A DEGREE THEORY FOR A CLASS OF PERTURBED FREDHOLM MAPS

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We define a notion of degree for a class of perturbations of nonlinear Fredholm maps of index zero between infinite dimensional real Banach spaces. Our notion extends the degree introduced by Nussbaum for locally α-contractive perturbations of the identity, as well as the recent degree for locally compact perturbations of Fredholm maps of index zero defined in [3].

1. Introduction

In this paper we define a concept of degree for a special class of perturbations of (nonlinear) Fredholm maps of index zero between (infinite dimensional real) Banach spaces, called *α-Fredholm maps*. The definition is based on the following two numbers (see e.g. [10]) associated with a map $f\colon \Omega \to F$ from an open subset of a Banach space E into a Banach space F:

$$\alpha(f) = \sup \left\{ \frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded, } \alpha(A) > 0 \right\},$$

$$\omega(f) = \inf \left\{ \frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded, } \alpha(A) > 0 \right\},$$

where α is the Kuratowski measure of noncompactness (in [10] $\omega(f)$ is denoted by $\beta(f)$, however, since ω is the last letter of the Greek alphabet, we prefer the notation $\omega(f)$ as in [8]).

Roughly speaking, the α-Fredholm maps are of the type $f = g - k$, where g is Fredholm of index zero and k satisfies, locally, the inequality

$$\alpha(k) < \omega(g).$$

These maps include locally compact perturbations of Fredholm maps (called *quasi-Fredholm maps*, for short) since, when g is Fredholm and k is locally compact, one has $\alpha(k) = 0$ and $\omega(g) > 0$, locally. Moreover, they also contain the α-contractive perturbations of the identity (called *α-contractive vector fields*), where, following Darbo [5], a map k is α-contractive if $\alpha(k) < 1$.

The degree obtained in this paper is a generalization of the degree for quasi-Fredholm maps defined for the first time in [14] by means of the Elworthy–Tromba theory. The latter degree has been recently redefined in [3] avoiding the use of the Elworthy–Tromba construction and using as a main tool a natural concept of orientation for nonlinear Fredholm maps introduced in [1] and [2]. Our construction is based on this new definition.

The paper ends by showing that for α-contractive vector fields our degree coincides with the degree defined by Nussbaum in [12] and [13].

2. Orientability for Fredholm maps

In this section we give a summary of the notion of orientability for nonlinear Fredholm maps of index zero between Banach spaces introduced in [1] and [2].

The starting point is a preliminary definition of a concept of orientation for linear Fredholm operators of index zero between real vector spaces (at this level no topological structure is needed).

Recall that, given two real vector spaces E and F, a linear operator $L\colon E \to F$ is said to be (*algebraic*) *Fredholm* if the spaces $\operatorname{Ker} L$ and $\operatorname{coKer} L = F/\operatorname{Im} L$ are finite dimensional. The *index* of L is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

Given a Fredholm operator of index zero L, a linear operator $A\colon E \to F$ is called a *corrector* of L if

i) $\operatorname{Im} A$ has finite dimension,
ii) $L + A$ is an isomorphism.

We denote by $\mathcal{C}(L)$ the nonempty set of correctors of L and we define in $\mathcal{C}(L)$ the following equivalence relation. Given $A, B \in \mathcal{C}(L)$, consider the automorphism

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)$$

of E. Clearly, the image of $K = (L+B)^{-1}(B-A)$ is finite dimensional. Hence, given any finite dimensional subspace E_0 of E containing the image of K, the restriction of T to E_0 is an automorphism of E_0. Therefore, its determinant is well defined and nonzero. It is easy to check that this value does not depend on E_0 (see [1]). Thus, the *determinant* of T, $\det T$ in symbols, is well defined as the determinant of the restriction of T to any finite dimensional subspace of E containing the image of K.

We say that A is *equivalent* to B or, more precisely, A is *L-equivalent* to B, if

$$\det\big((L + B)^{-1}(L + A)\big) > 0.$$

In [1] it is shown that this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes. This equivalence relation provides a concept of orientation of a linear Fredholm operator of index zero.

Definition 2.1. Let L be a linear Fredholm operator of index zero between two real vector spaces. An *orientation* of L is the choice of one of the two equivalence classes of $\mathcal{C}(L)$, and L is *oriented* when an orientation is chosen.

Given an oriented operator L, the elements of its orientation are called the *positive correctors* of L.

Definition 2.2. An oriented isomorphism L is said to be *naturally oriented* if the trivial operator is a positive corrector, and this orientation is called the *natural orientation* of L.

We now consider the notion of orientation in the framework of Banach spaces. From now on, and throughout the paper, E and F denote two real Banach spaces, $\mathrm{L}(E, F)$ is the Banach space of bounded linear operators from E into F, and $\Phi_0(E, F)$ is the open subset of $\mathrm{L}(E, F)$ of the Fredholm operators of index zero. Given $L \in \Phi_0(E, F)$, the symbol $\mathcal{C}(L)$ now denotes, with an abuse of notation, the set of bounded correctors of L, which is still nonempty.

Of course, the definition of orientation of $L \in \Phi_0(E,F)$ can be given as the choice of one of the two equivalence classes of bounded correctors of L, according to the equivalence relation previously defined.

In the context of Banach spaces, an orientation of a linear Fredholm operator of index zero induces, by a sort of stability, an orientation to any sufficiently close operator. Precisely, consider $L \in \Phi_0(E,F)$ and a corrector A of L. Since the set of the isomorphisms from E into F is open in $\mathrm{L}(E,F)$, A is a corrector of every T in a suitable neighborhood W of L. If, in addition, L is oriented and A is a positive corrector of L, then any T in W can be oriented by taking A as a positive corrector. This fact leads us to the following notion of orientation for a continuous map with values in $\Phi_0(E,F)$.

Definition 2.3. Let X be a topological space and $h\colon X \to \Phi_0(E,F)$ be continuous. An *orientation* of h is a continuous choice of an orientation $\alpha(x)$ of $h(x)$ for each $x \in X$, where 'continuous' means that for any $x \in X$ there exists $A \in \alpha(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is *orientable* when it admits an orientation and *oriented* when an orientation is chosen.

Remark 2.4. It is possible to prove (see [2, Proposition 3.4]) that two equivalent correctors A and B of a given $L \in \Phi_0(E,F)$ remain T-equivalent for any T in a neighborhood of L. This implies that the notion of 'continuous choice of an orientation' in Definition 2.3 is equivalent to the following one:

- *for any $x \in X$* ***and any*** *$A \in \alpha(x)$, there exists a neighborhood W of x such that $A \in \alpha(x')$ for all $x' \in W$.*

As a straightforward consequence of Definition 2.3, if $h\colon X \to \Phi_0(E,F)$ is orientable and $g\colon Y \to X$ is any continuous map, then the composition hg is orientable as well. In particular, if h is oriented, then hg *inherits* in a natural way an orientation from the orientation of h. Thus, if

$$H\colon X \times [0,1] \to \Phi_0(E,F)$$

is an oriented homotopy and $t \in [0,1]$ is given, the partial map $H_t = Hi_t$, where $i_t(x) = (x,t)$, inherits an orientation from H.

The following proposition shows an important property of the notions of orientation and orientability for maps into $\Phi_0(E,F)$. Such a property may be regarded as a sort of continuous transport of the orientation along a homotopy (see [2, Theorem 3.14]).

Proposition 2.5. *Let X be a topological space and consider a homotopy*

$$H\colon X \times [0,1] \to \Phi_0(E,F).$$

Assume that for some $t \in [0,1]$ the partial map $H_t = H(\cdot,t)$ is oriented. Then there exists and is unique an orientation of H such that the orientation of H_t is inherited from that of H.

Definition 2.3 and Remark 2.4 allow us to define a notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset Ω of E, a map $g\colon \Omega \to F$ is *Fredholm* if it is C^1 and its Fréchet derivative, $g'(x)$, is a Fredholm operator for all $x \in \Omega$. The *index* of g at x is the index of $g'(x)$ and g is said to be of *index* n if it is of index n at any point of its domain.

Definition 2.6. An *orientation* of a Fredholm map of index zero $g\colon \Omega \to F$ is an orientation of the derivative $g'\colon \Omega \to \Phi_0(E,F)$, and g is *orientable*, or *oriented*, if so is g' according to Definition 2.3.

The notion of orientability of Fredholm maps of index zero is mainly discussed in [1] and [2], where the reader can find examples of orientable and nonorientable maps and a comparison with an earlier notion given by Fitzpatrick, Pejsachowicz and Rabier in [9]. Here we recall a property (Theorem 2.8 below) that is the analogue for Fredholm maps of the continuous transport of an orientation along a homotopy stated in Proposition 2.5. We need first the following definition.

Definition 2.7. Let Ω be an open subset of E and $G\colon \Omega\times[0,1] \to F$ a C^1 homotopy. Assume that any partial map G_t is Fredholm of index zero. An *orientation* of G is an orientation of the partial derivative

$$\partial_1 G\colon \Omega \times [0,1] \to \Phi_0(E,F), \quad (x,t) \mapsto (G_t)'(x),$$

and G is *orientable*, or *oriented*, if so is $\partial_1 G$ according to Definition 2.3.

From the above definition it follows immediately that if G is oriented, any partial map G_t inherits an orientation from G.

Theorem 2.8 below is a straightforward consequence of Proposition 2.5.

Theorem 2.8. *Let $G\colon \Omega \times [0,1] \to F$ be a C^1 homotopy and assume that any G_t is a Fredholm map of index zero. If a given G_t is orientable, then G is orientable. If, in addition, G_t is oriented, then there exists and is unique an orientation of G such that the orientation of G_t is inherited from that of G.*

We conclude this section by showing how the orientation of a Fredholm map g is related to the orientations of domain and codomain of suitable restrictions of g. This argument will be crucial in the definition of the degree for quasi-Fredholm maps.

Let $g\colon \Omega \to F$ be an oriented map and Z a finite dimensional subspace of F transverse to g. By classical transversality results, $M = g^{-1}(Z)$ is a differentiable manifold of the same dimension as Z. In addition, M is orientable (see [1, Remark 2.5 and Lemma 3.1]). Here we show how the orientation of g and a chosen orientation of Z induce an orientation on any tangent space T_xM.

Let Z be oriented. Choose any $x \in M$ and let A be any positive corrector of $g'(x)$ with image contained in Z (the existence of such a corrector is ensured by the transversality of Z to g). Then, orient the tangent space T_xM in such a way that the isomorphism

$$(g'(x) + A)|_{T_xM}\colon T_xM \to Z$$

is orientation preserving. As proved in [3], the orientation of T_xM does not depend on the choice of the positive corrector A, but just on the orientation of Z and $g'(x)$. With this orientation, we call M the *oriented Fredholm g-preimage* of Z.

3. Orientability and degree for quasi-Fredholm maps

In this section we summarize the main ideas in the construction of a topological degree for quasi-Fredholm maps. See [3] for details. We start by recalling the construction of an orientation for this class of maps.

As before, E and F are real Banach spaces, and Ω is an open subset of E. A map $k\colon \Omega \to F$ is called locally compact if for any $x_0 \in \Omega$ the restriction of k to a convenient neighborhood of x_0 is a compact map (that is, a map whose image is contained in a compact subset of F).

Definition 3.1. A map $f\colon \Omega \to F$ is said to be *quasi-Fredholm* provided that $f = g - k$, where g is Fredholm of index zero and k is locally compact. The map g is called a *smoothing map* of f.

The following definition provides an extension to quasi-Fredholm maps of the concept of orientability.

Definition 3.2. A quasi-Fredholm map $f\colon \Omega \to F$ is *orientable* if it has an orientable smoothing map.

If f is an orientable quasi-Fredholm map, any smoothing map of f is orientable. Indeed, given two smoothing maps g^0 and g^1 of f, consider the homotopy

$$G(x,t) = (1-t)g^0(x) + tg^1(x), \quad (x,t) \in \Omega \times [0,1]. \tag{3.1}$$

Notice that any G_t is Fredholm of index zero, since it differs from g^0 by a C^1 locally compact map. By Theorem 2.8, if g^0 is orientable, then g^1 is orientable as well.

Let $f\colon \Omega \to F$ be an orientable quasi-Fredholm map. To define a notion of orientation of f, consider the set $\mathcal{S}(f)$ of the oriented smoothing maps of f. We introduce in $\mathcal{S}(f)$ the following equivalence relation. Given g^0, g^1 in $\mathcal{S}(f)$, consider, as in formula (3.1), the straight-line homotopy G joining g^0 and g^1. We say that g^0 is *equivalent* to g^1 if their orientations are inherited from the same orientation of G, whose existence is ensured by Theorem 2.8. It is immediate to verify that this is an equivalence relation.

Definition 3.3. Let $f\colon \Omega \to F$ be an orientable quasi-Fredholm map. An *orientation* of f is the choice of an equivalence class in $\mathcal{S}(f)$.

In the sequel, if f is an oriented quasi-Fredholm map, the elements of the chosen class of $\mathcal{S}(f)$ will be called *positively oriented smoothing maps* of f.

As for the case of Fredholm maps of index zero, the orientation of quasi-Fredholm maps verifies a homotopy invariance property, stated in Theorem 3.6 below. We need first some definitions.

Definition 3.4. A map $H\colon \Omega \times [0,1] \to F$ of the type

$$H(x,t) = G(x,t) - K(x,t)$$

is called a *homotopy of quasi-Fredholm maps* provided that G is C^1, any G_t is Fredholm of index zero, and K is locally compact. In this case G is said to be a *smoothing homotopy* of H.

We need a concept of orientability for homotopies of quasi-Fredholm maps. The definition is analogous to that given for quasi-Fredholm maps. Let $H\colon \Omega\times[0,1] \to F$ be a homotopy of quasi-Fredholm maps. Let $\mathcal{S}(H)$ be the set of oriented smoothing homotopies of H. Assume that $\mathcal{S}(H)$ is nonempty and define on this set an equivalence relation as follows. Given G^0 and G^1 in $\mathcal{S}(H)$, consider the map

$$\mathcal{G}\colon \Omega \times [0,1] \times [0,1] \to F$$

defined as

$$\mathcal{G}(x,t,s) = (1-s)G^0(x,t) + sG^1(x,t).$$

We say that G^0 is *equivalent* to G^1 if their orientations are inherited from an orientation of the map

$$(x,t,s) \mapsto \partial_1 \mathcal{G}(x,t,s).$$

The reader can easily verify that this is actually an equivalence relation on $\mathcal{S}(H)$.

Definition 3.5. A homotopy of quasi-Fredholm maps $H\colon \Omega \times [0,1] \to F$ is said to be *orientable* if $\mathcal{S}(H)$ is nonempty. An *orientation* of H is the choice of an equivalence class of $\mathcal{S}(H)$.

The following homotopy invariance property of the orientation of quasi-Fredholm maps is the analogue of Theorem 2.8 and a straightforward consequence of Proposition 2.5.

Theorem 3.6. *Let $H\colon \Omega \times [0,1] \to F$ be a homotopy of quasi-Fredholm maps. If a partial map H_t is oriented, then there exists and is unique an orientation of H such that the orientation of H_t is inherited from that of H.*

Let us now summarize the construction of the degree.

Definition 3.7. Let $f\colon \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. The triple $(f,U,0)$ is said to be *qF-admissible* provided that $f^{-1}(0) \cap U$ is compact.

The degree is defined as a map from the set of all qF-admissible triples into $\mathbb{Z}$. The construction is divided in two steps. In the first one we consider triples $(f,U,0)$ such that f has a smoothing map g with $(f-g)(U)$ contained in a finite dimensional subspace of F. In the second step this assumption is removed, the degree being defined for general qF-admissible triples.

Step 1. Let $(f,U,0)$ be a qF-admissible triple and let g be a positively oriented smoothing map of f such that $(f-g)(U)$ is contained in a finite dimensional subspace of F. As $f^{-1}(0) \cap U$ is compact, there exist a finite dimensional subspace Z of F and an open subset W of U containing $f^{-1}(0) \cap U$ and such that g is transverse to Z in W. We may assume that Z contains $(f-g)(U)$. Choose any orientation of Z and, as in Section 2, let the manifold $M = g^{-1}(Z) \cap W$ be the oriented Fredholm $g|_W$-preimage of Z. One can easily verify that $(f|_M)^{-1}(0) = f^{-1}(0) \cap U$. Thus $(f|_M)^{-1}(0)$ is compact, and the Brouwer degree of the triple $(f|_M, M, 0)$ is well defined.

Definition 3.8. Let $(f,U,0)$ be a qF-admissible triple and let g be a positively oriented smoothing map of f such that $(f-g)(U)$ is contained in a finite dimensional subspace of F. Let Z be a finite dimensional subspace of F and $W \subseteq U$ an open neighborhood of $f^{-1}(0) \cap U$ such that

(1) Z contains $(f-g)(U)$,
(2) g is transverse to Z in W.

Assume Z oriented and let M be the oriented Fredholm $g|_W$-preimage of Z. Then, the degree of $(f,U,0)$ is defined as

$$\deg_{qF}(f,U,0) = \deg(f|_M, M, 0), \tag{3.2}$$

where the right hand side of the above formula is the Brouwer degree of the triple $(f|_M, M, 0)$.

In [3] it is proved that the above definition is well posed, in the sense that the right hand side of (3.2) is independent of the choice of the smoothing map g, the open set W and the oriented subspace Z.

Step 2. Let us now extend the definition of degree to general qF-admissible triples.

Definition 3.9 (General definition of degree)**.** Let $(f, U, 0)$ be a qF-admissible triple. Consider:

(1) a positively oriented smoothing map g of f;
(2) an open neighborhood V of $f^{-1}(0) \cap U$ such that $\overline{V} \subseteq U$, g is proper on $\overline{V}$ and $(f - g)|_{\overline{V}}$ is compact;
(3) a continuous map $\xi \colon \overline{V} \to F$ having bounded finite dimensional image and such that
$$\|g(x) - f(x) - \xi(x)\| < \rho, \quad \forall x \in \partial V,$$
where ρ is the distance in F between 0 and $f(\partial V)$.

Then, the degree of $(f, U, 0)$ is given by
$$\deg_{qF}(f, U, 0) = \deg_{qF}(g - \xi, V, 0). \tag{3.3}$$

Observe that the right hand side of (3.3) is well defined since the triple $(g-\xi, V, 0)$ is qF-admissible. Indeed, $g - \xi$ is proper on $\overline{V}$ and thus $(g - \xi)^{-1}(0)$ is a compact subset of $\overline{V}$ which is actually contained in V by assumption (3). Moreover, as shown in [3], Definition 3.9 is well posed since $\deg_{qF}(g - \xi, V, 0)$ does not depend on g, ξ and V.

Theorem 3.10 below collects the most important properties of the degree for quasi-Fredholm maps (see [3] for the proof).

Theorem 3.10. *The following properties of the degree hold:*

1. *(Normalization) If the identity I of E is naturally oriented, then*
$$\deg_{qF}(I, E, 0) = 1.$$
2. *(Additivity) Given a qF-admissible triple $(f, U, 0)$ and two disjoint open subsets U_1, U_2 of U such that $f^{-1}(0) \cap U \subseteq U_1 \cup U_2$, one has*
$$\deg_{qF}(f, U, 0) = \deg_{qF}(f, U_1, 0) + \deg_{qF}(f, U_2, 0).$$
3. *(Excision) If $(f, U, 0)$ is qF-admissible and U_1 is an open subset of U containing $f^{-1}(0) \cap U$, then*
$$\deg_{qF}(f, U, 0) = \deg_{qF}(f, U_1, 0).$$
4. *(Existence) If $(f, U, 0)$ is qF-admissible and*
$$\deg_{qF}(f, U, 0) \neq 0,$$
then the equation $f(x) = 0$ has a solution in U.
5. *(Homotopy invariance) Let $H \colon U \times [0, 1] \to F$ be an oriented homotopy of quasi-Fredholm maps. If $H^{-1}(0)$ is compact, then $\deg_{qF}(H_t, U, 0)$ does not depend on $t \in [0, 1]$.*

4. Measures of noncompactness

In this section we recall the definition and properties of the Kuratowski measure of noncompactness [11], together with some related concepts. For general reference, see e.g. Deimling [6].

From now on the spaces E and F are assumed to be infinite dimensional. As before Ω is an open subset of E.

The *Kuratowski measure of noncompactness* $\alpha(A)$ of a bounded subset A of E is defined as the infimum of the real numbers $d > 0$ such that A admits a finite covering by sets of diameter less than d. If A is unbounded, we set $\alpha(A) = +\infty$. We summarize the following properties of the measure of noncompactness. Given $A \subseteq E$, by $\overline{\mathrm{co}}\, A$ we denote the closed convex hull of A.

Proposition 4.1. *Let $A, B \subseteq E$. Then*

1. $\alpha(A) = 0$ *if and only if* $\overline{A}$ *is compact;*
2. $\alpha(\lambda A) = |\lambda|\alpha(A)$ *for any* $\lambda \in \mathbb{R}$*;*
3. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$*;*
4. *if* $A \subseteq B$, *then* $\alpha(A) \leq \alpha(B)$*;*
5. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$*;*
6. $\alpha(\overline{\mathrm{co}}\, A) = \alpha(A)$.

Properties (1)–(5) are straightforward consequences of the definition, while the last one is due to Darbo [5].

Given a continuous map $f\colon \Omega \to F$, let $\alpha(f)$ and $\omega(f)$ be as in the Introduction. It is important to observe that $\alpha(f) = 0$ if and only if f is completely continuous (that is, the restriction of f to any bounded subset of Ω is a compact map) and $\omega(f) > 0$ only if f is proper on bounded closed sets. For a complete list of properties of $\alpha(f)$ and $\omega(f)$ we refer to [10]. We need the following one concerning linear operators.

Proposition 4.2. *Let $L\colon E \to F$ be a bounded linear operator. Then $\omega(L) > 0$ if and only if* $\operatorname{Im} L$ *is closed and* $\dim \operatorname{Ker} L < +\infty$.

As a consequence of Proposition 4.2 one gets that a bounded linear operator $L\colon E \to F$ is Fredholm if and only if $\omega(L) > 0$ and $\omega(L^*) > 0$, where L^* is the adjoint of L.

Let f be as above and fix $p \in \Omega$. We recall the definitions of $\alpha_p(f)$ and $\omega_p(f)$ given in [4]. Let $B(p, r)$ denote the open ball in E centered at p with radius r. Suppose that $B(p, r) \subseteq \Omega$ and consider

$$\alpha(f|_{B(p,r)}) = \sup\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq B(p,r),\, \alpha(A) > 0\right\}.$$

This is nondecreasing as a function of r. Hence, we can define

$$\alpha_p(f) = \lim_{r\to 0} \alpha(f|_{B(p,r)}).$$

Clearly $\alpha_p(f) \leq \alpha(f)$ for any $p \in \Omega$. In an analogous way, we define

$$\omega_p(f) = \lim_{r\to 0} \omega(f|_{B(p,r)}),$$

and we have $\omega_p(f) \geq \omega(f)$ for any p. It is easy to show that the main properties of α and ω hold, with minor changes, as well for α_p and ω_p (see [4]).

Proposition 4.3. *Let $f\colon \Omega \to F$ be continuous and $p \in \Omega$. Then*

(1) *if f is locally compact, $\alpha_p(f) = 0$;*
(2) *if $\omega_p(f) > 0$, f is locally proper at p.*

Clearly, for a bounded linear operator $L\colon E \to F$, the numbers $\alpha_p(L)$ and $\omega_p(L)$ do not depend on the point p and coincide, respectively, with $\alpha(L)$ and $\omega(L)$. Furthermore, for the C^1 case we get the following result.

Proposition 4.4 ([4]). *Let $f\colon \Omega \to F$ be of class C^1. Then, for any $p \in \Omega$ we have $\alpha_p(f) = \alpha(f'(p))$ and $\omega_p(f) = \omega(f'(p))$.*

Observe that if $f\colon \Omega \to F$ is a Fredholm map, as a straightforward consequence of Propositions 4.2 and 4.4, we obtain $\omega_p(f) > 0$ for any $p \in \Omega$.
As an application of Proposition 4.4 one could deduce the following result.

Proposition 4.5 ([4]). *Let $g\colon \Omega \to F$ and $\varphi\colon \Omega \to \mathbb{R}$ be of class C^1, with $\varphi(x) \geq 0$. Consider the product map $f\colon \Omega \to F$ defined by $f(x) = \varphi(x)g(x)$. Then, for any $p \in \Omega$ we have $\alpha_p(f) = \varphi(p)\alpha_p(g)$ and $\omega_p(f) = \varphi(p)\omega_p(g)$.*

By means of Proposition 4.5 one can easily find examples of maps f such that $\alpha(f) = \infty$ and $\alpha_p(f) < \infty$ for any p, and examples of maps f with $\omega(f) = 0$ and $\omega_p(f) > 0$ for any p (see [4]). Moreover, in [4] there is an example of a map f such that $\alpha(f) > 0$ and $\alpha_p(f) = 0$ for any p.

In the sequel we will deal with maps G defined on the product space $E \times \mathbb{R}$. In order to define $\alpha_{(p,t)}(G)$, we consider the norm

$$||(p,t)|| = \max\{||p||, |t|\}.$$

The natural projection of $E \times \mathbb{R}$ onto the first factor will be denoted by π_1.

Remark 4.6. With the above norm, π_1 is nonexpansive. Therefore $\alpha(\pi_1(X)) \leq \alpha(X)$ for any subset X of $E \times \mathbb{R}$. More precisely, since $\mathbb{R}$ is finite dimensional, if $X \subseteq E \times \mathbb{R}$ is bounded, we have $\alpha(\pi_1(X)) = \alpha(X)$.

5. Definition of degree

This section is devoted to the construction of a concept of degree for a class of triples that we shall call α-admissible. We start with two definitions.

Definition 5.1. Let $g\colon \Omega \to F$ be an oriented map, $k\colon \Omega \to F$ a continuous map and U an open subset of Ω. The triple (g, U, k) is said to be *α-admissible* if

i) $\alpha_p(k) < \omega_p(g)$ for any $p \in U$;
ii) the *solution set* $S = \{x \in U : g(x) = k(x)\}$ is compact.

Definition 5.2. Let (g, U, k) be an α-admissible triple and $\mathcal{V} = \{V_1, \ldots, V_N\}$ a finite covering of open balls of its solution set S. We say that $\mathcal{V}$ is an *α-covering* of S (relative to (g, U, k)) if for any $i \in \{1, \ldots, N\}$ the following properties hold:

i) the ball $\widetilde{V}_i$ of double radius and same center as V_i is contained in U;
ii) $\alpha(k|_{\widetilde{V}_i}) < \omega(g|_{\widetilde{V}_i})$.

Let (g, U, k) be an α-admissible triple and $\mathcal{V} = \{V_1, \dots, V_N\}$ an α-covering of the solution set S. We define the following sequence $\{C_n\}$ of convex closed subsets of E:

$$C_1 = \overline{\mathrm{co}} \left(\bigcup_{i=1}^{N} \{x \in V_i : g(x) \in k(\widetilde{V}_i)\} \right),$$

and, inductively,

$$C_n = \overline{\mathrm{co}} \left(\bigcup_{i=1}^{N} \{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_{n-1})\} \right), \quad n \geq 2.$$

Observe that, by induction, $C_{n+1} \subseteq C_n$ and $S \subseteq C_n$ for any $n \geq 1$. Then the set

$$C_\infty = \bigcap_{n \geq 1} C_n$$

turns out to be closed, convex, and containing S. Consequently, if S is nonempty, so is C_∞. To emphasize the fact that the set C_∞ is uniquely determined by the covering $\mathcal{V}$, sometimes it will be denoted by $C_\infty^{\mathcal{V}}$. Let us prove two other crucial properties of C_∞:

(1) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_\infty)\} \subseteq C_\infty$, for any $i = 1, \dots, N$;
(2) C_∞ is compact.

To verify the first one, fix $i \in \{1, \dots, N\}$ and let $x \in V_i$ be such that $g(x) \in k(\widetilde{V}_i \cap C_\infty)$. In particular, it follows $g(x) \in k(\widetilde{V}_i)$ and, consequently, $x \in C_1$. Moreover, for any $n \geq 1$ we have $g(x) \in k(\widetilde{V}_i \cap C_n)$ and this implies $x \in C_{n+1}$. Hence, $x \in C_\infty$, and the first property holds.

To check the compactness of C_∞, we prove that $\alpha(C_n) \to 0$ as $n \to \infty$. Let $n \geq 2$ be fixed. By the properties of the measure of noncompactness (see Section 4) we have

$$\begin{aligned} \alpha(C_n) &= \alpha \left(\bigcup_{i=1}^{N} \{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_{n-1})\} \right) \\ &= \max_{1 \leq i \leq N} \alpha \left(\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_{n-1})\} \right). \end{aligned}$$

Fix $i \in \{1, \dots, N\}$, and denote

$$A_{n,i} = \{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_{n-1})\}.$$

Since $A_{n,i} \subseteq \widetilde{V}_i$, by definition we have $\alpha(A_{n,i})\omega(g|_{\widetilde{V}_i}) \leq \alpha(g(A_{n,i}))$. Moreover, $g(A_{n,i}) \subseteq k(\widetilde{V}_i \cap C_{n-1})$. Therefore, as $\omega(g|_{\widetilde{V}_i}) > 0$, we have

$$\alpha(A_{n,i}) \leq \frac{1}{\omega(g|_{\widetilde{V}_i})} \alpha(g(A_{n,i})) \leq \frac{1}{\omega(g|_{\widetilde{V}_i})} \alpha(k(\widetilde{V}_i \cap C_{n-1})).$$

On the other hand, by definition, $\alpha(k(\widetilde{V}_i \cap C_{n-1})) \leq \alpha(k|_{\widetilde{V}_i})\alpha(\widetilde{V}_i \cap C_{n-1})$, thus

$$\alpha(A_{n,i}) \leq \frac{\alpha(k|_{\widetilde{V}_i})}{\omega(g|_{\widetilde{V}_i})} \alpha(\widetilde{V}_i \cap C_{n-1}) = \mu_i \alpha(\widetilde{V}_i \cap C_{n-1}) \leq \mu_i \alpha(C_{n-1}),$$

where by assumption $\mu_i = \alpha(k|_{\widetilde{V}_i})/\omega(g|_{\widetilde{V}_i}) < 1$. Finally,

$$\alpha(C_n) = \max_{1 \leq i \leq N} \alpha(A_{n,i}) \leq \max_{1 \leq i \leq N} \mu_i \alpha(C_{n-1}) = \mu \alpha(C_{n-1}),$$

where $\mu = \max_i \mu_i < 1$. Hence, $\alpha(C_n) \to 0$, and this implies that the set C_∞ is compact, as claimed.

Definition 5.3. Let (g, U, k) be an α-admissible triple, $\mathcal{V} = \{V_1, \dots, V_N\}$ an α-covering of the solution set S and C a compact convex set. We say that $(\mathcal{V}, C)$ is an α-*pair* (relative to (g, U, k)) if the following properties hold:

(1) $U \cap C \neq \emptyset$;
(2) $C_\infty^{\mathcal{V}} \subseteq C$;
(3) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C)\} \subseteq C$ for any $i = 1, \dots, N$.

Remark 5.4. Given any α-admissible triple (g, U, k), it is always possible to find an α-pair $(\mathcal{V}, C)$. Indeed, fix an α-covering $\mathcal{V}$ of the solution set S. If the corresponding compact set $C_\infty^{\mathcal{V}}$ is nonempty, then, clearly, the pair $(\mathcal{V}, C_\infty^{\mathcal{V}})$ verifies properties (1)–(3). If $C_\infty^{\mathcal{V}} = \emptyset$ (this can happen only if $S = \emptyset$), we may assume without loss of generality that

$$U \setminus \bigcup_{i=1}^{N} \widetilde{V}_i \neq \emptyset.$$

One can check that, given any $p \in U \setminus \bigcup_{i=1}^{N} \widetilde{V}_i$, the pair $(\mathcal{V}, \{p\})$ satisfies properties (1)–(3).

Let now $(\mathcal{V}, C)$ be an α-pair. Consider a retraction $r \colon E \to C$, whose existence is ensured by Dugundji's Extension Theorem [7]. Denote $V = \bigcup_{i=1}^{N} V_i$, and let W be a (possibly empty) open subset of V containing S such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. For example, if ρ denotes the minimum of the radii of the balls V_i, one may take as W the set

$$\{x \in V : \|x - r(x)\| < \rho\}.$$

Observe that property (3) above implies that the two equations $g(x) = k(x)$ and $g(x) = k(r(x))$ have the same solution set in W (notice that the composition kr is defined in $r^{-1}(U)$). The map kr is locally compact (even if not necessarily compact), hence the triple $(g - kr, W, 0)$ is admissible for the degree for quasi-Fredholm maps. We define the degree of (g, U, k) as follows:

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0),$$

where the right hand side is the degree defined in Section 3.

The following definition summarizes the above construction.

Definition 5.5. Let (g, U, k) be an α-admissible triple and $(\mathcal{V}, C)$ an α-pair. Consider a retraction $r \colon E \to C$. Let $\mathcal{V} = \{V_1, \dots, V_N\}$, denote $V = \bigcup_{i=1}^{N} V_i$, and let W be an open subset of V containing S such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. We put

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0).$$

In order to show that this definition is well posed, we have to prove that it is independent of the choice of the α-pair $(\mathcal{V}, C)$, of the retraction r and of the open set W. This is the purpose of the following proposition.

Proposition 5.6. *Let $(\mathcal{V}, C)$ and $(\mathcal{V}', C')$ be two α-pairs relative to an α-admissible triple (g, U, k), where*

$$\mathcal{V} = \{V_1, \dots, V_N\} \quad \text{and} \quad \mathcal{V}' = \{V_1', \dots, V_M'\}.$$

Consider two retractions $r\colon E \to C$ *and* $r'\colon E \to C'$. *Denote* $V = \bigcup_{i=1}^N V_i$, *and let* W *be an open subset of* V *containing* S *such that, for any* i, $x \in W \cap V_i$ *implies* $r(x) \in \widetilde{V}_i$. *Analogously, denote* $V' = \bigcup_{j=1}^M V'_j$, *and let* W' *be an open subset of* V' *containing* S *such that, for any* j, $x' \in W' \cap V'_j$ *implies* $r'(x') \in \widetilde{V}'_j$. *Then*

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr', W', 0).$$

Proof. Consider a third covering $\mathcal{V}'' = \{V''_1, \dots, V''_T\}$ of the solution set S of open balls such that for any $l \in \{1, \dots, T\}$ there exist i and j such that $V''_l \subseteq V_i \cap V'_j$. In particular, $\mathcal{V}''$ is still an α-covering of S. Consider the compact convex set $C_\infty^{\mathcal{V}''}$. We distinguish two different cases.

i) $C_\infty^{\mathcal{V}''} = \emptyset$. In this case $S = \emptyset$ and, consequently, by the existence property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(g - kr, W, 0) = 0 \quad \text{and} \quad \deg_{qF}(g - kr', W', 0) = 0.$$

ii) $C_\infty^{\mathcal{V}''} \neq \emptyset$. In this case, $(\mathcal{V}'', C_\infty^{\mathcal{V}''})$ is an α-pair. To simplify the notations, denote $C''_\infty = C_\infty^{\mathcal{V}''}$. Consider a retraction $r''\colon E \to C''_\infty$. Denote $V'' = \bigcup_{l=1}^T V''_l$, and let W'' be an open subset of V'' containing S such that, for any l, $x \in W'' \cap V''_l$ implies $r''(x) \in \widetilde{V}''_l$. Clearly, to prove the assertion it is sufficient to show that

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr'', W'', 0).$$

Now, denote $C_\infty = C_\infty^{\mathcal{V}}$ and let $\{C_n\}$ and $\{C''_n\}$ be the sequences of sets defining C_∞ and C''_∞, respectively. Since $C''_n \subseteq C_n$ for any $n \geq 1$, it follows $C''_\infty \subseteq C_\infty$. In particular, $C''_\infty \subseteq C$. Moreover, without loss of generality, we can assume that the open set W'' is contained in W. Thus, by the excision property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr, W'', 0). \tag{5.1}$$

Consider the following homotopy:

$$H\colon W'' \times [0,1] \to F,$$
$$H(x,t) = g(x) - k(tr(x) + (1-t)r''(x)).$$

Let $x \in W''$, let V''_l contain x for some l, and let V_i contain V''_l for some i. Since $x \in W'' \subseteq W$, we have $r(x) \in \widetilde{V}_i$ and $r''(x) \in \widetilde{V}''_l$. Hence, as $\widetilde{V}''_l \subseteq \widetilde{V}_i$, it follows $r''(x) \in \widetilde{V}_i$ and, consequently, H is well defined.

Let now $(x,t) \in W'' \times [0,1]$ be a pair such that $H(x,t) = 0$. If $x \in V''_l$ for some l, and $V''_l \subseteq V_i$ for some i, then both $r(x)$ and $r''(x)$ belong to $\widetilde{V}_i \cap C$, since $r''(x) \in C''_\infty$ and $C''_\infty \subseteq C$. Thus, $tr(x) + (1-t)r''(x) \in \widetilde{V}_i \cap C$ and, in particular, $g(x) \in k(\widetilde{V}_i \cap C)$. This implies $x \in C$ and, consequently, $r(x) = x$.

We want to show that, actually, $x \in C''_\infty$. Since $r(x) = x$, we have

$$tx + (1-t)r''(x) \in \widetilde{V}''_l \cap C$$

and, in particular, $g(x) \in k(\widetilde{V}''_l)$. Consequently, $x \in C''_1$. As $C''_\infty \subseteq C''_1$, we have $r''(x) \in C''_1$, and $tx + (1-t)r''(x) \in \widetilde{V}''_l \cap C''_1$ since this is convex. Thus, $g(x) \in k(\widetilde{V}''_l \cap C''_1)$, and this implies $x \in C''_2$. Inductively, we get $x \in C''_n$ for any $n \geq 1$. Hence, $x \in C''_\infty$ and, consequently, $r''(x) = x$.

Finally, $g(x) = k(x)$, that is, $x \in S$. Therefore, the solution set

$$\{(x,t) \in W'' \times [0,1] : H(x,t) = 0\}$$

coincides with $S\times[0,1]$. Hence, we can apply the homotopy invariance of the degree for quasi-Fredholm maps to get

$$\deg_{qF}(g-kr,W'',0)=\deg_{qF}(g-kr'',W'',0),$$

and the assertion follows taking into account formula (5.1). □

6. Properties of the degree

Theorem 6.1. *The following properties of the degree hold:*

1. *(Normalization) Let the identity I of E be naturally oriented. Then*
$$\deg(I,E,0)=1.$$
2. *(Additivity) Given an α-admissible triple (g,U,k) and two disjoint open subsets U^1,U^2 of U, assume that $S=\{x\in U : g(x)=k(x)\}$ is contained in $U^1\cup U^2$. Then*
$$\deg(g,U,k)=\deg(g,U^1,k)+\deg(g,U^2,k).$$
3. *(Homotopy invariance) Let $H\colon U\times[0,1]\to F$ be a homotopy of the form $H(x,t)=G(x,t)-K(x,t)$, where G is of class C^1, any $G_t=G(\cdot,t)$ is Fredholm of index zero, K is continuous, and $\alpha_{(p,t)}(K)<\omega_{(p,t)}(G)$ for any pair $(p,t)\in U\times[0,1]$. Assume that G is oriented and that $H^{-1}(0)$ is compact. Then $\deg(G_t,U,K_t)$ is well defined and does not depend on $t\in[0,1]$.*

Proof. 1. *(Normalization)* It follows easily from the normalization property of the degree for quasi-Fredholm maps.

2. *(Additivity)* Let $S^1=S\cap U^1$ and $S^2=S\cap U^2$, so that $S=S^1\cup S^2$. The fact that the triples (g,U^1,k) and (g,U^2,k) are α-admissible is clear from the definition.

Let $\mathcal{V}^1=\{V_1^1,\dots,V_N^1\}$ and $\mathcal{V}^2=\{V_2^2,\dots,V_M^2\}$ be two α-coverings of S^1 (relative to (g,U^1,k)) and of S^2 (relative to (g,U^2,k)), respectively. For simplicity, denote $C_\infty^1=C_\infty^{\mathcal{V}^1}$ and $C_\infty^2=C_\infty^{\mathcal{V}^2}$. Then, consider the family

$$\mathcal{V}=\{V_1^1,\dots,V_N^1,V_1^2,\dots,V_M^2\}.$$

Note that $\mathcal{V}$ is an α-covering of S. Consider the compact convex set $C_\infty=C_\infty^{\mathcal{V}}$. By definition, C_∞ contains both C_∞^1 and C_∞^2; moreover, it has the following properties:

$$\{x\in V_i^1 : g(x)\in k(\widetilde{V}_i^1\cap C_\infty)\}\subseteq C_\infty,\quad i=1,\dots,N;$$

and

$$\{x\in V_j^2 : g(x)\in k(\widetilde{V}_j^2\cap C_\infty)\}\subseteq C_\infty,\quad j=1,\dots,M.$$

We distinguish two different cases.

i) If $C_\infty=\emptyset$, then $S=\emptyset$, hence $S^1=\emptyset$ and $S^2=\emptyset$. Consequently, applying Definition 5.5, by the existence property of the degree for quasi-Fredholm maps it follows

$$\deg(g,U,k)=0;\quad \deg(g,U^1,k)=0;\quad \deg(g,U^2,k)=0.$$

ii) If $C_\infty\neq\emptyset$, consider a retraction $r\colon E\to C_\infty$. Denote $V^1=\bigcup_{i=1}^N V_i^1$, $V^2=\bigcup_{j=1}^M V_j^2$ and $V=V^1\cup V^2$. Let W be an open subset of V containing

S such that, for any i, $x \in W \cap V_i^1$ implies $r(x) \in \widetilde{V}_i^1$ and, for any j, $x' \in W \cap V_j^2$ implies $r(x') \in \widetilde{V}_j^2$. By definition we have

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0).$$

Since W is an open neighborhood of S in V, and V is the disjoint union of V^1 and V^2, we can assume $W = W^1 \cup W^2$, where $W^1 \subseteq V^1$ and $W^2 \subseteq V^2$. The open sets W^1 and W^2 are disjoint. In addition, W^1 contains S^1, and W^2 contains S^2. Therefore, by the additivity property of the degree for quasi-Fredholm maps, we have

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr, W^1, 0) + \deg_{qF}(g - kr, W^2, 0).$$

Now, observe that $(\mathcal{V}^\lambda, C_\infty)$ is an α-pair relative to (g, U^λ, k), for $\lambda = 1, 2$. Consequently,

$$\deg(g, U^\lambda, k) = \deg_{qF}(g - kr, W^\lambda, 0), \quad \lambda = 1, 2, \tag{6.1}$$

and the assertion follows.

3. *(Homotopy invariance)* For $t \in [0, 1]$, let Σ^t denote the compact set $\{x \in U : G_t(x) = K_t(x)\}$. Given any t, the fact that the triple (G_t, U, K_t) is α-admissible follows easily from the compactness of Σ^t and observing that $\alpha_p(K_t) \leq \alpha_{(p,t)}(K)$ and $\omega_p(G_t) \geq \omega_{(p,t)}(G)$ for all $p \in U$. Consequently, it is sufficient to show that the integer-valued function

$$t \mapsto \deg(G_t, U, K_t)$$

is locally constant. To this purpose, fix $\tau \in [0, 1]$ and, given $\delta > 0$, let I_δ denote the interval $[\tau - \delta, \tau + \delta] \cap [0, 1]$. It is possible to find $\delta > 0$ and a finite family of open balls $\mathcal{V} = \{V_1, \dots, V_N\}$ with the following properties:

i) $V = \bigcup_{i=1}^N V_i$ contains Σ^t for any $t \in I_\delta$;
ii) the ball $\widetilde{V}_i$ of double radius and same center as V_i is contained in U;
iii) $\alpha(K|_{\widetilde{V}_i \times I_\delta}) < \omega(G|_{\widetilde{V}_i \times I_\delta})$, for any $i = 1, \dots, N$.

In particular it follows that, for any $t \in I_\delta$, $\mathcal{V}$ is an α-covering of Σ^t. As in the construction of the sequence $\{C_n\}$ in Section 5, for any fixed $t \in I_\delta$ we define the following sequence of sets:

$$C_1^t = \overline{\text{co}} \left(\bigcup_{i=1}^N \{x \in V_i : G_t(x) \in K_t(\widetilde{V}_i)\} \right),$$

and, inductively,

$$C_n^t = \overline{\text{co}} \left(\bigcup_{i=1}^N \{x \in V_i : G_t(x) \in K_t(\widetilde{V}_i \cap C_{n-1}^t)\} \right), \quad n \geq 2.$$

Then we set $C_\infty^t = \bigcap_{n \geq 1} C_n^t$. We observe that C_∞^t is compact and convex, moreover it has the following property:

$$\{x \in V_i : G_t(x) \in K_t(\widetilde{V}_i \cap C_\infty^t)\} \subseteq C_\infty^t, \quad i = 1, \dots, N.$$

Now, we define the following sequence $\{\widehat{C}_n\}$ of convex closed subsets of E independent of t:

$$\widehat{C}_1 = \overline{\text{co}} \left(\pi_1 \left(\bigcup_{i=1}^N \{(x, t) \in V_i \times I_\delta : G(x, t) \in K(\widetilde{V}_i \times I_\delta)\} \right) \right),$$

and, inductively,

$$\widehat{C}_n = \overline{\mathrm{co}}\left(\pi_1\left(\bigcup_{i=1}^{N}\{(x,t) \in V_i \times I_\delta : G(x,t) \in K((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta)\}\right)\right), \quad n \geq 2.$$

Observe that, by induction, $\widehat{C}_{n+1} \subseteq \widehat{C}_n$ for any $n \geq 1$. Then the set

$$\widehat{C}_\infty = \bigcap_{n\geq 1} \widehat{C}_n$$

is closed and convex. We claim that the following properties of $\widehat{C}_\infty$ hold:

(1) $\widehat{C}_\infty$ is compact;
(2) $\widehat{C}_\infty$ contains C^t_∞ for any $t \in I_\delta$;
(3) $\{x \in V_i : G_t(x) \in K_t(\widetilde{V}_i \cap \widehat{C}_\infty)\} \subseteq \widehat{C}_\infty$ for any $i = 1, \dots, N$ and $t \in I_\delta$.

Let us prove that $\widehat{C}_\infty$ is compact. For simplicity, for any $n \geq 2$ and $i \in \{1, \dots, N\}$ we denote

$$\widehat{A}_{n,i} = \big\{(x,t) \in V_i \times I_\delta : G(x,t) \in K((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta)\big\},$$

and we set $\widehat{A}_n = \bigcup_{i=1}^N \widehat{A}_{n,i}$. Let $n \geq 2$ be fixed. Since $\widehat{A}_n \subseteq \widehat{C}_n \times I_\delta$, by Remark 4.6 we have $\alpha(\widehat{A}_n) \leq \alpha(\widehat{C}_n \times I_\delta) = \alpha(\widehat{C}_n)$. On the other hand,

$$\alpha(\widehat{C}_n) = \alpha(\overline{\mathrm{co}}\,(\pi_1(\widehat{A}_n))) = \alpha(\pi_1(\widehat{A}_n)) \leq \alpha(\widehat{A}_n),$$

the last inequality due to the fact that π_1 is nonexpansive. Consequently, we have

$$\alpha\left(\widehat{C}_n\right) = \alpha\left(\widehat{A}_n\right) = \alpha\left(\bigcup_{i=1}^{N} \widehat{A}_{n,i}\right) = \max_{1\leq i\leq N} \alpha\left(\widehat{A}_{n,i}\right).$$

Now, fix $i \in \{1, \dots, N\}$. Since $\widehat{A}_{n,i} \subseteq \widetilde{V}_i \times I_\delta$, by definition we have

$$\alpha(\widehat{A}_{n,i})\omega(G|_{\widetilde{V}_i\times I_\delta}) \leq \alpha(G(\widehat{A}_{n,i})).$$

Moreover, $G(\widehat{A}_{n,i}) \subseteq K((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta)$. Therefore,

$$\alpha(\widehat{A}_{n,i}) \leq \frac{1}{\omega(G|_{\widetilde{V}_i\times I_\delta})}\,\alpha(G(\widehat{A}_{n,i})) \leq \frac{1}{\omega(G|_{\widetilde{V}_i\times I_\delta})}\,\alpha(K((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta)).$$

On the other hand, by definition we have

$$\alpha(K((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta)) \leq \alpha(K|_{\widetilde{V}_i\times I_\delta})\alpha((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta),$$

and, by Remark 4.6, $\alpha((\widetilde{V}_i \cap \widehat{C}_{n-1}) \times I_\delta) = \alpha(\widetilde{V}_i \cap \widehat{C}_{n-1})$. Hence

$$\alpha(\widehat{A}_{n,i}) \leq \frac{\alpha(K|_{\widetilde{V}_i\times I_\delta})}{\omega(G|_{\widetilde{V}_i\times I_\delta})}\alpha(\widetilde{V}_i \cap \widehat{C}_{n-1}) = \nu_i\alpha(\widetilde{V}_i \cap \widehat{C}_{n-1}) \leq \nu_i\alpha(\widehat{C}_{n-1}),$$

where by assumption $\nu_i = \alpha(K|_{\widetilde{V}_i\times I_\delta})/\omega(G|_{\widetilde{V}_i\times I_\delta}) < 1$. Finally,

$$\alpha(\widehat{C}_n) = \max_{1\leq i\leq N} \alpha(\widehat{A}_{n,i}) \leq \max_{1\leq i\leq N} \nu_i\alpha(\widehat{C}_{n-1}) \leq \nu\alpha(\widehat{C}_{n-1}),$$

where $\nu = \max_i \nu_i < 1$. Thus, $\alpha(\widehat{C}_n) \to 0$ as $n \to \infty$, and this implies that the set $\widehat{C}_\infty$ is compact, as claimed.

For any fixed $t \in I_\delta$, the inclusion $C^t_\infty \subseteq \widehat{C}_\infty$ follows immediately from the fact that $C^t_n \subseteq \widehat{C}_n$ for any $n \geq 1$.

To verify the third property, fix $i \in \{1, \dots, N\}$ and $t \in I_\delta$, and let $x \in V_i$ be such that $G_t(x) \in K_t(\widetilde{V}_i \cap \widehat{C}_\infty)$. In particular, we have $G_t(x) \in K_t(\widetilde{V}_i)$, and this implies $x \in \widehat{C}_1$. Moreover, for any $n \geq 2$ we have $G_t(x) \in K_t(\widetilde{V}_i \cap \widehat{C}_{n-1})$. It follows $(x,t) \in \widehat{A}_{n,i}$, and, consequently, $x \in \pi_1(\widehat{A}_{n,i})$. Therefore, $x \in \widehat{C}_n$ for any $n \geq 2$. Hence, $x \in \widehat{C}_\infty$, and property (3) holds.

Since $\tau \in [0,1]$ is arbitrary, the assertion follows if we show that $\deg(G_t, U, K_t)$ is independent of $t \in I_\delta$. We distinguish two different cases.

i) $\widehat{C}_\infty = \emptyset$. In this case $C^t_\infty = \emptyset$ for any $t \in I_\delta$, hence $\Sigma^t = \emptyset$ for any t. Consequently, applying Definition 5.5, by the existence property of the degree for quasi-Fredholm maps we have $\deg(G_t, U, K_t) = 0$ for any $t \in I_\delta$.

ii) $\widehat{C}_\infty \neq \emptyset$. In this case, as properties (1)–(3) of $\widehat{C}_\infty$ hold, for any fixed $t \in I_\delta$ the pair $(\mathcal{V}, \widehat{C}_\infty)$ is an α-pair relative to the triple (G_t, U, K_t). Consider a retraction $r\colon E \to \widehat{C}_\infty$. Let W be an open subset of V containing $V \cap \widehat{C}_\infty$ such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. In particular, for any fixed $t \in I_\delta$ the open set W contains Σ^t. Thus, by definition we have

$$\deg(G_t, U, K_t) = \deg_{qF}(G_t - K_t r, W, 0), \quad t \in I_\delta.$$

Consider the following homotopy:

$$\widehat{H} : W \times I_\delta \to F$$

$$\widehat{H}(x,t) = G(x,t) - K(r(x),t).$$

This is a homotopy of quasi-Fredholm maps, since it is continuous and the map $(x,t) \mapsto K(r(x),t)$ is locally compact. Moreover, $\widehat{H}^{-1}(0)$ is compact, as it is closed in the compact set $H^{-1}(0)$. Then, the homotopy invariance property of the degree for quasi-Fredholm maps implies that $\deg_{qF}(G_t - K_t r, W, 0)$ does not depend on t. Hence, $\deg(G_t, U, K_t)$ is independent of $t \in I_\delta$, and we are done. □

7. Comparison with other degree theories

The purpose of this section is to show that our concept of degree extends the degree for quasi-Fredholm maps summarized in Section 3, and that it agrees with the Nussbaum degree [13] for the class of locally α-contractive vector fields.

7.1. **Degree for quasi-Fredholm maps.**

Let $f\colon \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. We recall that the triple $(f, U, 0)$ is qF-admissible provided that $f^{-1}(0) \cap U$ is compact.

Let $(f, U, 0)$ be a qF-admissible triple and let $f = g - k$, where g is a positively oriented smoothing map of f and k is locally compact. As pointed out in Section 4, we have $\omega_p(g) > 0$ and $\alpha_p(k) = 0$ for any $p \in U$. Hence, the triple (g, U, k) is α-admissible. We claim that

$$\deg(g, U, k) = \deg_{qF}(f, U, 0).$$

Indeed, let $\mathcal{V} = \{V_1, \dots, V_N\}$ be an α-covering of $S = \{x \in U : g(x) = k(x)\}$ relative to the triple (g, U, k), and consider the compact convex set $C_\infty = C^{\mathcal{V}}_\infty$. We distinguish two different cases.

i) If $C_\infty = \emptyset$, then $S = \emptyset$. Consequently, by the existence property of the degree for quasi-Fredholm maps and by Definition 5.5, we have

$$\deg_{qF}(f, U, 0) = 0 \quad \text{and} \quad \deg(g, U, k) = 0.$$

ii) If $C_\infty \neq \emptyset$, consider a retraction $r\colon E \to C_\infty$. Denote $V = \bigcup_{i=1}^N V_i$, and let W be a (possibly empty) open subset of V containing S such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. By definition we have

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0).$$

On the other hand, as $S \subseteq W$, by the excision property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(f, U, 0) = \deg_{qF}(f, W, 0).$$

Consider the following homotopy:

$$H\colon W \times [0,1] \to F,$$
$$H(x,t) = g(x) - k(tr(x) + (1-t)x).$$

Let $x \in W$, and let V_i contain x for some i. Since $r(x) \in \widetilde{V}_i$ and $x \in \widetilde{V}_i$, it follows $tr(x) + (1-t)x \in \widetilde{V}_i$ for any $t \in [0,1]$, and this shows that H is well defined.

As in the proof of Proposition 5.6 one gets

$$H^{-1}(0) \cap \big(W \times [0,1]\big) = S \times [0,1].$$

Hence, we can apply the homotopy invariance of the degree for quasi-Fredholm maps, obtaining

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - k, W, 0),$$

and the claim follows.

7.2. **Degree for locally α-contractive vector fields.** Let $f\colon \Omega \to F$ be a continuous map from an open subset of E into F. We recall the following definitions. The map f is said to be *α-Lipschitz* if $\alpha(f(A)) \leq \mu\alpha(A)$ for some $\mu \geq 0$ and any $A \subseteq \Omega$. If the α-Lipschitz constant μ is less than 1, then f is called *α-contractive*. The map f is said to be *α-condensing* if $\alpha(f(A)) < \alpha(A)$ for any $A \subseteq \Omega$ such that $0 < \alpha(A) < +\infty$. If for any $p \in \Omega$ there exists a neighborhood V_p of p such that $f|_{V_p}$ is α-contractive (resp. α-condensing), the map f is said to be *locally α-contractive* (resp. *locally α-condensing*).

In [12] and [13], Nussbaum developed a degree theory for triples of the form $(I - k, U, 0)$, where k is locally α-condensing. In particular, let U be an open subset of Ω and $k\colon \Omega \to E$ a locally α-condensing map. Assume that the set $S = \{x \in U : (I - k)(x) = 0\}$ is compact. Then, the triple $(I - k, U, 0)$ is admissible for the Nussbaum degree (N-admissible, for short). We will denote by $\deg_N(I - k, U, 0)$ the Nussbaum degree of an N-admissible triple.

We want to show that, in a sense to be specified, our degree and the Nussbaum degree coincide on the class of N-admissible triples of the form $(I - k, U, 0)$, where k is locally α-contractive.

Let $(I - k, U, 0)$ be a N-admissible triple and assume that the map k is locally α-contractive. Clearly, provided that I is oriented, the triple (I, U, k) is α-admissible. We claim that, if we assign the natural orientation to I, it follows

$$\deg(I, U, k) = \deg_N(I - k, U, 0).$$

Indeed, let $\mathcal{V} = \{V_1, \dots, V_N\}$ be an α-covering of S relative to the triple (I, U, k), and consider the (possibly empty) compact convex set $C_\infty = C_\infty^{\mathcal{V}}$.

Denote $\widetilde{V} = \bigcup_{i=1}^N \widetilde{V}_i$. As S is contained in $\widetilde{V}$, by the excision property of the Nussbaum degree we have

$$\deg_N(I - k, U, 0) = \deg_N(I - k, \widetilde{V}, 0).$$

Consider the following sequence $\{\widetilde{C}_n\}$ of convex closed subsets of E:

$$\widetilde{C}_1 = \overline{\mathrm{co}}\left(k(\widetilde{V})\right),$$

and, inductively,

$$\widetilde{C}_n = \overline{\mathrm{co}}\left(k(\widetilde{V} \cap \widetilde{C}_{n-1})\right), \quad n \geq 2.$$

Then the set

$$\widetilde{C}_\infty = \bigcap_{n \geq 1} \widetilde{C}_n$$

turns out to be closed, convex, and containing S. Moreover, the fact that k is locally α-contractive implies that $\widetilde{C}_\infty$ is compact. We observe that the following properties of $\widetilde{C}_\infty$ hold:

(1) $\widetilde{C}_\infty$ contains C_∞;

(2) $\{x \in V_i : x \in k(\widetilde{V}_i \cap \widetilde{C}_\infty)\} \subseteq \widetilde{C}_\infty$ for any $i = 1, \dots, N$.

The inclusion $C_\infty \subseteq \widetilde{C}_\infty$ follows immediately from the fact that $C_n \subseteq \widetilde{C}_n$ for any $n \geq 1$, where $\{C_n\}$ is the sequence of sets which defines C_∞, as in Section 5. On the other hand, property (2) follows from the trivial inclusion

$$\{x \in V_i : x \in k(\widetilde{V}_i \cap \widetilde{C}_n)\} \subseteq k(\widetilde{V} \cap \widetilde{C}_n),$$

which holds for any $n \geq 1$ and $i \in \{1, \dots, N\}$.

To prove the assertion, we distinguish two different cases.

i) $\widetilde{C}_\infty = \emptyset$. In this case, $C_\infty = \emptyset$ by (1), and $S = \emptyset$. Consequently, by the existence property of the Nussbaum degree and by Definition 5.5, we have

$$\deg_N(I - k, U, 0) = 0 \quad \text{and} \quad \deg(I, U, k) = 0.$$

ii) $\widetilde{C}_\infty \neq \emptyset$. In this case, as properties (1) and (2) of $\widetilde{C}_\infty$ hold, $(\mathcal{V}, \widetilde{C}_\infty)$ is an α-pair relative to the triple (I, U, k). Consider a retraction $r \colon E \to \widetilde{C}_\infty$. Denote $V = \bigcup_{i=1}^N V_i$, and let W be a (possibly empty) open subset of V containing S such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. By definition we have

$$\deg(I, U, k) = \deg_{qF}(I - kr, W, 0).$$

On the other hand (see [12] and [13]), we have

$$\deg_N(I - k, \widetilde{V}, 0) = \deg_{LS}(I - kr, r^{-1}(\widetilde{V}) \cap \widetilde{V}, 0).$$

Finally, let $W' = W \cap r^{-1}(\widetilde{V}) \cap \widetilde{V}$. As S is contained in W', by the excision property of the Leray–Schauder degree we have

$$\deg_{LS}(I - kr, r^{-1}(\widetilde{V}) \cap \widetilde{V}, 0) = \deg_{LS}(I - kr, W', 0),$$

and by the excision property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(I - kr, W, 0) = \deg_{qF}(I - kr, W', 0).$$

The claim now follows from the fact that the degree for quasi-Fredholm maps is an extension of the Leray–Schauder degree (see [3]).

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