ON GENERAL PROPERTIES OF N-TH ORDER RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Consider the second order RFDE (retarded functional differential equation) $x''(t) = f(t, x_t)$, where f is a continuous real-valued function defined on the Banach space $\mathbb{R} \times C^1([-r, 0], \mathbb{R})$. The weak assumption of continuity on f (due to the strong topology of $C^1([-r, 0], \mathbb{R})$) makes not convenient to transform this equation into a first order RFDE of the type $z'(t) = g(t, z_t)$. In fact, in this case, the associated $\mathbb{R}^2$-valued function g could be discontinuous (with the C^0-topology) and, in addition, not necessarily defined on the whole space $\mathbb{R} \times C([-r, 0], \mathbb{R}^2)$. Consequently, in spite of what happens for ODEs, the classical results regarding existence, uniqueness, and continuous dependence on data for first order RFDEs could not apply.

Motivated by this obstruction, we provide results regarding general properties, such as existence, uniqueness, continuous dependence on data and continuation of solutions of RFDEs of the type $x^{(n)}(t) = f(t, x_t)$, where f is an $\mathbb{R}^k$-valued continuous function on the Banach space $\mathbb{R} \times C^{(n-1)}([-r, 0], \mathbb{R}^k)$. Actually, for the sake of generality, our investigation will be carried out in the case of infinite delay.

1. Introduction

Delay differential equations and retarded functional differential equations (called RFDEs for short) represent a well-studied subject in view of many applications (see e.g. [1, 9, 11]). Recently, we devoted a series of papers to first and second order RFDEs on possibly noncompact manifolds, allowing also the case of infinite delay (see [2–8]). We mostly focused on the problem of existence of periodic solutions, as well as on the structure of the set of solutions of parameterized RFDEs. For such equations, we obtained global continuation results by means of topological methods. In this framework, we also performed a preliminary study in the paper [6] in which we investigated general properties of RFDEs with infinite delay on differentiable manifolds.

Here we settle in the context of Euclidean spaces, and we tackle a different but related problem regarding higher order RFDEs whose reduction to first order equations is not convenient, in spite of what happens for ODEs.

Consider, for example, the second order RFDE

$$x''(t) = -\varepsilon x'(t) + g(x_t), \tag{1.1}$$

where $\varepsilon > 0$ and $g\colon C([-r, 0], \mathbb{R}^k) \to \mathbb{R}^k$ $(r > 0)$ is a continuous function. Here, as usual when dealing with RFDEs, if $x\colon J \to \mathbb{R}^k$ is a function defined on an interval,

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given $t \in J$, x_t denotes the map $\theta \in [-r, 0] \mapsto x(t + \theta)$, whenever it makes sense, that is, whenever $[t - r, t] \subseteq J$.

Obviously, an equation (no matter how it is written) is well-defined if it is clear what is a solution. For the equation (1.1), as well as for a broader class of second order RFDEs, two different notions are prominent. The first one is the following.

Definition (C^0-solution)**.** A function $x\colon J \to \mathbb{R}^k$, defined on an interval J, is a C^0*-solution* of (1.1) if it is continuous and satisfies eventually the equality

$$x''(t) = -\varepsilon x'(t) + g(x_t),$$

meaning that there exists $\tau \in J$ ($\tau < \sup J$) such that $[\tau - r, \tau] \subseteq J$ and the equality is verified for each $t \in (\tau, +\infty) \cap J$.

The second definition of solution is given by modifying the previous one just by additionally requiring x to be of class C^1.

Definition (C^1-solution)**.** A function $x\colon J \to \mathbb{R}^k$, defined on an interval J, is a C^1*-solution* of (1.1) if it is of class C^1 and satisfies eventually the equality

$$x''(t) = -\varepsilon x'(t) + g(x_t).$$

Obviously, with any one of these notions, a solution turns out to be eventually of class C^2.

In spite of the similarity of the two notions of solution, when dealing with an initial value problem such as

$$\begin{cases} x''(t) &= -\varepsilon x'(t) + g(x_t), \quad t > \tau \\ x_\tau &= \eta, \end{cases} \tag{1.2}$$

with $\eta \in C^1([-r, 0], \mathbb{R}^k)$, the above definitions yield very divergent consequences.

If one seeks for a C^1-solution, the problem is, in some sense, well-posed, since any C^1-solution must satisfy the additional initial condition $x'(\tau) = \eta'(0)$. Therefore, under suitable assumptions on g (such as Lipschitz continuity), one gets the uniqueness in the future (i.e. for $t \geq \tau$). To see this, it is sufficient to transform the above problem into the following first order initial value problem in $\mathbb{R}^k \times \mathbb{R}^k$:

$$\begin{cases} x'(t) &= y(t), \quad t > \tau \\ y'(t) &= -\varepsilon y(t) + g(x_t), \quad t > \tau \\ x_\tau &= \eta, \\ y(\tau) &= \eta'(0). \end{cases} \tag{1.3}$$

For existence, as well as uniqueness, results regarding initial value problems such as (1.3) we suggest [14].

On the other hand, if one seeks for solutions of (1.2) according to the first definition, the problem is under-determined: it becomes well-posed for any additional condition $x'(\tau^+) = c$, with $c \in \mathbb{R}^k$ (where $x'(\tau^+)$ denotes the right derivative of x at τ), so that the uniqueness of the solution of problem (1.2) is never obtained.

Contrary to the above first notion of solution (the C^0 one), the second one is suitable for the following general second order RFDE in $\mathbb{R}^k$:

$$x''(t) = h(t, x_t, x'_t), \tag{1.4}$$

where $h\colon \mathbb{R} \times C([-r, 0], \mathbb{R}^k) \times C([-r, 0], \mathbb{R}^k) \to \mathbb{R}^k$ is a continuous function, and x'_t is a shortened form of $(x')_t$. Of course, (1.4) includes as a particular case the equation (1.1). To see this, put $h(t, \varphi, \psi) = -\varepsilon\psi(0) + g(\varphi)$.

Anyhow, if one is interested in the C^1-solutions of (1.4), it is convenient to consider the graphically simpler and more general equation

$$x''(t) = f(t, x_t), \tag{1.5}$$

where f is an $\mathbb{R}^k$-valued continuous function defined on $\mathbb{R} \times C^1([-r,0],\mathbb{R}^k)$. To see that (1.5) is, in fact, more general than (1.4), it is sufficient to define $f(t,\varphi) = h(t,\varphi,\varphi')$.

Apart its graphic simplicity, the equation (1.5) has two advantages: one is that f may be defined on $\mathbb{R} \times C^1([-r,0],\mathbb{R}^k)$ and not necessarily on $\mathbb{R} \times C([-r,0],\mathbb{R}^k)$; the other one is that the assumption of continuity of f on the Banach space $C^1([-r,0],\mathbb{R}^k)$ is a mild condition, due to the fact that the topology of this space is stronger than the one induced by $C([-r,0],\mathbb{R}^k)$.

Of course if f is defined and continuous on $\mathbb{R} \times C([-r,0],\mathbb{R}^k)$, it is, in particular, defined and continuous on the Banach space $\mathbb{R} \times C^1([-r,0],\mathbb{R}^k)$.

However, dealing with the equation (1.5) has a disadvantage: when (1.5) is converted into the first order equation $z'(t) = g(t,z_t)$ by putting $z(t) = (x(t),y(t))$ and $g(t,z_t) = (y(t), f(t,x_t))$, the associated continuous function

$$g\colon \mathbb{R} \times C^1([-r,0],\mathbb{R}^k \times \mathbb{R}^k) \to \mathbb{R}^k \times \mathbb{R}^k, \text{ given by } (t,\varphi,\psi) \mapsto (\psi(0), f(t,\varphi)),$$

could not be compatible with any $(\mathbb{R}^k \times \mathbb{R}^k)$-valued continuous function defined on $\mathbb{R} \times C([-r,0],\mathbb{R}^k \times \mathbb{R}^k)$. In other words, g could be discontinuous with the coarse topology induced on $\mathbb{R} \times C^1([-r,0],\mathbb{R}^k \times \mathbb{R}^k)$ by the containing Banach space $\mathbb{R} \times C([-r,0],\mathbb{R}^k \times \mathbb{R}^k)$. Thus, the classical existence, as well as uniqueness, results regarding initial value problems for first order RFDEs could not apply.

Our purpose is to alleviate this disadvantage by proving general properties of the equation (1.5), as well as higher order equations of the type

$$x^{(n)}(t) = f(t,x_t), \tag{1.6}$$

with $f\colon \mathbb{R} \times C^{(n-1)}([-r,0],\mathbb{R}^k) \to \mathbb{R}^k$ continuous. These equations will sometimes be associated with an initial condition of Cauchy type as $x_\tau = \eta$, with $\tau \in \mathbb{R}$ and $\eta \in C^{(n-1)}([-r,0],\mathbb{R}^k)$, obtaining results regarding existence, uniqueness and continuous dependence on data.

Actually, for the sake of generality, we will investigate the case of infinite delay, which includes the equation (1.6) as a special case.

As already pointed out, delay equations and RFDEs in Euclidean spaces have been studied by many authors from different points of view. For a general reference about RFDEs with finite delay, we suggest the monograph by Hale and Verduyn Lunel [14]. Among others, we refer also to the works of Gaines and Mawhin [12], Nussbaum [17, 18] and Mallet-Paret, Nussbaum and Paraskevopoulos [16]. For RFDEs with infinite delay we recommend the articles of Hale and Kato [13] and, more recently, of Oliva and Rocha [19], and the book by Hino, Murakami and Naito [15].

In the above papers and books, the basic properties of RFDEs in $\mathbb{R}^k$ have been investigated, as well as other related issues (e.g. characterizing the space of initial functions of RFDEs with infinite delays, see [15]). In spite of this, to the best of our knowledge, our particular point of view on higher order RFDEs has been never pursued. The main purpose of this paper is to fill this gap.

2. Preliminaries

Given $m \in \{0,1,2,\dots\}$ and $b \in \mathbb{R}$, we will denote by $BU^m((-\infty,b],\mathbb{R}^k)$ the space of all functions $x\colon (-\infty,b] \to \mathbb{R}^k$ which are bounded and uniformly continuous with their derivatives up to the order m. This is a Banach space, being a closed subset of the space $BC^m((-\infty,b],\mathbb{R}^k)$ of the C^m-functions which are bounded with their derivatives up to the order m. As usual, $BU((-\infty,b],\mathbb{R}^k)$ and $BC((-\infty,b],\mathbb{R}^k)$ stand for $BU^0((-\infty,b],\mathbb{R}^k)$ and $BC^0((-\infty,b],\mathbb{R}^k)$, respectively.

When $m > 0$, in $BC^m((-\infty,b],\mathbb{R}^k)$, and consequently in $BU^m((-\infty,b],\mathbb{R}^k)$, among the many equivalent Banach norms we consider the following:

$$\|x\| = \sup_{t\in(-\infty,b]} |x(t)| + \sup_{t\in(-\infty,b]} |x^{(m)}(t)|,$$

where here, and throughout the paper, $|\cdot|$ is the Euclidean norm of $\mathbb{R}^k$.

For simplicity's sake, the norm of any infinite-dimensional Banach space will be denoted uniquely by $\|\cdot\|$. No confusion should arise: the space whose norm is considered will be apparent from the context.

We recall that a subset Q of $BC((-\infty,b],\mathbb{R}^k)$ is precompact (i.e. totally bounded) if and only if it is bounded and given any $\varepsilon > 0$ there exists a finite covering $\mathcal{F}$ of arbitrary subsets of $(-\infty,b]$ such that the oscillation of any $\varphi \in Q$ in each $S \in \mathcal{F}$ is less than ε (see e.g. [10, Part 1, IV.6.5]). Of course, the same holds true for the subspace $BU((-\infty,b],\mathbb{R}^k)$ of $BC((-\infty,b],\mathbb{R}^k)$. Consequently, a subset Q of $BU^m((-\infty,b],\mathbb{R}^k)$ is precompact if and only if it is bounded and given any $\varepsilon > 0$ there exists a finite covering $\mathcal{F}$ of $(-\infty,b]$ such that for any $\varphi \in Q$, the oscillation of $\varphi^{(m)}$ in each $S \in \mathcal{F}$ is less than ε. Clearly, the space being complete, any precompact subset of $BU^m((-\infty,b],\mathbb{R}^k)$ is relatively compact.

Remark 2.1. Due to the fact that in $BU^m((-\infty,b],\mathbb{R}^k)$ the derivative of order m of any function is uniformly continuous, a subset of this space is totally bounded only if it is bounded and made up of functions whose m-th derivatives are equi-uniformly continuous. The converse is not true even when $m = 0$: think about a traveling wave with compact support that goes to $-\infty$ and preserves its shape.

Let $x\colon J \to \mathbb{R}^k$ be a continuous function defined on an unbounded below real interval (that is, J is either a left, open or closed, half-line, or it coincides with $\mathbb{R}$). As usual, given any $t \in J$, by $x_t\colon (-\infty,0] \to \mathbb{R}^k$ we mean the function $\theta \mapsto x(t+\theta)$.

Remark 2.2. One can easily check that the function that associates to any $(t,x) \in (-\infty,b] \times BU^m((-\infty,b],\mathbb{R}^k)$ the element $x_t \in BU^m((-\infty,0],\mathbb{R}^k)$ is continuous. Thus, in particular, given $x \in BU^m((-\infty,b],\mathbb{R}^k)$, the map $t \in (-\infty,b] \mapsto x_t$ is a continuous curve in $BU^m((-\infty,0],\mathbb{R}^k)$.

Let n be a positive integer and $f\colon \Omega \to \mathbb{R}^k$ a continuous function defined on an open subset Ω of $\mathbb{R} \times BU^{n-1}((-\infty,0],\mathbb{R}^k)$. Let us consider in $\mathbb{R}^k$ a retarded functional differential equation of order n of the type

$$x^{(n)}(t) = f(t,x_t). \tag{2.1}$$

Definition 2.3 (Solution of an equation)**.** A *solution* of the equation (2.1) is a function $x\colon J \to \mathbb{R}^k$, defined on an unbounded below interval, such that $x_t \in BU^{n-1}((-\infty,0],\mathbb{R}^k)$ for all $t \in J$, which verifies eventually the equality $x^{(n)}(t) = f(t,x_t)$. That is, x is a solution of (2.1) if there exists $\tau < \sup J$, such that, for each $t \in (\tau,+\infty) \cap J$, one has $(t,x_t) \in \Omega$ and $x^{(n)}(t) = f(t,x_t)$.

Obviously, according to Remark 2.2, any solution of the equaltion (2.1) is eventually of class C^n.

A solution of (2.1) is said to be *maximal* if it is not a proper restriction of another solution. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

Given an initial function $\eta \in BU^{n-1}((-\infty,0],\mathbb{R}^k)$ and an instant τ such that $(\tau,\eta) \in \Omega$, we will be interested in the following initial value problem:

$$\begin{cases} x^{(n)}(t) = f(t,x_t), & t > \tau \\ x_\tau = \eta. \end{cases} \tag{2.2}$$

Definition 2.4 (Solution of an initial value problem)**.** A *solution* of problem (2.2) is a solution $x\colon J \to \mathbb{R}^k$ of the equation (2.1) such that $\sup J > \tau$, $x^{(n)}(t) = f(t, x_t)$ for all $t \in (\tau, +\infty) \cap J$, and $x_\tau = \eta$.

Clearly, a function $x\colon J \to \mathbb{R}^k$, defined on an unbounded below interval, is a solution of (2.2) if and only if $\sup J > \tau$ and for all $t \in J$ one has

$$x(t) = \begin{cases} \sum_{j=0}^{n-1} \frac{(t-\tau)^j}{j!} \eta^{(j)}(0) + \int_\tau^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x_s)\, ds, & \text{if } t \geq \tau \\ \eta(t-\tau), & \text{if } t \leq \tau, \end{cases} \tag{2.3}$$

where $\eta^{(0)}(0) := \eta(0)$.

Remark 2.5. In spite of the fact that the n-th order derivative of a solution $x\colon J \to \mathbb{R}^k$ of (2.2) may not exist at $t = \tau$, the right n-th derivative $x^{(n)}(\tau_+)$ of x at τ always exists and is equal to $f(\tau, \eta)$. In fact, by definition, $x^{(n)}(\tau_+)$ is the n-th derivative at τ of the restriction x_+ of x to the interval $[\tau, +\infty) \cap J$, and this restriction, because of (2.3), is a C^n function such that $x_+^{(n)}(t) = f(t, x_t)$ for all $t \in [\tau, +\infty) \cap J$.

3. Existence

Here, our attention is devoted to the existence, global or local, of the solutions of the initial value problem (2.2).

A continuous function $f\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ will be called *strictly retarded* if there exists $\varepsilon > 0$ such that the value $f(t, \varphi)$ depends only on t and the restriction of φ to $(-\infty, -\varepsilon]$. That is, $\varphi_1(\theta) = \varphi_2(\theta)$ for all $\theta \in (-\infty, -\varepsilon]$ implies $f(t, \varphi_1) = f(t, \varphi_2)$ for all $t \in \mathbb{R}$.

Lemma 3.1. *Let $f\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ be a strictly retarded function and choose $\tau \in \mathbb{R}$. Then,*

(1) *any $\xi \in BU^{n-1}((-\infty, \tau], \mathbb{R}^k)$ is the restriction of a unique maximal solution of equation* (2.1)*;*
(2) *any maximal solution of equation* (2.1) *is defined on the whole real line.*

Proof of (1). Choose any $\xi \in BU^{n-1}((-\infty, \tau], \mathbb{R}^k)$ and let $\hat{\xi}$ be the C^{n-1} extension of ξ to the whole real line given by

$$\hat{\xi}(t) = \sum_{j=0}^{n-1} \frac{(t-\tau)^j}{j!} \xi^{(j)}(\tau), \quad \text{for } t \geq \tau$$

and, of course, $\hat{\xi}(t) = \xi(t)$ for $t \leq \tau$. Clearly $\hat{\xi}_s \in BU^{n-1}((-\infty, 0], \mathbb{R}^k)$, for any $s \in \mathbb{R}$, and the map $s \mapsto \hat{\xi}_s$ is continuous (see Remark 2.2). Thus, we may define the C^{n-1} function

$$\hat{x}(t) = \begin{cases} \hat{\xi}(t) + \int_\tau^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, \hat{\xi}_s)\, ds, & \text{if} \quad t \geq \tau \\ \hat{\xi}(t), & \text{if} \quad t \leq \tau\,. \end{cases}$$

Since f is strictly retarded, there exists $\varepsilon > 0$ such that $f(t, \varphi)$ depends only on t and the restriction of φ to $(-\infty, -\varepsilon]$. Hence, for any t in the interval $(\tau, \tau + \varepsilon]$, one has $f(t, \hat{\xi}_t) = f(t, \hat{x}_t)$ and, thus, $\hat{x}^{(n)}(t) = f(t, \hat{x}_t)$. This proves that the restriction x of $\hat{x}$ to $(-\infty, \tau + \varepsilon]$ is a solution of (2.1). Therefore, by Zorn's lemma, x is the restriction of a maximal solution of (2.1), still denoted for simplicity by x, and defined on an interval J containing $(-\infty, \tau + \varepsilon]$. Now, taking into account (2.3), again from the fact that f is strictly retarded it follows that, for any $t \in J$, the value $x(t)$ depends only on the restriction of x to $(-\infty, t - \varepsilon]$. This implies the uniqueness of the maximal solution of (2.1).

Proof of (2). Let $x\colon J \to \mathbb{R}^k$ be a maximal solution of (2.1) and, by contradiction, assume that $\sup J < +\infty$. Due to the fact that f is strictly retarded, the same argument used in the proof of (1) shows that x can be extended to a solution defined on $(-\infty, \sup J + \varepsilon)$, contradicting the maximality of x. □

Notice that, as a consequence of both the assertions of Lemma 3.1, if f is a strictly retarded function, then the initial value problem (2.2) admits exactly one *global solution* (i.e. a solution defined on the whole real line).

Theorem 3.2 (Global existence)**.** *Let* $f\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ *be a continuous function with bounded image. Then, problem* (2.2) *has a global solution.*

Proof. Let $\{\varepsilon_j\}$ be a sequence of positive numbers converging to 0 and consider the following auxiliary problem depending on $j \in \mathbb{N}$:

$$\begin{cases} x^{(n)}(t) = f(t, x_{t-\varepsilon_j}), & t > \tau \\ x_\tau = \eta. \end{cases} \tag{3.1}$$

Let $f_j\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ be defined by $f_j(t, \varphi) = f(t, \varphi_{-\varepsilon_j})$; so that in problem (3.1) the expression $f(t, x_{t-\varepsilon_j})$ can be replaced by $f_j(t, x_t)$. Clearly, f_j is a strictly retarded function. Hence, because of Lemma 3.1, problem (3.1) has a unique solution x^j defined on the whole real line. Now, observe that the restrictions to the half line $[\tau, +\infty)$ of the functions x^j are all of class C^n (see Remark 2.5) and, f having bounded image, these restrictions have equibounded n-th derivatives. Consequently, in any compact interval $[\tau, b]$ these functions are also equibounded, due to the fact that their values at τ are all equal to $\eta(0)$. Thus, taking into account that in the left half line $(-\infty, \tau]$ the functions x^j do not depend on j, applying Ascoli's Theorem to any compact interval $[\tau, b]$, and by using a standard diagonal procedure, we may assume, without loss of generality, that the sequence $\{x^j\}$ has the following properties:

- there exists a function $x\colon \mathbb{R} \to \mathbb{R}^k$ such that $x^j(t) \to x(t)$, for any $t \in \mathbb{R}$;
- $x_s \in BU^{n-1}((-\infty, 0], \mathbb{R}^k)$, for all $s \in \mathbb{R}$;
- $x^j_s \to x_s$ in the space $BU^{n-1}((-\infty, 0], \mathbb{R}^k)$, for all $s \in \mathbb{R}$.

Now observe that, as $j \to +\infty$, one actually has $x^j_{s-\varepsilon_j} \to x_s$, for any $s \in \mathbb{R}$ (see Remark 2.2). Consequently, by applying Lebesgue's Dominated Convergence Theorem in equality (2.3), with $f_j(t, x_t)$ in place of $f(t, x_t)$, we get that x is a solution of the initial value problem (2.2), proving the assertion. □

Theorem 3.3 (Local existence)**.** *Let* $f\colon \Omega \to \mathbb{R}^k$ *be a continuous function on an open subset* Ω *of* $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$ *and* $(\tau, \eta) \in \Omega$*. Then, the initial value problem* (2.2) *admits at least one solution. In particular, any maximal solution is defined on an open interval.*

Proof. Let $N \subseteq \Omega$ be a closed neighborhood of (τ, η) whose image under f is bounded. Due to the Tietze extension Theorem, the restriction $f|_N$ has a continuous extension $\hat{f}\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ with bounded image. By applying Theorem 3.2 to the problem

$$\begin{cases} x^{(n)}(t) = \hat{f}(t, x_t), & t > \tau \\ x_\tau = \eta, \end{cases}$$

we get the existence of a solution $\hat{x}$ defined on the whole real line. Because of the continuity of the map $t \mapsto (t, \hat{x}_t)$, and taking into account that $(\tau, \hat{x}_\tau) = (\tau, \eta) \in N$, one can find $\delta > 0$ such that $(t, \hat{x}_t) \in N$ for all $t \in [\tau, \tau + \delta)$. Since $f = \hat{f}$ in N, the restriction of $\hat{x}$ to the half line $(-\infty, \tau + \delta)$ is a solution of problem (2.2).

It remains to show that the domain of a maximal solution, call it x, cannot be a closed interval of the type $(-\infty, b]$. In fact, if this were the case, by applying the above argument to problem (2.2) with initial data $(\tau, \eta) = (b, x_b)$ we would get a contradiction. $\square$

4. Uniqueness

In this section we will give conditions ensuring the unique dependence on the past of the solutions of equation (2.1). We need the following folk result, whose proof is given here for the sake of completeness.

Lemma 4.1. *Let $\alpha\colon [\tau, \tau+h) \to \mathbb{R}^k$ $(0 < h \leq +\infty)$ be a C^1 function such that $\alpha(\tau) = 0$ and $|\alpha'(t)| \leq c \sup_{s\in[\tau,t]} |\alpha(s)|$ for some constant $c \geq 0$ and all $t \in [\tau, \tau+h)$. Then, $\alpha(t) \equiv 0$ in $[\tau, \tau+h)$.*

Proof. Assume the contrary. Then, without loss of generality, we may suppose that τ is such that α is nontrivial in any interval $[\tau, \tau+\delta]$, with $0 < \delta < h$. Take δ such that $\delta c < 1$ and let $t_0 \in [\tau, \tau+\delta]$ satisfy the condition $|\alpha(t_0)| = \max\limits_{s\in[\tau,\tau+\delta]} |\alpha(s)| > 0$. We have

$$|\alpha(t_0)| = |\alpha(t_0) - \alpha(\tau)| \leq (t_0 - \tau) \sup_{s\in[\tau,t_0]} |\alpha'(s)| \leq \delta c |\alpha(t_0)|.$$

Being $\delta c < 1$, the above inequality implies $\alpha(t_0) = 0$, and this is a contradiction. $\square$

Let $f\colon \Omega \to \mathbb{R}^k$ be continuous on an open subset of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Given an open set $U \subseteq \Omega$, we will say that f is *compactly Lipschitz in U with respect to the second variable* or, for short, *c-Lipschitz in U*, if, given any compact subset Q of U, there exists $L \geq 0$ such that

$$|f(t, \varphi) - f(t, \psi)| \leq L \|\varphi - \psi\|$$

for all $(t, \varphi), (t, \psi) \in Q$.

Moreover, we will say that f is *locally c-Lipschitz in Ω* if for any $(\tau, \eta) \in \Omega$ there exists an open neighborhood of (τ, η) in Ω in which f is c-Lipschitz. In spite of the fact that a locally Lipschitz function is not necessarily (globally) Lipschitz, one could actually show that if f is locally c-Lipschitz in Ω, then it is also (globally) c-Lipschitz in Ω. As a consequence, if f is C^1 or, more generally, locally Lipschitz in the second variable, then it is additionally c-Lipschitz.

Roughly speaking, Theorem 4.2 below shows that, if f is c-Lipschitz in Ω, then the future of the solutions of equation (2.1) is uniquely determined by the past. In particular, under this assumption, the initial value problem (2.2) has a unique maximal solution, which is necessarily defined on an open (unbounded below) interval, as stated in Theorem 3.3.

Theorem 4.2 (Uniqueness)**.** *Let Ω be an open subset of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$ and let $f\colon \Omega \to \mathbb{R}^k$ be c-Lipschitz. Let $x^1\colon J_1 \to \mathbb{R}^k$, $x^2\colon J_2 \to \mathbb{R}^k$ be two maximal solutions of equation* (2.1)*. If there exists $\tau \in J_1 \cap J_2$ such that $x^1(t) = x^2(t)$ for $t \leq \tau$ and $(x^i)^{(n)}(t) = f(t, x^i_t)$ for $t \in J_i$ (i=1,2), $t > \tau$, then $J_1 = J_2$ and $x^1 = x^2$.*

Proof. Since, according to Theorem 3.3, $J_1 \cap J_2$ is an open interval, there exists $h > 0$ such that $[\tau, \tau+h] \subseteq J_1 \cap J_2$. Then, each one of the sets

$$Q_i = \big\{(t, x^i_t) \in \Omega \subseteq \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) :\ t \in [\tau, \tau+h]\big\}, \quad i = 1, 2,$$

is compact, as the image of the continuous curve $t \mapsto (t, x_t^i) \in \Omega$ defined on $[\tau, \tau+h]$. Since f is c-Lipschitz in Ω, there exists $L \geq 0$, corresponding to the compact set $Q = Q_1 \cup Q_2$, such that for any $t \in [\tau, \tau+h]$ we have

$$|f(t, x_t^2) - f(t, x_t^1)| \leq L\|x_t^2 - x_t^1\|.$$

Set $\beta(t) = x^2(t) - x^1(t)$, for $t \in J_1 \cap J_2$. Then, choosing any $t \in [\tau, \tau+h]$, we get

$$|\beta^{(n)}(t)| \leq L\|\beta_t\| = L\big(\sup_{\theta \leq 0} |\beta(t+\theta)| + \sup_{\theta \leq 0} |\beta^{(n-1)}(t+\theta)|\big).$$

Consequently, since $\beta(s) = 0$ for $s \leq \tau$, one has

$$|\beta^{(n)}(t)| \leq L\big(\sup_{s \in [\tau,t]} |\beta(s)| + \sup_{s \in [\tau,t]} |\beta^{(n-1)}(s)|\big). \tag{4.1}$$

Moreover, the fact that $\beta(\tau) = \beta'(\tau) = \cdots = \beta^{(n-1)}(\tau) = 0$ implies

$$|\beta(s)| = \Big|\int_\tau^s \frac{(s-\sigma)^{n-2}}{(n-2)!}\beta^{(n-1)}(\sigma)\,d\sigma\Big| \leq M \sup_{\sigma \in [\tau,t]} |\beta^{(n-1)}(\sigma)|, \ \text{ for } s \in [\tau, t], \tag{4.2}$$

where

$$M := \int_\tau^{\tau+h} \frac{(\tau+h-\sigma)^{n-2}}{(n-2)!}\,d\sigma = \frac{h^{n-1}}{(n-1)!}.$$

Hence, by (4.1) and (4.2), we get

$$|\beta^{(n)}(t)| \leq L(M+1) \sup_{s \in [\tau,t]} |\beta^{(n-1)}(s)|.$$

Now, by applying Lemma 4.1 with $\alpha = \beta^{(n-1)}$, we obtain $\beta^{(n-1)}(t) = 0$ for all $t \in [\tau, \tau+h)$ and, thus, again by (4.2), $\beta(t) = 0$ for all $t \in [\tau, \tau+h)$.

This shows that x^1 and x^2 coincide in any right neighborhood of τ contained in the open interval $J_1 \cap J_2$. This implies $J_1 = J_2$ and, consequently, $x_1 = x_2$. In fact, if one of the intervals were strictly contained in the other, the corresponding solution would admit an extension to the bigger interval, contradicting its maximality. □

5. Continuous dependence on data

Below we will be concerned with upper semicontinuous multivalued maps. We recall that a multivalued map Ψ between two metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is said to be *upper semicontinuous* if it is compact valued and for any open subset U of $\mathcal{Y}$ the upper inverse image of U, i.e. the set $\Psi^{-1}(U) = \{x \in \mathcal{X} : \Psi(x) \subseteq U\}$, is an open subset of $\mathcal{X}$. Equivalently, Ψ is upper semicontinuous if and only if it has closed graph and sends compact sets into relatively compact sets.

The next lemma regards the continuous dependence on the initial data of the set of solutions of problem (2.2) in the case when f is globally defined with bounded image.

Lemma 5.1. *Let $f \colon \mathbb{R} \times BU^{n-1}((-\infty,0],\mathbb{R}^k) \to \mathbb{R}^k$ be a continuous function with bounded image. Then, for any $b \in \mathbb{R}$, the multivalued map*

$$\Sigma_b^f \colon (-\infty, b) \times BU^{n-1}((-\infty,0],\mathbb{R}^k) \multimap BU^{n-1}((-\infty,b],\mathbb{R}^k)$$

that associates to any (τ,η) the set

$$\Sigma_b^f(\tau,\eta) = \big\{x \in BU^{n-1}((-\infty,b],\mathbb{R}^k) : x \text{ is a solution of problem } (2.2)\big\}$$

is upper semicontinuous.

Proof. According to the characterization stated above, it is enough to show that Σ_b^f has closed graph and sends compact sets into relatively compact sets.

Let us prove first that Σ_b^f has closed graph. To this end, take (τ, η, x) in the graph G of Σ_b^f. This means that x belongs to $\Sigma_b^f(\tau, \eta)$ and, by (2.3), satisfies

$$x(t) = \begin{cases} \sum_{j=0}^{n-1} \frac{(t-\tau)^j}{j!} \eta^{(j)}(0) + \int_\tau^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x_s)\, ds, & \text{if } \tau \le t \le b \\ \eta(t-\tau), & \text{if } t \le \tau. \end{cases} \tag{5.1}$$

Define the subset F of the space

$$(-\infty, b] \times (-\infty, b) \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \times BU^{n-1}((-\infty, b], \mathbb{R}^k)$$

consisting of the quadruples (t, τ, η, x) which satisfy (5.1). Notice that F is closed, because of the continuity of the following four $\mathbb{R}^k$-valued functions involved in (5.1): $(t, \tau, \eta, x) \mapsto x(t)$, $(t, \tau, \eta, x) \mapsto \eta(t-\tau)$, $(t, \tau, \eta, x) \mapsto \sum_{j=0}^{n-1} \frac{(t-\tau)^j}{j!} \eta^{(j)}(0)$ and $(t, \tau, \eta, x) \mapsto \int_\tau^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x_s)\, ds$. The continuity of the last one, the integral function, can be deduced from the Dominated Convergence Theorem. Thus, the slices $F_t = \{(\tau, \eta, x) : (t, \tau, \eta, x) \in F\}$ of F are all closed. Consequently, so is the graph $G = \bigcap_{t \le b} F_t$ of Σ_b^f.

It remains to show that Σ_b^f sends compact sets into relatively compact sets. Take a compact set of $(-\infty, b) \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$ and observe that it is contained in a set of the type $[\alpha, \beta] \times A$, with $\beta < b$ and A a compact subset of $BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Thus, it is enough to show that the subset $K = \Sigma_b^f([\alpha, \beta] \times A)$ of $BU^{n-1}((-\infty, b], \mathbb{R}^k)$ is relatively compact. To this end, observe that K is relatively compact if and only if so are both $T_1(K)$ and $T_2(K)$, where

$$T_1 \colon BU^{n-1}((-\infty, b], \mathbb{R}^k) \to BU^{n-1}((-\infty, \alpha], \mathbb{R}^k),$$

and

$$T_2 \colon BU^{n-1}((-\infty, b], \mathbb{R}^k) \to C^{n-1}([\alpha, b], \mathbb{R}^k),$$

denote the restriction operators to the intervals $(-\infty, \alpha]$ and $[\alpha, b]$, respectively.

Let us consider first $T_1(K)$. According to Remark 2.2, the map

$$(\tau, \eta) \in [\alpha, +\infty) \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \mapsto \eta_{\alpha-\tau} \in BU^{n-1}((-\infty, 0], \mathbb{R}^k)$$

is continuous. Therefore, A being compact, so is the set

$$\big\{\eta_{\alpha-\tau} : \tau \in [\alpha, \beta], \eta \in A\big\},$$

which, up to the isometry

$$x \in BU^{n-1}((-\infty, \alpha], \mathbb{R}^k) \mapsto x_\alpha \in BU^{n-1}((-\infty, 0], \mathbb{R}^k),$$

can be identified with $T_1(K)$. Thus, $T_1(K)$ is compact.

To complete the proof, let us show that $T_2(K)$ is relatively compact. To this end, consider in $C^{n-1}([\alpha, b], \mathbb{R}^k)$ the Banach norm

$$\|x\| = |x(\alpha)| + \max_{t \in [\alpha, b]} |x^{(n-1)}(t)|.$$

Observe first that, because of the continuity of the evaluation map $x \mapsto x(\alpha)$, there exists $C > 0$ such that $|x(\alpha)| \le C$ for all $x \in K$. According to Ascoli's theorem, it remains to prove that the functions of $T_2(K)$ have equicontinuous derivatives of order $(n-1)$ on $[\alpha, b]$.

To this purpose, choose $\varepsilon > 0$ and take any $x \in K$. According to Remark 2.1, the compactness of A implies that A is bounded and its elements have equi-uniformly continuous derivatives of order $n-1$. Hence, there exists $\sigma > 0$ such that, if

$\theta_1, \theta_2 \in (-\infty, 0]$ with $|\theta_1 - \theta_2| < \sigma$ and $\eta \in A$, then $|\eta^{(n-1)}(\theta_1) - \eta^{(n-1)}(\theta_2)| < \varepsilon$. Therefore, recalling that $x(t) = \eta(t - \tau)$ for some $\eta \in A$ and all $t \le \tau$, one has

$$|x^{(n-1)}(t_1) - x^{(n-1)}(t_2)| < \varepsilon,$$

for any $t_1, t_2 \in [\alpha, \tau]$, with $|t_1 - t_2| < \sigma$.

Now, let us consider the function x in the interval $[\tau, b]$. Since f is bounded, there exists $L > 0$ such that $|x^{(n)}(t)| \le L$ for all $t \in [\tau, b]$, and, consequently, $x^{(n-1)}$ is Lipschitz continuous on $[\tau, b]$, with constant L.

Now, by taking $\delta = \min\{\sigma, \varepsilon/L\}$, we obtain

$$|x^{(n-1)}(t_1) - x^{(n-1)}(t_2)| < 2\varepsilon,$$

for any $t_1, t_2 \in [\alpha, b]$ with $|t_1 - t_2| < \delta$. Since $x \in K$ is arbitrary, this proves that $T_2(K)$ is relatively compact in $C^{n-1}([\alpha, b], \mathbb{R}^k)$. □

Our purpose now is to remove the assumptions, in Lemma 5.1, that the function f is defined on the whole space $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$ and has bounded image. More precisely, we will prove a result concerning the continuous dependence of the solutions of problem (2.2) on the initial data, in the general case in which the function f is merely continuous on an open subset Ω of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. To this end we will previously extend the validity of the notation Σ_b^f introduced in Lemma 5.1.

Take $b \in \mathbb{R}$ and assume that $(\tau, \eta) \in \Omega$, with $\tau < b$, is such that any maximal solution of (2.2) is defined up to b. In this case, and only in this case, we define the set

$$\Sigma_b^f(\tau, \eta) = \big\{x \in BU^{n-1}((-\infty, b], \mathbb{R}^k) : x \text{ is a solution of problem (2.2)}\big\}.$$

Notice that $\Sigma_b^f(\tau, \eta)$ is nonempty, whenever it is defined. In fact, in this case, the vacuous truth does not apply, due to the existence of at least one maximal solution of the initial value problem (2.2) ensured by Theorem 3.3.

As a special example, we observe that, under the assumptions of Lemma 5.1, the set $\Sigma_b^f(\tau, \eta)$ is defined for any (τ, η) in the open subset

$$\mathcal{D}_b^f = (-\infty, b) \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$$

of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Moreover, in this case, $\Sigma_b^f(\tau, \eta)$ is always compact.

Theorem 5.2 (Continuous dependence). *Let $f\colon \Omega \to \mathbb{R}^k$ be a continuous function on an open subset Ω of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Then, given $b \in \mathbb{R}$, the set*

$$\mathcal{D}_b^f = \big\{(\tau, \eta) \in \Omega : \Sigma_b^f(\tau, \eta) \text{ is defined and compact}\big\}$$

is open and the multivalued map $(\tau, \eta) \in \mathcal{D}_b^f \multimap \Sigma_b^f(\tau, \eta)$ is upper semicontinuous.

Proof. Let us show first that $\mathcal{D}_b^f$ is open. To this end, consider the multivalued map

$$\Gamma\colon (-\infty, b) \times BU^{n-1}((-\infty, b], \mathbb{R}^k) \multimap \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$$

defined by $(a, x) \multimap \big\{(t, x_t) : a \le t \le b\big\}$. Notice that Γ is compact valued, since, given (a, x) in its domain, the curve $t \in [a, b] \mapsto (t, x_t)$ is continuous, according to Remark 2.2. We claim that Γ is actually upper semicontinuous. To this purpose, let us prove that the graph of Γ is a closed subset of

$$\mathcal{T} = (-\infty, b) \times BU^{n-1}((-\infty, b], \mathbb{R}^k) \times \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$$

and that Γ maps compact sets into relatively compact sets.

Observe that the graph of Γ is equal to

$$\big\{(a, x, t, \varphi) \in \mathcal{T} : a \le t \le b, \varphi = x_t\big\},$$

which is closed because of the continuity of the maps

$$(a, x, t, \varphi) \mapsto \varphi \quad \text{and} \quad (a, x, t, \varphi) \mapsto x_t.$$

Now, any compact set in the domain of Γ is contained in another compact set of the type $[c, b] \times K$, and $\Gamma([c, b] \times K) = \{(t, x_t) : t \in [c, b], x \in K\}$ is as well compact, being the image of $[c, b] \times K$ under the continuous map $(t, x) \mapsto (t, x_t)$. Thus, Γ is upper semicontinuous, as claimed.

As a consequence of this, we get that the multivalued map

$$P_b^f : \mathcal{D}_b^f \multimap \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$$

that to any $(\tau, \eta) \in \mathcal{D}_b^f$ associates the *brush of f starting at* (τ, η),

$$P_b^f(\tau, \eta) := \Gamma\big(\{\tau\} \times \Sigma_b^f(\tau, \eta)\big),$$

is compact valued (recall that an upper semicontinuous multivalued map transforms compact sets into compact sets).

Observe that the initial point (τ, η) belongs to $P_b^f(\tau, \eta)$ whatever is $(\tau, \eta) \in \mathcal{D}_b^f$. Moreover, one has

$$P_b^f(\tau, \eta) = \big\{(t, x_t) : \tau \le t \le b,\ x \text{ is a solution in } (-\infty, b] \text{ of problem (2.2)}\big\}.$$

Thus, recalling that any solution $x \in \Sigma_b^f(\tau, \eta)$ must satisfy the condition $(t, x_t) \in \Omega$ for all $t \in [\tau, b]$, one gets $P_b^f(\tau, \eta) \subseteq \Omega$ for all $(\tau, \eta) \in \mathcal{D}_b^f$. Notice also that any brush $P_b^f(\tau, \eta)$ is a connected set. In fact, it is actually path connected, since any element in it can be joined with the initial point (τ, η). This fact will be useful later.

In order to show that $\mathcal{D}_b^f$ is open, take an arbitrary element $(\check\tau, \check\eta) \in \mathcal{D}_b^f$. We need to find a neighborhood of $(\check\tau, \check\eta)$ which is contained in $\mathcal{D}_b^f$. To this purpose consider the compact set $P_b^f(\check\tau, \check\eta)$ and let V be an open neighborhood of $P_b^f(\check\tau, \check\eta)$ whose closure $\overline{V}$ is contained in Ω and whose image $f(V)$ is bounded. Because of the Tietze Extension Theorem, the restriction $f|_{\overline{V}}$ of f to the closure of V admits a continuous extension $g \colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ with bounded image.

Now, we can apply Lemma 5.1 with the function f replaced by g, getting the upper semicontinuous multivalued map

$$\Sigma_b^g \colon \mathcal{D}_b^g \multimap BU^{n-1}((-\infty, b], \mathbb{R}^k),$$

defined on $\mathcal{D}_b^g = (-\infty, b) \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Thus, taking into account that the composition of upper semicontinuous maps is upper semicontinuous, the multivalued map

$$P_b^g \colon \mathcal{D}_b^f \multimap \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k),$$

given by $P_b^g(\tau, \eta) := \Gamma\big(\{\tau\} \times \Sigma_b^g(\tau, \eta)\big)$, is as well upper semicontinuous.

Let $(\tau, \eta) \in V$. We claim that, whenever one of the two brushes, $P_b^f(\tau, \eta)$ and $P_b^g(\tau, \eta)$, is contained in V, then so is the other one and $P_b^f(\tau, \eta) = P_b^g(\tau, \eta)$.

For sure we have $P_b^f(\tau, \eta) \cap V = P_b^g(\tau, \eta) \cap V$, since the two functions f and g agree in V. Therefore, assume, for example, that $P_b^f(\tau, \eta)$ is contained in V. Then $P_b^g(\tau, \eta) \cap V$ is a compact set, since so is $P_b^f(\tau, \eta)$. Consequently $P_b^g(\tau, \eta) \cap V = P_b^g(\tau, \eta)$, since otherwise it would be disconnected. This proves our claim.

As a first consequence of this we get that the set $P_b^g(\check\tau, \check\eta)$ is equal to $P_b^f(\check\tau, \check\eta)$ and is contained in V. Due to the upper semicontinuity of P_b^g, there exists a neighborhood U in $\mathcal{D}_b^g$ such that, for any $(\tau, \eta) \in U$, one has $P_b^g(\tau, \eta) \subseteq V$ and, consequently, $P_b^f(\tau, \eta) = P_b^g(\tau, \eta)$.

This implies that, whenever $(\tau, \eta) \in U$, any solution in $(-\infty, b]$ of problem (2.2) is as well a solution of the same problem with f replaced by g, and viceversa. Thus, because of the arbitrariness of $(\check{\tau}, \check{\eta})$ and taking into account of Lemma 5.1, one gets both that $\mathcal{D}_b^f$ is open and the multivalued map Σ_b^f is upper semicontinuous. □

Remark 5.3. In the case when the uniqueness of the solution of problem (2.2) is assumed (as e.g. in [6]), the map Σ_b^f turns out to be single valued and, by the above theorem, it is in fact continuous.

6. Continuation property

Let us now discuss the continuation property of the solutions. Our first result states that, given a solution $x\colon J \to \mathbb{R}^k$ of equation (2.1), if the curve $t \mapsto (t, x_t)$ lies eventually in a bounded and complete subset $\mathcal{C}$ of Ω such that $f(\mathcal{C})$ is bounded, then x is not a maximal solution. In this way we include some special cases that can be found in [14, Chapter 12] and [15, Chapter 2].

Theorem 6.1 (Continuation of solutions)**.** *Let $f\colon \Omega \to \mathbb{R}^k$ be a continuous function on an open subset Ω of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Let $x\colon (-\infty, b) \to \mathbb{R}^k$, $b < +\infty$, be a solution of equation* (2.1) *such that (t, x_t) belongs eventually to a complete subset $\mathcal{C}$ of Ω. If $f(\mathcal{C})$ is bounded, then x is continuable.*

Proof. Since (t, x_t) belongs eventually to $\mathcal{C}$, $f(t, x_t)$ is eventually bounded, and so is $x^{(n)}(t)$. Thus,

$$\lim_{t \to b^-} x^{(n-1)}(t)$$

exists and is finite. This implies that all the functions $x^{(j)}(t)$, $j = 1, \dots, n-1$, are eventually bounded as well, so that the limits

$$\lim_{t \to b^-} x^{(j)}(t), \quad j = 0, \dots, n-2,$$

exist and are finite. Therefore, x admits a C^{n-1} extension, call it $\bar{x}$, to the closed interval $(-\infty, b]$. Clearly, $\bar{x}_b$ belongs to $BU^{n-1}((-\infty, 0], \mathbb{R}^k)$ and

$$\lim_{t \to b^-} (t, x_t) = (b, \bar{x}_b).$$

Since $\mathcal{C}$ is complete, $(b, \bar{x}_b)$ belongs to $\mathcal{C}$ and, thus, to Ω. Consequently, because of the continuity of f, we get $\bar{x}^{(n)}(b) = f(b, \bar{x}_b)$, so that $\bar{x}$ is a solution of (2.1), which cannot be maximal according to Theorem 3.3. □

The following two corollaries are straightforward consequences of Theorem 6.1. Therefore, their proofs will be omitted.

Corollary 6.2. *Let $f\colon \Omega \to \mathbb{R}^k$ be a continuous function on an open subset Ω of $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Let $x\colon (-\infty, b) \to \mathbb{R}^k$ be a solution of equation* (2.1) *and assume that (t, x_t) belongs eventually to a compact subset of Ω. Then x is continuable.*

Corollary 6.3. *Let $f\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ be a continuous function sending bounded sets into bounded sets. If $x\colon J \to \mathbb{R}^k$ is a solution of equation* (2.1) *such that (t, x_t) is eventually bounded, then x is continuable.*

The continuation property of the solutions may fail if, in Corollary 6.3, the assumption that f sends bounded sets into bounded sets is removed (see [6] for an example of a first order RFDE).

The following consequence of Theorem 6.1 is an extension of Corollary 6.3 and can be regarded as a Kamke-type result for RFDEs.

Corollary 6.4. *Let W be an open subset of $\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k$ and set*

$$\Omega = \{(s, \varphi) \in \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) : (s, \varphi(0), \varphi^{(n-1)}(0)) \in W\}.$$

Let $f\colon \Omega \to \mathbb{R}^k$ be a continuous function sending bounded sets into bounded sets. If $x\colon (-\infty, b) \to \mathbb{R}^k$ is a solution of equation (2.1) such that $(t, x(t), x^{(n-1)}(t))$ belongs eventually to a compact subset of W, then x is continuable.

Proof. Denote by $\Phi\colon \mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k) \to \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k$ the continuous map $(s, \varphi) \mapsto (s, \varphi(0), \varphi^{(n-1)}(0))$, and observe that $\Omega = \Phi^{-1}(W)$, so that Ω is an open set. By assumption, there exist $\tau < b$ and a compact subset K of W such that $(t, x(t), x^{(n-1)}(t)) \in K$ when $\tau \le t < b$.

Let $\mathcal{C}$ denote the closure in $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$ of the set

$$\{(t, x_t), \tau \le t < b\}.$$

Notice that $\mathcal{C}$ is bounded, because of the assumption $(t, x(t), x^{(n-1)}(t)) \in K$ for t in a left neighborhood of b, and it is complete, being contained in the closed subset $\Phi^{-1}(K)$ of the Banach space $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Moreover, $\mathcal{C}$ is contained in Ω, since so is $\Phi^{-1}(K)$, and $f(\mathcal{C})$ is bounded, since f maps bounded sets into bounded sets. Therefore, all the assumptions in Theorem 6.1 are satisfied, and x is continuable. □

We point out that, if in Corollary 6.4 we take $W = \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k$, then Ω turns out to be the entire space $\mathbb{R} \times BU^{n-1}((-\infty, 0], \mathbb{R}^k)$. Consequently, in this special case, given a solution $x\colon (-\infty, b) \to \mathbb{R}^k$ of (2.1), the following two assumptions are equivalent:

- (t, x_t) is eventually bounded;
- $(t, x(t), x^{(n-1)}(t))$ belongs eventually to a compact set.

This shows that Corollary 6.3 is a special case of Corollary 6.4, as claimed above.

7. Examples

Here we give two examples showing how some initial value problems for higher order ODEs, as well as higher order RFDEs with finite delay, can be interpreted in the framework of RFDEs with infinite delay.

Example 7.1 (From ODEs to RFDEs)**.** Let $g\colon W \to \mathbb{R}^k$ be a continuous function defined on an open subset W of $\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k$ and consider the initial value problem

$$\begin{cases} x''(t) = g(t, x(t), x'(t)), & t \ge \tau \\ x(\tau) = a, \\ x'(\tau) = b, \end{cases} \tag{7.1}$$

where (τ, a, b) is a given element of W.

Let us show how this problem can be interpreted as an initial value problem of a second order RFDE with infinite delay defined on an open subset of the space $\mathbb{R} \times BU^1((-\infty, 0], \mathbb{R}^k)$.

To this end, consider the open set

$$\Omega = \big\{(t, \varphi) \in \mathbb{R} \times BU^1((-\infty, 0], \mathbb{R}^k) : (t, \varphi(0), \varphi'(0)) \in W\big\}$$

and define $f\colon \Omega \to \mathbb{R}^k$ by $f(t, \varphi) = g(t, \varphi(0), \varphi'(0))$. Choose any function η in $BU^1((-\infty, 0], \mathbb{R}^k)$ such that $\eta(0) = a$ and $\eta'(0) = b$. For example, take $\eta(\theta) = (a + \theta b)\exp(-\theta^2)$. Then, any solution $x\colon J \to \mathbb{R}^k$ of the system

$$\begin{cases} x''(t) = f(t, x_t), & t > \tau \\ x_\tau = \eta, \end{cases} \tag{7.2}$$

if restricted to the interval $J \cap [\tau, +\infty)$, is as well a solution of the initial value problem (7.1). In fact, for $t > \tau$ one has

$$x''(t) = f(t, x_t) = g(t, x_t(0), x_t'(0)) = g(t, x(t), x'(t)),$$

and for $t = \tau$ we get $x(\tau) = x_\tau(0) = \eta(0) = a$ and $x'(\tau) = x_\tau'(0) = \eta'(0) = b$.

The same argument shows that, in some sense, the converse is also true. More precisely, if $x\colon I \to \mathbb{R}^k$ is a solution of (7.1), then x can be extended to a solution of (7.2) defined on the interval $(-\infty, \tau] \cup I$. Thus, the two problems, (7.1) and (7.2), may be regarded as equivalent.

Example 7.2 (From finite to infinite delay)**.** Let $g\colon W \to \mathbb{R}^k$ be a continuous function defined on an open subset W of $\mathbb{R} \times C^{(n-1)}([-r, 0], \mathbb{R}^k)$, $r > 0$, and consider, in W, the following initial value problem with finite delay:

$$\begin{cases} x^{(n)}(t) = g(t, x_t), & t > \tau \\ x_\tau = \psi, \end{cases} \tag{7.3}$$

where $\tau \in \mathbb{R}$ and $\psi \in C^{(n-1)}([-r, 0], \mathbb{R}^k)$ are given.

The above system can also be viewed as an initial value problem with infinite delay. To see this, consider the subset of $\mathbb{R} \times BU^{(n-1)}((-\infty, 0], \mathbb{R}^k)$ given by

$$\Omega = \big\{(t, \varphi) : (t, \varphi|_{[-r,0]}) \in W\big\},$$

where $\varphi|_{[-r,0]}$ denotes the restriction of φ to the interval $[-r, 0]$. The continuity of the map $(t, \varphi) \mapsto (t, \varphi|_{[-r,0]})$ implies that Ω is an open set.

Now, define $f\colon \Omega \to \mathbb{R}^k$ by $f(t, \varphi) = g(t, \varphi|_{[-r,0]})$ and choose any function $\eta \in BU^{(n-1)}((-\infty, 0], \mathbb{R}^k)$ such that $\eta|_{[-r,0]} = \psi$. Then, as one can easily check, problem (7.3) and

$$\begin{cases} x^{(n)}(t) = f(t, x_t), & t > \tau \\ x_\tau = \eta \end{cases}$$

may be regarded as equivalent, in the sense that any solution $x\colon J \to \mathbb{R}^k$ of one of them coincides, for $t \in [\tau - r, +\infty) \cap J$, with a solution of the other.

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