ON THE DEGREE FOR ORIENTED QUASI-FREDHOLM MAPS: ITS UNIQUENESS AND ITS EFFECTIVE EXTENSION OF THE LERAY–SCHAUDER DEGREE

PIERLUIGI BENEVIERI, ALESSANDRO CALAMAI, AND MASSIMO FURI

ABSTRACT. In a previous paper, the first and third author developed a degree theory for oriented locally compact perturbations of C^1 Fredholm maps of index zero between real Banach spaces. In the spirit of a celebrated Amann–Weiss paper, we prove that this degree is unique if it is assumed to satisfy three axioms: Normalization, Additivity and Homotopy invariance. Taking into account that any compact vector field has a canonical orientation, from our uniqueness result we shall deduce that the above degree provides an effective extension of the Leray–Schauder degree.

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1. Introduction

In [6] the first and third author, by means of the celebrated "finite reduction method" (see [9] and references therein), developed a degree theory for (oriented) locally compact perturbations of C^1 Fredholm maps of index zero between real Banach spaces, called (oriented) *quasi-Fredholm maps* for short. Fundamental for the construction of this degree is the simple notion of orientation that they introduced in [4, 5] for C^1 Fredholm maps of index zero between real Banach manifolds and the adaptation in [6] of this concept to quasi-Fredholm maps.

In their celebrated paper [1] of 1973, Amann and Weiss showed that both the Brouwer degree and the Leray–Schauder degree are uniquely determined by three properties, namely *Normalization*, *Additivity* and *Homotopy invariance*, which they considered as axioms. As pointed out in [1], the uniqueness of the Brouwer degree had been previously established by Führer (see [14] and [15]).

In this paper, following the general spirit of Amann–Weiss, we obtain an analogous result concerning the degree for oriented quasi-Fredholm maps. Namely, Theorem 6.1 below, which asserts that there exists at most one integer-valued map, defined on the class of the admissible pairs, satisfying a

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specific Normalization property (stated for naturally oriented invertible linear operators) with the more classical Additivity and Homotopy invariance properties. Actually, our uniqueness result holds true even if the degree is regarded as a real-valued function. In fact, the image of such a map must be contained in $\mathbb{Z}$ (precisely, it turns out to coincide with $\mathbb{Z}$).

We point out that several authors tried to extend the validity of the Amann–Weiss axioms to different degree theories. Let us mention the paper [22] by Nussbaum about noncompact vector fields, the article [7] by the first and third author regarding oriented C^1 Fredholm maps of index zero between real Banach manifolds, and the paper by the third author with Pera and Spadini [16] about the fixed point index on differentiable manifolds.

As far as we know, the question of determining a degree for nonlinear Fredholm maps of index zero traced back to the pioneering works of Caccioppoli [10] and Smale [29], who independently defined a modulo 2 degree.

Since the decade of 1970, many authors addressed the problem of defining an integer-valued degree for Fredholm maps. Among them we cite Elworthy and Tromba [11, 12] and Fitzpatrick, Pejsachowicz and Rabier [13] who defined a notion of degree for C^2 Fredholm maps between real Banach manifolds. The definition in [11, 12] was obtained by introducing a concept of orientation for real Banach manifolds (based on the rather unnatural concept of Fredholm structure), and the one in [13] by defining, for the first time, a notion of orientation for Fredholm maps between real Banach manifolds.

Regarding the integer-valued degree in the C^1 case, as far as we know, the first approach was presented by Borisovic, Zvjagin and Sapronov in the survey paper [9] (see also the papers [18, 32, 33, 34, 36]). The construction in [9] is based on a finite-dimensional reduction method developed by Sapronov [28] and which goes back to Caccioppoli [10]. Such an approach avoids the use of the Sard-Smale theorem and hence needs only the assumption of C^1-differentiability. Yet, in [9] the crucial concept of orientation for Banach manifolds still relies on Fredholm structures.

Later, an integer-valued degree for oriented C^1 Fredholm maps of index zero was introduced in two independent papers: in [26] by Pejsachowicz and Rabier for maps between Banach spaces and in [4] by the first and third author for maps between Banach manifolds. Concerning the notions of orientability in [13, 26] and in [4], we stress that these concepts are not equivalent in the non-flat framework of Banach manifolds, even if, as shown in [24] by Pejsachowicz, they agree in Banach spaces. However, regarding the orientation (and not merely the orientability), the two concepts are not equivalent even in the flat case. In fact, according to the definition in [4], any orientable map has at least two different orientations (and exactly two if its domain is connected), but following the notion in [13] this is not so for a constant map from $\mathbb{R}^n$ into itself. Moreover, we point out that, in [4], thanks to the simplicity of the notion of orientation, the construction of degree does not require any Leray–Schauder theory, and the invariance of degree under a homotopy $(x, \lambda) \mapsto H(x, \lambda)$ holds under the minimal assumption that H is continuous and continuously differentiable with respect to the first variable, plus the hypothesis that the partial derivative with respect to the first variable $(x, \lambda) \mapsto \partial_1 H(x, \lambda)$ is an oriented map (in the sense of Definition 3.5 below).

Recall that a compact vector field is a Fredholm map of index zero only if it is C^1. Hence, a remarkable motivation to consider quasi-Fredholm maps is to provide a full extension of the Leray–Schauder degree. In [19] Mawhin extended the Leray–Schauder approach by defining a coincidence degree for compact perturbations of a linear Fredholm operator of index zero. As far as we know, in his paper, a purely algebraic notion of orientation of a non-invertible linear Fredholm operator of index zero appears for the first time.

Independent definitions of degree for quasi-Fredholm maps have been given in [35] by Zvyagin and Ratiner (making use of the notion of Fredholm structure), in [27] by Rabier and Salter (for oriented maps), and in the already mentioned article [6] that, inexplicably, was kept frozen for too long by the journal (it was received on december 16, 2003). More recently, a further generalization has been developed by Väth [31] in the framework of multivalued maps (see also [23]).

In the setting of quasi-Fredholm maps, a crucial point is the lack of a universally accepted notion of orientation. Here we will follow the simple approach introduced in [4, 5, 6] and pursued in [31].

As already pointed out, in the very general and quite comprehensive monograph [31], Väth extends the degree for quasi-Fredholm maps in the setting of multivalued maps. This degree is defined (see [31, Definition 13.1.13]) by means of three properties which can be thought as axioms, one of them being a finite-dimensional reduction. However, these properties, from which Väth deduces the uniqueness of the degree, are different from the axioms that we consider here, which are a natural extension of the Amann–Weiss ones.

In the last section of this paper we make use of Theorem 6.1 as well as the uniqueness result by Amann and Weiss to prove that the degree for oriented quasi-Fredholm maps introduced in [6] provides an effective generalization of the Leray–Schauder degree. For this purpose, we show that it is possible to identify, in a canonical way (related to what in [4, 5, 6] is called the *natural orientation* of the identity map), the class of the Leray–Schauder admissible pairs with a subclass of the pairs which are admissible for the degree of the oriented quasi-Fredholm maps. With this identification, the restriction of the last degree to this subclass coincides with the Leray–Schauder degree.

As a final remark, we stress that in [2, 3] we defined a concept of topological degree for a special class of oriented noncompact perturbations of nonlinear Fredholm maps of index zero, called *α-Fredholm maps*. The definition of these maps is related to the Kuratowski measure of noncompactness and extends the degree in [21] by Nussbaum. The α-Fredholm maps are of the type $f = g - k$, where we require the noncompact perturbation k to have some suitable property of relative compactness with respect to the oriented C^1 Fredholm map g. While the degree of an oriented quasi-Fredholm map f is independent of the representation $f = g - k$, g being a smoothing map of f, it is not clear if the degree for α-Fredholm maps depends on the representation. We leave this study to further investigation.

2. Preliminaries

Let $f\colon X \to Y$ be a continuous map between topological spaces. We recall that f is said to be *compact* if its image, $\operatorname{Img} f := f(X)$, is relatively compact in Y. Thus, f is called *locally compact* if for any $x \in X$ there exists a neighborhood U of x such that the restriction $f|_U$ is compact.

Assume now that X and Y are metric spaces and $f\colon X \to Y$ is continuous. The map f is called *completely continuous* if it is compact on any bounded subset of X. The map f is said to be *proper* if $f^{-1}(K)$ is compact for any compact subset K of Y and *locally proper* if any $x \in X$ admits a closed neighborhood in which f is proper. It is easy to check that f is proper if and only if it is closed (i.e. it maps closed sets to closed sets) and $f^{-1}(y)$ is compact for any $y \in Y$.

One can verify that (when it makes sense) the sum of a proper map plus a compact map is a proper map. Thus, adding a locally proper map with a locally compact map, one gets a locally proper map.

By abuse of terminology, if E and F are Banach spaces and $L\colon E \to F$ is linear, then L is said to be a *compact (linear) operator* if it is locally compact or, equivalently (in this special case), completely continuous. Notice that a compact operator is necessarily bounded.

Let now E be a real vector space (here no additional structure is needed) and denote by I the identity on E. Let T be an endomorphism of E which is a finite dimensional linear perturbation of the identity; that is, the image of the linear operator $K = I - T$ is contained in a finite dimensional subspace E_0 of E. Thus T maps E_0 into itself and, consequently, the determinant of its restriction $T_0\colon E_0 \to E_0$ is well defined. It is easy to check that such a determinant does not depend on the choice of E_0. Thus, it makes sense to define the *determinant* of T, $\det(T)$, as the determinant of the restriction of T to any finite dimensional subspace of E containing the image of K (see [17, § III-4] and references therein). One can easily check that, as in the case when E is finite dimensional, T is invertible if and only if $\det(T) \neq 0$.

Let $L(E)$ be the vector space of the endomorphisms of E and denote by $\Psi(E)$ the affine subspace of $L(E)$ of the operators that are *admissible for the determinant*. Namely,

$$\Psi(E) = \big\{T \in L(E) : \operatorname{Img}(I - T) \text{ is finite dimensional}\big\}.$$

In many cases, a practical method for computing the determinant of an operator $T \in \Psi(E)$ is given by the following result (see [8]).

Proposition 2.1. *Let $T \in L(E)$ and let $E = E_1 \oplus E_2$. Assume that, with this decomposition of E, the matrix representation of T is of the type*

$$\begin{pmatrix} I_1 & U \\ V & S \end{pmatrix}$$

where I_1 is the identity operator on E_1. If $\dim(E_2) < +\infty$ (or, more generally, if $S \in \Psi(E_2)$ and the operators U and V have finite dimensional image), then $T \in \Psi(E)$ and $\det(T) = \det(S - VU)$.

3. Oriented Fredholm maps

In this section we recall the concept of orientability and orientation for Fredholm maps of index zero between real Banach spaces introduced by the first and third author in [4, 5]. The starting point is a concept of orientation for Fredholm linear operators of index zero between real vector spaces.

Let E and F be real vector spaces (yet, here no additional structure is needed). Let us recall that a linear operator $L\colon E \to F$ is said to be *Fredholm* (see e.g. [30]) if both $\operatorname{Ker} L$ and $\operatorname{coKer} L := F/\operatorname{Img} L$ have finite dimension. The *index* of L is defined as

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

Of course, any linear operator from $\mathbb{R}^k$ to $\mathbb{R}^s$ is Fredholm of index $k - s$.

For short, a Fredholm operator of index n will be also called a Φ_n*-operator*, or a Φ*-operator* if its index is not specified.

Given a Fredholm operator of index zero $L\colon E \to F$, a linear operator $A\colon E \to F$ is called a *corrector* of L if the following conditions hold:

- the image of A is finite-dimensional,
- $L + A$ is an isomorphism.

Notice that the set of correctors of L is nonempty. This is true, and of crucial importance in what follows, even when L does not need to be corrected (i.e. when it is invertible). On the set $\mathcal{C}(L)$ of correctors of L one has an equivalence relation as follows. Let $A, B \in \mathcal{C}(L)$ be given and consider the following automorphism of E:

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A).$$

Since the operator $K = (L+B)^{-1}(B-A)$ has finite-dimensional image, T is a finite dimensional (linear) perturbation of the identity. Thus, as pointed out in Section 2, its determinant is well defined. We say that A is *equivalent* to B or, more precisely, A is L*-equivalent* to B if

$$\det\big((L + B)^{-1}(L + A)\big) > 0.$$

As shown in [4], this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes, and this provides a concept of orientation for Fredholm operators of index zero between vector spaces.

Definition 3.1 (Algebraic orientation of a Φ_0-operator)**.** Let $L\colon E \to F$ be a Fredholm linear operator of index zero. An *orientation* of L is the choice of one of the two equivalence classes of $\mathcal{C}(L)$, and L is *oriented* when an orientation is chosen. Any of the two orientations of L is called *opposite* to the other. If L is oriented, the elements of its orientation are called the *positive correctors* of L.

The following notion of natural (and unnatural) orientation of an isomorphism will be useful throughout the paper.

Definition 3.2 (Natural algebraic orientation of an isomorphism)**.** An oriented isomorphism L is said to be *naturally oriented* if the trivial operator is a positive corrector, and we will refer to this orientation as the *natural orientation* of L. Conversely, L is *unnaturally oriented* if the trivial operator is not a positive corrector; in this case L assumes the *unnatural orientation*.

Definition 3.3 (Sign of an oriented Φ_0-operator)**.** Let $L\colon E \to F$ be a Φ_0-operator. Its *sign* is the integer

$$\operatorname{sign} L = \begin{cases} +1 & \text{if } L \text{ is invertible and naturally oriented,} \\ -1 & \text{if } L \text{ is invertible and not naturally oriented,} \\ 0 & \text{if } L \text{ is not invertible.} \end{cases}$$

From now on, E and F will denote two real Banach spaces. Any Fredholm operator between Banach spaces will be assumed to be bounded. Moreover, $L(E, F)$ will denote the Banach space of bounded linear operators from E into F and $\Phi_0(E, F)$ will be the open subset of $L(E, F)$ of the Fredholm operators of index zero. Given $L \in \Phi_0(E, F)$, the symbol $\mathcal{C}(L)$ now denotes, with a slight abuse of notation, the set of bounded correctors of L, which is still nonempty. Of course, the definition of algebraic orientation of $L \in \Phi_0(E, F)$ can be given as the choice of one of the two equivalence classes of bounded correctors of L, according to the above equivalence relation.

In the context of Banach spaces an orientation of a Fredholm operator of index zero induces an orientation to any sufficiently close operator. Precisely, consider $L \in \Phi_0(E, F)$ and a corrector A of L. Suppose that L is oriented with A positive corrector. Since the set of the isomorphisms of E into F is open in $L(E, F)$, then A is a corrector of every T in a suitable neighborhood W of L in $\Phi_0(E, F)$. Thus, any $T \in W$ can be oriented by taking A as a positive corrector. This fact allows us to give the following definition.

Definition 3.4. Let X be a topological space and $h\colon X \to \Phi_0(E, F)$ a continuous map. An *orientation* of h is a continuous choice of an orientation $\alpha(x)$ of $h(x)$ for each $x \in X$, where 'continuous' means that for any $x \in X$ there exists $A \in \alpha(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is *orientable* when it admits an orientation and *oriented* when an orientation is chosen.

It is possible to prove (see [5, Proposition 3.4]) that two equivalent correctors A and B of a given $L \in \Phi_0(E, F)$ remain T-equivalent for any T in a neighborhood of L. This implies that the notion of 'continuous choice of an orientation' in Definition 3.4 is equivalent to the following one:

> *for any $x \in X$ **and any** $A \in \alpha(x)$, there exists a neighborhood U of x such that $A \in \alpha(x')$ for all $x' \in U$.*

According to [5], the notion of continuity in the definition of oriented map can be regarded as a true continuity by introducing the following topological space (which is actually a real Banach manifold). Let $\widehat{\Phi}_0(E,F)$ denote the set of pairs (L,α) with $L \in \Phi_0(E,F)$ and α one of the two equivalence classes of $\mathcal{C}(L)$. Given an open subset W of $\Phi_0(E,F)$ and an operator $A \in L(E,F)$ with finite dimensional image, consider the set

$$O_{(W,A)} = \big\{(L,\alpha) \in \widehat{\Phi}_0(E,F) : L \in W,\ A \in \alpha\big\}.$$

The collection of sets obtained in this way is a basis for a topology on $\widehat{\Phi}_0(E,F)$ and the natural projection $p\colon (L,\alpha) \mapsto L$ is a double covering of $\Phi_0(E,F)$. Observe also that the family of the restrictions of p to the open subsets of $\widehat{\Phi}_0(E,F)$ in which p is injective is an atlas for a Banach manifold structure modeled on $L(E,F)$.

It is easy to check that the following is an alternative definition of orientation, and has the advantage that many properties of the orientable maps can be directly deduced from well known results in covering space theory.

Definition 3.5 (Topological orientation for $\Phi_0(E,F)$-valued maps)**.** Let $h\colon X \to \Phi_0(E,F)$ be a continuous map defined on a topological space X. An *orientation* of h is a lift $\widehat{h}$ of h, that is, a continuous map $\widehat{h}\colon X \to \widehat{\Phi}_0(E,F)$ such that $p\widehat{h} = h$. The map h is called *orientable* when it admits a lift, and *oriented* when one of its lifts has been chosen.

According to Definition 3.5, an orientation of h is a continuous map $\widehat{h}\colon X \to \widehat{\Phi}_0(E,F)$ of the form $\widehat{h}\colon x \mapsto (h(x),\alpha(x))$. Thus $\widehat{h}$ is completely determined by its second component α. For this reason, when it is convenient, we shall merely call α an orientation of h, which is in the spirit of Definition 3.4.

In order to make clear (as well as formally correct) our axiomatic treatment of the degree for oriented quasi-Fredholm maps, it is important to recall that, given any set Y, there exists only one function from the empty set into Y, and this is called the *empty function* to Y. This notion derives from the definition of function f from a set X to a set Y (written $f\colon X \to Y$) as a triple $f = (X,Y,\Gamma)$, where Γ, the graph of f, is a subset of $X \times Y$ with the following property:

$$\text{If } x \in X \text{ then } \exists!\, y \in Y, \text{ denoted by } f(x), \text{ such that } (x,y) \in \Gamma. \tag{3.1}$$

The first set of (X,Y,Γ) is the *domain* of f and the second is the *codomain.* The empty function into Y is the triple $(\emptyset,Y,\emptyset)$, whose graph is necessarily empty, as the unique subset of the Cartesian product $\emptyset \times Y$. Notice that, for this triple, (3.1) is satisfied, being a *vacuous truth.*

One can easily show that if $h\colon X \to \Phi_0(E,F)$ is orientable with nonempty domain, then it admits at least two orientations. If, in addition, X is connected, then h admits exactly two orientations (one opposite to the other). Of course, if X is empty, then h admits only one orientation: the empty function $\widehat{h}\colon X \to \widehat{\Phi}_0(E,F)$. Moreover, from the theory of covering spaces, one can deduce that if X is simply connected and locally path connected, then h is orientable (see e.g. [5]).

As a straightforward consequence of Definition 3.5, if $h\colon X \to \Phi_0(E,F)$ is oriented and $h'\colon Y \to X$ is any continuous map, then the composition hh' *inherits* in a natural way an orientation from h. In this case we say that the two oriented maps, as well as the corresponding orientations, are *compatible* among them. This is the case, for example, for the restriction of h to any subset X' of X, since $h|_{X'}$ is the composition of h with the inclusion $X' \hookrightarrow X$. In this case, the orientation of $h|_{X'}$ inherited by h will be called the *oriented restriction* of h to X'. Another important example occurs when $H\colon X \times [0,1] \to \Phi_0(E,F)$ is an oriented homotopy and $\lambda \in [0,1]$ is given. In this case the partial map $H_\lambda = H(\cdot,\lambda)$ inherits an orientation from H, being the composition HJ_λ, where $J_\lambda(x) = (x,\lambda)$. The following theorem shows that, in some sense, the converse is true. Such a result can be seen as a sort of continuous transport of an orientation along a homotopy, and it is a straightforward consequence of the theory of covering spaces (see [5, Theorem 3.14]).

Theorem 3.6 (Orientation transport for $\Phi_0(E,F)$-valued maps)**.** *Given a homotopy $H\colon X \times [0,1] \to \Phi_0(E,F)$, assume that for some $\lambda \in [0,1]$ the partial map $H_\lambda = H(\cdot,\lambda)$ is oriented. Then, there exists and is unique an orientation of H which is compatible with (the oriented map) H_λ. In particular, H_0 and H_1 are either both orientable or both non-orientable.*

The following consequence of Theorem 3.6, together with Corollary 3.14, will be useful in the proof of Proposition 4.3 below.

Corollary 3.7. *Let $G\colon X \times [0,1] \times [0,1] \to \Phi_0(E,F)$ be a continuous map. Assume that the partial map $G_{(0,0)} = G(\cdot,0,0)$ is oriented. Then, there exists and is unique an orientation of G which is compatible with $G_{(0,0)}$.*

Proof. Consider the homotopy $H\colon X \times [0,1] \to \Phi_0(E,F)$ given by

$$H(x,\lambda) = G(x,\lambda,0).$$

According to Theorem 3.6, H can be, and will be, oriented with the unique orientation compatible with $H_0 = G_{(0,0)}$. Now, denote by Y the product $X \times [0,1]$ and consider the homotopy $K\colon Y \times [0,1] \to \Phi_0(E,F)$ given by

$$K(y,\mu) = G(x,\lambda,\mu), \quad \text{where we put } y = (x,\lambda).$$

Observe that $K_0 = K(\cdot,0) = H$. Hence, again because of Theorem 3.6, K admits a unique orientation compatible with K_0. Consequently, identifying G with K, we get the assertion. □

Let us now recall the notion of orientability for Fredholm maps of index zero between real Banach spaces, given in [4, 5].

Recall that, given an open subset Ω of E, a map $g\colon \Omega \to F$ is a *Fredholm map* if it is C^1 and its Fréchet derivative, $Dg(x)$, is a Fredholm operator for all $x \in \Omega$. The *index* of g at x is the index of $Dg(x)$ and g is said to be of *index* n if it is of index n at any point of its domain.

Through this paper, the empty function $g\colon \emptyset \to F$ will be regarded as Fredholm of index zero. This is formally correct and convenient for us, even if it seems peculiar. In fact, the assertion "if $x \in \emptyset$ then $Dg(x)$ is Fredholm of index zero" is true, being a vacuous truth.

Hereafter, a nonlinear Fredholm map of index zero will be also called a Φ_0-*map*. Notice that a Φ_0-operator $L\colon E \to F$ is also a Φ_0-map, being differentiable at any $x \in E$ with $DL(x) = L$.

According to a result of Smale (see [29]), a Fredholm map defined on an open subset of a Banach space is locally proper.

Definition 3.8 (Topological orientation of a Φ_0-map)**.** An *orientation* of a Fredholm map of index zero $g\colon \Omega \to F$ is an orientation of the derivative $Dg\colon x \mapsto Dg(x)$, in the sense of Definition 3.5. Moreover, g is *orientable*, or *oriented*, if so is Dg.

Observe that, if $g\colon \Omega \to F$ in an oriented Φ_0-map and V is an open subset of Ω, then the restriction $g|_V$ of g to V inherits, in a natural way, an orientation from g. This restriction, with the orientation inherited by g, will be called the *oriented restriction* of g to V.

Unless otherwise stated, to avoid cumbersome notation, if g is oriented, the symbol $g|_V$ will denote the oriented restriction of g to V. Of course, if $V = \emptyset$, then $g|_V$ has only one possible orientation: the empty function

$$\widehat{Dg|_\emptyset}\colon \emptyset \to \widehat{\Phi}_0(E,F),$$

which is the unique lift of $Dg|_\emptyset\colon \emptyset \to \Phi_0(E,F)$.

We point out that if $L\colon E \to F$ is a Φ_0-operator, then it is orientable if regarded as a Φ_0-map; that is, in the sense of Definition 3.8. In fact, at any $x \in E$, the derivative $DL(x)$ coincides with L, which can be 'constantly' oriented according to Definition 3.1. Unless otherwise stated, for such an operator the two notions of orientation, the algebraic and the topological, will be identified. The same convention is assumed even when one considers the restriction $L|_\Omega$ of L to any open subset Ω of E.

Definition 3.9 (Natural topological orientation of a diffeomorphism)**.** An oriented deffeomorphism $\phi\colon U \to V$ between two open subsets $U \subseteq E$ and $V \subseteq F$ is said to be *naturally oriented* if the trivial operator is a positive corrector of $D\phi(x)$ for any $x \in U$. We will refer to this orientation as the *natural orientation* of ϕ. Conversely, ϕ is *unnaturally oriented* if the trivial operator is not a positive corrector of $D\phi(x)$ for all $x \in U$; in this case ϕ assumes the *unnatural orientation*.

The following result, in particular, gives a sufficient condition for the orientability of a Φ_0-map (see [4]).

Proposition 3.10. *Let $g\colon \Omega \to F$ be a Fredholm map of index zero. If g is orientable and Ω is nonempty, then g admits at least two orientations. If, in addition, Ω is connected, then g admits exactly two orientations (one opposite to the other). If Ω is empty, then g is orientable and admits only one orientation. If Ω is simply connected, then g is orientable.*

Let Ω be open in E and let $H\colon \Omega \times [0,1] \to F$ be a continuous map. We say that H is a *homotopy of Φ_0-maps* or, simply, a *Φ_0-homotopy* if it is continuously differentiable with respect to the first variable and any partial map H_λ is a Φ_0-map.

Definition 3.11 (Topological orientation of a Φ_0-homotopy)**.** An *orientation* of a Φ_0-homotopy $H\colon \Omega \times [0,1] \to F$ is an orientation of the partial derivative map

$$\partial_1 H\colon \Omega \times [0,1] \to \Phi_0(E,F), \quad (x,\lambda) \mapsto DH_\lambda(x),$$

according to Definition 3.5. Moreover, H is *orientable*, or *oriented*, if so is $\partial_1 H$.

Notice that, if a Φ_0-homotopy $H\colon \Omega\times[0,1] \to F$ is oriented and $\lambda \in [0,1]$ is given, then the partial map $H_\lambda\colon \Omega \to F$ inherits an orientation which is compatible with H. The following straightforward consequence of Theorem 3.6 shows that the converse is true.

Theorem 3.12 (Orientation transport for Φ_0-maps)**.** *Let $H\colon \Omega\times[0,1] \to F$ be a Φ_0-homotopy. Given $\lambda \in [0,1]$, assume that the partial map H_λ is oriented. Then there exists and is unique an orientation of H which is compatible with H_λ.*

Definition 3.13 (Induced orientation)**.** Due to Theorem 3.12, if two maps, f_0 and f_1, are joined by a Φ_0-homotopy H and f_0 is oriented, then f_1 can be oriented according to the unique orientation of H which is compatible with f_0. In this case we shall say that *"f_1 has the orientation induced by f_0 through H"* or simply that *"f_1 has the orientation directly induced by f_0"* when H is the straight-line homotopy $(x,\lambda) \mapsto \lambda f_1(x) + (1-\lambda) f_0(x)$.

The following direct consequence of Corollary 3.7 will be useful in the proof of Proposition 4.3 below.

Corollary 3.14. *Let $G\colon \Omega \times [0,1] \times [0,1] \to F$ be C^1 and assume that any $G_{(\lambda,\mu)}$ is Fredholm of index zero. If $G_{(0,0)}$ is oriented, then there exists and is unique an orientation of G which is compatible with $G_{(0,0)}$.*

We conclude this section by showing that the orientation of a Fredholm map g is related to the orientations of domain and codomain of suitable restrictions of g. This argument is crucial in the definition of the degree for oriented quasi-Fredholm maps.

Let $g\colon \Omega \to F$ be a Fredholm map of index zero and Z a finite dimensional subspace of F, transverse to g. We recall that Z is said to be *transverse* to g *at* $x \in \Omega$ if $\operatorname{Img} Dg(x) + Z = F$. The space Z is *transverse* to g if it is transverse at any point of the domain of g.

From classical transversality results, it follows that $M = g^{-1}(Z)$ is a differentiable manifold of the same dimension as Z. Assume that g is orientable. It is possible to prove that M is orientable. The proof can be found in [4, Remark 2.5 and Lemma 3.1]. Here, let us show how, given any $x \in M$, an orientation of g and an orientation of Z induce an orientation on the tangent space T_xM of M at x.

Assume that g is oriented and let Z be oriented too. Consider $x \in M$ and a positive corrector A of $Dg(x)$ with image contained in Z (the existence of such a corrector is ensured by the transversality of Z to g). Then, orient T_xM in such a way that the isomorphism

$$(Dg(x) + A)|_{T_xM}\colon T_xM \to Z$$

is orientation preserving. As proved in [6], the orientation of T_xM does not depend on the choice of the positive corrector A, but only on the orientations of Z and $Dg(x)$. With this orientation, we call M the *oriented g-preimage* of Z.

4. Oriented quasi-Fredholm maps

In this section we recall the concept of orientability and orientation, introduced by the first and third author in [6], for locally compact perturbations of Fredholm maps of index zero between real Banach spaces.

Let $f\colon \Omega \to F$ be a continuous map defined on an open subset of E. For short, we say that f is a *quasi-Fredholm map* (or a *qF-map*) if there exists a Fredholm map of index zero $g\colon \Omega \to F$, called a *smoothing map* of f, such that the difference $k = g - f$ is a locally compact map.

Notice that, if $F = E$ and f admits the identity among its smoothing maps, then f is a *locally compact vector field*.

In what follows, unless otherwise stated, f will denote a quasi-Fredholm map from an open subset Ω of E to F, and $\mathcal{S}(f)$ will stand for the family of smoothing maps of f.

Remark 4.1. Observe that, if g_0 is a smoothing map of f, then any other element of the family $\mathcal{S}(f)$ is obtained by adding to g_0 an arbitrary C^1 locally compact map $h\colon \Omega \to F$. Therefore, $\mathcal{S}(f)$ is an affine subspace of the real vector space $C(\Omega, F)$ of the continuous maps from Ω to F. Precisely, $\mathcal{S}(f) = g_0 + \mathcal{H}$, where $\mathcal{H}$ is the subspace of $C(\Omega, F)$ consisting of the C^1 locally compact maps.

The following definition provides an extension to quasi-Fredholm maps of the concept of orientability.

Definition 4.2. A quasi-Fredholm map $f\colon \Omega \to F$ is *orientable* if it has an orientable smoothing map.

If f is orientable, then any smoothing map of f is orientable. Indeed, given $g_0, g_1 \in \mathcal{S}(f)$, consider the homotopy $H\colon \Omega \times [0,1] \to F$ defined by

$$H(x,\lambda) = (1-\lambda)g_0(x) + \lambda g_1(x). \tag{4.1}$$

Since $\mathcal{S}(f)$ is a convex set (see Remark 4.1), H is Φ_0-homotopy. Thus, because of Theorem 3.12, if g_0 is orientable, then g_1 is orientable as well.

Observe also that, if a given map $g_0 \in \mathcal{S}(f)$ is oriented, then any other map $g \in \mathcal{S}(f)$ can be oriented by transporting the orientation of g_0 up to g along the line segment joining g_0 with g (i.e. applying Theorem 3.12 to the straight-line homotopy joining g_0 with g). Is it correct to define this collection of oriented maps an *orientation of the family* $\mathcal{S}(f)$? The answer is yes if for any pair of oriented maps of the collection, say g_1 and g_2, the unique orientation of the straight-line homotopy joining g_1 with g_2 which is compatible with g_1 (ensured by Theorem 3.12) is compatible also with g_2. This, as we shall see, is a direct consequence of Proposition 4.3 below. Therefore, from now on, by an *orientation of the family* $\mathcal{S}(f)$ we shall mean that to any map in $\mathcal{S}(f)$ is assigned an orientation with the following property: the orientations of any pair of smoothing maps of f are compatible with an orientation of the straight-line homotopy joining these two maps.

To define a notion of orientation of f, consider the set $\widehat{\mathcal{S}}(f)$ of the oriented smoothing maps of f. We introduce in $\widehat{\mathcal{S}}(f)$ the following equivalence relation. Given g_0, g_1 in $\widehat{\mathcal{S}}(f)$, consider, as in formula (4.1), the straight-line homotopy H joining g_0 and g_1. We say that g_0 is *equivalent* to g_1, $g_0 \sim g_1$ in symbols, if the unique orientation of H which is compatible with g_0 (ensured by Theorem 3.12) is as well compatible with g_1. In other words, according to Definition 3.13, g_0 is equivalent to g_1 if the second map has the orientation *directly induced* by the first one.

For the sake of completeness we give here the proof of Proposition 4.3 below, that in [6] was omitted.

Proposition 4.3. *The above is an equivalence relation in* $\widehat{\mathcal{S}}(f)$.

Proof. Reflexivity and symmetry are immediate to be verified. To prove transitivity, let g_0, g_1 and g_2 belong to $\widehat{\mathcal{S}}(f)$, and suppose $g_0 \sim g_1$ and $g_1 \sim g_2$.

Consider the C^1 map $G\colon \Omega \times [0,1] \times [0,1] \to F$ defined as

$$G(x,\lambda,\mu) = (1-\mu)\big((1-\lambda)g_0(x) + \lambda g_1(x)\big) + \mu g_2(x).$$

Notice that any partial map $G_{(\lambda,\mu)} = G(\cdot,\lambda,\mu)$ is a convex combination of g_0, g_1 and g_2. Consequently, because of Remark 4.1, it lies in $\mathcal{S}(f)$ and, in particular, it is Fredholm of index zero. Therefore, according to Corollary 3.14, we may assume that G is oriented with the unique orientation compatible with $G_{(0,0)} = g_0$. Let us show that this orientation is compatible with g_1 and g_2, as well.

Consider the straight-line homotopy $\overline{G}\colon \Omega \times [0,1] \to F$, given by

$$\overline{G}(x,\lambda) = (1-\lambda)g_0(x) + \lambda g_1(x) = G(x,\lambda,0),$$

and orient it with the orientation inherited from G. Because of Theorem 3.12, $\overline{G}$ is the unique oriented homotopy compatible with g_0. Since $g_0 \sim g_1$, $\overline{G}$ is as well compatible with g_1. Consequently, G, being compatible with $\overline{G}$, is compatible also with $g_1 = G_{(1,0)}$.

The same argument applies to the homotopy $\widehat{G}\colon \Omega \times [0,1] \to F$, given by

$$\widehat{G}(x,\mu) = (1-\mu)g_1(x) + \mu g_2(x) = G(x,1,\mu),$$

showing that G is compatible with $g_2 = G_{(1,1)}$.

Finally, since G is compatible with both g_0 and g_2, orienting the straight-line homotopy joining these two smoothing maps with the orientation inherited from G, we get that $g_0 \sim g_2$. □

Observe that, because of of Theorem 3.12 and Proposition 4.3, an equivalence class of $\widehat{\mathcal{S}}(f)$ may be regarded as an orientation of the family $\mathcal{S}(f)$.

The following definition provides an extension to quasi-Fredholm maps of the concept of orientation given in Definition 3.8.

Definition 4.4. Let $f\colon \Omega \to F$ be an orientable quasi-Fredholm map. An *orientation* of f is an equivalence class of $\widehat{\mathcal{S}}(f)$. In particular, if f is a locally compact vector field (i.e. $F = E$ and the identity of E is a smoothing map of f), then it admits a distinguished orientation, called *canonical*: the one determined by the natural orientation of the identity.

Obviously, if the Banach space E is finite dimensional, then any continuous map $f\colon \Omega \to E$ on an open subset of E is quasi-Fredholm. More precisely, it is a locally compact vector field and, consequently, it is orientable and, unless otherwise stated, we will assume it as canonically oriented.

In the sequel, if a quasi-Fredholm map f is oriented, any element in the chosen class of $\widehat{\mathcal{S}}(f)$ will be called a *positively oriented smoothing map* of f.

Observe that if two oriented smoothing maps of $f\colon \Omega \to F$ are equivalent and V is an open subset of Ω, then the oriented restrictions to V of these two smoothing maps are equivalent in the set $\widehat{\mathcal{S}}(f|_V)$. Thus, if f is oriented, the restriction $f|_V$ inherits, in a natural way, an orientation by f. This restriction, together with the orientation inherited by f, will be called the *oriented restriction* of f to V. Notice that if V is empty, the oriented restriction of f to V is unique, and this does not depend on the orientation of f.

Hereafter, unless otherwise stated, if f is an oriented quasi-Fredholm map defined on Ω and $V \subseteq \Omega$ is open, with the symbol $f|_V$ we shall mean the oriented restriction of f to V.

Here is the analogue for quasi-Fredholm maps of Proposition 3.10.

Proposition 4.5. *Let $f\colon \Omega \to F$ be a quasi-Fredholm map. If f is orientable and Ω is nonempty, then f admits at least two orientations. If, in addition,*

Ω is connected, then f admits exactly two orientations (one opposite to the other). If Ω is empty, then f is orientable and admits only one orientation. If Ω is simply connected, then f is orientable.

As for Fredholm maps of index zero, the property of quasi-Fredholm maps of being or not being orientable is homotopic invariant, as shown in Theorem 4.8 below. We need first some definitions.

Definition 4.6. Let $H\colon \Omega \times [0,1] \to F$ be a map of the form

$$H(x,\lambda) = G(x,\lambda) - K(x,\lambda),$$

where G is C^1, any G_λ is Fredholm of index zero and K is locally compact. We call H a *quasi-Fredholm homotopy* and G a *smoothing homotopy* of H.

The definition of orientability for quasi-Fredholm homotopies is analogous to that given for quasi-Fredholm maps. Let $H\colon \Omega \times [0,1] \to F$ be a quasi-Fredholm homotopy. Let $\widehat{\mathcal{S}}(H)$ be the set of oriented smoothing homotopies of H. Assume that $\widehat{\mathcal{S}}(H)$ is nonempty and define on this set an equivalence relation as follows. Given G_0 and G_1 in $\widehat{\mathcal{S}}(H)$, consider the map

$$\mathcal{H}\colon \Omega \times [0,1] \times [0,1] \to F$$

defined as

$$\mathcal{H}(x,\lambda,\mu) = (1-\mu)G_0(x,\lambda) + \mu G_1(x,\lambda).$$

We say that G_0 is *equivalent* to G_1 if their orientations are inherited from an orientation of the map

$$(x,\lambda,\mu) \mapsto \partial_1 \mathcal{H}(x,\lambda,\mu).$$

As in Proposition 4.3, it is possible to prove that this is actually an equivalence relation on $\widehat{\mathcal{S}}(H)$.

Definition 4.7. A quasi-Fredholm homotopy $H\colon \Omega \times [0,1] \to F$ is said to be *orientable* if $\widehat{\mathcal{S}}(H)$ is nonempty. An *orientation* of H is an equivalence class of $\widehat{\mathcal{S}}(H)$.

The following result regarding the continuous transport of an orientation of a quasi-Fredholm map along a homotopy is the analogue of Theorem 3.12.

Theorem 4.8 (Orientation transport for qF-maps)**.** *Let $H\colon \Omega \times [0,1] \to F$ be a quasi-Fredholm homotopy. If a partial map H_λ is oriented, then there exists and is unique an orientation of H which is compatible with H_λ.*

5. Degree for oriented quasi-Fredholm maps

In this section we summarize the construction of the degree for oriented quasi-Fredholm maps introduced by the first and third author in [6]. See also [2] and [3] for extensions of this degree to a class of maps involving the Kuratowski measure of non-compactness.

Definition 5.1. Let $f\colon \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. The pair (f, U) is said to be *admissible* for the degree (of f in U at $0 \in E$) provided that $f^{-1}(0) \cap U$ is compact.

The degree defined in [6] is a map that to every admissible pair (f, U) assigns an integer, $\deg(f, U)$ in symbols, verifying the following three *fundamental properties*. Recall that if $V \subseteq \Omega$ is open, $f|_V$ stands for the oriented restriction of f to V.

- (*Normalization*) Let $L\colon E \to F$ be a naturally oriented isomorphism. Then
$$\deg(L, E) = 1.$$
- (*Additivity*) Let (f, U) be an admissible pair, and U_1, U_2 two disjoint open subsets of U such that $f^{-1}(0) \cap U \subseteq U_1 \cup U_2$. Then,
$$\deg(f, U) = \deg(f|_{U_1}, U_1) + \deg(f|_{U_2}, U_2).$$
- (*Homotopy invariance*) Let $H\colon U \times [0, 1] \to F$ be an oriented quasi-Fredholm homotopy. If $H^{-1}(0)$ is compact, then $\deg(H_\lambda, U)$ does not depend on $\lambda \in [0, 1]$.

With the notation of Definition 4.6, we do not know if the Homotopy invariance property still holds by replacing the assumption that the smoothing homotopy G of H is C^1 with the weaker hypothesis that it is continuous and continuously differentiable with respect to the first variable. For sure this is true when the locally compact perturbation K is zero (see [4]).

We observe that if two maps $f_1, f_2\colon \Omega \to F$ differ by a locally compact map and one is quasi-Fredholm, so is the other one. More precisely, they have the same family of smoothing maps. Therefore, if one is oriented, f_1 for example, then f_2 can be oriented by choosing the same equivalence class of oriented smoothing maps defining the orientation of f_1. As in Definition 3.13 we shall say that f_2 *has the orientation directly induced by* f_1.

Definition 5.2. Let $f\colon \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. Given $y \in F$, the triple (f, U, y) is said to be *admissible* for the degree (of f in U at $y \in E$) if so is the pair $(f - y, U)$ with the orientation of $f - y$ directly induced by f. In this case $\deg(f, U, y)$ is a convenient alternative notation for the integer $\deg(f - y, U)$.

The construction of the degree for admissible pairs consists of two steps. In the first one we consider pairs (f, U) such that f has a smoothing map g with $(f - g)(U)$ contained in a finite dimensional subspace of F. In the second step we remove this assumption, defining the degree for all admissible pairs.

Step 1. Let (f, U) be an admissible pair and let g be a positively oriented smoothing map of f such that $(f - g)(U)$ is contained in a finite dimensional subspace of F. As $f^{-1}(0) \cap U$ is compact, there exist a finite dimensional subspace Z of F and an open neighborhood $W \subseteq U$ of $f^{-1}(0) \cap U$ such that

g is transverse to Z in W. Assume that Z contains $(f-g)(U)$. Choose any orientation of Z and orient the C^1-manifold $M = g^{-1}(Z) \cap W$ in such a way that it is the oriented $g|_W$-preimage of Z. Let $f|_M$ denote the restriction of f to M, as domain, and to Z, as codomain. One can easily verify that $(f|_M)^{-1}(0) = f^{-1}(0) \cap U$. Thus $(f|_M)^{-1}(0)$ is compact and, consequently, the Brouwer degree, $\deg_B(f|_M, M, 0)$, of the triple $(f|_M, M, 0)$ is well defined. With this notation, we can give the following definition of degree for (f, U).

Definition 5.3. The *degree* of the admissible pair (f, U) is the integer

$$\deg(f, U) = \deg_B(f|_M, M, 0). \tag{5.1}$$

In [6] it is proved that the above definition is well posed, in the sense that the right hand side of (5.1) is independent of the choice of the smoothing map g, the open set W and the oriented subspace Z.

Step 2. Let us now extend the definition of degree to general admissible pairs.

Definition 5.4. Let (f, U) be an admissible pair. Consider:

(1) a positively oriented smoothing map g of f;
(2) an open neighborhood V of $f^{-1}(0) \cap U$ such that $\overline{V} \subseteq U$ and g is proper on $\overline{V}$ and $(f-g)|_{\overline{V}}$ has relatively compact image;
(3) a continuous map $\xi \colon \overline{V} \to F$ having bounded finite dimensional image and such that

$$\|g(x) - f(x) - \xi(x)\| < \rho, \quad \forall x \in \partial V,$$

where ρ is the distance in F between 0 and the set $f(\partial V)$, which is closed, since f is proper on $\overline{V}$ as the sum of the proper map $g|_{\overline{V}}$ and the compact map $(f-g)|_{\overline{V}}$.

Then,

$$\deg(f, U) = \deg(g - \xi, V). \tag{5.2}$$

Observe that the right hand side of (5.2) is well defined since the pair $(g - \xi, V)$ is admissible. Indeed, $g - \xi$ is proper on $\overline{V}$ and thus $(g-\xi)^{-1}(0)$ is a compact subset of $\overline{V}$ which is actually contained in V by assumption (3) above.

In [6] it is proved that Definition 5.4 is well posed since formula (5.2) does not depend on g, ξ and V.

6. Uniqueness of the degree for oriented quasi-Fredholm maps

In this section we prove the main result of the paper. Namely, Theorem 6.1 below, which asserts that *there exists at most one real function defined on the class of quasi-Fredholm admissible pairs verifying the three fundamental properties: Normalization, Additivity and Homotopy invariance.* Thus, this function coincides with the degree for oriented quasi-Fredholm maps and is integer valued.

Theorem 6.1. *Let $\mathcal{T}$ denote the class of all admissible pairs and assume that $d\colon \mathcal{T} \to \mathbb{R}$ is a function verifying the following three axioms:*

- (Normalization) *Let $L\colon E \to F$ be a naturally oriented isomorphism. Then*
$$d(L, E) = 1.$$
- (Additivity) *Let (f, U) be an admissible pair, and U_1, U_2 two disjoint open subsets of U such that $f^{-1}(0) \cap U \subseteq U_1 \cup U_2$. Then,*
$$d(f, U) = d(f|_{U_1}, U_1) + d(f|_{U_2}, U_2).$$
- (Homotopy invariance) *Let $H\colon U \times [0,1] \to F$ be an oriented quasi-Fredholm homotopy. If $H^{-1}(0)$ is compact, then $d(H_\lambda, U)$ does not depend on $\lambda \in [0,1]$.*

Then $d = \deg$.

The proof of Theorem 6.1 will proceed as follows. First of all, we will show that if $L\colon E \to F$ is an unnaturally oriented isomorphism, then
$$d(L, E) = -1. \tag{6.1}$$
Afterwards, using the above equality and the Homotopy invariance property, we will prove that, given a diffeomorphism $\phi\colon U \to V$ between two open sets $U \subseteq E$ and $V \subseteq F$, if $0 \in V$ and $\overline{x} = \phi^{-1}(0)$, then
$$d(\phi, U) = \begin{cases} +1 & \text{if } D\phi(\overline{x}) \text{ is naturally oriented,} \\ -1 & \text{otherwise.} \end{cases} \tag{6.2}$$

Hence, as a consequence of formulas (6.1) and (6.2), and of the first two fundamental properties, we will show that, for every admissible pair (f, U) such that $f|_U$ is C^1 and 0 is a regular value of f in U, we have
$$d(f, U) = \sum_{x \in f^{-1}(0) \cap U} \operatorname{sign} Df(x). \tag{6.3}$$

The next step will be the proof of the uniqueness of d on the subclass of $\mathcal{T}$ of the pairs (f, U) such that f is C^1 on U. This will be obtained by the Homotopy invariance property and the local properness of nonlinear Fredholm maps.

At that point, to prove the uniqueness of d on $\mathcal{T}$ we will use as a crucial tool an aproximation result for compact maps in Banach spaces (see Proposition 6.8 below) which is based on a result by Pejsachowicz and Rabier (Lemma 6.7).

Finally, since the function deg verifies the three fundamental properties, we will get $d = \deg$.

This process will be developed in a number of steps.

Step 1. This is a preliminary part in which we show some properties of d which follow from the Additivity and the Homotopy invariance properties.

Given any oriented quasi-Fredholm map f, the pair $(f,\emptyset)$ is admissible, being the empty set compact. By the Additivity property, we get

$$d(f,\emptyset) = d(f|_\emptyset,\emptyset) + d(f|_\emptyset,\emptyset),$$

and

$$d(f|_\emptyset,\emptyset) = d(f|_\emptyset,\emptyset) + d(f|_\emptyset,\emptyset).$$

Hence, one has

$$d(f,\emptyset) = d(f|_\emptyset,\emptyset) = 0.$$

By the above equality and the Additivity we obtain the following (often neglected) Localization property.

Proposition 6.2 (Localization)**.** *Let $f\colon \Omega \to F$ be an oriented quasi-Fredholm map, U an open subset of Ω. If (f,U) is an admissible pair, then*

$$d(f,U) = d(f|_U,U),$$

where $f|_U$ denotes the oriented restriction of f to U.

Proof. By the Additivity one has

$$d(f,U) = d(f|_U,U) + d(f|_\emptyset,\emptyset).$$

Then, the assertion follows being $d(f|_\emptyset,\emptyset) = 0$. □

Another consequence of the Additivity (and of the Localization) is the Excision property, which basically assert that $d(f,U)$ depends only on the behavior of f in any neighborhood of $f^{-1}(0) \cap U$.

Proposition 6.3 (Excision)**.** *If (f,U) is admissible and V is an open subset of U such that $f^{-1}(0) \cap U \subseteq V$, then (f,V) is admissible and*

$$d(f,U) = d(f,V).$$

Proof. The pair (f,V) is clearly admissible. From the Additivity and the fact that $d(f|_\emptyset,\emptyset) = 0$ one gets

$$d(f,U) = d(f|_V,V).$$

On the other hand, the Localization implies that

$$d(f,V) = d(f|_V,V),$$

and the assertion follows. □

From the Excision we obtain the Existence property.

Proposition 6.4 (Existence)**.** *Let $d(f,U)$ be nonzero. Then, the equation $f(x) = 0$ admits at least one solution in U.*

Proof. Assume that $f^{-1}(0) \cap U$ is empty. By the Excision property, taking $V = \emptyset$, we get

$$d(f,U) = d(f,\emptyset) = 0,$$

which contradicts the assumption. □

The following is an immediate consequence of the Additivity and the Localization properties.

Proposition 6.5 (Classical Additivity)**.** *Given an admissible pair* (f, U) *and two disjoint open subsets* U_1, U_2 *of* U *such that* $f^{-1}(0) \cap U \subseteq U_1 \cup U_2$, *one has*

$$d(f, U) = d(f, U_1) + d(f, U_2).$$

The reader who is familiar with the degree theory probably observes that the above property is the classical version of the Additivity which is usually mentioned in the literature. Actually, we believe not possible to deduce the Localization property of $d\colon \mathcal{T} \to \mathbb{R}$ by replacing the Additivity property with the above classical version.

As in Definition 5.2, given an oriented quasi-Fredholm map $f\colon \Omega \to F$, an open subset U of Ω and $y \in F$, if the pair $(f - y, U)$ is admissible (with the orientation of $f - y$ directly induced by f), then $d(f, U, y)$ denotes the number $d(f - y, U)$. In this case the triple (f, U, y) is said to be *admissible* and y is called the *target point*. Notice that, because of Proposition 6.4, if $d(f, U, y) \neq 0$, then the equation $f(x) = y$ has at least one solution in U.

The next property shows that, for an important class of admissible triples, the degree depends continuously on the target point $y \in F$.

Proposition 6.6 (Continuous dependence)**.** *Let* $f\colon \Omega \to F$ *be an oriented quasi-Fredholm map and* U *an open set whose closure* $\overline{U}$ *is contained in* Ω. *Assume that* f *is proper on* $\overline{U}$ *and let* y *belong to the open set* $F \setminus f(\partial U)$. *Then* (f, U, y) *is admissible and* $d(f, U, y)$ *depends only on the connected component of* $F \setminus f(\partial U)$ *containing* y. *In particular, if* $U = \Omega = E$ *and* $f\colon E \to F$ *is proper, then* $d(f, E, y)$ *does not depend on* $y \in F$.

Proof. The set $F \setminus f(\partial U)$ is open, since f, being proper, maps the closed set ∂U onto the closed set $f(\partial U)$. Thus, there exists a ball B centered at y which does not intersect $f(\partial U)$.

Fix any $z \in B$ and let C denote the line segment joining y with z. Since f is proper on $\overline{U}$, $f^{-1}(C) \cap \overline{U}$ is a compact set, and it is contained in U being $C \cap f(\partial U) = \emptyset$. In particular, (f, U, y) is admissible.

Now, the Homotopy invariance property implies $d(f, U, z) = d(f, U, y)$, since the closed set

$$\big\{(x, \lambda) \in \overline{U} \times [0, 1]\colon f(x) = (1 - \lambda)y + \lambda z\big\}$$

is contained in the compact subset $(f^{-1}(C) \cap \overline{U}) \times [0, 1]$ of $U \times [0, 1]$. Consequently, $z \in B$ being arbitrary, the map that to any $q \in F \setminus f(\partial U)$ assigns $d(f, U, q) \in \mathbb{R}$ is locally constant, and this implies the assertion. □

Step 2. Let $L\colon E \to F$ be an oriented isomorphism. In this step we prove that

$$d(L, E) = \operatorname{sign} L.$$

If L is naturally oriented, from the Normalization property we get

$$d(L,E) = 1 = \operatorname{sign} L.$$

Thus, we assume that L is unnaturally oriented, and we need to show that $d(L,E) = -1$.

Fix any nonzero vector $\overline{w} \in E$ and call E_2 the one-dimensional subspace of E spanned by $\overline{w}$. As a consequence of the Hahn–Banach Theorem there exists a closed subspace E_1 of E such that

$$E = E_1 \oplus E_2.$$

Thus, any element x of E can be uniquely written as

$$x = v + t\overline{w}, \tag{6.4}$$

with $v \in E_1$ and $t \in \mathbb{R}$.

Taking into account (6.4), define $f\colon E \to F$ by

$$f(v + t\overline{w}) = L\left(v + |t|\overline{w}\right).$$

Observe that f is a quasi-Fredholm map and L is one of its smoothing maps. In fact, one has

$$f(v + t\overline{w}) = L(v + t\overline{w}) - h(t\overline{w}),$$

where

$$h(t\overline{w}) = \begin{cases} 0 & \text{if } t \geq 0 \\ 2ty & \text{if } t < 0. \end{cases}$$

Notice that f coincides with L in the closure $\overline{E}_+$ of the open half space $E_+ := \{v + t\overline{w}\colon t > 0\}$ and with the isomorphism $\widetilde{L} \in L(E,F)$, defined as

$$\widetilde{L}(v + t\overline{w}) = L(v - t\overline{w}),$$

in the closure $\overline{E}_-$ of $E_- := \{v + t\overline{w}\colon t < 0\}$.

Clearly, $\widetilde{L}$ is another smoothing map of f, since the difference $L - \widetilde{L}$ is the linear operator $v + t\overline{w} \mapsto 2t\overline{w}$, which is locally compact (having finite dimensional image) and C^1 (being linear).

Orient f according to the assumed unnatural orientation of the smoothing map L. This implies that $\widetilde{L}$, as another smoothing map of f, receives an orientation which is compatible with that of f, and this can be obtained by transporting the orientation of L along the straight-line homotopy joining L with $\widetilde{L}$. Le us show that this orientation of $\widetilde{L}$ is the natural one.

Since L and $\widetilde{L}$ are isomorphisms, they are, in particular, proper maps. Thus, their respective restrictions to the closed sets $\overline{E}_+$ and $\overline{E}_-$ are proper as well. Consequently, so is the map f, which can be regarded as obtained by glueing the above two restrictions. Because of Proposition 6.6, this implies that $d(f,E,y)$ does not depend on the target point $y \in F$ (recall that $d(f,U,y)$ is an alternative notation for the number $d(f-y,U)$). Therefore, as f is not surjective, from the Existence property (Proposition 6.4) we obtain $d(f,E,y) = 0$ for all $y \in F$. Thus, in particular, $d(f,E,\overline{y}) = 0$, where $\overline{y} = f(\overline{w})$.

Observe that $f^{-1}(\overline{y}) = \{-\overline{w}, \overline{w}\}$, with $-\overline{w} \in E_-$ and $\overline{w} \in E_+$. Hence, by the the Additivity property, we get

$$0 = d(f, E, \overline{y}) = d(f|_{E_-}, E_-, \overline{y}) + d(f|_{E_+}, E_+, \overline{y}). \tag{6.5}$$

Now consider $d(f|_{E_+}, E_+, \overline{y})$. Since the oriented restrictions $f|_{E_+}$ and $L|_{E_+}$ coincide, we obtain

$$d(f|_{E_+}, E_+, \overline{y}) = d(L|_{E_+}, E_+, \overline{y}).$$

Thus, because of the Localization and Excision properties (propositions 6.2 and 6.3, respectively), we get the equality

$$d(f|_{E_+}, E_+, \overline{y}) = d(L, E, \overline{y}).$$

Analogously, we have

$$d(f|_{E_-}, E_-, \overline{y}) = d(\widetilde{L}, E, \overline{y}).$$

Consequently, taking into account (6.5), we obtain

$$d(L, E, \overline{y}) + d(\widetilde{L}, E, \overline{y}) = 0,$$

and recalling that the maps L and $\widetilde{L}$ are proper on E, we equivalently have

$$d(L, E) + d(\widetilde{L}, E) = 0.$$

Hence, to prove our assertion we need to show that $\widetilde{L}$ is naturally oriented, so that $d(\widetilde{L}, E) = 1$. For this purpose, consider the homotopy

$$G\colon E \times [0, 1] \to F, \qquad G(x, \lambda) = (1 - \lambda)Lx + \lambda\widetilde{L}x,$$

and orient it with the unique orientation which is compatible with L. Thus any operator

$$G_\lambda = (1 - \lambda)L + \lambda\widetilde{L}, \quad \lambda \in [0, 1]$$

is oriented, and $G_1 = \widetilde{L}$, as a smoothing map of f, receives the orientation which is compatible with that of f.

Let us show that the corrector A of L, defined by

$$A(v + t\overline{w}) = -4t\overline{y},$$

is a positive corrector. In fact, the composition

$$L^{-1}(L + A)\colon E \to E$$

acts as follows: $v + t\overline{w} \mapsto v - 3t\overline{w}$. Hence, its restriction to the one-dimensional space E_2 has determinant -3. Thus, A is not L-equivalent to the trivial operator of E and, since L is unnaturally oriented, A is a positive corrector of L.

Now, observe that A is a corrector of any linear operator

$$G_\lambda = (1 - \lambda)L + \lambda\widetilde{L}, \quad \lambda \in [0, 1].$$

In fact, one can verify that

$$(G_\lambda + A)(v + t\overline{w}) = Lv - (3 + 2\lambda)t\overline{y}$$

and, consequently, $G_\lambda + A$ is an isomorphism for all $\lambda \in [0,1]$. This implies that A is a positive corrector of any oriented operator G_λ, $\lambda \in [0,1]$.

Finally, consider the composition

$$\widetilde{L}^{-1}(\widetilde{L} + A)\colon E \to E, \quad \text{given by} \quad \widetilde{L}^{-1}(\widetilde{L} + A)(v + t\overline{w}) = v + 5t\overline{w}.$$

The determinant of the restriction $\widetilde{L}^{-1}(\widetilde{L} + A)|_{E_2}\colon E_2 \to E_2$ is 5, which means that $\widetilde{L}$ is naturally oriented. Thus, the proof of step 2 is complete.

Step 3. Let $f\colon U \to F$ be a diffeomorphism of an open subset U of E onto an open subset V of F such that $0 \in V$. In this step we prove that

$$d(f, U) = \operatorname{sign} Df(\overline{x}), \tag{6.6}$$

where $\overline{x} = f^{-1}(0)$. Consider the homotopy $H\colon U \times [0,1] \to F$ defined as

$$H(x, \lambda) = (1 - \lambda)f(x) + \lambda Df(\overline{x})(x - \overline{x}).$$

It is immediate to see that H is C^1, $H(\overline{x}, \lambda) = 0$ for every $\lambda \in [0,1]$ and the Fréchet derivative at $\overline{x}$ of any partial map $H_\lambda = H(\cdot, \lambda)$ is

$$DH_\lambda(\overline{x}) = Df(\overline{x}).$$

As $Df(\overline{x})$ is an isomorphism, the Implicit Function Theorem and the compactness of $[0,1]$ imply the existence of a neighborhood W of $\overline{x}$ in U such that, for any $\lambda \in [0,1]$, the equation $H(x, \lambda) = 0$ admits in W the unique solution $\overline{x}$. This implies that $H^{-1}(0) \cap (W \times [0,1])$ is compact and, consequently, by the Homotopy invariance property, one has

$$d(f, W) = d(Df(\overline{x}) - q, W),$$

where $q = Df(\overline{x})\overline{x}$.

Notice that $Df(\overline{x})\colon E \to F$ is an isomorphism, being the Fréchet derivative at a given point of a diffeomorphism. Thus, from the Excision property, we get

$$d(Df(\overline{x}) - q, W) = d(Df(\overline{x}) - q, E)$$

Moreover, since $Df(\overline{x})$ is a proper map, $d(Df(\overline{x}) - y, E)$ does not depend on $y \in F$ (see Proposition 6.6). Consequently,

$$d(f, W) = d(Df(\overline{x}), E),$$

and the equality (6.6) follows from the previous step 2 (and the Excision property).

Step 4. We are now in the position to prove formula (6.3). Let (f, U) be an admissible pair such that f is C^1 on U and 0 is a regular value for $f|_U$. We know that $f^{-1}(0) \cap U$ is a finite set, say $\{x_1, \dots, x_n\}$. Since $Df(x_i)$ is an isomorphism for any $i = 1, \dots, n$, we can apply the Inverse Function Theorem, obtaining that there exist n pairwise disjoint neighborhoods $U_1, \dots, U_n$ of $x_1, \dots, x_n$, respectively, such that each restriction $f|_{U_i}$ is a diffeomorphism onto an open neighborhood of 0. By the Classical Additivity property (Proposition 6.5) we have

$$d(f, U) = \sum_{i=1}^{n} d(f, U_i).$$

On the other hand, by the above step 3, we obtain

$$d(f, U_i) = \operatorname{sign} Df(x_i)$$

and this proves formula (6.3).

Step 5. Here we show the uniqueness of d on the class $\mathcal{T}_2$ ($\subseteq \mathcal{T}$) of the pairs (f, U) with f of class C^1 on U.

Let (f, U) be an admissible pair such that f is C^1 on U. Since f is locally proper, there exists an open subset W of U, containing $f^{-1}(0) \cap U$, such that $\overline{W} \subseteq U$ and f is proper on $\overline{W}$. By the Excision property we have

$$d(f, U) = d(f, W).$$

By the Continuous dependence property (Proposition 6.6), we see that $d(f - y, W)$ depends only on the connected component V of $F \setminus f(\partial W)$ containing 0. This component is an open set, since, as f is proper in $\overline{W}$, $f(\partial W)$ is closed in F. Therefore, taking into account the Sard–Smale theorem [29], we may compute $d(f, W)$ by choosing a regular value $y \in V$ for $f|_W$. In fact, because of formula (6.3), one has

$$d(f, W) = d(f - y, W) = \sum_{x \in f^{-1}(y) \cap W} \operatorname{sign} Df(x),$$

and this shows the uniqueness of d on $\mathcal{T}_2$.

Step 6. Here we show that the uniqueness of d on $\mathcal{T}_2$ implies the uniqueness of d on the class $\mathcal{T}_1$ ($\subseteq \mathcal{T}$) of those admissible pairs (f, U) such that f has a smoothing map g with the property that $(f - g)(U)$ is contained in a finite dimensional subspace of F. This step contains one of the most important difficulties of our process.

As is well know, a continuous real map, defined in a compact subset of $\mathbb{R}^n$, can be uniformly approximated by a smooth map defined on the whole $\mathbb{R}^n$. As far as we know, an analogous result does not hold if $\mathbb{R}^n$ is replaced by a general Banach space E, unless the compact domain of the map is contained in a finite-dimensional subspace of E (recall that any finite-dimensional subspace of E is the image of a bounded linear projector). Thanks to the following lemma by Pejsachowicz and Rabier (see [25, Theorem 7.1]), an approximation result like the one in the finite dimensional case holds true even when the domain of the map *is contained in a finite-dimensional submanifold of* E.

Lemma 6.7 (Pejsachowicz–Rabier)**.** *Let M be a finite-dimensional C^1 submanifold of E, and K a compact subset of M. Then, there exist a finite-dimensional subspace E_1 of E and a C^1 diffeomorphism $\phi \colon E \to E$ such that $\phi(K) \subseteq E_1$.*

Proposition 6.8. *Let K be a compact subset of E. Assume that there exists a finite dimensional submanifold M of E containing K. Let $\gamma\colon K \to \mathbb{R}$ be a continuous map. Then, given a positive ε, there exists a bounded C^1 map $\eta\colon E \to \mathbb{R}$ such that*

$$\sup_{x\in K} |\gamma(x) - \eta(x)| < \varepsilon.$$

Proof. According to Lemma 6.7, let $\phi\colon E \to E$ be a C^1 diffeomorphism such that $\widetilde{K} = \phi(K)$ is contained in a finite-dimensional subspace E_1 of E. Consider $\widetilde{\gamma}\colon \widetilde{K} \to \mathbb{R}$ defined as $\widetilde{\gamma}(x) = \gamma(\phi^{-1}(x))$. As $\widetilde{K}$ is compact, given a positive ε, there exists a bounded C^1 map $\widetilde{\xi}\colon E_1 \to \mathbb{R}$ such that

$$\sup_{y\in \widetilde{K}} |\widetilde{\gamma}(y) - \widetilde{\xi}(y)| < \varepsilon.$$

Define $\xi\colon E \to \mathbb{R}$ by $\xi(x) = \widetilde{\xi}(P(x))$, where P is a bounded linear projector onto E_1, and let $\eta\colon E \to \mathbb{R}$ be such that $\eta(x) = \xi(\phi(x))$. It follows that

$$\sup_{x\in K} |\gamma(x) - \eta(x)| = \sup_{x\in K} |\widetilde{\gamma}(\phi(x)) - \widetilde{\xi}(\phi(x))| = \sup_{y\in \widetilde{K}} |\widetilde{\gamma}(y) - \widetilde{\xi}(y)| < \varepsilon,$$

and this proves the assertion. □

Let (f, U) be an admissible pair and let g be a positively oriented smoothing map of f such that $(f - g)(U)$ is contained in a finite dimensional subspace of F. As $f^{-1}(0) \cap U$ is compact, there exist a finite dimensional subspace Z of F and an open neighborhood W of $f^{-1}(0) \cap U$ in U such that the following conditions hold:

- g is transverse to Z in W;
- Z contains $(f - g)(U)$;

As already seen, the set $M = g^{-1}(Z) \cap W$ is a boundaryless C^1 manifold of the same dimension as Z. Let us now consider an open subset V of W such that

- $f^{-1}(0) \cap U \subseteq V \subseteq \overline{V} \subseteq W$;
- g is proper and bounded on $\overline{V}$;
- $f - g$ is a compact map on $\overline{V}$.

The subset

$$S = g^{-1}(g(\overline{V}) \cap Z) \cap \overline{V}$$

of E turns out to be compact, due, in particular, to the fact that Z is finite-dimensional and that g is proper on $\overline{V}$. In addition, S is contained in the manifold M.

Now, let δ be the positive distance between 0 and $f(\partial V)$, and let $k = g - f$. We are in the position to apply a straightforward consequence of Proposition 6.8 to the restriction $k|_S$, obtaining a compact C^1 map $k^*\colon E \to F$, having image contained in Z and such that

$$\sup_{x\in S} \|k^*(x) - k(x)\| < \delta/2.$$

Consider the homotopy $\overline{H}\colon \overline{V} \times [0,1] \to F$ defined as

$$\overline{H}(x,\lambda) = g(x) - \lambda k(x) - (1-\lambda)k^*(x).$$

Notice that the restriction H of $\overline{H}$ to $V \times [0,1]$ is a quasi-Fredholm homotopy, which is orientable since so is its partial map $H_1 = f$. Orient H with the unique orientation compatible H_1. Our purpose is to apply the Homotopy invariance property and show that

$$d(g-k,V) = d(g-k^*,V). \tag{6.7}$$

To obtain the above equality it is sufficient to verify that the set

$$C = \big\{(x,\lambda) \in V \times [0,1]\colon H(x,\lambda) = 0\big\}$$

is compact. Since g is proper on $\overline{V}$ and k and k^* are compact on $\overline{V}$, it follows that $\overline{H}$ is proper. To prove the compactness of the above set C, it is sufficient to check that $\overline{H}^{-1}(0)$ does not intersect $\partial V \times [0,1]$, i.e. that C coincides with $\overline{H}^{-1}(0)$. Let $(x,\lambda) \in \overline{H}^{-1}(0)$ be given. Observe that $k(x)$ and $k^*(x)$ belong to Z. Hence, $g(x)$ belongs to Z too. Therefore, $x \in S$ by the definition of this set. Consequently,

$$\|k^*(x) - k(x)\| < \delta/2.$$

Suppose now, by contradiction, that $x \in \partial V$. We have

$$\begin{aligned}&\|H(x,\lambda)\| = \|g(x) - \lambda k(x) - (1-\lambda)k^*(x)\| \geq \\ &\|f(x)\| - (1-\lambda)\|k(x) - k^*(x)\| \geq \\ &\delta - \delta/2 = \delta/2 > 0.\end{aligned}$$

Therefore, we obtain that, if (x,λ) is any element in $\overline{H}^{-1}(0)$, then x does not belong to ∂V. Consequently, C coincides with the compact set $\overline{H}^{-1}(0)$. Thus, by the Homotopy invariance property we get the computation formula (6.7), which, because of the previous step, implies the uniqueness of d on the subclass $\mathcal{T}_1$ of $\mathcal{T}$.

Step 7. In this final step we conclude the process, showing the uniqueness of d on the whole class $\mathcal{T}$.

Let (f,U) be an admissible pair. Consider:

- a positively oriented smoothing map g of f;
- an open neighborhood V of $f^{-1}(0)$ such that $\overline{V} \subseteq U$, g is proper on $\overline{V}$ and $(f-g)|_{\overline{V}}$ is compact;
- a continuous map $\xi\colon \overline{V} \to F$ having bounded finite dimensional image and such that

  $$\|g(x) - f(x) - \xi(x)\| < \delta, \quad \forall x \in \partial V,$$

  where δ is the distance in F between 0 and $f(\partial V)$.

Consider the homotopy $\overline{H}\colon \overline{V} \times [0,1] \to F$, defined as

$$\overline{H}(x,\lambda) = (1-\lambda)f(x) + \lambda\big(g(x) - \xi(x)\big).$$

The restriction H of $\overline{H}$ to $V \times [0,1]$ is a quasi-Fredholm homotopy, being a compact perturbation of the map $(x,\lambda) \mapsto g(x)$. In addition, $H^{-1}(0)$ is a compact subset of $V \times [0,1]$. Assume that H is oriented with the unique orientation compatible with f. Therefore we are in the position to apply the Homotopy invariance property showing that

$$d(f,V) = d(g-\xi,V).$$

This proves the uniqueness of d.

7. The Leray–Schauder case

In this section we show that the degree for oriented quasi-Fredholm maps provides a generalization of the Leray–Schauder degree, in the sense that there exists a "canonical" embedding j of the class of the Leray–Schauder admissible pairs into the class $\mathcal{T}$ (of the quasi-Fredholm admissible pairs) such that the composition of j with $\deg\colon \mathcal{T} \to \mathbb{Z}$ coincides with the Leray–Schauder degree $\deg_{LS}$ (see Theorem 7.1 below).

Let, as before, E denote any real Banach space. Recall that a continuous map $f\colon X \to E$, defined on a subset of E, is called a *compact vector field* (on X) if it differs from the identity by a completely continuous map. Given a compact vector field $f\colon \overline{U} \to E$ on the closure of a bounded open subset of E, the pair (f,U) is said to be *Leray–Schauder admissible* (*LS-admissible*, for short) provided that $f^{-1}(0) \subseteq U$ (i.e. $0 \notin f(\partial U)$). In this case, an integer, $\deg_{LS}(f,U)$, is defined and called *Leray–Schauder degree of f in U*. Denoting by $\mathcal{C}_{LS}$ the class of the LS-admissible pairs, this degree is a function $\deg_{LS}\colon \mathcal{C}_{LS} \to \mathbb{Z}$ that is known to satisfy the following three basic properties:

- (*LS-normalization*) Let U be a bounded open subset of E and let I denote the identity of E. If $0 \in U$, then
$$\deg_{LS}(I|_{\overline{U}}, U) = 1.$$
- (*LS-additivity*) Let (f,U) be an LS-admissible pair, and U_1, U_2 two disjoint open subsets of U such that $f^{-1}(0) \subseteq U_1 \cup U_2$. Then,
$$\deg_{LS}(f,U) = \deg_{LS}(f|_{\overline{U}_1}, U_1) + \deg_{LS}(f|_{\overline{U}_2}, U_2).$$
- (*LS-homotopy invariance*) Let $H\colon \overline{U} \times [0,1] \to F$ be a homotopy of compact vector fields on $\overline{U}$. If $H^{-1}(0)$ is contained in $U \times [0,1]$, then $\deg_{LS}(H(\cdot,\lambda),U)$ does not depend on $\lambda \in [0,1]$.

We recall that a famous result of Amann and Weiss [1] asserts that $\deg_{LS}$ is the unique map satisfying the above three properties. This fact will be crucial in the proof of Theorem 7.1 below.

We define now a "canonical" one-to-one map j from the class $\mathcal{C}_{LS}$ of the LS-admissible pairs onto a subclass $\mathcal{T}_{LS}$ of $\mathcal{T}$.

Notice that a compact vector field on an open subset of a real Banach space is, in particular, a locally compact vector field (which is, we recall, a quasi-Fredholm map having the identity among its smoothing maps). Thus, according to Definition 4.4, a compact vector field has a distinguished orientation: the canonical one (that is, the one directly induced by the natural orientation of the identity).

Let $j\colon \mathcal{C}_{LS} \to \mathcal{T}$ be the map defined by $j(f,U) = (f|_U, U)$, where the orientation of the restriction $f|_U$ is the canonical one.

Clearly, $j(f,U)$ belongs to the class $\mathcal{T}$ of the admissible pairs. In fact, with the notation of Definition 5.1, Ω coincides with U and $f^{-1}(0) \cap U$ is compact since f is proper on its domain $\overline{U}$ and such that $f^{-1}(0) \subseteq U$.

The map j is clearly one-to-one, since if two continuous maps defined on $\overline{U}$ do not coincide, they necessarily have different restrictions to U. Moreover, one can check that the image of j is the subclass $\mathcal{T}_{LS}$ of $\mathcal{T}$ of the pairs (g,U) with the following properties:

- U is a bounded open subset of a real Banach space E;
- $g\colon U \to E$ is a canonically oriented compact vector field on U;
- g admits a continuous extension $f\colon \overline{U} \to E$;
- $0 \notin f(\partial U)$.

The following result shows that if in $\mathcal{T}$ we identify $\mathcal{C}_{LS}$ with its image $j(\mathcal{C}_{LS}) = \mathcal{T}_{LS}$, then $\deg\colon \mathcal{T} \to \mathbb{Z}$ may be regarded as an extension of the Leray–Schauder degree $\deg_{LS}\colon \mathcal{C}_{LS} \to \mathbb{Z}$.

Theorem 7.1. *For any LS-admissible pair* (f,U) *one has*

$$\deg\big(j(f,U)\big) = \deg_{LS}(f,U).$$

Proof. Because of the uniqueness result due to Amann and Weiss [1], it is enough to check that that the map $d_*\colon \mathcal{C}_{LS} \to \mathbb{Z}$ defined by

$$d_*(f,U) = \deg\big(j(f,U)\big)$$

verifies the above three basic properties of the Laray–Schauder degree: LS-normalization, LS-additivity, and LS-homotopy invariance.

This follows easily from the properties of $\deg\colon \mathcal{T} \to \mathbb{Z}$. Indeed, observe that the LS-normalization property is a consequence of the Normalization property of deg, the Localization property of d (Proposition 6.2), and the fact that (because of Theorem 6.1) $d = \deg$. The LS-additivity property is trivially satisfied. Moreover, regarding the LS-homotopy invariance property, notice that if $H\colon \overline{U} \times [0,1] \to E$ is a homotopy of compact vector fields on $\overline{U}$, then its restriction to $U \times [0,1]$ is an oriented quasi-Fredholm homotopy, provided that any partial map $H_\lambda|_U\colon U \to E$ is canonically oriented (as it is in the pair $(H_\lambda|_U, U) = j(H_\lambda, U)$). Finally, since U is bounded, H is a proper map. Consequently, $H^{-1}(0)$ is compact. □

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