GLOBAL BRANCHES OF PERIODIC SOLUTIONS FOR FORCED DELAY DIFFERENTIAL EQUATIONS ON COMPACT MANIFOLDS

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Abstract. We prove a global bifurcation result for T-periodic solutions of the T-periodic delay differential equation $x'(t) = \lambda f(t, x(t), x(t-1))$ depending on a real parameter $\lambda \geq 0$. The approach is based on the fixed point index theory for maps on ANR's.

1. Introduction

Let $M \subseteq \mathbb{R}^k$ be a smooth manifold with (possibly empty) boundary, and let

$$f : \mathbb{R} \times M \times M \to \mathbb{R}^k$$

be a continuous map which is T-periodic in the first variable and tangent to M in the second one; that is

$$f(t+T, p, q) = f(t, p, q) \in T_pM\,, \quad \forall\, (t, p, q) \in \mathbb{R} \times M \times M\,,$$

where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p. Consider the delay differential equation

$$x'(t) = \lambda f(t, x(t), x(t-1)) \tag{1.1}$$

depending on a nonnegative real parameter λ. By a *T-periodic pair* of the above equation we mean a pair (λ, x), where $\lambda \geq 0$ and $x : \mathbb{R} \to M$ is a T-periodic solution of (1.1) corresponding to λ. The set of the T-periodic pairs of (1.1) is regarded as a subset of $[0, +\infty) \times C_T(M)$, where $C_T(M)$ is the set of the continuous T-periodic maps from $\mathbb{R}$ to M with the metric induced by the Banach space $C_T(\mathbb{R}^k)$ of the continuous T-periodic $\mathbb{R}^k$-valued maps (with the standard supremum norm). A T-periodic pair (λ, x) will be called *trivial* if $\lambda = 0$. In this case x is a constant M-valued map and will be identified with a point of M.

Under the assumptions that M is compact with nonzero Euler–Poincaré characteristic, that $T \geq 1$, and that f satisfies a natural inward condition along the boundary of M (when nonempty), we prove the existence of an unbounded – with respect to λ – connected branch of nontrivial T-periodic pairs whose closure intersects the set of the trivial T-periodic pairs in a nonempty set called *set of bifurcation points*. Our result extends an analogous one of the last two authors for the undelayed case (see [6] and [7]).

This unusual notion of bifurcation goes back to Ambrosetti and Prodi: in [14] they used the expression *atypical bifurcation*, also called *co-bifurcation* in [5].

We point out that the assumption $T \geq 1$ is crucial for the method used here, based on fixed point index theory for locally compact maps on ANR's and applied to a Poincaré-type T-translation operator. In a forthcoming paper we will tackle the case $0 < T < 1$, in which the T-translation operator is not locally compact (actually, not locally condensing).

2. Preliminary results

Let M be an arbitrary subset of $\mathbb{R}^k$. We recall the notions of tangent cone and tangent space of M at a given point p in the closure $\overline{M}$ of M. The definition of tangent cone is equivalent to the classical one introduced by Bouligand in [2].

Definition 2.1. A vector $v \in \mathbb{R}^k$ is said to be *inward* to M at $p \in \overline{M}$ if there exist two sequences $\{\alpha_n\}$ in $[0, +\infty)$ and $\{p_n\}$ in M such that

$$p_n \to p \quad \text{and} \quad \alpha_n(p_n - p) \to v.$$

The set C_pM of the vectors which are inward to M at p is called the *tangent cone* of M at p. The *tangent space* T_pM of M at p is the vector subspace of $\mathbb{R}^k$ spanned by C_pM. A vector v of $\mathbb{R}^k$ is said to be *tangent* to M at p if $v \in T_pM$.

To simplify some statements and definitions we put $C_pM = T_pM = \emptyset$ whenever $p \in \mathbb{R}^k$ does not belong to $\overline{M}$ (this can be regarded as a consequence of Definition 2.1 if one replaces the assumption $p \in \overline{M}$ with $p \in \mathbb{R}^k$). Observe that T_pM is the trivial subspace $\{0\}$ of $\mathbb{R}^k$ if and only if p is an isolated point of M. In fact, if p is an accumulation point, then, given any $\{p_n\}$ in $M \backslash \{p\}$ such that $p_n \to p$, the sequence $\big\{\alpha_n(p_n - p)\big\}$, with $\alpha_n = 1/\|p_n - p\|$, admits a convergent subsequence whose limit is a unit vector.

One can show that in the special and important case when M is a *∂-manifold*, i.e. a smooth manifold with (possibly empty) boundary ∂M, then T_pM has the same dimension as M for all $p \in M$. Moreover, C_pM is a closed half-space in T_pM (delimited by $T_p\partial M$) if $p \in \partial M$, and $C_pM = T_pM$ if $p \in M \backslash \partial M$.

Let, as above, M be a subset of $\mathbb{R}^k$, and let $g : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a continuous map. We say that g is *tangent to M in the second variable* or, for short, that g is a *vector field on M* if $g(t, p, q) \in T_pM$ for all $(t, p, q) \in \mathbb{R} \times M \times M$. In particular, g will be said *inward* (to M) if $g(t, p, q) \in C_pM$ for all $(t, p, q) \in \mathbb{R} \times M \times M$. If M is a closed subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, we will say that g is *away from* $N \backslash M$ if $g(t, p, q) \notin C_p(N \backslash M)$ for all $(t, p, q) \in \mathbb{R} \times M \times M$.

Given a vector field $g : \mathbb{R} \times M \times M \to \mathbb{R}^k$ (on M), consider the following delay differential equation:

$$x'(t) = g(t, x(t), x(t-1)). \tag{2.1}$$

By a *solution* of (2.1) we mean a continuous function $x : J \to M$, defined on a (possibly unbounded) real interval with length greater than 1, which is of class C^1 on the subinterval $(\inf J + 1, \sup J)$ of J and verifies $x'(t) = g(t, x(t), x(t-1))$ for all $t \in J$ with $t > \inf J + 1$.

Given g as above and given a continuous map $\varphi : [-1, 0] \to M$, consider the following initial value problem:

$$\begin{cases} x'(t) = g(t, x(t), x(t-1)), \\ x(t) = \varphi(t), \quad t \in [-1, 0]. \end{cases} \tag{2.2}$$

A solution of this problem is a solution $x : J \to M$ of (2.1) such that $J \supseteq [-1, 0]$ and $x(t) = \varphi(t)$ for all $t \in [-1, 0]$.

The following technical lemma regards the existence of a persistent solution of problem (2.2).

Lemma 2.2. *Let M be a compact subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$ and assume that g is a vector field on M which is away from $N \backslash M$. Then problem* (2.2) *admits a solution defined on the whole half line* $[-1, +\infty)$.

Proof. First of all, notice that we may extend g to a vector field g_1 on N. Indeed, since M is closed in N, because of the Tietze Extension Theorem, g has an $\mathbb{R}^k$-valued (continuous) extension to $\mathbb{R}\times N\times N$. It is sufficient to consider the component of this extension which is tangent to N in the second variable.

Now, let us use g_1 to define a suitable new extension $\widetilde{g}:\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}^k$ of g. Let $U\subseteq\mathbb{R}^k$ be a tubular neighborhood of N and let $r:U\to N$ be the associated retraction (if N is an open set of $\mathbb{R}^k$, then $U=N$ and r is the identity). Let $\sigma:\mathbb{R}^k\to[0,1]$ be a continuous function with compact support, $\operatorname{supp}\sigma$, contained in U and such that $\sigma(p)=1$ if $p\in M$ (observe that U is an open neighborhood of M in $\mathbb{R}^k$). Define $\widetilde{g}$ by

$$\widetilde{g}(t,p,q)=\begin{cases}\sigma(p)\sigma(q)g_1(t,r(p),r(q)) & \text{if } p,q\in U,\\ 0 & \text{otherwise.}\end{cases}$$

Now, consider the following auxiliary problem depending on $n\in\mathbb{N}$:

$$\begin{cases} x'(t)=\widetilde{g}(t,x(t-\frac{1}{n}),x(t-1)), & t>0,\\ x(t)=\varphi(t), & t\in[-1,0].\end{cases} \tag{2.3}$$

Clearly problem (2.3) has a solution defined on $[-1,1/n]$ and, given a solution on $[-1,\beta]$, one can extend it to the interval $[-1,\beta+1/n]$. Thus, problem (2.3) has a global solution $x_n:[-1,+\infty)\to\mathbb{R}^k$.

Define $\mu:[0,+\infty)\to\mathbb{R}$ by

$$\mu(t)=\max\big\{\|\widetilde{g}(\tau,p,q)\|:\tau\in[0,t],\ p,q\in\operatorname{supp}\sigma\big\}.$$

Notice that μ is continuous because of the compactness of $\operatorname{supp}\sigma$. For all $n\in\mathbb{N}$ and all $t>0$, we have $\|x_n'(t)\|\le\mu(t)$ and, consequently,

$$\|x_n(t)\|\le\|\varphi(0)\|+\int_0^t\mu(s)\,ds,\quad t\ge 0.$$

Thus, by Ascoli's Theorem, we may assume that, as $n\to\infty$, $\{x_n(t)\}$ converges to a continuous function $x(t)$, uniformly on compact subsets of $[-1,+\infty)$. Because of this, $\{x_n'(t)\}$ converges to $\widetilde{g}(t,x(t),x(t-1))$, uniformly on compact subsets of $(0,+\infty)$. Therefore, by classical results, one gets $x'(t)=\widetilde{g}(t,x(t),x(t-1))$ for all $t>0$. Thus, the assertion follows if we show that $x(t)$ lies entirely in M.

Let us show first that $x(t)\in N$ for all $t\ge 0$ (this could be false if $\widetilde{g}$ were an arbitrary continuous extension of g). Clearly $x(t)$ belongs (for all $t\ge0$) to the compact subset $\operatorname{supp}\sigma$ of the tubular neighborhood U. Thus, the C^1 function

$$\delta(t)=\|x(t)-r(x(t))\|^2$$

is well defined for $t\ge0$ and verifies $\delta(0)=0$. Assume, by contradiction, that $x(t)\notin N$ for some $t>0$. This means that $\delta(t)>0$ for some $t>0$ and, consequently, its derivative must be positive at some $\tau>0$. That is,

$$\delta'(\tau)=2\big\langle x(\tau)-r(x(\tau)),\ \widetilde{g}(\tau,x(\tau),x(\tau-1))-w(\tau)\big\rangle>0,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^k$, and $w(\tau)$ is the derivative at $t=\tau$ of the curve $t\mapsto r(x(t))$. This is a contradiction since both the vectors $\widetilde{g}(\tau,x(\tau),x(\tau-1))$ and $w(\tau)$ are tangent to N at $r(x(\tau))$ and, consequently, orthogonal to $x(\tau)-r(x(\tau))$.

It remains to show that $x(t)\in M$ for all $t>0$. Let $s=\inf\{t>0:x(t)\notin M\}$, and assume by contradiction $s<+\infty$ (here we adopt the convention $\inf\emptyset=+\infty$). Let $\{t_n\}$ be a sequence converging to s and such that $x(t_n)\in N\backslash M$. Clearly $x(s)\in M$ and $t_n>s$ for all $n\in\mathbb{N}$. We have

$$\lim_{n\to\infty}\frac{x(t_n)-x(s)}{t_n-s}=x'(s)=g(s,x(s),x(s-1))\,.$$

This implies, because of the definition of tangent cone, that the vector $g(s, x(s), x(s-1))$ belongs to $C_{x(s)}(N \backslash M)$, contradicting the fact that the vector field g is away from $N \backslash M$. □

From now on M will be a compact ∂-manifold in $\mathbb{R}^k$. In this case one may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M (see e.g. [11]). It is not hard to show that a vector field g on M is away from the complement $N \backslash M$ if and only if it is *strictly inward*; meaning that g is inward and $g(t, p, q) \notin T_p \partial M$ for all $(t, p, q) \in \mathbb{R} \times \partial M \times M$.

Proposition 2.3. *Let $M \subseteq \mathbb{R}^k$ be a compact ∂-manifold and let g be an inward vector field on M. Then, problem* (2.2) *admits a solution defined on the whole half line* $[-1, +\infty)$.

Proof. As already pointed out, we may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M. Let $\nu : M \to \mathbb{R}^k$ be any strictly inward tangent vector field on M. For example, define $\nu(p)$ for any $p \in \partial M$ as the unique unitary vector belonging to $C_p M \cap T_p \partial M^{\perp}$, and then extend ν to a tangent vector field on the whole manifold M (by removing the normal component of the extension ensured by the Tietze Extension Theorem). For any $n \in \mathbb{N}$, define the strictly inward vector field $g_n : \mathbb{R} \times M \times M \to \mathbb{R}^k$ by $g_n(t, p, q) = g(t, p, q) + \nu(p)/n$, and let $x_n : [-1, +\infty) \to M$ be a solution of the initial value problem

$$\begin{cases} x'(t) = g_n(t, x(t), x(t-1)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0], \end{cases}$$

whose existence is ensured by Lemma 2.2. As in the proof of Lemma 2.2, one can show that $\{x_n(t)\}$ has a subsequence which converges (uniformly on compact subsets of $[-1, +\infty)$) to a solution of problem (2.2), and we are done. □

The following result regards uniqueness and continuous dependence on data of the solutions of problem (2.2). Its proof is standard and, therefore, will be omitted.

Proposition 2.4. *Let g be as in Proposition 2.3 and assume, moreover, that it is of class C^1. Then, problem* (2.2) *admits a unique solution on $[-1, +\infty)$. Moreover, if $\{g_n\}$ is a sequence of C^1 inward vector fields on M which converges uniformly to g and $\{\varphi_n\}$ is a sequence of continuous maps from $[-1, 0]$ to M which converges uniformly to φ, then the sequence of the solutions of the initial value problems*

$$\begin{cases} x'(t) = g_n(t, x(t), x(t-1)), & t > 0, \\ x(t) = \varphi_n(t), & t \in [-1, 0]. \end{cases}$$

converges uniformly on compact subsets of $[-1, +\infty)$ to the solution of (2.2).

3. Fixed point index

This section is devoted to summarizing the main properties of the fixed point index in the context of ANR's. Let X be a metric ANR and consider a locally compact (continuous) X-valued map k defined on a subset $\mathcal{D}(k)$ of X. Given an open subset U of X contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. It is known that to any admissible pair (k, U) we can associate an integer $\mathrm{ind}_X(k, U)$ - the *fixed point index* of k in U - which satisfies properties analogous to those of the classical Leray–Schauder degree [10]. The reader can see for instance [1], [9], [12] or [13] for a comprehensive presentation of the index theory for ANR's. As regards the connection with the homology theory we refer to standard algebraic topology textbooks (e.g. [3], [15]).

Let us summarize the main properties of the index.

i) (*Existence*) If $\mathrm{ind}_X(k,U) \neq 0$, then k admits at least one fixed point in U.
ii) (*Normalization*) If X is compact, then $\mathrm{ind}_X(k,X) = \Lambda(k)$, where $\Lambda(k)$ denotes the Lefschetz number of k.
iii) (*Additivity*) Given two open disjoint subsets U_1, U_2 of U such that any fixed point of k in U is contained in $U_1 \cup U_2$, then $\mathrm{ind}_X(k,U) = \mathrm{ind}_X(k,U_1) + \mathrm{ind}_X(k,U_2)$.
iv) (*Excision*) Given an open subset U_1 of U such that k has no fixed point in $U \backslash U_1$, then $\mathrm{ind}_X(k,U) = \mathrm{ind}_X(k,U_1)$.
v) (*Commutativity*) Let X and Y be metric ANR's. Suppose that U and V are open subsets of X and Y respectively and that $k : U \to Y$ and $h : V \to X$ are locally compact maps. Assume that one of the sets of fixed points of hk in $k^{-1}(V)$ or kh in $h^{-1}(U)$ is compact. Then, the other set is compact as well and $\mathrm{ind}_X(hk, k^{-1}(V)) = \mathrm{ind}_Y(kh, h^{-1}(U))$.
vi) (*Generalized homotopy invariance*) Let I be a compact real interval and Ω an open subset of $X \times I$. For any $\lambda \in I$, denote $\Omega_\lambda = \{x \in X : (x,\lambda) \in \Omega\}$. Let $H : \Omega \to X$ be a locally compact map such that the set $\{(x,\lambda) \in \Omega : H(x,\lambda) = x\}$ is compact. Then $\mathrm{ind}_X(H(\cdot,\lambda),\Omega_\lambda)$ is independent of λ.

The last property is actually a slight generalization (and a consequence) of the standard homotopy invariance which deals with maps defined on Cartesian products $U \times I$ (U open in X).

4. Branches of periodic solutions

From now on we will adopt the following notation. By M we mean a compact ∂-manifold in $\mathbb{R}^k$ and by $C([-1,0],M)$ the (complete) metric space of the M-valued (continuous) functions defined on $[-1,0]$ with the metric induced by the Banach space $C([-1,0],\mathbb{R}^k)$. Given $T > 0$, by $C_T(\mathbb{R}^k)$ we denote the Banach space of the continuous T-periodic maps $x : \mathbb{R} \to \mathbb{R}^k$ (with the standard supremum norm) and by $C_T(M)$ we mean the metric subspace of $C_T(\mathbb{R}^k)$ of the M-valued maps.

Let $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be an inward vector field on M which is T-periodic in the first variable. Consider the following delay differential equation depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f(t, x(t), x(t-1)). \tag{4.1}$$

We will say that $(\lambda, x) \in [0,+\infty) \times C_T(M)$ is a *T-periodic pair* (of (4.1)) if $x : \mathbb{R} \to M$ is a T-periodic solution of (4.1) corresponding to λ. A T-periodic pair of the type $(0,x)$ is said to be *trivial.* In this case the function x is constant and will be identified with a point of M, and viceversa.

A pair $(\lambda,\varphi) \in [0,+\infty) \times C([-1,0],M)$ will be called a *T-starting pair* (of (4.1)) if there exists $x \in C_T(M)$ such that $x(t) = \varphi(t)$ for all $t \in [-1,0]$ and (λ,x) is a T-periodic pair. A T-starting pair of the type $(0,\varphi)$ will be called *trivial.* Clearly, the map $\rho : (\lambda,x) \mapsto (\lambda,\varphi)$ which associates to a T-periodic pair (λ,x) the corresponding T-starting pair (λ,φ) is continuous (φ being the restriction of x to the interval $[-1,0]$). Moreover, if f is C^1, from Proposition 2.4 it follows that ρ is actually a homeomorphism between the set of T-periodic pairs and the set of T-starting pairs.

Given $p \in M$, it is convenient to regard the pair $(0,p)$ both as a trivial T-periodic pair and as a trivial T-starting pair. With this in mind, notice that the restriction of the map ρ to $\{0\}\times M \subseteq [0,+\infty)\times C_T(M)$ as domain and to $\{0\} \times M \subseteq [0,+\infty) \times C([-1,0],M)$ as codomain is the identity.

An element $p_0 \in M$ will be called a *bifurcation point* of the equation (4.1) if every neighborhood of $(0,p_0)$ in $[0,+\infty)\times C_T(M)$ contains a nontrivial T-periodic pair (i.e. a T-periodic pair (λ,x) with $\lambda > 0$). The following result provides a necessary condition for a point $p_0 \in M$ to be a bifurcation point.

Proposition 4.1. *Assume that $p_0 \in M$ is a bifurcation point of the equation* (4.1)*. Then the tangent vector field $w : M \to \mathbb{R}^k$ defined by*

$$w(p) = \frac{1}{T}\int_0^T f(t,p,p)\,dt$$

vanishes at p_0.

Proof. By assumption there exists a sequence $\{(\lambda_n, x_n)\}$ of T-periodic pairs such that $\lambda_n > 0$, $\lambda_n \to 0$, and $x_n(t) \to p_0$ uniformly on $\mathbb{R}$. Given $n \in \mathbb{N}$, since $x_n(T) = x_n(0)$ and $\lambda_n \neq 0$, we get

$$\int_0^T f(t, x_n(t), x_n(t-1))\,dt = 0,$$

and the assertion follows passing to the limit. □

Our main result (Theorem 4.6 below) provides a sufficient condition for the existence of a bifurcation point in M. More precisely, under the assumption that the Euler–Poincaré characteristic of M is nonzero, we will prove the existence of a *global bifurcating branch* for the equation (4.1); that is, an unbounded and connected set of nontrivial T-periodic pairs whose closure intersects the set $\{0\} \times M$ of the trivial T-periodic pairs. We point out that, $C_T(M)$ being bounded, a global bifurcating branch is necessarily unbounded with respect to λ. In particular, the existence of such a branch ensures the existence of a T-periodic solution of the equation (4.1) for each $\lambda \geq 0$.

Since M is an ANR, it is not difficult to show (see e.g. [4]) that the metric space $C([-1,0], M)$ is an ANR as well (clearly of the same homotopy type as M). For the sake of simplicity, from now on, the metric space $C([-1,0], M)$ will be denoted by X.

Suppose, for the moment, that f is C^1 (this assumption will be removed in Theorem 4.6). Given $\lambda \geq 0$ and $\varphi \in X$, consider in M the following delay differential (initial value) problem:

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t-1)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases} \tag{4.2}$$

When necessary, the unique solution of problem (4.2), ensured by Proposition 2.4, will be denoted by $x_{(\lambda,\varphi)}(\cdot)$ to emphasize the dependence on (λ, φ). Given $\lambda \in [0, +\infty)$, consider the Poincaré-type operator

$$P_\lambda : X \to X$$

defined as $P_\lambda(\varphi)(s) = x_{(\lambda,\varphi)}(s + T)$, $s \in [-1, 0]$. The following two propositions regard some crucial properties of P_λ.

Proposition 4.2. *The fixed points of P_λ correspond to the T-periodic solutions of the equation* (4.1) *in the following sense: φ is a fixed point of P_λ if and only if it is the restriction to* $[-1, 0]$ *of a T-periodic solution.*

Proof. (If) Obvious.

(Only if) Let $\varphi \in X$ be such that $P_\lambda(\varphi)(s) = x_{(\lambda,\varphi)}(s + T) = \varphi(s)$ for any $s \in [-1, 0]$. Define $\eta : [-1, +\infty) \to M$ by $\eta(t) = x_{(\lambda,\varphi)}(t + T)$. Then, if $t \in [-1, 0]$ we have

$$\eta(t) = x_{(\lambda,\varphi)}(t + T) = \varphi(t),$$

and, if $t > 0$,

$$\begin{aligned} \eta'(t) &= x'_{(\lambda,\varphi)}(t+T) \\ &= \lambda f(t+T, x_{(\lambda,\varphi)}(t+T), x_{(\lambda,\varphi)}(t+T-1)) \\ &= \lambda f(t, \eta(t), \eta(t-1)). \end{aligned}$$

That is, the function η is a solution of problem (4.2) and, because of the uniqueness of the solution, it follows that

$$x_{(\lambda,\varphi)}(t+T) = \eta(t) = x_{(\lambda,\varphi)}(t), \qquad t \in [-1, +\infty).$$

Consequently, the T-periodic extension of $x_{(\lambda,\varphi)}$ to $\mathbb{R}$ is a solution of (4.1). □

Proposition 4.3. *The map $P : [0, +\infty) \times X \to X$, defined by $(\lambda, \varphi) \mapsto P_\lambda(\varphi)$, is continuous. Moreover, if $T \geq 1$, then P is locally compact.*

Proof. The continuity of P is a consequence of Proposition 2.4. If $T \geq 1$, the local compactness follows from Ascoli's Theorem. □

Let us remark that in the case when $0 < T < 1$ the operator P is still continuous but not locally compact.

If $\lambda = 0$, given $\varphi \in X$, problem (4.2) becomes

$$\begin{cases} x'(t) = 0, & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases}$$

In the interval $[0, +\infty)$ the solution of this problem is the constant map $t \mapsto \varphi(0)$. Thus,

$$P_0(\varphi)(s) = \varphi(0), \quad s \in [-1, 0].$$

Hence, P_0 sends X into the subset of the constant functions (which can be identified with M), and its restriction $P_0|_M : M \to M$ coincides with the identity. By the commutativity property of the fixed point index, using the identification introduced above, we get

$$\mathrm{ind}_X(P_0, X) = \mathrm{ind}_M(P_0|_M, M).$$

Moreover, the normalization property of the fixed point index implies that

$$\mathrm{ind}_M(P_0|_M, M) = \mathrm{ind}_M(I|_M, M) = \Lambda(I|_M) = \chi(M).$$

The latter equality follows from the fact that the Lefschetz number of the identity on a compact ANR coincides with its Euler–Poincaré characteristic. Consequently,

$$\mathrm{ind}_X(P_0, X) = \chi(M). \tag{4.3}$$

The following result (see Lemma 1.4 of [8]) will play a crucial role in the proof of Lemma 4.5 and Theorem 4.6 below.

Lemma 4.4. *Let K be a compact subset of a locally compact metric space Z. Assume that any compact subset of Z containing K has nonempty boundary. Then $Z \backslash K$ contains a connected set whose closure is not compact and intersects K.*

Lemma 4.5 below regards the existence of an unbounded connected branch of nontrivial T-starting pairs for equation (4.1) which emanates from the set of the trivial T-starting pairs. In the undelayed case, the analogue of Lemma 4.5 (see [6, Theorem 1]) is in a finite dimensional context since, in that case, the Poincaré operator P_λ maps M into itself.

Since we identify M with the subset of X of the constant maps, from now on $\{0\} \times M$ will be regarded as a subset of $[0, +\infty) \times X$. Given a set $G \subseteq [0, +\infty) \times X$ and $\lambda \geq 0$, we will denote by G_λ the slice $\{x \in X : (\lambda, x) \in G\}$.

Lemma 4.5. *Let M be a compact ∂-manifold with nonzero Euler–Poincaré characteristic, and let f be a C^1 inward vector field on M which is T-periodic in the first variable, with $T \geq 1$. Then, the equation* (4.1) *admits a connected branch of nontrivial T-starting pairs whose closure in the set of the T-starting pairs is not compact and intersects* $\{0\} \times M$.

Proof. Let

$$S = \big\{(\lambda, \varphi) \in [0, +\infty) \times X : (\lambda, \varphi) \text{ is a } T\text{-starting pair of (4.1)}\big\}.$$

Notice that, as a consequence of Proposition 4.3, the set S is locally compact. Moreover, the slice S_0 coincides with M (regarded as the set of constant functions from $[-1, 0]$ to M).

We apply Lemma 4.4 with $\{0\} \times M$ in place of K and with S in place of Z. Assume, by contradiction, that there exists a compact set $\widehat{S} \subseteq S$ containing $\{0\} \times M$ and with empty boundary in S. Thus, $\widehat{S}$ is also an open subset of the metric space S. Hence, there exists a bounded open subset U of $[0, +\infty) \times X$ such that $\widehat{S} = U \cap S$. Since $\widehat{S}$ is compact, the generalized homotopy invariance property of the fixed point index implies that $\mathrm{ind}_X(P_\lambda, U_\lambda)$ does not depend on $\lambda \in [0, +\infty)$. Moreover, the slice $\widehat{S}_\lambda = U_\lambda \cap S_\lambda$ is empty for some λ. This implies that $\mathrm{ind}_X(P_\lambda, U_\lambda) = 0$ for any $\lambda \in [0, +\infty)$ and, in particular, $\mathrm{ind}_X(P_0, U_0) = 0$.

Now, since U_0 is an open subset of X containing M, by the excision property of the fixed point index, taking into account equality (4.3), we get that

$$\mathrm{ind}_X(P_0, U_0) = \mathrm{ind}_X(P_0, X) = \chi(M) \neq 0,$$

which is a contradiction. Therefore, because of Lemma 4.4, there exists a connected subset of S whose closure in S intersects $\{0\} \times M$ and is not compact. □

Let S denote the set of the T-starting pairs of (4.1) and let $A \subseteq S$ be a connected branch of nontrivial T-starting pairs as in the assertion of Lemma 4.5. Since the map $P : (\lambda, \varphi) \mapsto P_\lambda(\varphi)$ is continuous, S is a closed subset of $[0, +\infty) \times X$ and, consequently, the closure $\overline{A}$ of A in S is the same as in $[0, +\infty) \times X$. Thus, $\overline{A}$ cannot be bounded since, otherwise, it would be compact because of Ascoli's Theorem. Moreover, since X is bounded, the set A is necessarily unbounded in λ. This implies, in particular, that, under the assumption that f is C^1, the equation (4.1) has a T-periodic solution for any $\lambda \geq 0$.

In Theorem 4.6 below, which deals with T-periodic pairs instead of T-starting pairs, the inward vector field f is assumed to be merely continuous. Under the assumption that the Euler–Poincaré characteristic of M is nonzero, the result asserts the existence of a global bifurcating branch of nontrivial T-periodic pairs, which, $C_T(M)$ being bounded, must be unbounded with respect to λ.

Theorem 4.6. *Let M be a compact ∂-manifold with nonzero Euler–Poincaré characteristic, and let f be an inward vector field on M, T-periodic in the first variable, with $T \geq 1$. Then, the equation* (4.1) *admits an unbounded connected set of nontrivial T-periodic pairs whose closure meets the set of the trivial T-periodic pairs.*

Proof. The proof will be divided into two steps. In the first one f is assumed to be C^1 (so that Lemma 4.5 applies) and in the second one f is merely continuous.

Step 1. Assume that f is of class C^1. Let $\Sigma \subseteq [0,+\infty) \times C_T(M)$ denote the set of the T-periodic pairs of (4.1) and $S \subseteq [0,+\infty) \times X$ the set of the T-starting pairs (of the same equation). Let $A \subseteq S$ be a connected branch of nontrivial T-starting pairs as in the assertion of Lemma 4.5. As already pointed out, the map $\rho : \Sigma \to S$, which associates to any T-periodic pair (λ, x) the corresponding T-starting pair (λ, φ), is a homeomorphism. Moreover, the restriction of ρ to $\{0\} \times M \subseteq \Sigma$ as domain and to $\{0\} \times M \subseteq S$ as codomain is the identity. Thus, the subset $\rho^{-1}(A)$ of Σ is connected, made up of nontrivial T-periodic pairs, its closure in Σ is not compact and meets the set $\{0\} \times M$ of the trivial T-periodic pairs. One can easily check that Σ is closed in $[0,+\infty) \times C_T(M)$ and, because of Ascoli's Theorem, any bounded subset of Σ is relatively compact. Thus $\rho^{-1}(A)$ must be unbounded and its closure in Σ is the same as in $[0,+\infty) \times C_T(M)$.

Step 2. Suppose now that f is continuous and let, as in the previous step, Σ denote the set of the T-periodic pairs of (4.1). As already pointed out, Σ is a closed, locally compact subset of $[0,+\infty)\times C_T(M)$.

We apply Lemma 4.4 with $\{0\} \times M$ in place of K and with Σ in place of Z. Assume, by contradiction, that there exists a compact set $\widehat{\Sigma} \subseteq \Sigma$ containing $\{0\} \times M$ and with empty boundary in the metric space Σ. Thus, $\widehat{\Sigma}$ is also an open subset of Σ and, consequently, both $\widehat{\Sigma}$ and $\Sigma \backslash \widehat{\Sigma}$ are closed in $[0,+\infty)\times C_T(M)$. Hence, there exists a bounded open subset W of $[0,+\infty) \times C_T(M)$ such that $\widehat{\Sigma} \subseteq W$ and $\partial W \cap \Sigma = \emptyset$.

Let now $\{f_n\}$ be a sequence of C^1 inward vector fields on M, T-periodic in the first variable, and such that $\big\{f_n(t,p,q)\big\}$ converges to $f(t,p,q)$ uniformly on $[0,T] \times M \times M$. Given any $n \in \mathbb{N}$, let Σ_n denote the set of the T-periodic pairs of the equation

$$x'(t) = \lambda f_n(t, x(t), x(t-1)).$$

Since W is bounded and contains $\{0\} \times M$, the previous step implies that for any $n \in \mathbb{N}$ there exists a pair $(\lambda_n, x_n) \in \Sigma_n \cap \partial W$. We may assume $\lambda_n \to \lambda_0$ and, by Ascoli's Theorem, $x_n(t) \to x_0(t)$ uniformly. Since $\big\{\lambda_n f_n(t,p,q)\big\}$ converges to $\lambda_0 f(t,p,q)$ uniformly on $[0,T] \times M \times M$, $x_0(t)$ is a T-periodic solution of the equation

$$x'(t) = \lambda_0 f(t, x(t), x(t-1)).$$

That is, (λ_0, x_0) is a T-periodic pair of (4.1) and, consequently, (λ_0, x_0) belongs to $\partial W \cap \Sigma$, which is a contradiction. Therefore, by Lemma 4.4 one can find a connected branch C of nontrivial T-periodic pairs of (4.1) whose closure in Σ (which is the same as in $[0,+\infty) \times C_T(M)$) intersects $\{0\} \times M$ and is not compact. Finally, C cannot be bounded since, otherwise, because of Ascoli's Theorem, its closure would be compact. This completes the proof. □

Observe that from Proposition 4.1 and Theorem 4.6 we can deduce the following well known consequence of the Poincaré–Hopf Theorem: *If w is an inward tangent vector field on a compact ∂-manifold with nonzero Euler–Poincaré characteristic, then w must vanish at some point.*

5. Examples

In this section we give three examples illustrating how our main result applies. In the first one $M \subseteq \mathbb{R}^k$ is the closure of an open ball; in the second one M is an annulus in $\mathbb{R}^{2n+1}$; and in the third one M is a (two dimensional) sphere in $\mathbb{R}^3$. As before, any point $p \in M$ will be identified with the constant function which assigns p to any $t \in \mathbb{R}$. All the maps are tacitly assumed to be continuous.

Example 5.1. Let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be T-periodic in the first variable, with $T \geq 1$. Assume that the inner product $\langle f(t,p,q), p\rangle$ is negative for $\|p\|$ large and all $(t,q) \in \mathbb{R} \times \mathbb{R}^k$.

Let us prove that the equation

$$x'(t) = \lambda f(t, x(t), x(t-1)) \tag{5.1}$$

admits a connected branch of T-periodic pairs $(\lambda, x) \in (0,+\infty) \times C_T(\mathbb{R}^k)$ which is unbounded with respect to λ and whose closure in $[0,+\infty) \times C_T(\mathbb{R}^k)$ contains a pair of the type $(0, p_0)$ with $p_0 \in \mathbb{R}^k$ such that $w(p_0) = 0$, where $w : \mathbb{R}^k \to \mathbb{R}^k$ is the average wind velocity defined by

$$w(p) = \frac{1}{T}\int_0^T f(t,p,p)\,dt.$$

By assumption, there exists $r > 0$ such that $\langle f(t,p,q), p\rangle$ is negative for $\|p\| = r$ and all $(t,q) \in \mathbb{R}\times\mathbb{R}^k$. Let $M = \overline{B(0,r)}$, where $B(0,r)$ denotes the open ball in $\mathbb{R}^k$ centered at 0 with radius r. Clearly, f is an inward vector field on M (it is actually strictly inward). Moreover, $\chi(M) = 1$ since M is contractible. Hence, Proposition 4.1 and Theorem 4.6 apply to the equation (5.1).

Example 5.2. Let $k \in \mathbb{N}$ be odd and let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be T-periodic in the first variable, with $T \geq 1$. Assume that $f(t,p,q)$ is centrifugal for $\|p\| > 0$ small and centripetal for $\|p\|$ large.

Let us show how Theorem 4.6 applies to prove that the equation

$$x'(t) = f(t, x(t), x(t-1))$$

has a T-periodic solution $x(t)$ satisfying the condition $x(t) \neq 0$ for all $t \in \mathbb{R}$. Incidentally, observe that the above equation admits the trivial solution since, f being continuous, as a consequence of the centrifugal hypothesis on f we must have $f(t,0,q) = 0$ for all $(t,q) \in \mathbb{R} \times \mathbb{R}^k$.

Because of the centrifugal and centripetal assumptions, there exist $r_1, r_2 > 0$, with $r_1 < r_2$, such that for all $(t,q) \in \mathbb{R}\times\mathbb{R}^k$ the inner product $\langle f(t,p,q), p\rangle$ is positive when $\|p\| = r_1$ and negative when $\|p\| = r_2$. Let M be the annulus $\overline{B(0,r_2)}\backslash B(0,r_1)$. Clearly, f is an inward vector field on M. Moreover, $\chi(M) = 2$ since M is homotopically equivalent to the (even dimensional) sphere S^{k-1}. Hence, Theorem 4.6 implies that, for any $\lambda \geq 0$, the equation

$$x'(t) = \lambda f(t, x(t), x(t-1))$$

has a solution lying on the annulus M.

In the above example, the assumption that the dimension k is odd cannot be removed. In fact, if k is any even natural number, we may define a centrifugal-centripetal vector field $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ by

$$f(t,p,q) = Ap + (1 - \|p\|)p,$$

where A is the $k\times k$ matrix associated with the linear operator $(p_1, p_2, \ldots, p_k) \mapsto (-p_2, p_1, \ldots, -p_k, p_{k-1})$. Observe that f is an autonomous (and undelayed) vector field; therefore, given any $T > 0$, it may be regarded as T-periodic. However, all the periodic solutions of

$$x' = Ax + (1 - \|x\|)x$$

have period 2π since they are as well solutions of the linear differential equation $x' = Ax$. In fact, because of the centrifugal-centripetal property of f, they must lie in the unit sphere S^{k-1}.

Example 5.3. Consider the following system of delay differential equations:

$$\begin{cases} x_1'(t) = -x_2(t)x_3(t-1) \\ x_2'(t) = x_1(t)x_3(t-1) - x_3(t)\sin t \\ x_3'(t) = x_2(t)\sin t\,. \end{cases}$$

Let us show that this system has a 2π-periodic solution lying on the unit sphere S^2 of $\mathbb{R}^3$.

Let $f : \mathbb{R} \times S^2 \times S^2 \to \mathbb{R}^3$ be defined by

$$f(t, p, q) = (-p_2 q_3,\, p_1 q_3 - p_3 \sin t,\, p_2 \sin t),$$

where $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ belong to S^2. Clearly, f is an inward vector field on S^2, since $\partial S^2 = \emptyset$ and $\langle f(t, p, q), p\rangle = 0$ for all $(t, q) \in \mathbb{R} \times S^2$. Moreover, it is 2π-periodic with respect to $t \in \mathbb{R}$. We need to prove that the equation

$$x'(t) = \lambda f(t, x(t), x(t - 1))$$

admits a 2π-periodic solution (on S^2) for $\lambda = 1$. This is a consequence of Theorem 4.6, since $\chi(S^2) = 2$.

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