon W \times [0,1] \to F$ be a homotopy. We say that H is a *homotopy of Fredholm maps of index zero* provided that any partial map $H_t = H(\cdot,t)$ is Fredholm of index zero and the map $(x,t) \mapsto DH_t(x)$, from $W \times [0,1]$ into $\Phi_0(E,F)$, is continuous. If this map is orientable (resp. oriented), then H is said to be *orientable* (resp. *oriented*). Our notion of orientability is, in some sense, stable under small perturbations. More precisely, the following result holds (see [1]).

Theorem 2.5. *Let W be an open subset of E and $H\colon W\times[0,1] \to F$ be a homotopy of Fredholm maps of index zero. If H_0 is orientable, then H is orientable as well and, in particular, any partial map H_t is orientable.*

We end this section by sketching the construction of the degree in the context of Banach spaces (for a complete discussion see [1]).

Definition 2.6. Let $g\colon W \to F$ be an oriented (Fredholm) map (of index zero) and $y \in F$. We say that the triple (g,W,y) is *admissible* if $g^{-1}(y)$ is compact. More generally, if U is an open subset of W, the triple (g,U,y) is admissible if so is $(g|_U,U,y)$.

The *degree* is a map from the family of all the admissible triples into $\mathbb{Z}$ with the following properties:

i) (*Normalization*) Let $g\colon U \to F$ be a diffeomorphism from an open subset U of E onto an open subset V of F, and let $y \in V$. If g is naturally oriented (this means that, at any $x \in U$, $Dg(x)$ is naturally oriented), then

$$\deg(g,U,y) = 1.$$

ii) (*Additivity*) If (g,W,y) is an admissible triple and U_1, U_2 are two disjoint open subsets of W such that $g^{-1}(y) \subset U_1 \cup U_2$, then

$$\deg(g,W,y) = \deg(g,U_1,y) + \deg(g,U_2,y).$$

iii) (*Homotopy invariance*) Let $H\colon W \times [0,1] \to F$ be an oriented homotopy (of Fredholm maps of index zero), and let $y\colon [0,1] \to F$ be a (continuous) path. If the set

$$\{(x,t) \in W \times [0,1] \colon H(x,t) = y(t)\}$$

is compact, then $\deg(H_t,W,y(t))$ does not depend on t.

The degree of an admissible triple (g, U, y) is preliminarily defined when y is a regular value for g in U. In this case we put

$$\deg(g, U, y) = \sum_{x \in g^{-1}(y)} \mathrm{sign} Dg(x).$$

To define the degree of an admissible triple (g, W, y) in the case when y is not (necessarily) a regular value, we consider an open subset U of W, containing $g^{-1}(y)$ and such that g is proper in the closure $\overline{U}$ of U (such an U exists since, as pointed out in [13], Fredholm maps are locally proper). Then the degree of (g, W, y) is defined as

$$\deg(g, W, y) := \deg(g, U, z),$$

where z is any regular value for g in U, sufficiently close to y. The following lemma justifies this definition.

Lemma 2.7. *Let (g, W, y) be admissible and let U_1 and U_2 be two open neighborhoods of $g^{-1}(y)$ such that $\overline{U}_1 \cup \overline{U}_2 \subset W$ and g is proper in $\overline{U}_1 \cup \overline{U}_2$. Then, there exists a neighborhood V of y such that for any pair of regular values $y_1, y_2 \in V$ one has*

$$\deg(g, U_1, y_1) = \deg(g, U_2, y_2).$$

3. Bifurcation results

In this section we apply our notions of orientation and degree to obtain some bifurcation results for nonlinear Fredholm maps of index zero between Banach spaces. As before, E and F will denote real Banach spaces.

Let $f\colon \Omega \to F$ be a continuous map from an open subset of $\mathbb{R} \times E$ into F. We say that f is a *(continuous) family of Fredholm maps of index zero* if the following conditions are satisfied:

a) for any $\lambda \in \mathbb{R}$, the partial map $f_\lambda := f(\lambda, \cdot)$, which is defined on the (possibly empty) open subset $\Omega_\lambda = \{x \in E \colon (\lambda, x) \in \Omega\}$ of E, is Fredholm of index zero;
b) the map $D_2 f\colon \Omega \to \Phi_0(E, F)$, which associates to any $(\lambda, x) \in \Omega$ the partial derivative $D_2 f(\lambda, x) := Df_\lambda(x)$, is continuous.

As in the case of a homotopy of Fredholm maps of index zero, an orientation of $f\colon \Omega \to F$ is just an orientation of $D_2 f\colon \Omega \to \Phi_0(E, F)$, according to Definition 2.3.

Consider now a family of Fredholm maps of index zero $f\colon \Omega \to F$ with the property that $f(\lambda, 0) = 0$ for all $(\lambda, 0) \in \Omega$. A real number λ_0 is a *bifurcation point* for the equation

$$f(\lambda, x) = 0 \tag{3.1}$$

if $(\lambda_0, 0) \in \Omega$ and any neighborhood of $(\lambda_0, 0)$ contains *nontrivial solutions* of (3.1); i.e. solutions (λ, x) with $x \neq 0$.

It is well known that a necessary condition for λ_0 to be a bifurcation point is that $Df_{\lambda_0}(0)$ is a singular operator (i.e. noninvertible). It is an easy consequence of the Brouwer degree theory that, when E and F are finite dimensional, then λ_0 is a bifurcation point provided that, given any isomorphism $L\colon E \to F$, the real function $\lambda \mapsto \det(L^{-1} Df_\lambda(0))$ changes sign at λ_0 (meaning that in any neighborhood of

λ_0 there are two points λ_1 and λ_2, with $\lambda_1 < \lambda_0 < \lambda_2$, in which the function has opposite sign). Theorem 3.1 below represents the natural extension of this sufficient condition to the infinite dimensional framework. The statement does not require any notion of orientation or degree. The proof, which is based on our degree theory, is the same as in the finite dimensional case, and will be given just for the sake of completeness.

Theorem 3.1. *Let $f\colon \Omega \to F$ be a family of Fredholm maps of index zero such that $f(\lambda, 0) = 0$ for all $(\lambda, 0) \in \Omega$, and let $(\lambda_0, 0) \in \Omega$. Given any corrector A of $Df_{\lambda_0}(0)$, if the real function*

$$\lambda \mapsto \det((Df_\lambda(0) + A)^{-1} Df_\lambda(0))$$

(which is defined in a neighborhood of λ_0) changes sign at λ_0, then λ_0 is a bifurcation point.

Proof. Assume that λ_0 is not a bifurcation point. Then there exists an open neighborhood

$$U = (\lambda_0 - \delta, \lambda_0 + \delta) \times B_r(0)$$

of $(\lambda_0, 0)$ in Ω such that

$$U \cap f^{-1}(0) = (\lambda_0 - \delta, \lambda_0 + \delta) \times \{0\}.$$

Let A be a corrector of $Df_{\lambda_0}(0)$. Since the set of the isomorphisms from E into F is open in $L(E, F)$, we may assume that U is such that A is a corrector of any operator $Df_\lambda(x)$ with $(\lambda, x) \in U$. Thus A induces an orientation of f in U. Since $B_r(0) \cap f_\lambda^{-1}(0) = \{0\}$ for any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $\deg(f_\lambda, B_r(0), 0)$ is well defined and, by the homotopy invariance, independent of λ. On the other hand, whenever $Df_\lambda(0)$ is an isomorphism, from the definition of degree we get

$$\deg(f_\lambda, B_r(0), 0) = \operatorname{sign} Df_\lambda(0) = \operatorname{sign}\det((Df_\lambda(0) + A)^{-1} Df_\lambda(0));$$

and this is impossible since, by assumption,

$$\det((Df_\lambda(0) + A)^{-1} Df_\lambda(0))$$

changes sign at λ_0. □

From now on, to simplify the statements, given a (continuous) path $\gamma\colon (a, b) \to \Phi_0(E, F)$ of Fredholm operators of index zero, we say that γ has a *sign jump* at some $\lambda_0 \in (a, b)$ if, given a corrector A of $\gamma(\lambda_0)$, the real function $\lambda \mapsto \det((\gamma(\lambda) + A)^{-1}\gamma(\lambda))$ changes sign at λ_0. This means that any neighborhood of λ_0 contains two points, λ_1 and λ_2, such that $\lambda_1 < \lambda_0 < \lambda_2$ and $\operatorname{sign}\gamma(\lambda_1)\operatorname{sign}\gamma(\lambda_2) < 0$. Notice that this product does not depend on the choice of one of the two orientations of γ.

Under the extra assumption that the family $f\colon \Omega \to F$ is orientable we get the following global bifurcation result.

Theorem 3.2. *Let Ω be an open subset of $\mathbb{R} \times E$ and $f\colon \Omega \to F$ be a family of Fredholm maps of index zero such that $f(\lambda, 0) = 0$ for all $(\lambda, 0) \in \Omega$. Assume that Ω is simply connected (or, more generally, that f is orientable). If $\lambda \mapsto Df_\lambda(0)$ has a sign jump at some λ_0, then there exists a connected set of nontrivial solutions of $f(\lambda, x) = 0$ whose closure in Ω contains $(\lambda_0, 0)$ and is either noncompact or contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_0$.*

The proof of the theorem is based on the three preliminary results. The following one is due to [9].

Lemma 3.3. *Let Y be a locally compact Hausdorff topological space and Y_0 be a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then there exists a connected subset C of $Y \setminus Y_0$ whose closure is noncompact and intersects Y_0.*

The next result is a straightforward extension of the homotopy invariance of the degree. The simple proof is based on the excision property of the degree (which is a consequence of the additivity) and the homotopy invariance.

Theorem 3.4. *Let $f\colon \Omega \to F$ be an oriented family of Fredholm maps of index zero and let $y\colon [a,b] \to F$ be a (continuous) path. If the set*

$$\{(\lambda, x) \in \Omega\colon f(\lambda, x) = y(\lambda),\ \lambda \in [a,b]\}$$

is compact, then

$$\deg(f_\lambda, \Omega_\lambda, y(\lambda))$$

does not depend on λ.

The following lemma is an extension of a result due to Smale in [13], regarding the local properness of Fredholm nonlinear maps. We recall that a map is proper if the inverse image of any compact set is compact. A map is locally proper if any point of its domain admits a closed neighborhood in which the restriction is proper.

Lemma 3.5. *Let $f\colon \Omega \to F$ be a continuous map from an open subset of $\mathbb{R}^k \times E$ into F. Assume that f is continuously differentiable with respect to the second variable, with Fredholm derivative $D_2 f(\lambda, x)$ for all $(\lambda, x) \in \Omega$. Then f is locally proper.*

Proof. Let $(\lambda_0, x_0) \in \Omega$. We need to show that there exists a closed neighborhood of (λ_0, x_0) in which f is proper. Since $L := D_2 f(\lambda_0, x_0)$ is Fredholm, there exist two subspaces E_1 and F_1 of E and F, respectively, such that $E = \mathrm{Ker} L \oplus E_1$ and $F = F_1 \oplus \mathrm{Range} L$. Denote by P_1 and P_2 the projections associated with the decomposition $F = F_1 \oplus \mathrm{Range} L$. Any element x in E can be written in the form (p, q), with $p \in \mathrm{Ker} L$ and $q \in E_1$, and any element y in F can be written in the form (v, w) with $v \in F_1$ and $w \in \mathrm{Range} L$. Define the maps $f_1\colon \Omega \times F_1 \to F_1$ and $f_2\colon \Omega \times \mathrm{Range} L \to \mathrm{Range} L$ by

$$f_1(\lambda, p, q, v) = P_1(f(\lambda, p, q)) - v, \quad f_2(\lambda, p, q, w) = P_2(f(\lambda, p, q)) - w.$$

Put $(p_0, q_0) = x_0$ and $(v_0, w_0) = f(\lambda_0, x_0)$, and observe that the derivative of f_2 at $(\lambda_0, p_0, q_0, w_0)$ with respect to the third variable is an isomorphism. Thus, by the Implicit Function Theorem, there exists a closed neighborhood $A \times B \times C \times D$ of $(\lambda_0, p_0, q_0, w_0)$ in which $f_2^{-1}(0)$ is the graph of a continuous map $\eta\colon A \times B \times D \to C$. Let us prove that f is proper on $A \times B \times C$. To this aim consider any sequence $\{(\lambda_n, p_n, q_n)\}$ in $A \times B \times C$ such that $\{f(\lambda_n, p_n, q_n)\} = \{(v_n, w_n)\}$ is convergent. To prove that f is proper on $A \times B \times C$ it is sufficient to verify that $\{(\lambda_n, p_n, q_n)\}$ admits a convergent subsequence. Since A and B are finite dimensional, we can assume that $\{\lambda_n\}$ and $\{p_n\}$ are convergent. Then, from the continuity of η it follows that $\{q_n\} = \{\eta(\lambda_n, p_n, w_n)\}$ is convergent as well, and this ensures the properness of f. □

Proof of Theorem 3.2. Since f is orientable, we may assume it is oriented. Denote by S the set of nontrivial solutions of $f(\lambda, x) = 0$. We apply first Lemma 3.3 with $Y = S \cup \{(\lambda_0, 0)\}$ and $Y_0 = \{(\lambda_0, 0)\}$. By Lemma 3.5 the map f is locally proper and, consequently, S is locally compact. This implies that Y is locally compact as well. If the pair (Y, Y_0) does not verify the assumptions of Lemma 3.3, there exists a compact subset K of Y containing $(\lambda_0, 0)$, with empty boundary in Y. Thus K is an open neighborhood of $(\lambda_0, 0)$ in Y and, consequently, there exists an open subset U of Ω such that $U \cap Y = K$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 < \lambda_0 < \lambda_2$, $[\lambda_1, \lambda_2] \times \{0\} \subset U$, and

$$\mathrm{sign} D_2 f(\lambda_1, 0)\, \mathrm{sign} D_2 f(\lambda_2, 0) < 0.$$

Observe that, given $\lambda \in \mathbb{R}$, we have $f_\lambda^{-1}(0) \cap U_\lambda = K_\lambda \cup (U_\lambda \cap \{0\})$, where $f_\lambda = f(\lambda, \cdot)$, $U_\lambda = \{x \in E \colon (\lambda, x) \in U\}$, and $K_\lambda = \{x \in E \colon (\lambda, x) \in K\}$. Thus, choosing an orientation of f in U, $\deg(f_\lambda, U_\lambda, 0)$ turns out to be well defined for all $\lambda \in \mathbb{R}$. Moreover, since $[\lambda_1, \lambda_2] \times \{0\} \subset U$, by Theorem 3.4 we get

$$\deg(f_{\lambda_1}, U_{\lambda_1}, 0) = \deg(f_{\lambda_2}, U_{\lambda_2}, 0).$$

Let $B_\delta(0) \subset E$ be an open ball centered at 0 with radius δ such that no points of K are in $((-\infty, \lambda_1] \cup [\lambda_2, +\infty)) \times \overline{B_\delta(0)}$, where $\overline{B_\delta(0)}$ stands for the closure of $B_\delta(0)$. Assume also that δ is such that $(f_{\lambda_i})^{-1}(0) \cap \overline{B_\delta(0)} = \{0\}$, $i = 1, 2$. For any real λ, let $V_\lambda = U_\lambda \backslash \overline{B_\delta(0)}$. By the additivity of the degree,

$$\deg(f_{\lambda_i}, U_{\lambda_i}, 0) = \deg(f_{\lambda_i}, V_{\lambda_i}, 0) + \deg(f_{\lambda_i}, B_\delta(0), 0),$$

$i = 1, 2$. Since K is compact, there exist $\mu_1 < \lambda_1$ and $\mu_2 > \lambda_2$ such that $(\{\mu_i\} \times V_{\mu_i}) \cap K$ is empty, $i = 1, 2$, and, consequently, $\deg(f_{\mu_i}, V_{\mu_i}, 0) = 0$. Therefore, because of Theorem 3.4, one has $\deg(f_{\lambda_i}, V_{\lambda_i}, 0) = 0$, for $i = 1, 2$. Thus

$$\deg(f_{\lambda_1}, B_\delta(0), 0) = \deg(f_{\lambda_2}, B_\delta(0), 0).$$

On the other hand, recalling the definition of degree, we have

$$\deg(f_{\lambda_i}, B_\delta(0), 0) = \mathrm{sign} D_2 f(\lambda_i, 0),$$

$i = 1, 2$, and this is impossible since $\mathrm{sign} D_2 f(\lambda_1, 0) \neq \mathrm{sign} D_2 f(\lambda_2, 0)$. Thus, Lemma 3.3 applies to get the existence of a connected subset C of S whose closure in Y (i.e. $\overline{C} \cap Y$) contains $(\lambda_0, 0)$ and is not compact. If the closure of C in Ω is not compact, the assertion is proved. Suppose instead that $\overline{C} \cap \Omega$ is compact. It follows that $\overline{C} \cap Y$, being not compact, is strictly contained in $\overline{C} \cap \Omega$. Thus $\overline{C}$ contains a solution to the equation $f(\lambda, x) = 0$ which is not in $Y = S \cup \{(\lambda_0, 0)\}$, and this is a trivial solution $\{(\lambda_*, 0)\}$ with $\lambda_* \neq \lambda_0$. □

A straightforward generalization of the above theorem leads to the next result. The proof, which is similar to that of Theorem 3.2, will be omitted.

Theorem 3.6. *Let $f \colon \Omega \to F$ be an oriented family of Fredholm maps of index zero such that $f(\lambda, 0) = 0$ for all $(\lambda, 0) \in \Omega$. Let also λ_1, $\lambda_2 \in \mathbb{R}$ be such that $[\lambda_1, \lambda_2] \times \{0\} \subset \Omega$ and that $\mathrm{sign} Df_{\lambda_1}(0) \neq \mathrm{sign} Df_{\lambda_2}(0)$. Then there exists a connected set of nontrivial solutions of $f(\lambda, x) = 0$ whose closure in Ω has nonempty intersection with $[\lambda_1, \lambda_2] \times \{0\}$ and either is not compact or contains a point $(\lambda_*, 0)$ with $\lambda_* \notin [\lambda_1, \lambda_2]$.*

Bifurcation results for nonlinear Fredholm maps between Banach spaces have been obtained in recent years by Fitzpatrick, Pejsachowicz and Rabier. We give here a short comparison between their results and the contents of this paper.

In [6] Fitzpatrick and Pejsachowicz defined the notion of *parity* of a path of Fredholm maps of index zero. Let us briefly summarize this concept. Consider two Banach spaces E and F and a path $\gamma\colon [a,b] \to \Phi_0(E,F)$. Then, there exists a continuous path $k\colon [a,b] \to L(E,F)$ such that $k(t)$ is compact for every $t \in [a,b]$, and $\gamma(t)+k(t)$ is an isomorphism. Let $Iso(E,F)$ denote the space of the isomorphisms from E into F and define $\beta\colon [a,b] \to Iso(E,F)$ by $\beta(t) = \gamma(t)+k(t)$. One has $\beta(t)^{-1} \circ \gamma(t) = I - h(t)$, where $h(t)$ is a compact operator from E into itself. The path $t \mapsto \beta(t)^{-1}$ is called a *parametrix* of γ. Assume now that $\gamma(a)$ and $\gamma(b)$ are isomorphisms. Given a parametrix $\beta^{-1}\colon [a,b] \to Iso(F,E)$ of γ, the number

$$\sigma(\gamma,[a,b]) = \deg_{LS}(\beta^{-1}(a) \circ \gamma(a)) \deg_{LS}(\beta^{-1}(b) \circ \gamma(b)),$$

which is either 1 or -1, does not depend on the parametrix β^{-1} and, consequently, can be actually associated to γ (as before $\deg_{LS}$ stands for the Leray-Schauder degree). The number $\sigma(\gamma,[a,b])$ is called *parity* of γ.

The following is a local bifurcation theorem for C^1 Fredholm maps proved by Fitzpatrick and Pejsachowicz in [5].

Theorem 3.7. *Let $f\colon \mathbb{R}\times E \to F$ be a C^1 Fredholm map of index 1 with $f(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$. Assume that, for some $a < b$, $D_2f(a,0)$ and $D_2f(b,0)$ are invertible and*

$$\sigma(Df_2(\cdot,0),[a,b]) = -1.$$

Then every neighborhood of $[a,b]\times\{0\}$ contains a nontrivial solution of $f(\lambda,0) = 0$.

More recently, Fitzpatrick, Pejsachowicz and Rabier, by means of the notion of parity, introduced an oriented degree theory for (nonlinear) C^2 Fredholm maps between Banach spaces, later extended to the C^1 case by Pejsachowicz and Rabier in [11] (see [2] for a comparison between our theory and the construction of Fitzpatrick, Pejsachowicz and Rabier). The following is a global bifurcation theorem proved in [11] by Pejsachowicz and Rabier by means of their degree theory for C^1 maps (a similar result for the C^2 case was previously obtained by Fitzpatrick, Pejsachowicz and Rabier in [7]).

Theorem 3.8. *Let $f\colon \mathbb{R}\times E \to F$ be a C^1 Fredholm map of index 1 with $f(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$. Assume that, for some $a < b$, $D_2f(a,0)$ and $D_2f(b,0)$ are invertible and*

$$\sigma(Df_2(\cdot,0),[a,b]) = -1.$$

Denote by S the closure in $\mathbb{R}\times E$ of $f^{-1}(0) \setminus (\mathbb{R}\times\{0\})$ and by C the connected component of $S \cup ([a,b]\times\{0\})$ containing $[a,b]\times\{0\}$. Then, either C is noncompact or C contains a point $(\lambda_*,0)$ with $\lambda_* \notin [a,b]$.

There is a strong connection between our approach and that of Fitzpatrick, Pejsachowicz and Rabier. This is firstly due to the fact that many techniques are standard in bifurcation theory and secondly to the close link between their notion of parity and our concept of sign of an oriented operator. The following result enlightens this link (see [2]).

Proposition 3.9. *Let $\gamma\colon [a,b] \to \Phi_0(E,F)$ be continuous and such that $\gamma(a)$ and $\gamma(b)$ are isomorphisms. Then, given any orientation of γ, one has*

$$\sigma(\gamma,[a,b]) = \operatorname{sign}\gamma(a)\operatorname{sign}\gamma(b).$$

In the light of the above proposition it is clear that the results in this paper extend Theorems 3.7 and 3.8 to the not necessarily C^1 case. Moreover, the assertion about the existence of a global branch of nontrivial solutions (based on Lemma 3.3) is sharper than that in the results of Fitzpatrick, Pejsachowicz and Rabier. In our opinion, however, the interest of our approach to bifurcation, which is based on a simple notion of orientation, is that it can be easily adapted to general Banach manifolds (without additional structures). This is the argument of the next section.

4. Extension to manifolds

This final section is devoted to a generalization of the previous bifurcation results to the framework of Banach manifolds. This can be obtained by means of the extension to this context of the notions of orientability and degree given in [1] and [2]. We summarize briefly these notions.

In the sequel M and N will denote two real Banach manifolds (modeled on Banach spaces E and F respectively).

Definition 4.1. Let $g\colon M \to N$ be a C^1 Fredholm map of index zero. An *orientation* α of g is a continuous choice of an orientation $\alpha(x)$ of $Dg(x)$ for any $x \in M$; where continuous means that, given a selection of positive correctors $\{A_x \in \alpha(x)\}_{x\in M}$, and two local charts $\varphi\colon U \to E$ and $\psi\colon V \to F$ of M and N respectively, with $g(U) \subset V$, the family of linear operators

$$\left\{D\psi(g(\varphi^{-1}(z))) \circ A_{\varphi^{-1}(z)} \circ D\varphi^{-1}(z)\right\}_{z\in\varphi(U)}$$

defines an orientation of the composite map $\psi g\varphi^{-1}\colon \varphi(U) \to F$. The map g will be called *orientable* if it admits an orientation and *oriented* if an orientation is assigned.

Let Ω be an open subset of $\mathbb{R}\times M$. A continuous map $f\colon \Omega \to N$ is a *(continuous) family of Fredholm maps of index zero* if it is continuously differentiable with respect to the second variable and the partial derivative $D_2f(\lambda,x)\colon T_xM \to T_{f(\lambda,x)}N$ is Fredholm of index zero for every $(\lambda,x) \in \Omega$. The map f will be called *orientable* if it is possible to assign, for every $(\lambda,x) \in \Omega$, an orientation $\alpha(\lambda,x)$ to $D_2f(\lambda,x)$, which depends continuously on (λ,x). This means that, given any two local charts $\varphi\colon U \to E$ and $\psi\colon V \to F$ of M and N respectively, the map

$$(\lambda,z) \mapsto D\psi(f(\lambda,\varphi^{-1}(z))) \circ D_2f(\lambda,\varphi^{-1}(z)) \circ D\varphi^{-1}(z) \in \Phi_0(E,F),$$

which is defined in an open (possibly empty) subset $\mathcal{A}$ of $\mathbb{R}\times E$, turns out to be oriented by assigning to any $(\lambda,z) \in \mathcal{A}$ the following equivalence class of correctors:

$$(\lambda,z) \mapsto \left\{D\psi(f(\lambda,\varphi^{-1}(z))) \circ A \circ D\varphi^{-1}(z)\colon A \in \alpha(\lambda,\varphi^{-1}(z))\right\}.$$

Let us now sketch the construction of the degree for the class of oriented maps. Consider an oriented map $g\colon M \to N$. Given an element $y \in N$, we call the triple (g,M,y) *admissible* if $g^{-1}(y)$ is compact. The degree is an integer valued map whose domain is the class of all the admissible triples, and which verifies the most important properties of the classical degree theory, such as normalization, additivity and homotopy invariance (see [1] for the exact statements).

As in the flat situation, the degree is firstly defined in the special case when (g, M, y) is a *regular triple*; that is, when (g, M, y) is admissible and y is a regular value for g in M. In this case

$$\deg(g, M, y) := \sum_{x \in g^{-1}(y)} \mathrm{sign} Dg(x),$$

where, according to Definition 2.2, $\mathrm{sign} Dg(x) = 1$ if $Dg(x) \colon T_x M \to T_y N$ is naturally oriented, and $\mathrm{sign} Dg(x) = -1$ otherwise.

After this preliminary definition, the assumption that $y \in N$ is a regular value for g is removed using the same strategy as in the flat case. The details can be found in [1].

Now, given a continuous family of Fredholm maps of index zero $f \colon \Omega \to F$, assume that for $x_0 \in M$ and $y_0 \in N$ one has $f(\lambda, x_0) = y_0$ for all $(\lambda, x_0) \in \Omega$. We say that a solution (λ, x) to the equation

$$f(\lambda, x) = y_0$$

is *trivial* if $x = x_0$ and *nontrivial* otherwise. A bifurcation point of the above equation is a real number λ_0 such that $(\lambda_0, x_0) \in \Omega$ and any neighborhood of (λ_0, x_0) contains nontrivial solutions.

We close with the following global bifurcation result for continuous families of Fredholm maps of index zero between Banach manifolds. The proof, based on degree theory in Banach manifolds, is similar to that of Theorem 3.2 and, therefore, will be omitted.

Theorem 4.2. *Let $f \colon \Omega \to N$ be an oriented family of Fredholm maps of index zero such that $f(\lambda, x_0) = y_0$ for all $(\lambda, x_0) \in \Omega$. If the function $\lambda \mapsto \mathrm{sign} D_2 f(\lambda, x_0)$ changes sign at some λ_0, then there exists a connected set of nontrivial solutions of $f(\lambda, x) = y_0$ whose closure in Ω contains (λ_0, x_0) and either is noncompact or contains a point (λ_*, x_0) with $\lambda_* \neq \lambda_0$.*

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