Introductory topics in derived algebraic geometry

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Abstract

We give a quick introduction to derived algebraic geometry (DAG) sampling basic constructions and techniques. We discuss affine derived schemes, derived algebraic stacks, and the Artin-Lurie representability theorem. Through the example of deformations of smooth and proper schemes, we explain how DAG sheds light on classical deformation theory. In the last two sections, we introduce differential forms on derived stacks, and then specialize to shifted symplectic forms, giving the main existence theorems proved in [PTVV].

Contents

1 Introduction

Derived Algebraic Geometry (DAG) starts with the idea of replacing the affine objects of Algebraic Geometry, i.e. commutative rings, by some kind of "derived commutative rings" whose internal homotopy theory is non trivial. This can be achieved over $\mathbb{Q}$ by considering commutative differential non-positively graded algebras (cdga's), while in general one might instead consider simplicial commutative algebras.[1] For simplicity, we will stick to the case of cdga's (i.e. we will assume to work over $\mathbb{Q}$). As in classical

[1]Note that DAG based on cdga's over $\mathbb{Q}$ or on simplicial commutative $\mathbb{Q}$-algebras are equivalent theories.

Algebraic Geometry, the first step is to develop the local or affine theory, i.e to define and study finiteness conditions, flat, smooth, étale properties for morphisms between cdga's. This is the content of Section 2. Once this is set up, we will define the analog in DAG of being a sheaf or stack with respect to a derived version of a topology on these derived affine objects (Section 3), and then specify which sheaves or stacks are of geometric type (these will be called derived algebraic or derived Artin stacks). This last step will be done by using atlases and a recursion on the "level of algebraicity" in Section 4 where we will also list the main basic properties of derived algebraic stacks. It's not always easy to decide whether a given derived stack is algebraic by using only the definition, and a powerful criterion is given by J. Lurie's DAG version of M. Artin's representability theorem: this is explained in Section 5, where we also give a simplified but often very useful version of Lurie's representability.
In Section 6 we explain, through the example of deformations of a given smooth proper scheme, how DAG fills some conceptual gaps in classical deformation theory, the idea being that once we allow ourselves to consider also deformations over an affine derived base, then deformation theory becomes completely transparent.
The second part of this article describes symplectic geometry in DAG. We start by describing differential forms and closed differential forms on a derived stack (Section 7). These forms have two new features with respect to the classical case: first of all they have a degree (this new degree of freedom comes from the fact that in DAG the module of K'ahler differentials is replaced by a complex, the so-called cotangent complex), secondly the notion of a closed form in DAG consists of a datum rather than a property. Once these notions are in place, the definition of a derived version of symplectic structure is easy. The final Section 8 reviews some the main examples and existence results in the theory of derived symplectic geometry taken from [PTVV], namely the derived symplectic structure on the mapping derived stack of maps from a $\mathcal{O}$-compact oriented derived stack to a symplectic derived stack, and the derived symplectic structure on a lagrangian intersection.

2 Affine derived geometry

2.1 Homotopical algebra of dg-modules and cdga in characteristic 0

Let k be a commutative $\mathbb{Q}$-algebra.

Definition 2.1 *We will write*

- $\mathrm{dgmod}_k^{\leqslant 0}$ *for the model category of non-positively graded dg-modules over k (with differential increasing the degree), with weak equaivalences $W=$ "quasi-isomorphisms" and fibrations $Fib=$ "surjections in* $\deg < 0$ *", endowed with the usual symmetric $\otimes$ structure compatible with the model structure (i.e. a symmetric monoidal model category, [Hov]).*

- $\mathbf{dgmod}_k^{\leqslant 0}$ *for the symmetric monoidal ∞-category obtained by inverting (in the ∞-categorical sense[2]) quasi-isomorphisms in* $\mathrm{dgmod}_k^{\leqslant 0}$.

- $\mathrm{cdga}_k^{\leqslant 0} := CAlg(\mathrm{dgmod}_k^{\leqslant 0})$ *for the model category of commutative unital monoids in* $\mathrm{dgmod}_k^{\leqslant 0}$*, with $W=$ "quasi-isomorphisms" and $Fib=$ "surjections in* $\deg < 0$ *" (here we need that k is a $\mathbb{Q}$-algebra). These are non-positively graded commutative differential graded k-algebras (cdga's for short).*

[2] See M. Robalo's paper in this volume.

- $\mathbf{cdga}_k^{\leqslant 0}(k) := \mathbf{CAlg}(\mathbf{dgmod}_k^{\leqslant 0})$ *for the ∞-category of non-positively graded commutative differential graded k-algebras (derived k-algebras) that can also be obtained by inverting (in the ∞-categorical sense) quasi-isomorphisms in* $cdga_k^{\leqslant 0}$.

- $\mathbf{S}$ *will denote the ∞-category of spaces.*

We will use analogous notations for unbounded dg-modules over k, and unbounded cdga's over k, by simply omitting the $(-)^{\leqslant 0}$ suffix. Same convention for non necessarily commutative algebras (by writing **Alg** instead of **CAlg**).

There is an ∞-adjunction:

$$\mathbf{dgmod}_k^{\leqslant 0} \underset{U}{\overset{Sym_k}{\rightleftarrows}} \mathbf{cdga}_k^{\leqslant 0}$$

(induced by a Quillen adjunction on the corresponding model categories) where Sym_k denotes the free cdga functor, and U the forgetful functor.

Mapping spaces. We have the following explicit models for mapping spaces in $\mathbf{cdga}_k^{\leqslant 0}$. Let $\Omega_n^{\bullet}$ be the algebraic de Rham complex of $k[t_0, t_1, \cdots, t_n]/(\sum_i t_i - 1)$ over k. Consider $[n] \mapsto \Omega_n^{\bullet}$ as a simplicial object in $\mathbf{cdga}_k$. Thus, if $B \in \mathbf{cdga}_k^{\leqslant 0}$, then the assignment $[n] \mapsto \tau_{\leqslant 0}(\Omega_n^{\bullet} \otimes_k B)$ defines a simplicial object in $\mathbf{cdga}_k^{\leqslant 0}$.

For any pair (A, B) in $\mathbf{cdga}_k^{\leqslant 0}$, there is an equivalence of of spaces (simplicial sets)

$$\mathsf{Map}_{\mathbf{cdga}_k^{\leqslant 0}}(A, B) \simeq ([n] \mapsto Hom_{cdga_k^{\leqslant 0}}(Q_k A, \tau_{\leqslant 0}(\Omega_n^{\bullet} \otimes_k B)))$$

where $Q_k A$ is a cofibrant replacement of A in the model category $cdga_k^{\leqslant 0}$.

Cautionary exercises. The following exercises are meant to make the reader aware of the boundaries of the territory where we will be working.

Exercise 2.2 *Let $char(k) = p$. Show that $CAlg(dgmod_k)$ cannot have a model structure with W="quasi-isomorphisms" and Fib=" degreewise surjections".*

Hint: 1) Construct a map $f : A \to B$ in $CAlg(dgmod_k)$ with the following property: $\exists i, \alpha \in H^i(B)$ such that α^p is not in the image of $H^{ip}(f)$.
2) Prove that no such f can be factored as $Fib \circ (W \cap Cof)$.

Exercise 2.3 *Let $char(k) = 0$. Show that the obvious ∞-functor $\mathbf{CAlg}(\mathbf{dgmod}_k) \to \mathbf{Alg}(\mathbf{dgmod}_k)$, though conservative, is not fully faithful.*

For $A \in \mathbf{cdga}_k^{\leqslant 0}$ we consider $\mathbf{dgmod}(A)$ the symmetric monoidal ∞-category of (unbounded) dg-modules over A, and we define

$$\mathbf{CAlg}(\mathbf{dgmod}(A)) \simeq A/\mathbf{cdga}_k^{\leqslant 0}.$$

Given any map $f : A \to B \in \mathbf{cdga}_k^{\leqslant 0}$, there is an induced ∞-adjunction

$$\mathbf{dgmod}(B) \underset{f_*}{\overset{f^*=(-)\otimes_A B}{\rightleftarrows}} \mathbf{dgmod}(A)$$

which is an equivalence of ∞-categories if f is an equivalence in $\mathbf{cdga}_k^{\leqslant 0}$.

2.2 Cotangent complex

Let $f : A \to B$ be a map in $\mathbf{cdga}_k^{\leqslant 0}$. For any $M \in \mathbf{dgmod}^{\leqslant 0}(B)$, let $B \oplus M \in \mathbf{cdga}_k^{\leqslant 0}$ be the trivial square zero extension of B by M (i.e. B acts on itself and on M in the obvious way, and $M \cdot M = 0$). $B \oplus M$ is naturally an A-algebra and it has a natural projection map $pr_B : B \oplus M \to B$ of A-algebras.

Definition 2.4 *The space of derivations from B to M over A is defined as*

$$\mathsf{Der}_A(B, M) := \mathsf{Map}_{A/\mathbf{cdga}_k^{\leqslant 0}/B}(B, B \oplus M).$$

Equivalently,

$$\mathsf{Der}_A(B, M) = fib\left(\mathsf{Map}_{A/\mathbf{cdga}_k^{\leqslant 0}}(B, B \oplus M) \xrightarrow{pr_{B,*}} \mathsf{Map}_{A/\mathbf{cdga}_k^{\leqslant 0}}(B, B)\ ;\ \mathrm{id}_B\right).$$

$M \to \mathsf{Der}_A(B, M)$ can be constructed as an ∞ functor $\mathbf{dgmod}^{\leqslant 0}(B) \to \mathbf{S}$, and we have the following derived analog of the classical existence results for Kähler differentials.

Proposition 2.5 *The ∞-functor $M \mapsto \mathsf{Der}_A(B, M)$ is corepresentable, i.e. $\exists\, \mathbb{L}_{B/A} \in \mathbf{dgmod}^{\leqslant 0}(B)$ and a canonical equivalence $\mathsf{Der}_A(B, -) \simeq \mathsf{Map}_{\mathbf{dgmod}(B)}(\mathbb{L}_{B/A}, -)$.*

Proof. Let $Q_A B \simeq B$ be a cofibrant replacement of B in $A/cdga_k^{\leqslant 0}$. Then $\mathbb{L}_{B/A} := \Omega^1_{Q_A B/A} \otimes_{Q_A B} B$ does the job. $\square$

Let us list the most useful properties of the cotangent complex construction (for details, see [HAG-II]).

(1) Given a commutative square in $\mathbf{cdga}_k^{\leqslant 0}$

$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "u"'] & B' \end{tikzcd}$$

there is an induced map

$$\mathbb{L}_{B/A} \to u_* \mathbb{L}_{B'/A'} \Leftrightarrow \mathbb{L}_{B/A} \otimes_B B' \to \mathbb{L}_{B'/A'}.$$

In particular, we get a canonical map $\mathbb{L}_{B/A} \to \mathbb{L}_{H^0(B)/H^0(A)}$. If $B = S, A = R$ are discrete[3], the canonical map $H^0(\mathbb{L}_{S/R}) \to \Omega^1_{S/R}$ is an isomorphism.

[3] An object $D \in \mathbf{cdga}_k^{\leqslant 0}$ is *discrete* if the canonical map $D \to H^0(D)$ is a quasi-isomorphism.

(2) If the commutative diagram in (1) is a pushout square, then the map $\mathbb{L}_{B/A} \otimes_B B' \to \mathbb{L}_{B'/A'}$ is an equivalence.

(3) If $C \to A \to B$ are maps in $\mathbf{cdga}_k^{\leqslant 0}$, then there is an induced cofiber sequence

$$\mathbb{L}_{A/C} \otimes_A B \to \mathbb{L}_{B/C} \to \mathbb{L}_{B/A}$$

in $\mathbf{dgmod}(B)$, where the first map is as in (1) via $u' = id_C$, $u : A \to B$, and the second map is as in (1) with $u = id_B$ and $u' = (C \to A)$).

(4) If

$$\begin{tikzcd} A \arrow[r, "u'"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "u"'] & B' \end{tikzcd}$$

is a pushout in $\mathbf{cdga}_k^{\leqslant 0}$, then we have a fiber sequence

$$\mathbb{L}_{A/k} \otimes_A B' \to \mathbb{L}_{B/k} \otimes_B B' \oplus \mathbb{L}_{A'/k} \otimes_A B' \to \mathbb{L}_{B'/k}$$

in $\mathbf{dgmod}(B')$.

Exercise 2.6 *Compute $\mathbb{L}_{A/k}$ for $A := k[x_1, \cdots, x_n]/(f)$, for $f \neq 0$*

Hint: Use the associated Koszul resolution.

Exercise 2.7 *Compute the cotangent complex of $k[\varepsilon]$ over k, where ε has cohomological degree -1, and use this to classify explicitly all derivations $k[\varepsilon] \to k$.*

Postnikov towers. We will describe here the Postnikov tower of a cdga and its relation to the cotangent complex.

Definition 2.8 *Let $A \in \mathbf{cdga}_k^{\leqslant 0}$. A Postnikov tower for A is the data of a sequence $\{P_n(A)\}_{n \geqslant 0}$ in $A/\mathbf{cdga}_k^{\leqslant 0}$, and of maps*

$$\cdots \longrightarrow P_n(A) \xrightarrow{\pi_n} P_{n-1}(A) \xrightarrow{\pi_{n-1}} \cdots \longrightarrow P_1(A) \xrightarrow{\pi_1} P_0(A)$$

in $A/\mathbf{cdga}_k^{\leqslant 0}$, satisfying the following properties:

1. *$P_n(A)$ is n-truncated (i.e. $H^{-i}(P_n(A)) = 0$ for $i > n$);*
2. *$H^j(A) \to H^j(P_n(A))$ is an iso, for all $0 \leqslant j \leqslant n$;*
3. *For any n-truncated $B \in \mathbf{cdga}_k^{\leqslant 0}$, the map $\mathsf{Map}_{\mathbf{cdga}_k^{\leqslant 0}}(P_n(A), B) \to \mathsf{Map}_{\mathbf{cdga}_k^{\leqslant 0}}(A, B)$ induced by $A \to P_n(A)$, is an equivalence of spaces.*

Proposition 2.9 *A Postnikov tower is unique in the ∞-category $\mathsf{Fun}_\infty(\mathbb{N}^{op}, A/\mathbf{cdga}_k^{\leqslant 0})$ of ∞-functors. Moreover, the canonical map $A \to \lim_{n\geqslant 0} P_n(A)$ is an equivalence in $\mathbf{cdga}_k^{\leqslant 0}$.*

Exercise 2.10 *Construct a Postnikov tower such that*

$$\begin{aligned} P_0(A) &= A^0/d(A^{-1})[0] \\ P_1(A) &= (\cdots \to 0 \to A^{-1}/d(A^{-2}) \to A^0\,) \\ &\quad etc. \end{aligned}$$

The following result explains how the cotangent complex controls the Postnikov tower of a cdga.

Proposition 2.11 *Let*

$$\cdots \longrightarrow P_n(A) \xrightarrow{\pi_n} P_{n-1}(A) \xrightarrow{\pi_{n-1}} \cdots \longrightarrow P_1(A) \xrightarrow{\pi_1} P_0(A)$$

be a Postnikov tower for A. For any n, there exists a map $\phi_n : \mathbb{L}_{P_n(A)/A} \to H^{-n-1}(A)[n+2]$ such that the following square is cartesian in $A/\mathbf{cdga}_k^{\leqslant 0}$

$$\begin{tikzcd} P_{n+1}(A) \arrow[r] \arrow[d, "\pi_n"'] & P_n(A) \arrow[d, "d_0"] \\ P_n(A) \arrow[r, "d_{\phi_n}"'] & P_n(A) \oplus H^{-n-1}(A)[n+2] \end{tikzcd} ,$$

where d_{ϕ_n} is the derivation corresponding to ϕ_n via the identification

$$\begin{aligned} &\pi_0(\mathsf{Map}_{A/\mathbf{cdga}_k^{\leqslant 0}/P_n(A)}(P_n(A), P_n(A) \oplus H^{-n-1}(A)[n+2])) \\ &\qquad\qquad \simeq Hom_{Ho(\mathbf{dgmod}(P_n(A)))}(\mathbb{L}_{P_n(A)/A}, H^{-n-1}(A)[n+2]), \end{aligned}$$

Corollary 2.12 *A map $A \to B$ in $\mathbf{cdga}_k^{\leqslant 0}$ is an equivalence iff the following two properties hold*

- *$H^0(A) \to H^0(B)$ is an isomorphism (of discrete k-algebras)*
- *$\mathbb{L}_{B/A} \simeq 0$.*

The previous corollary is the first of several examples of the following general Principle:

($\diamond$) $\boxed{\text{derived algebraic geometry} = \text{algebraic geometry (of the truncation)} + \text{deformation theory}}$

Remark 2.13 Note this principle seriously fails for derived geometry over unbounded cdga's or non-connective commutative ring spectra.

Definition 2.14 *Let $n \in \mathbb{N}$. A non-positively graded dg-module M is said to be n-*connective *if $H^{-i}M = 0$ for every $0 \leqslant i < n$. A morphism $f\colon M \to N$ is said to be n-*connected *if $\mathsf{fib}(f)$ is n-connective.*

The following result gives more precise and useful relations between the degree of connectedness of a map between cdga's and the degree of connectivity of its cotangent complex.

Proposition 2.15 *Let $f : A \to B$ be a map in $\mathbf{cdga}_k^{\leqslant 0}$, and $n \in \mathbb{N}$.*

1. *If f is n-connected, then $\mathbb{L}_{B/A}$ is $(n+1)$-connective.*
2. *If $\mathbb{L}_{B/A}$ is $(n+1)$-connective and $H^0(f)$ is an isomorphism, then f is n-connected.*

2.3 Derived commutative algebra: finiteness, flat, smooth, étale.

In order to motivate the basic definitions in derived commutative algebra, we will give here also the classical definitions in commutative algebra in a form that is appropriate for our generalization.

Definition 2.16
- *(classical) A map $R \to S$ of discrete commutative k-algebras is* **finitely presented** *if $Hom_{R/CAlg(k)}(S, -)$ commutes with filtered colimits.*
- *(derived) A map $A \to B$ in $\mathbf{cdga}_k^{\leqslant 0}$ is* **derived finitely presented** *if $\mathsf{Map}_{A/\mathbf{cdga}_k^{\leqslant 0}}(B, -)$ commutes with (homotopy) filtered colimits.*

Proposition 2.17 *A map $A \to B$ in $\mathbf{cdga}_k^{\leqslant 0}$ is fp iff*
- *$H^0(A) \to H^0(B)$ is classically finitely presented, and*
- *$\mathbb{L}_{B/A}$ is a perfect B-dg module (i.e. is a dualizable object in $(\mathbf{dgmod}(B), \otimes_B)$).*

See [HAG-II, 2.2.] or [Lu-HA, Theorem 7.4.3.18] for a proof.

Thus the fact that a map between discrete rings is classically finitely presented does not imply that the map is derived finitely presented. However, "classical finitely presented + lci " $\Longrightarrow$ "derived finitely presented".

Definition 2.18
- *(classical) A map $R \to S$ of discrete commutative k-algebras is* **flat** *if $(-) \otimes_R S : mod(R) \to mod(S)$ preserves pullbacks (i.e. preserves kernels).*
- *(derived) A map $A \to B$ in $\mathbf{cdga}_k^{\leqslant 0}$ is* **derived flat** *if $(-) \otimes_A B : \mathbf{dgmod}^{\leqslant 0}(A) \to \mathbf{dgmod}^{\leqslant 0}(B)$ preserves pullbacks.*

Hence, a map of discrete commutative k-algebras is classically flat iff it is derived flat.

Remark 2.19 Note that a map $A \to B$ in $\mathbf{cdga}_k^{\leqslant 0}$ is derived flat iff for any discrete A-module M, the derived tensor product $M \otimes_A^L B$ is discrete (i.e of zero tor-amplitude).

Definition 2.20
- *(classical) A map $R \to S$ of discrete commutative k-algebras is* **formally étale** *if $\tau_{\geqslant -1}\mathbb{L}_{S/R} \simeq 0$.*

- *(derived) A map $A \to B$ in* $\mathbf{cdga}_k^{\leqslant 0}$ *is* **derived formally étale** *if* $\mathbb{L}_{B/A} \simeq 0$.
- *(classical) A map $R \to S$ of discrete commutative k-algebras is* formally smooth *if* $Hom_{D(S)}(\tau_{\geqslant -1}\mathbb{L}_{S/R}, M) = 0$ *for any $M \in dgmod^{\leqslant 0}(S)$ s.t. $H^0(M) = 0$.*
- *(derived) A map $A \to B$ in* $\mathbf{cdga}_k^{\leqslant 0}$ *is* **derived formally smooth** *if* $Hom_{D^{\leqslant 0}(B)}(\mathbb{L}_{B/A}, M) = 0$ *for any $S \in dgmod^{\leqslant 0}(B)$ s.t. $H^0(M) = 0$.*

Exercise 2.21 *Show that the above definition of classical formally smooth (respectively, étale) coincides with the usual one given by the infinitesimal lifting property ([EGA-IV]).*

Hint (for the formally smooth case): show that $f : R \to S$ is formally smooth in the sense of [EGA-IV] iff $\tau_{\geqslant -1}\mathbb{L}_f \simeq P[0]$ with P projective over S.

Remark 2.22 Note that considering $\tau_{\geqslant -1}\mathbb{L}_f$ and not all of $\mathbb{L}_f$, in Definition 2.20 is strictly necessary. In fact there is an example ([Stacks-Project, TAG 06E5]) of a map $f : k \to S$, where k is a field, such that f is formally étale (hence formally smooth) but $H^{-2}(\mathbb{L}_f) \neq 0$. Note that such an f is necessarily not finitely presented.

Definition 2.23
- *(classical) A map $R \to S$ of discrete commutative k-algebras is* **étale** *(respectively* **smooth***) if it is finitely presented and formally étale (respectively formally smooth).*
- *(derived) A map $A \to B$ in* $\mathbf{cdga}_k^{\leqslant 0}$ *is* **derived étale** *(respectively* **derived smooth***) if it is derived finitely presented and derived formally étale (respectively derived formally smooth).*
- *(classical) A map $R \to S$ of discrete commutative k-algebras is a Zariski* **open immersion** *if it is flat, finitely presented and the product map $S \otimes_R S \to S$ is an isomorphism.*
- *(derived) A map $A \to B$ in* $\mathbf{cdga}_k^{\leqslant 0}$ *is a Zariski* **derived open immersion** *if it is flat, finitely presented, and the product map $B \otimes_A B \to B$ is an equivalence.*

There is a characterization of the above derived properties which is very useful in practice.

Definition 2.24 *A map $A \to B$ in* $\mathbf{cdga}_k^{\leqslant 0}$ *is* **strong** *if the canonical map $H^0(B) \otimes_{H^0(A)} H^i(A) \to H^i(B)$ is an isomorphism for all i.*

Theorem 2.25 *A map $A \to B$ in* $\mathbf{cdga}_k^{\leqslant 0}$ *is derived flat (respectively derived smooth, derived étale, a Zariski derived open immersion) iff it is strong and $H^0(A) \to H^0(B)$ is flat (respectively smooth, étale, a Zariski open immersion).*

Note that if $A \to B$ is derived flat and A is discrete, then B is discrete as well.

Exercise 2.26 *Assume Theorem 2.25. Show that* $f : A \to B$ *in* $\mathbf{cdga}_k^{\leqslant 0}$ *is derived smooth iff the following three conditions hold:*

- f *is derived finitely presented;*
- f *is derived flat;*
- B *is a perfect* $B \otimes_A B$*-dg module.*

Convention. In the rest of this article, we will omit the adjective "derived" when writing any of the above properties for morphisms of derived algebras, i.e. "derived étale" will be replaced by "étale", etc.

3 Étale topology and derived stacks

Definition 3.1

- *A family* $\{A \to A_i\}_i$ *of maps in* $\mathbf{cdga}_k^{\leqslant 0}$ *is an* **étale covering family if**
  - *(i) each* $A \to A_i$ *is étale;*
  - *(ii) the family* $\{H^0(A) \to H^0(A_i)\}_i$ *is a classical étale covering family (of discrete commutative* k*-algebras).*

  This defines a topology on the ∞*-category* $\mathbf{dAff}(k) := (\mathbf{cdga}_k^{\leqslant 0})^{op}$ *(i.e., by definition, a Grothendieck topology on* $Ho(\mathbf{dAff}(k))$*). This topology is called the* **étale topology** *on derived rings and denoted by* **(ét)**.

- *The* ∞*-category* $Sh(\mathbf{dAff}(k), (ét))$ *of sheaves of spaces on this* ∞*-site, is denoted by* $\mathbf{dSt}(k)$ *and called the* ∞*-category of* **derived stacks** *over* k.

- *An* ∞*-functor* $F : \mathbf{dAff}(k)^{op} \to \mathbf{S}$ *is a derived stack iff for any* $A \in \mathbf{cdga}_k^{\leqslant 0}$ *and any étale hypercover* $A \to B^\bullet$*, the induced map* $F(A) \to \lim F(B^\bullet)$ *is an equivalence of spaces (in this case, we say that* F *has* **étale hyperdescent***).*

The étale topology on derived rings is ***subcanonical***, i.e. for any $A \in \mathbf{cdga}_k^{\leqslant 0}$, the ∞-functor

$$\mathbf{Spec}\, A : \mathbf{dAff}(k) \longrightarrow \mathbf{S} \ : \ B \longmapsto \mathsf{Map}_{\mathbf{cdga}_k^{\leqslant 0}}(A, B)$$

is a sheaf for (ét) (i.e. it has étale hyperdescent). Any derived stack equivalent to $\mathbf{Spec}\, A$ will be called a ***derived affine scheme***.

A consequence of Theorem 2.25 is the following result saying that topologically (étale or Zariski) a derived affine scheme is indistiguishable from its truncation.

Proposition 3.2 H^0 *induces an equivalence of étale or Zariski sites of* A *and of* $H^0(A)$*, for any* $A \in \mathbf{cdga}_k^{\leqslant 0}$.

This statement is analogous to the equivalence between the small étale or Zariski site of a scheme and of its reduced subscheme.

Remark 3.3 Čech nerves of étale coverings are special étale hypercovers. One may also consider ∞-functors $F : \mathbf{dAff}(k)^{op} \to \mathbf{S}$ having descent just for these special Cech étale hypercovers: we obtain, in general, different categories of derived stacks. However, the full subcategories of ***truncated*** stacks (i.e. whose values are truncated homotopy types) are in fact equivalent.

Next we list a few basic facts concerning the ∞-category of derived stacks. We will denote by $\mathbf{St}(k) := Sh(\mathbf{Aff}(k), \text{ét})$ the ∞-category of (underived, higher) stacks over k, for the classical étale topology (see also [HAG-II, 2.1]).

- There is an ∞-adjunction

$$\mathbf{St}(k) \underset{t_0}{\overset{i}{\rightleftarrows}} \mathbf{dSt}(k)$$

between derived and underived stacks (for the étale topology), i being the left adjoint. The ***truncation functor*** t_0 is determined by the property $t_0(\mathbf{Spec}\, A) = Spec\, H^0(A)$, and the fact that it preserves limits (being right adjoint). Note that i is fully faithful.

- The ∞-category $\mathbf{dSt}(k)$ is cartesian closed, i.e. there are internal Hom's, denoted as $\mathsf{MAP}_{\mathbf{dSt}(k)}(F, G) \in \mathbf{dSt}(k)$, and equivalences of spaces

$$\mathsf{Map}_{\mathbf{dSt}(k)}(F \times G, H) \simeq \mathsf{Map}_{\mathbf{dSt}(k)}(F, \mathsf{MAP}(G, H)).$$

More generally, we have an equivalence in $\mathbf{dSt}(k)$:

$$\mathsf{MAP}_{\mathbf{dSt}(k)}(F \times G, H) \simeq \mathsf{MAP}_{\mathbf{dSt}(k)}(F, \mathsf{MAP}_{\mathbf{dSt}(k)}(G, H)).$$

- The ∞-functor i does ***not*** preserve pullbacks nor internal Hom's (in fact, a lucky feature).

Definition 3.4 *If $F \in \mathbf{dSt}(k)$, we define*

$$\mathbf{QCoh}(F) := \lim_{\mathbf{Spec}\, A \to F} \mathbf{dgmod}(A)$$

(the limit being taken inside the ∞-category of k-linear stable, symmetric monoidal ∞-categories). This is called the symmetric monoidal ∞-category of **quasi-coherent complexes on** F*.*

We can globalize the definition of cotangent complex to stacks, as follows.

Let $F \in \mathbf{dSt}(k)$, $x : S = Spec\, A \to F$, and $M \in dgmod^{\leqslant 0}(A)$. We have a projection map $\mathrm{pr} : A \oplus M \to A$.

Definition 3.5

- *We say that a derived stack F* **has a cotangent complex at** x*, if there is a $(-n)$-connective A-module $\mathbb{L}_{F,x}$ (for some n) such that the ∞-functor*

$$\begin{aligned} \mathsf{Der}_{F,x} : \quad \mathbf{dgmod}^{\leqslant 0}(A) &\longrightarrow \mathbf{S} \\ M &\longmapsto \mathsf{fib}\left(F(A \oplus M) \xrightarrow{F(pr)} F(A); x \right) \end{aligned}$$

is equivalent to the functor $\mathsf{Map}_{\mathbf{dgmod}(A)}(\mathbb{L}_{F,x}, -)$ (restricted to $\mathbf{dgmod}^{\leqslant 0}(A)$). If this is the case, $\mathbb{L}_{F,x}$ is called a cotangent complex of F at x[4]*.*

- *We say that F* **has a (global) cotangent complex** *if $\exists\ \mathbb{L}_F \in \mathbf{QCoh}(F)$, such that for any*

$$x : S = \mathbf{Spec}\, A \to F\,,$$

$x^*\mathbb{L}_F$ is a cotangent complex for F at x

[4]Obviously, any two cotangent complexes of F at x are canonically equivalent.

4 Derived algebraic stacks

In this section we will single out, among all derived stacks, the "geometric" ones, called derived algebraic stacks. The definition will involve a long induction on $n \geqslant 0$.

We start with the case $n = 0$.

Definition 4.1 *A map $f : F \to G$ in $dSt(k)$ is defined to be an* **epimorphism** *(respectively, a* **monomorphism***) if the induced map $\pi_0(F) \to \pi_0(G)$ of sheaves of sets on the usual site* $(\mathsf{Ho}(\mathbf{dAff}(k)), (\acute{e}t))$ *is an epimorphism (respectively, if the diagonal $\Delta_f : F \to F \times_G F$ is an equivalence).*

Definition 4.2 (Derived schemes)

- *A map $u : F \to S = \mathbf{Spec}\, A$ of derived stacks is a Zariski open immersion if it is a monomorphism and there is a family $\{p_i : \mathbf{Spec}\, A_i \to F\}$ of morphisms in $dSt(k)$ such that $\coprod_i p_i : \coprod_i \mathbf{Spec}\, A_i \to \mathbf{Spec}\, F$ is an epimorphism, and each composite $u_i : \mathbf{Spec}\, A_i \to F \to \mathbf{Spec}\, A$ is a Zariski open immersion of cdga's (so that we already know what this means).*

- *A morphism $F \to G$ of derived stacks is a* **Zariski open immersion** *if for any $S \to G$ with S affine, the induced map $F \times_G S \to S$ is a Zariski open immersion (as defined in the previous item).*

- *A derived stack F is a* **derived scheme** *if there exists a family $\{\mathbf{Spec}\, A_i \to F\}_i$ of Zariski open immersions, such that the induced map $\coprod_i \mathbf{Spec}\, A_i \to F$ is an epimorphism. Such a family is called a* **Zariski atlas** *for F.*

Note that if $F \to \mathbf{Spec}\, A$ is a Zariski open immersion, then F is automatically a derived scheme. Once derived schemes are defined, we can extend the notion of smooth, flat, étale to maps between them.

Definition 4.3 *A morphism of derived schemes $f : X \to Y$ is* **smooth** *(respectively* **flat***, respectively* **étale***) if there are Zariski atlases $\{U_i \to X\}_{i \in I}$, $\{V_j \to Y\}_{j \in J}$ and, for any $i \in I$ there is $j(i) \in J$ and a commutative diagram*

$$\begin{tikzcd} U_i \arrow[r, "f_{i,j(i)}"] \arrow[d] & V_{j(i)} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

such that $f_{i,j(i)}$ is smooth (resp. flat, resp. étale) between derived rings.

Having defined derived schemes, we may give the general inductive definition of *derived algebraic n-stacks*, for $n \geqslant 0$.

Definition 4.4 (Derived 0-algebraic stacks)

- *A derived stack F is* **0-algebraic** *if it is (equivalent to) a derived scheme.*

- *A morphism of derived stacks $F \to G$ is* **0-representable** *if for any $S \to G$ where S is 0-algebraic, the pullback $S \times_G F$ is 0-algebraic.*

- *a 0-representable morphism of derived stacks $F \to G$ is* **smooth** *if for any $S \to G$ where S is 0-algebraic, the morphism of derived schemes $S \times_G F \to S$ is smooth.*

Definition 4.5 (Derived n-algebraic stacks) *Let $n > 0$, and suppose we have already defined the notions of* **derived** $(n-1)$**-algebraic stack**, *of* $(n-1)$**-representable morphism** *(between arbitrary derived stacks), and of* **smooth** $(n-1)$**-representable morphism** *(between arbitrary derived stacks). Then*

- *A derived stack F is* n**-algebraic** *if there exists a smooth $(n-1)$-algebraic morphism $p : U \to F$ of derived stacks, where U is 0-algebraic and p is an epimorphism. Such a p is called a* **smooth** n**-atlas** *for F.*
- *A morphism of derived stacks $F \to G$ is* n**-representable** *if for any $S \to G$ where S is 0-algebraic, the pullback $S \times_G F$ is n-algebraic.*
- *An n-representable morphism of derived stacks $F \to G$ is* **smooth** *(resp.* **flat**, *resp.* **étale***) if for any $S \to G$ where S is 0-algebraic, there exists a smooth n-atlas $U \to S \times_G F$, such that the composite $U \to S$, between 0-algebraic stacks, is smooth (resp. flat, resp. étale).*
- *A derived stack is* **algebraic** *if it is m-algebraic for some $m \geqslant 0$.*

Remark 4.6 Exactly the same definitions 4.4 and 4.5, with "derived scheme" replaced by (classical) "scheme" and "affine derived scheme" by (classical) "affine scheme", give us a notion of underived algebraic (higher) n-stack, for each $n \geqslant 0$. This notion was first proposed by C. Simpson and C. Walter.

Exercise 4.7 *If F is n-algebraic, then the diagonal map $F \to F \times F$ is $(n-1)$-representable.*

We list below some important and useful properties of derived algebraic stacks.

- The full sub-∞-category $\mathbf{dSt}^{\mathrm{alg}}(k) \subset \mathbf{dSt}(k)$ of algebraic stacks is closed under pullbacks and finite disjoint unions.
- Representable morphisms are stable under composition and arbitrary pullbacks.
- If $F \to G$ is a smooth epimorphism of derived stacks, then F is algebraic iff G is algebraic.
- A non-derived stack $\mathcal{X}$ is algebraic iff the derived stack $i(\mathcal{X})$ is algebraic.
- If F is a derived algebraic stack, then its truncation $t_0(F)$ is an algebraic (underived) stack.
- If F is a derived algebraic stack and $t_0(F)$ is an m-truncated[5] (underived) stack, then for any n-truncated $A \in \mathbf{cdga}_k^{\leqslant 0}$ (i.e. $H^i(A) = 0$ for $i < -n$), the space $F(A)$ is $(n+m)$-truncated.
- If $F \to G$ is a flat morphism of derived algebraic stacks, then: G underived (i.e. $\simeq i(\mathcal{X})$) implies that F is underived.
- A $(n-1)$-representable morphism is n-representable.
- A $(n-1)$- representable smooth (resp. étale, flat, Zariski open immersion) is n-representable smooth (resp. étale, flat, Zariski open immersion).

[5] I.e. it sends any discrete commutative k-algebra to an m-truncated space.

- If $F \to G$ is a map of derived stack, G is n-algebraic and there is a smooth atlas $S \to G$ such that $S \times_G F$ is n-algebraic, then F is n-algebraic (i.e. being n-algebraic is smooth-local on the target).

- A map $A \to B$ in $\mathbf{cdga}_k^{\leqslant 0}$ is quasi finitely presented (qfp, for short) if $H^0(A) \to H^0(B)$ is (classically) finitely presented. Flat qfp covers define a topology (qfpf) on $\mathbf{dAff}(k)$. We can replace (ét) by (qfpf), and smooth (atlases) by flat (atlases), and we get another full subcategory $\mathbf{dSt}^{\text{qfpf, alg}}(k)$ of derived stacks which are algebraic for this new pair (qfpf, flat). A deep result of B. Toën [To-flat] says that in fact $\mathbf{dSt}^{\text{qfpf, alg}}(k) = \mathbf{dSt}^{\text{alg}}(k)$. The analogous statement for underived stacks in groupoids was proven by M. Artin [Ar]).

- Derived algebraic stacks admitting ***étale*** atlases are called ***derived Deligne-Mumford*** stacks, and often, by analogy with the classical case, general derived algebraic stacks (i.e. with smooth atlases) are also called *derived Artin stacks.*

- An (underived) algebraic n-stack locally of finite presentation over k (with bounded cotangent complex) has a cotangent complex of amplitude $\subseteq [-1, n]$[6] On the opposite side, a derived affine scheme always have a cotangent complex with amplitude $\subseteq (-\infty, 0]$. Therefore, for an arbitrary derived algebraic stack F, the negative degrees of $\mathbb{L}_F$ are often referred to as its "derived" degrees, while the positive ones as its "stacky" degrees.

5 Lurie's Representability Theorem

Let us introduce some fundamental ***deformation-theoretic properties*** of a derived stack.

Definition 5.1 *A derived stack F is*

- **nilcomplete** *if for any $A \in \mathbf{cdga}_k^{\leqslant 0}$, the canonical map $F(A) \to \lim_{n \geqslant 0} F(P_n(A))$ is an equivalence in $\mathbf{S}$ (recall that $\{P_n(A)\}_{n \geqslant 0}$ is a Postnikov tower for A).*

- **infinitesimally cohesive** *if the F-image of any cartesian square in $\mathbf{cdga}_k^{\leqslant 0}$*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ B' \arrow[r, "q"'] & A \end{tikzcd}$$

where $H^0(p)$ and $H^0(q)$ are surjective with nilpotent kernels, is cartesian in $\mathbf{S}$.

- **infinitesimally cartesian** *if for any $A \in \mathbf{cdga}_k^{\leqslant 0}$, any $M \in \mathbf{dgmod}^{\leqslant 0}(A)$, s.t. $H^0(M) = 0$, and any derivation d from A to M, the F-image of the pullback*

$$\begin{tikzcd} A \oplus_d \Omega M \arrow[r] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r, "triv"'] & A \oplus M \end{tikzcd}$$

is a pullback in $\mathbf{S}$.

[6]By a famous result of L. Avramov, a scheme locally of finite type over k either has a perfect cotangent complex with amplitude $\subseteq [-1, 0]$ (and this happens iff the scheme is lci), or has an unbounded cotangent complex.

- **discretely integrable**[7] *if for any classical complete local noetherian k-algebra $(R, \mathfrak{m})$, the canonical map $F(\widehat{R}) \to \lim_n F(R/\mathfrak{m}^n)$ is an equivalence in $\mathbf{S}$.*

Obviously, being infinitesimally cohesive implies being infinitesimally cartesian, and the notion of infinitesimal cohesiveness is a derived version of the classical Schlessinger condition [Stacks-Project, Tag 06J1].

We are now able to state the following important result that is the most useful known criterion for checking algebraicity of a derived stack.

Theorem 5.2 (Lurie's Representability Theorem) *Let k be a noetherian G-ring (e.g. noetherian and excellent). A derived stack F over k is algebraic iff the following conditions hold:*

- *F is nilcomplete.*
- *F is infinitesimally cohesive.*
- *F is discretely integrable.*
- *F has a cotangent complex*

Remark 5.3 There is also a more general version of Theorem 5.2 where the base k is a derived ring ([Lu-SAG, Theorem 18.4.0.1]).

Corollary 5.4 ("Easy" Representability Theorem) *Let k be a noetherian commutative ring. A derived stack F over k is n-algebraic k iff the following conditions hold:*

- *The truncation $t_0(F)$ is an (underived) algebraic n-stack.*
- *F is nilcomplete, infinitesimally cartesian and has a cotangent complex.*

In this corollary, we have separated the first condition which is a global one but only concerns the truncation, and the second condition which is purely deformation-theoretic. This is another instance of Principle ($\diamond$). A proof of Corollary 5.4, independent of Theorem 5.2, can be found in [HAG-II, Appendix C].

Remark 5.5 Note that the G-ring condition in Theorem 5.2 re-appears when we want to further unzip the first condition in Corollary 5.4.

Exercise 5.6 *Let X/k be a flat and proper underived scheme and Y/k an underived smooth scheme. Let $\mathsf{MAP}_{\mathbf{dSt}(k)}(X,Y) : A \longmapsto \mathsf{Map}_{\mathbf{dSt}(k)}(X \times \mathbf{Spec}\, A, Y)$ the internal mapping derived stack.*

- *Show that $t_0(\mathsf{MAP}_{\mathbf{dSt}(k)}(X,Y))$ is the classical scheme of morphisms from X to Y*
- *Let $\mathsf{MAP}_{\mathbf{dSt}(k)}(X,Y) \xleftarrow{p} X \times \mathsf{MAP}_{\mathbf{dSt}(k)}(X,Y) \xrightarrow{ev} Y$.*
  Show that $\mathbb{T} := p_* ev^*(T_{Y/k}[0])$ is a tangent complex for $\mathsf{MAP}_{\mathbf{dSt}(k)}(X,Y)$.
- *Apply Lurie's representability theorem (in the easy case, if the reader so wishes) to deduce that $\mathsf{MAP}_{\mathbf{dSt}(k)}(X,Y)$ is a derived geometric stack (actually, a derived scheme).*

[7] This is the same as ***integrable*** (as in [Lu-SAG, Definition 17.3.3.1]) if F is already nilcomplete, infinitesimally cohesive and admits a cotangent complex.

6 DAG "explains" classical deformation theory

In this section, we illustrate how derived deformation theory makes classical deformation theory completely transparent, by working out an explicit example of a very classical deformation problem: the *infinitesimal deformations of a proper smooth scheme over* $k = \mathbb{C}$.

We will first describe what classical deformation theory tells us in this special case, point out some weak points in this approach, and finally describe how derived algebraic geometry fixes these issues. The reader will find all of the omitted details in [Po-Ve].

Recall that the objects of study of classical (formal) deformation theory are reduced functors

$$F \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Grpd} \hookrightarrow \mathbf{S}$$

Here, $\mathbf{Art}_{\mathbb{C}}$ denotes the category of *Artin rings* over $\mathbb{C}$ (i.e. artinian local $\mathbb{C}$-algebras with residue field isomorphic to $\mathbb{C}$, or equivalently, augmented over $\mathbb{C}$), and "reduced" means that $F(\mathbb{C})$ is weakly contractible. For example, if $\mathbf{calg}_{\mathbb{C}}$ denotes the usual category of (discrete) commutative $\mathbb{C}$-algebras, and $F \colon \mathbf{calg}_{\mathbb{C}} \to \mathbf{Grpd}$ is a classical moduli problem, then the choice of any point $\xi \in F(\mathbb{C})$ determines a functor

$$\widehat{F}_{\xi} := F \times_{F(\mathbb{C})} * : \mathbf{Art}_{\mathbb{C}} \longrightarrow \mathbf{Grpd}\,,\ R \longmapsto F(A) \times_{F(\mathbb{C})} *$$

by forming the homotopy pullback: the morphism $F(A) \to F(\mathbb{C})$ is the F-image of the augmentation $a : A \to \mathbb{C}$, while the map $* \to F(\mathbb{C})$ is the chosen point ξ). Equivalently,

$$\widehat{F}_{\xi}(R) = \mathsf{fib}(F(a) : F(R) \to F(\mathbb{C})\,; \xi)$$

where fib denotes the homotopy fiber in $\mathbf{Grpd}$. The functor $\widehat{F}_{\xi}$ is obviously reduced, and is called the *formal completion of* F *at* ξ.

Let us apply this formal completion construction to our chosen, well known moduli functor

$$F \colon \mathbf{calg}_{\mathbb{C}} \longrightarrow \mathbf{Grpd} \hookrightarrow \mathbf{S}$$

sending a commutative $\mathbb{C}$-algebra R into the groupoid of proper smooth morphisms

$$Y \longrightarrow \mathrm{Spec}(R)$$

and isomorphisms between them. In this case, if we fix a proper smooth $\mathbb{C}$-scheme

$$\xi \colon X_0 \longrightarrow \mathrm{Spec}(\mathbb{C}),$$

this is a $\mathbb{C}$-point of F, and the corresponding homotopy base change $\widehat{F}_{\xi}$ is exactly the usual reduced functor, classically denoted as Def_{X_0}.

Definition 6.1 *The groupoid* $\widehat{F}_{\xi}(\mathbb{C}[t]/t^{n+1})$ *is called the* groupoid of n-th order infinitesimal deformations of *of* F *at* ξ. *The elements of the connected component* $\pi_0(\widehat{F}_{\xi}(\mathbb{C}[t]/t^{n+1}))$ *are called* n-th order infinitesimal deformations *of* F *at* ξ.

The following properties are classically well known (see e.g. [Har-def]):

1. if $\xi_1 \in \widehat{F}_{\xi}(\mathbb{C}[\varepsilon])$ is a first order deformation of ξ, then $\mathrm{Aut}_{\widehat{F}_{\xi}(\mathbb{C}[\varepsilon])}(\xi_1) \simeq H^0(X_0, T_{X_0})$;

2. $\pi_0(\widehat{F}_\xi(\mathbb{C}[\varepsilon])) \simeq H^1(X_0, T_{X_0})$;

3. if ξ_1 is a first order deformation, then there exists a class $\mathrm{obs}(\xi_1) \in H^2(X_0, T_{X_0})$ such that $\mathrm{obs}(\xi_1) = 0$ if and only if ξ_1 extends to a second order deformation. The class $\mathrm{obs}(\xi_1)$ is called an *obstruction class* for ξ_1.

The first two properties are completely satisfactory: they give algebraical interpretations (the rhs's) of deformation theoretic objects (the lhs's). Or conversely, as the reader prefers. This is not quite true for the third property, and it raises two *natural questions* :

A What is the deformation-theoretic meaning of the *entire* $H^2(X_0, T_{X_0})$?

B How can we intrinsically identify the space of *all* obstructions[8] inside $H^2(X_0, T_{X_0})$?

Derived algebraic geometry gives a more general perspective on the subject, and answers both questions. It allows a natural interpretation of $H^2(X_0, T_{X_0})$ as the group of *derived deformations* i.e. (isomorphism classes of) deformations over a specific *derived* ring, and it identifies, consequently, the obstructions space in a natural way. Let's work these answers out.

Define

$$\mathbb{F}\colon \mathbf{cdga}_{\mathbb{C}}^{\leqslant 0} \longrightarrow \mathbf{S}$$

sending a cdga A to the maximal ∞-groupoid of equivalences inside the ∞-category of proper[9] and smooth maps of derived schemes

$$Y \longrightarrow \mathbf{Spec}\, A.$$

It is clear that $\mathbb{F}$ is a derived stack. Moreover, since a derived scheme that is (derived) smooth over an underived scheme is automatically an underived scheme, we easily conclude that $\mathbb{F}(R) \simeq F(R)$ for any discrete commutative $\mathbb{C}$-algebra R, and more generally, that $\mathrm{t}_0(\mathbb{F}) \simeq F$.

Exercise 6.2 *Show that $\mathbb{F}$ is infinitesimally cohesive (Definition 5.1).*

Consider the full ∞-subcategory $\mathbf{dArt}_{\mathbb{C}}$ of $\mathbf{cdga}_{\mathbb{C}}^{\leqslant 0}$ of cdga's A such that

- $H^0(A) \in \mathbf{Art}_{\mathbb{C}}$;
- for all $i < 0$, $H^i(A)$ is a module of finite type over $H^0(A)$;
- $H^i(A) = 0$ for $i \ll 0$.

The objects of $\mathbf{dArt}_{\mathbb{C}}$ will simply be called *derived Artin rings.*

Our choice of a $\xi \in F(\mathbb{C}) \simeq \mathbb{F}(\mathbb{C})$, identified with a morphism $\xi : * \to \mathbb{F}(\mathbb{C})$, allows us to consider the *formal completion* of $\mathbb{F}$ at ξ, by taking the pullback:

$$\widehat{\mathbb{F}}_\xi := \mathbb{F} \times_{\mathbb{F}(\mathbb{C})} * : \mathbf{dArt}_{\mathbb{C}} \longrightarrow \mathbf{S}.$$

Equivalently,

$$\mathbf{dArt}_{\mathbb{C}} \ni A \longmapsto \widehat{\mathbb{F}}_\xi(A) = \mathsf{fib}(\mathbb{F}(a) : \mathbb{F}(A) \to \mathbb{F}(\mathbb{C})\,; \xi)$$

[8] It can happen that every obstruction is trivial but $H^2(X_0, T_{X_0}) \neq 0$. An example is given by a smooth projective surface $X_0 \subseteq \mathbb{P}^3_{\mathbb{C}}$ of degree $\geqslant 6$.

[9] By definition, a morphism of derived schemes or stacks is proper if it is so on the truncations.

where α is the derived version of the augmentation, i.e. the composite

$$A \longrightarrow H^0(A) \xrightarrow{a} \mathbb{C},$$

and fib denotes the (homotopy) fiber in $\mathbf{S}$.

The following result answers to Question A above: the entire $H^2(X_0, T_{X_0})$ can be interpreted as a space of *derived deformations*.

Proposition 6.3 *There is a canonical isomorphism of $\mathbb{C}$-vector spaces*

$$\pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X_0, T_{X_0}).$$

Proof. First of all, $\mathbb{F}$ has a cotangent complex at ξ in the sense of [HAG-II, Definition 1.4.1.5], and it can be shown that

$$\mathbb{T}_{\mathbb{F},\xi} \simeq \mathbb{R}\Gamma(X_0, T_{X_0}[1])$$

(note that since X_0 is smooth $\mathbb{T}_{X_0} \simeq T_{X_0}$). Therefore (using e.g. [HAG-II, Proposition 1.4.1.6]), we obtain

$$\mathbb{L}_{\mathbb{F},\xi} \simeq \mathbb{T}^\vee_{\mathbb{F},\xi} \simeq \mathbb{R}\Gamma(X_0, \Omega_{X_0}[-1]).$$

As a consequence (recall Definition 3.5), we get

$$\pi_0(\mathsf{Der}_{\mathbb{F},\xi}(\mathbb{C}[1])) \simeq \pi_0(\mathsf{Map}_{\mathbf{dgmod}_\mathbb{C}}(\mathbb{L}_{\mathbb{F},\xi}, \mathbb{C}[1])) \simeq \mathrm{Ext}^0(\mathbb{L}_{\mathbb{F},\xi}, \mathbb{C}[1]) \simeq$$

$$\simeq \mathrm{Ext}^0(\mathbb{L}_{\mathbb{F},\xi}[-1], \mathbb{C}) \simeq H^0(\mathbb{T}_{\mathbb{F},\xi}[1]) \simeq H^0(\mathbb{R}\Gamma(X_0, T_{X_0}[2])) \simeq H^2(X_0, T_{X_0}).$$

We conclude that

$$\pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq \pi_0(\mathsf{fib}(\mathbb{F}(\mathbb{C} \oplus \mathbb{C}[1]) \to \mathbb{F}(\mathbb{C}), \xi)) \simeq \pi_0(\mathsf{Der}_{\mathbb{F},\xi}(\mathbb{C}[1])) \simeq H^2(X_0, T_{X_0})$$

$\square$

Remark 6.4 Proposition 6.3 also explains why a classical deformation theoretic interpretation of the full $H^2(X_0, T_{X_0})$ is impossible: $H^2(X_0, T_{X_0})$ is the vector space of deformations (of $\mathbb{F}$ at ξ) over the base $\mathbb{C} \oplus \mathbb{C}[1]$ which is *not* a classical Artin ring but a derived one.

Now that we have answered Question A, i.e. we have a derived deformation-theoretic interpretation of the entire $H^2(X_0, T_{X_0})$ at hand, we can proceed by answering Question B above.
We begin by an auxiliary result (for a more general version the reader is invited to consult [Po-Ve, Thm. 3.1]).

Lemma 6.5 *Let*

$$J \longrightarrow R \xrightarrow{f} S$$

be a square zero extension of (augmented) classical Artin rings over $\mathbb{C}$ (i.e. f is surjective, $J = \ker f$, and $J^2 = 0$). Then, there exists a derived derivation $d\colon R \to R \oplus J[1]$ and a homotopy cartesian diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d, "\pi \circ d"] \\ \mathbb{C} \arrow[r] & \mathbb{C} \oplus J[1] \end{tikzcd}$$

where $\pi\colon S \oplus J[1] \to \mathbb{C} \oplus J[1]$ is the natural map induced by the augmentation $S \to \mathbb{C}$.

Proof. Note that $S \oplus J[1]$ can be represented by the obvious cdga

$$0 \longrightarrow J \xrightarrow{0} S \longrightarrow 0$$

where S sits in degree 0. The trivial derivation d_0 is then represented by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & S \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & J \arrow[r, "0"] & S \arrow[r] & 0 \end{tikzcd}$$

Observe that we may represent S also by the cdga

$$0 \longrightarrow J \xrightarrow{i} R \longrightarrow 0,$$

where i denotes the inclusion map. Then, we can define a derived derivation $d : S \to S \oplus J[1]$ by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & R \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & J \arrow[r, "0"] & S \arrow[r] & 0, \end{tikzcd}$$

and remark that d is a fibration of cdga's. Since the model category of cdga's is proper, the ordinary pullback of the zero derivation d_0 and of d computes the homotopy pullback $S \oplus_d J$. But the ordinary pullback is given by just

$$0 \longrightarrow 0 \longrightarrow R \longrightarrow 0$$

(i.e., by just R sitting in degree 0). So, we conclude that the square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d, "d_0"] \\ S \arrow[r, "d"] & S \oplus J[1] \end{tikzcd}$$

is a (homotopy) pullback of cdga's over $\mathbb{C}$. So, we are left to show that

$$\begin{tikzcd} S \arrow[r] \arrow[d, "d_0"'] & \mathbb{C} \arrow[d] \\ S \oplus J[1] \arrow[r] & \mathbb{C} \oplus J[1] \end{tikzcd}$$

is a (homotopy) pullback. However, the map $S \oplus J[1] \to C \oplus J[1]$ is a fibration, hence it is enough to show that this diagram is a strict pullback, which is a straightforward verification. $\square$

If, in particular, we take the square-zero extension $R = \mathbb{C}[t]/(t^3) \to S = \mathbb{C}[t]/(t^2)$, by Lemma 6.5 we obtain a (homotopy) pullback

$$\begin{tikzcd} \mathbb{C}[t]/(t^3) \arrow[r] \arrow[d] & \mathbb{C}[t]/(t^2) \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} \oplus \mathbb{C}[1] \end{tikzcd}$$

Exercise 6.6 *Construct the rightmost vertical map in the above diagram as in the very first part of the proof of Lemma 6.5, and prove directly (i.e. without using Lemma 6.5) that the above square is cartesian.*

Observe that both maps $\mathbb{C} \to \mathbb{C}\oplus\mathbb{C}[1]$ and $\mathbb{C}[t]/(t^2) \to \mathbb{C}\oplus\mathbb{C}[1]$ are surjective on H^0, with nilpotent kernels. Since $\mathbb{F}$ is infinitesimally cohesive (Exercise 6.2), the diagram

$$\begin{tikzcd} \mathbb{F}(\mathbb{C}[t]/t^3) \simeq F(\mathbb{C}[t]/t^3) \arrow[r] \arrow[d] & \mathbb{F}(\mathbb{C}[t]/t^2) \simeq F(\mathbb{C}[t]/t^2) \arrow[d] \\ \mathbb{F}(\mathbb{C}) \simeq F(\mathbb{C}) \arrow[r] & \mathbb{F}(\mathbb{C}\oplus\mathbb{C}[1]) \end{tikzcd}$$

is cartesian in $\mathbf{S}$. Via the augmentation maps, this whole diagram maps to $F(\mathbb{C})$, and taking fibers at ξ gives us a diagram

$$\begin{tikzcd} \widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/t^3) \simeq \widehat{F}_\xi(\mathbb{C}[t]/t^3) \arrow[r] \arrow[d] & \widehat{F}_\xi(\mathbb{C}[t]/t^2) \simeq \widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/t^2) \arrow[d] \\ * \arrow[r] & \widehat{\mathbb{F}}_\xi(\mathbb{C}\oplus\mathbb{C}[1]) \end{tikzcd}$$

which is, obviously, again cartesian in $\mathbf{S}_*$ (the ∞-category of pointed spaces or simplicial sets). In other words, we obtain a fiber sequence of pointed spaces

$$\widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/(t^3)) \longrightarrow \widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/(t^2)) \longrightarrow \widehat{\mathbb{F}}_\xi(\mathbb{C}\oplus\mathbb{C}[1]),$$

and therefore a corresponding exact sequence on π_0's

$$\pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/(t^3))) \longrightarrow \pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/(t^2))) \xrightarrow{\mathsf{obs}} \pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C}\oplus\mathbb{C}[1])) \simeq H^2(X_0, T_{X_0}) \qquad (*)$$

of pointed sets. As a consequence, we see that

- a first order deformation (i.e. an element in $\pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/(t^2))) = \pi_0(\widehat{F}_\xi(\mathbb{C}[t]/(t^2))))$ extends to a second order deformation (i.e to an element in $\pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C}[t]/(t^3))) = \pi_0(\widehat{F}_\xi(\mathbb{C}[t]/(t^3))))$, if and only if its image via obs vanishes.

In other words, the set $\mathsf{Obs}_2(F;\xi)$ of all obstructions to extending a first order deformation to a second order one is given by the *image of the obstruction map*

$$\mathsf{obs} : \pi_0(\widehat{F}_\xi(\mathbb{C}[t]/(t^2))) \longrightarrow \pi_0(\widehat{\mathbb{F}}_\xi(\mathbb{C}\oplus\mathbb{C}[1])) \simeq H^2(X_0, T_{X_0}).$$

This is a complete answer to Question B.

Remark 6.7 Although strictly speaking this is not necessary, one can furthermore observe that the middle and the rightmost pointed sets in sequence $(*)$ are actually $\mathbb{C}$-vector spaces, pointed at 0, and that the obs is a morphism of vector spaces.

Exercise 6.8 *Prove the statements in Remark 6.7.*

Exercise 6.9 *Extend the previous arguments and results to higher order infinitesimal deformations and obstructions.*

7 Forms and closed forms

In this section we describe the theory of differential forms on derived algebraic stacks and its implications for the local geometry of derived stacks in general and for derived moduli stacks in particular.

7.1 The case of affine derived schemes

As explained in sections 2.2 and 3, given $A \in \mathbf{cdga}_k^{\leqslant 0}$, and a cofibrant (quasi-free) replacement $QA \to A$ of A, the affine derived scheme $\mathbf{Spec}\, A \in \mathbf{dSt}(k)$ has a cotangent complex:

$$\mathbb{L}_{\mathbf{Spec}\, A} = \mathbb{L}_A = \left(\begin{array}{l} \Omega^1_{QA} \otimes_{QA} A, \text{ where} \\ \Omega^1_{QA} \text{ is the module} \\ \text{of Kähler differen-} \\ \text{tials of } QA \end{array} \right)$$

Remark 7.1 We will often use implicitly the equivalence of ∞-category $\mathbf{dgmod}(QA) \simeq \mathbf{dgmod}(A)$, and consequently *identify* Ω^1_{QA} with $\Omega^1_{QA} \otimes_{QA} A = \mathbb{L}_A$.

This definition globalizes to general (algebraic) derived stacks X as in Definition 3.5. Moreover, X is locally of finite presentation if and only if $\mathbb{L}_X$, which is a priori an object in $\mathbf{QCoh(X)}$, is in fact perfect. In this case the intrinsically defined tangent complex of X can be computed as $\mathbb{T}_X = \mathbb{L}_X^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)$.

In the affine case the complex of p-forms on $\mathbf{Spec}\, A$ is defined to be $\wedge^p_{QA}\mathbb{L}_A = \Omega^p_{QA}$. We will write $\Omega^{p,i}_{QA}$ for the terms of the complex Ω^p_{QA}.

The condition that QA is quasi-free means that if we ignore the differential, then as a graded algebra QA is a polynomial algebra in a finite number of variables $\{t_i\}$ of various grading degrees. By definition the module Ω^1_{QA} of Kähler differentials for QA is the graded module over the graded algebra QA which is freely generated by formal generators dt_i where the grading degree of dt_i is set to be equal to the grading degree of t_i. The differential on Ω^1_{QA} is the unique k-linear map $d : \Omega^{1,a}_{QA} \to \Omega^{1,a+1}_{QA}$ which makes Ω^1_{QA} a dg module over QA and such that the de Rham differential $d_{DR} : QA \to \Omega^1_{QA}$ assigning the formal derivative to each polynomial is a map of dg modules over QA.

As usual $d_{DR} : QA \to \Omega^1_{QA}$ extends to a de Rham differential on p-forms $d_{DR} : \Omega^p_{QA} \to \Omega^{p+1}_{QA}$ by the graded Leibniz rule. Thus the sum $\oplus_{p\geqslant 0} \wedge^p_{QA} \mathbb{L}_A = \oplus_{p\geqslant 0}\Omega^p_{QA}$ is a fourth quadrant bicomplex with

vertical differential $d : \Omega^{p,i}_{QA} \to \Omega^{p,i+1}_{QA}$ induced by d_{QA}, and

horizontal differential $d_{DR} : \Omega^{p,i}_{QA} \to \Omega^{p+1,i}_{QA}$ given by the de Rham differential.

In analogy with the underived setting, the differential forms on an affine derived scheme admit a **Hodge filtration** with steps $F^p(A) := \oplus_{q\geqslant p}\Omega^q_{QA}$, each of which is still a fourth quadrant bicomplex. This Hodge filtration turns out to be the correct vessel for defining closed p-forms and working with the closedness condition in the derived setting.

Motivation: If X is a smooth scheme$/k$, then the sheaf $\Omega_X^{p,cl} = \ker\left[\Omega_X^p \xrightarrow{d_{DR}} \Omega_X^{p+1}\right]$ of closed p-forms on X is naturally quasi-isomorphic to the stupid truncation of the algebraic de Rham complex, i.e.

$$\Omega_X^{p,cl} \cong \left(\Omega_X^{\geqslant p}[p], d_{DR}\right).$$

Up to a shift this truncation is precisely the p-the step of the Hodge filtration of the complex of algebraic forms, and so we can use the hypercohomology of the complex $F^p\Omega_X^\bullet[p] = (\Omega_X^{\geqslant p}[p], d_{DR})$ as a model for closed p forms in general.

Thus we have the following natural

Definition 7.2 *If* $\mathbf{Spec}\, A$ *is an affine derived scheme, the* ***complex of closed*** p***-forms*** *on* $\mathbf{Spec}\, A$ *is the complex* $\mathbf{A}^{p,cl}(A) := \mathrm{tot}^{\prod}(F^p(A))[p]$*, where* $\mathrm{tot}^{\prod}(F^p(A))$ *denotes the completed (i.e. product) totalization of the double complex* $F^p(A)$*.*

Furthermore, in the derived setting all these notions admit natural refinements which account for the freedom of performing a shift in the internal homological grading of the cotangent complex. This leads to the notions of shifted forms and shifted closed forms.

Definition 7.3 *For an affine derived scheme* $\mathbf{Spec}\, A$ *define*

- ***the complex of*** n***-shifted*** p***-forms on*** $\mathbf{Spec}\, A$*:* $\mathbf{A}^p(A;n) := \bigwedge^p \mathbb{L}_A[n] = \Omega^p_{QA}[n]$
- ***the complex of*** n***-shifted closed*** p***-forms on*** $\mathbf{Spec}\, A$*:* $\mathbf{A}^{p,cl}(A;n) := \mathrm{tot}^{\prod}(F^p(A))[n+p]$
- ***the Hodge tower of*** $\mathbf{Spec}\, A$*:* $\cdots \to \mathbf{A}^{p,cl}(A)[-p] \to \mathbf{A}^{p-1,cl}(A)[1-p] \to \cdots \to \mathbf{A}^{0,cl}(A)$

Explicitly an n-shifted closed p-form ω on $\mathbf{Spec}\, A$ is an infinite collection

$$\omega = \{\omega_i\}_{i\geqslant 0}, \qquad \omega_i \in \Omega_A^{p+i,n-i}, \qquad \text{satisfying} \qquad d_{DR}\omega_i = -d\omega_{i+1}.$$

Notation. We write $|E|$ for $\mathsf{Map}_{\mathbf{dgmod}_k^{\leqslant 0}}(k, \tau_{\leqslant 0}E)$, i.e. the Dold-Kan construction applied to the $\leqslant 0$-truncation of a complex E.

Remark 7.4 **(i)** It is clear from this description that the notion of an n-shifted closed p-form is much more flexible than the naive notion of an n-shifted strictly closed p-form, i.e. an element $\omega_0 \in \Omega_A^{p,n}$ satisfying $d_{DR}\omega_0 = 0$. Given an n-shifted closed p-form $\omega = \{\omega_i\}_{i\geqslant 0}$ we will call the collection $\{\omega_i\}_{i\geqslant 1}$ the **key** closing ω.

(ii) By definition we have an **underlying** p**-form map**

$$\mathbf{A}^{p,cl}(A;n) \to \wedge^p\mathbb{L}_A[n]$$

which induces a map on cohomology

$$H^0(\mathbf{A}^{p,cl}(A)[n]) \to H^n(X, \wedge^p\mathbb{L}_A).$$

(iii) The homotopy fiber of the underlying p-form map is the **complex of keys** for a given n-shifted p-form and can be very complicated. Thus for a p-form in derived geometry being closed is *not* a property but rather a list of coherent data.

(iv) The complex $\mathbf{A}^{0,cl}(A)$ of closed 0-forms on $X = \mathbf{Spec}\, A$ is exactly Illusie's derived de Rham complex of A [Ill, ch. VIII].

(v) If $A \in \mathbf{cdga}_k^{\leqslant 0}$ is quasi-free, then

$$\begin{aligned}\mathbf{A}^{p,cl}(A;n) &= \left|\prod_{i\geqslant 0}\left(\Omega_A^{p+1}[n-i], d+d_{DR}\right)\right| \\ &= \left|\mathrm{tot}^{\Pi}(F^p(A))[n]\right| \\ &= \left|NC^w(A)(p)[n+p]\right|,\end{aligned}$$

where $NC^w(A)$ denotes the weighted negative cyclic complex for A. Hence

$$\pi_0\mathbf{A}^{p,cl}(A;n) = HC_-^{n-p}(A)(p).$$

7.2 Functoriality and gluing

To globalize the definitions of forms and closed forms we consider:

- The ∞-functor of n-shifted p-forms

$$\mathcal{A}^p(-;n) : \mathbf{cdga}_k^{\leqslant 0} \to \mathbf{S}, \qquad A \mapsto |\mathbf{A}^p(A;n)|.$$

- The ∞-functor n-shifted closed p-forms

$$\mathcal{A}^{p,cl}(-;n) : \mathbf{cdga}_k^{\leqslant 0} \to \mathbf{S}, \qquad A \mapsto \left|\mathbf{A}^{p,cl}(A)[n]\right|.$$

One can check [PTVV] that the functors $\mathcal{A}^p(-;n)$ and $\mathcal{A}^{p,\mathrm{cl}}(-;n)$ are derived stacks for the étale topology and that the assignment of an underlying p-form

$$\mathcal{A}^{p,cl}(-;n) \to \mathcal{A}^p(-;n)$$

is a map of derived stacks. With this in mind one can now give the following general

Definition 7.5 *Let $X \in \mathbf{dSt}(k)$ be a derived algebraic stack locally of finite presentation. Then we define:*

- $\mathcal{A}^p(X) := \mathsf{Map}_{\mathbf{dSt}(k)}(X, \mathcal{A}^p(-))$ *to be the space of p-forms on X;*
- $\mathcal{A}^{p,cl}(X) := \mathsf{Map}_{\mathbf{dSt}(k)}(X, \mathcal{A}^{p,cl}(-))$ *to be the space of closed p-forms on X;*
- *the corresponding n-shifted versions :* $\mathcal{A}^p(X;n) := \mathsf{Map}_{\mathbf{dSt}(k)}(X, \mathcal{A}^p(-;n))$
  $\mathcal{A}^{p,cl}(X;n) := \mathsf{Map}_{\mathbf{dSt}(k)}(X, \mathcal{A}^{p,cl}(-;n))$

- *an n-shifted (respectively closed) p-form on X is an element in $\pi_0\mathcal{A}^p(X;n)$ (respectively in $\pi_0\mathcal{A}^{p,cl}(X;n)$)*

Remark 7.6 The definition has some straighforward consequences:

1) If X is a smooth underived scheme, then there are no negatively shifted forms.
2) When $X = \mathbf{Spec}\, A$ is an affine derived scheme, then there are no positively shifted forms.

For a general derived stack X shifted differential forms might exist for any $n \in \mathbb{Z}$.

As in the affine case the underlying p-form map of simplicial sets

$$\mathcal{A}^{p,cl}(X;n) \to \mathcal{A}^p(X;n)$$

will not typically be a monomorphism. Its homotopy fiber at a given p-form ω_0 is the space of keys of ω_0. However, if X is a smooth and proper scheme then this map is indeed a monomorphism, i.e. its homotopy fibers are either empty or contractible [PTVV]. Thus we have no new phenomena in this case.

In the general case the following theorem provides a concrete and expected algebraic model for global forms.

Theorem 7.7 (Proposition 1.14 in [PTVV]) *For a derived algebraic stack X the n-shifted p-forms satisfy smooth descent, i.e.*

$$\mathcal{A}^p(X;n) \simeq \mathsf{Map}_{\mathbf{QCoh}(X)}(\mathcal{O}_X, (\wedge^p\mathbb{L}_X)[n]).$$

In particular an n-shifted p-form on X is an element in $H^n(X, \wedge^p\mathbb{L}_X)$

Guided by the classical case we can also define algebraic de Rham cohomology for derived stacks.

Definition 7.8 *Given a derived algebraic stack X the n-th algebraic de Rham cohomology of X is defined to be $H^n_{DR}(X) = \pi_0\mathcal{A}^{0,cl}(X;n)$.*

Remark 7.9
- This notion agrees with Illusie's definition in the case of affine schemes.
- if X is a algebraic derived stack locally of finite presentation, then [To-EMS, Proposition 5.2] $H^\bullet_{DR}(X) \cong H^\bullet_{DR}(t_0X)$ = algebraic de Rham cohomology of the underived higher stack t_0X defined by the standard Hartshorne's completion formalism [Har-dR].

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The previous remark combined with the canonical resolution of singularities in characteristic zero and smooth and proper descent for algebraic de Rham cohomology have the follwoing important consequence

Corollary 7.10 (Corollary 5.3 in [To-EMS]) *Let X be a algebraic derived stack which is locally of finite presentation. Suppose ω is an n-shifted closed p-form on X with $n < 0$. Then ω is exact, i.e. $[\omega] = 0 \in H^{n+p}_{DR}(X)$.*

7.3 Examples

In this section we describe several natural examples where forms and closed forms can be computed exactly.

(1) If $X = \mathbf{Spec}\, A$ is an usual (underived) smooth affine scheme, then

$$\mathcal{A}^{p,cl}(X;n) = (\tau_{\leqslant n}(\underset{0}{\Omega_A^p} \xrightarrow{d_{DR}} \underset{1}{\Omega_A^{p+1}} \xrightarrow{d_{DR}} \cdots))[n],$$

and hence

$$\pi_0 \mathcal{A}^{p,cl}(X;n) = \begin{cases} 0, & n < 0 \\ \Omega_A^{p,cl}, & n = 0 \\ H_{DR}^{n+p}(X), & n > 0 \end{cases}$$

(2) If X is a smooth and proper scheme, then $\pi_i \mathcal{A}^{p,cl}(X;n) = F^p H_{DR}^{n+p-i}(X)$.

(3) If X is a (underived, higher) algebraic stack, and $X_\bullet \to X$ is a smooth affine simplicial groupoid[10] presenting X, then $\pi_0 \mathcal{A}^p(X;n) = H^n(\Omega^p(X_\bullet), \delta)$ with $\delta =$ Čech differential.

In particular if G is a complex reductive group, then

$$\pi_0 \mathcal{A}^p(BG;n) = \begin{cases} 0, & n \neq p \\ (\mathrm{Sym}^p \mathfrak{g}^\vee)^G, & n = p. \end{cases}$$

(4) Similarly

$$\mathcal{A}^{p,cl}(BG;n) = \left| \prod_{i \geqslant 0} \left(\mathrm{Sym}^{p+i} \mathfrak{g}^\vee\right)^G [n+p-2i] \right|,$$

and so

$$\pi_0 \mathcal{A}^{p,cl}(BG;n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (\mathrm{Sym}^p \mathfrak{g}^\vee)^G, & \text{if } n \text{ is even.} \end{cases}$$

(5) Derived schemes naturally arise as derived intersections of ordinary schemes. A *dg scheme* ([CioFo-Ka]) (over k) is a scheme X equipped with a sheaf $\mathcal{A}_X^\bullet$ of non-positively graded quasi-coherent k-dg algebras such that $\mathcal{A}_X^0 = \mathcal{O}_X$. There is an obvious notion of morphism between dg schemes, and the equivalences are defined as those morphisms inducing quasi-isomorphisms between the corresponding sheaves of cdga's. If we localize the ∞-category of dg schemes with respect to such equivalences, we get an ∞-category $\mathbf{dgSch}_k$ admitting a functor Θ to the ∞-category of derived schemes: for an affine dg scheme $(X = \mathrm{Spec}\, R, \mathcal{A}_X)$, $\mathcal{A}_X$ is given by a k-cdga A_X, and the functor sends $(X, \mathcal{A}_X)$ to $\mathbf{Spec}\, A_X$.

[10] E.g. the nerve of a smooth affine atlas.

For more general, non affine dg schemes $(X, \mathcal{A}_X)$, the functor is defined using an affine Zariski covering of X (see [To-Ve-hagdag] for more details). The functor Θ is conservative but neither full, nor faithful, nor essentially surjective. The reason for not being essentially surjective is that, for a dg scheme $(X, \mathcal{A}_X)$, $\Theta(X, \mathcal{A}_X)$ is always an globally *embeddable* derived scheme, i.e. there exists an underived scheme X and a closed immersion of derived schemes $\Theta(X, \mathcal{A}_X) \to X$. Not all derived schemes have this property. However, for the purpose of this example, we will stick to dg schemes, and tacitly identify them as derived schemes by using (though not writing) the functor Θ.
In a typical setup one considers a smooth variety M over k and two smooth subvarieties $L_1, L_2 \subset M$. The **derived intersection** X of L_1 and L_2 is defined as a dg scheme, i.e. a space equipped with a sheaf of non-positively graded dg algebras:

$$X := L_1 \overset{h}{\underset{M}{\times}} L_2 = \left(L_1 \cap L_2, \, \mathcal{O}_{L_1} \overset{L}{\underset{\mathcal{O}_M}{\otimes}} \mathcal{O}_{L_2} \right).$$

The tangent complex of X is a a perfect complex concentrated in degrees 0 and 1 and is explicitly given by

$$\mathbb{T}_{X,x} = \qquad \underset{}{[}\ \underset{0}{T_{L_1,x} \oplus T_{L_2,x}} \longrightarrow \underset{1}{T_{M,x}}\ ],$$

In particular we have

- $H^0(\mathbb{T}_{X,x}) = T_{L_1 \cap L_2, x}$, and
- $H^1(\mathbb{T}_{X,x}) =$ failure of transversality of the intersection $L_1 \cap L_2$.

An important special case of a derived intersection is the derived zero locus of a section of a vector bundle. Let L be a smooth variety over k, $E \to L$ an algebraic vector bundle on L, and $s \in H^0(L, E)$ and a section in E. The **derived zero locus** X of s is defined as the derived intersection of s with the zero section of E inside $M = \mathrm{tot}(E)$:

$$X := \mathsf{Rzero}(s) = L \overset{h}{\underset{s,M,o}{\times}} L = \left(Z, \, i_L^{-1}(\mathrm{Sym}^\bullet(E^\vee[1]), s^\flat) \right),$$

where:

- $Z = t_0 X = \mathsf{zero}(s)$ is the scheme theoretic zero locus of s,
- $i_L : Z \to L$ is the natural inclusion, and
- $s^\flat$ is the contraction with s.

In particular $\mathbb{T}_X = \quad [\ \underset{0}{i_L^* T_L \oplus i_L^* T_L} \xrightarrow{i_L^* do + i_L^* ds} \underset{1}{i_M^* T_M}\ ],$ where

- $M = \mathrm{tot}(E)$, and
- i_M, o, and s are the natural maps

$$\begin{tikzcd} & L \arrow[dr, "o"] & \\ Z \arrow[ur, "i_L"] \arrow[rr, "i_M"] \arrow[dr, "i_L"'] & & M. \\ & L \arrow[ur, "s"'] & \end{tikzcd}$$

Exercise 7.11 *Let $\nabla : E \to E \otimes \Omega^1_L$ be an algebraic connection on E. Show that there is a natural quasi-isomorphism*

$$\mathbb{T}_X = \left[i_L^* T_L \xrightarrow{(\nabla s)^\flat} i_L^* E \right] = \left[T_L \xrightarrow{(\nabla s)^\flat} E \right]_{|Z} .$$

The algebraic connection $\nabla : E \to E \otimes \Omega^1_L$ may exists only locally on L and is not unique. Check that $(\nabla s)_{|Z}$ is well defined globally and independent of the choice of ∇.

Suppose again $X = \mathsf{Rzero}(s)$ for $s \in H^0(L, E)$ on a smooth L, then

$$\begin{array}{ccc} \Omega^1_X = & E^\vee_{|Z} \xrightarrow{(\nabla s)^\flat} & \Omega^1_{L|Z}, \\ & -1 & 0 \end{array}$$

Assume we have chosen[11] an algebraic connection $\nabla : E \to E \otimes \Omega^1_L$ which is also flat, i.e. $\nabla^2 = 0$. Using such a ∇ we can explicate Ω^1_X as a module over the Koszul dga of s:

$$(\mathrm{Sym}^\bullet(E^\vee[1]), s^\flat) = \left[\cdots \xrightarrow{s^\flat} \wedge^2 E^\vee \xrightarrow{s^\flat} E^\vee \xrightarrow{s^\flat} \mathcal{O}_L \right].$$

In other words using ∇ we can resolve Ω^1_X by a double complex of vector bundles on L so that this double complex is on-the-nose a module over $(\mathrm{Sym}^\bullet(E^\vee[1]), s^\flat)$:

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^1_L \arrow[r, "s^\flat"] & E^\vee \otimes \Omega^1_L \arrow[r, "s^\flat"] & \Omega^1_L \arrow[r] & \Omega^1_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \arrow[r, "s^\flat"] \arrow[u] & E^\vee \otimes E^\vee \arrow[r, "s^\flat"] \arrow[u, "{[\nabla, s^\flat]}"'] & E^\vee \arrow[r] \arrow[u, "{[\nabla, s^\flat]}"'] & E^\vee_{|Z} \arrow[u, "(\nabla s)^\flat"'] \arrow[d, equal] & -1 \\
 & & & & \Omega^1_X &
\end{tikzcd}$$

In the same way we can describe Ω^2_X as a module over the Koszul dga:

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^2_L \arrow[r] & E^\vee \otimes \Omega^2_L \arrow[r] & \Omega^2_L \arrow[r] & \Omega^2_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & (E^\vee \otimes \Omega^1_L)_{|Z} \arrow[u] & -1 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee_{|Z} \arrow[u] \arrow[d, equal] & -2 \\
 & & & & \Omega^2_X &
\end{tikzcd}$$

[11] Such a connection always exists Zariski locally on L.

-1 shifted 2 forms

In particular the -1 shifted 2-forms on the derived critical locus X will be given by pairs of sections $\alpha \in E^\vee \otimes \Omega^2_L$, and $\phi \in E^\vee \otimes \Omega^1_L$.

Note also that in terms of these resolutions, the de Rham differential $d_{DR} : \Omega^1_X \to \Omega^2_X$ is the map that term-by-term is given by the sum $d_{DR} = \nabla + \kappa$. Here κ is the Koszul contraction

$$\kappa : \wedge^a E^\vee \otimes S^b E^\vee \to \wedge^{a-1} E^\vee \otimes S^{b+1} E^\vee,$$

i.e. the contraction with $\mathrm{id}_E \in E \otimes E^\vee$ followed by the multiplication $E^\vee \otimes S^b E^\vee \to S^{b+1} E^\vee$.

The case when $X = \mathsf{Rzero}(s)$ is the derived zero locus of an algebraic one form $s \in \Omega^1_L$ plays a special role in Donaldson-Thomas theory [Be]. The explicit Koszul model of forms on such an X leads to the following important description of the -1 shifted 2-forms due to Behrend:

Remark 7.12 (K. Behrend) If $E = \Omega^1_L$ and so s is a 1-form on L, then a (-1)-shifted 2-form on $X = \mathsf{Rzero}(s)$ corresponds to a pair of elements

$$\alpha \in (\Omega^1_L)^\vee \otimes \Omega^2_L \text{ and } \phi \in (\Omega^1_L)^\vee \otimes \Omega^1_L \text{ such that } [\nabla, s^\flat](\phi) = s^\flat(\alpha).$$

Take $\phi = \mathrm{id} \in (\Omega^1_L)^\vee \otimes \Omega^1_L$. Suppose the local ∇ is chosen so that $\nabla(\mathrm{id}) = 0$ (i.e. ∇ is a torsion free connection). Then $[\nabla, s^\flat](\mathrm{id}) = ds$. In other words the pair (α, id) gives a (-1)-shifted 2-form on X if and only if $ds = s^\flat(\alpha) = -s \wedge \alpha$. Equivalently (α, id) gives a -1-shifted 2-form on X when $ds_{|Z} = 0$, i.e. if and only if s is an almost closed 1-form on L.

Exercise 7.13 *Suppose s is almost closed one form on L and let (α, id) be an associated (-1)-shifted 2-form. Describe the complex of keys for (α, id).*

8 Shifted symplectic geometry

To illustrate the power of the general theory we will review a geometric concept that is inherently derived in nature - the notion of a shifted symplectic structure. Here we only sketch the highlights of the theory. Full details can be found in [PTVV].

Recall that for an ordinary smooth scheme X over k a **symplectic structure** on X is a non-degenerate closed algebraic 2-form. In other words a symplectic strucure is an element $\omega \in H^0(X, \Omega^{2,cl}_X)$ such that its adjoint $\omega^\flat : T_X \to \Omega^1_X$ is a sheaf isomorphism. This straightforward definition does not work when X/k is a derived stack for at least two reasons

- The tangent complex $\mathbb{T}_X$ of X need not have amplitude which is symmetric around zero. When the amplitude of $\mathbb{T}_X$ is not symmetric, the tangent and cotangent complex of X will not have the same amplitude so they can not be quasi-isomorphic.
- A form being closed is not just a condition but rather an extra structure implemented by the key closing the form.

Taking these comments into account we arrive at the following natural

Definition 8.1 *Let X be a derived algebraic stack/k locally of finite presentation (so that $\mathbb{L}_X$ is perfect).*

- *A n-shifted 2-form $\omega : \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ - i.e. $\omega \in \pi_0(\mathcal{A}^2(X; n))$ - is* ***nondegenerate*** *if its adjoint $\omega^\flat : \mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism in $\mathbf{QCoh}(X)$.*
- *The* ***space of n-shifted symplectic forms*** $\mathrm{Sympl}(X; n)$ *on X/k is the subspace of $\mathcal{A}^{2,cl}(X; n)$ of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces*

$$\begin{tikzcd} \mathrm{Sympl}(X,n) \arrow[r] \arrow[d] & \mathcal{A}^{2,cl}(X,n) \arrow[d] \\ \mathcal{A}^2(X,n)^{nd} \arrow[r] & \mathcal{A}^2(X,n) \end{tikzcd}$$

Remark 8.2

- The nondegeneracy condition in the definition of shifted symplectic structure should be viewed as a duality between the *stacky* (positive degree) and the *derived* (negative degree) parts of $\mathbb{L}_X$ (see the end of §4).
- Suppose the perfect complex $\mathbb{L}_X$ has amplitude $[-m, n]$ for $m, n \geqslant 0$. Then at best X can admit an $(m-n)$-shifted symplectic structure.
- If $G = GL_n$, then BG has a canonical 2-shifted symplectic form (see Example (4) section 7.3) :

$$k \to (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee[-1] \wedge \mathfrak{g}^\vee[-1])[2] = \mathrm{Sym}^2\mathfrak{g}^\vee$$

  given by the dual of the trace map $(A, B) \mapsto \mathrm{tr}(AB)$.
- Similarly, for a reductive G/k choosing a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$ gives rise (see Example (4) section 7.3) to a 2-shifted symplectic form on BG.
- For a smooth variety the n-shifted cotangent bundle $T^\vee X[n] := \mathbf{Spec}_X(\mathrm{Sym}(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form. The same holds for the shifted cotangent bundle of a general derived algebraic stack locally of finite presentation over k [Ca-Cot].

Beyond these elementary examples shifted symplectic structures frequently arise on derived moduli stacks. A common method for producing such structures comes from a algebraic version of the AKSZ formalism in quantum field theory [AKSZ]. Before we can explain this we need to recall some basic facts about $\mathcal{O}$-orientations and push-forwards in derived geometry.

Definition 8.3 *Let X be a derived stack/k and let $A \in \mathbf{cdga}_k^{\leqslant 0}$. Let X_A denote the derived A-stack $X \times \mathbf{Spec}\, A$. We will say that X is* ***$\mathcal{O}$-compact over*** *k if for every $A \in \mathbf{cdga}_k^{\leqslant 0}$ we have that $\mathcal{O}_{X_A}$ is a compact object in $\mathbf{QCoh}(X_A)$ and for any perfect complex $E \in \mathbf{QCoh}(X_A)$, the cochain module $C(X_A, E) = \mathbb{R}\underline{Hom}(\mathcal{O}, E)$ is a perfect A-dg module.*

Suppose X is derived stack over k. For every $A \in \mathbf{cdga}_k^{\leqslant 0}$, the cup product $\cup$ on $C(X_A, \mathcal{O})$ turns this cochain complex into a commutative dga over A.

Given any morphism $\eta : C(X, \mathcal{O}) \to k[-d]$ of k-cdga, we get a morphism

$$\begin{tikzcd} C(X_A, \mathcal{O}) \arrow[r, "\eta_A"] \arrow[d, equal] & A[-d] \arrow[d, equal] \\ C(X, \mathcal{O}) \otimes_k A \arrow[r, "\eta \otimes \mathrm{id}_A"'] & k[-d] \otimes_k A \end{tikzcd}$$

of A-cdga and an induced morphism

$$C(X_A, \mathcal{O}) \otimes_A C(X_A, \mathcal{O}) \xrightarrow{\cup} C(X_A, \mathcal{O}) \xrightarrow{\eta_A} A[-d]. \tag{1}$$

If X is also $\mathcal{O}$-compact over k, then writing $C(X_A, \mathcal{O})^\vee = \mathbb{R}\underline{Hom}(C(X_A, \mathcal{O}), A)$ for the dual of $C(X_A, \mathcal{O})$ over A then (1) can be rewritten as the ajoint map:

$$C(X_A, \mathcal{O}) \xrightarrow{-\cup\eta} C(X_A, \mathcal{O})^\vee[-d].$$

More generally, for any perfect complex $E \in \mathbf{QCoh}(X_A)$ we can compose the natural pairing $C(X_A, E) \otimes_A C(X_A, E^\vee) \to C(X_A, \mathcal{O})$ with η_A to obtain a morphism

$$C(X_A, E) \xrightarrow{-\cup\eta_A} C(X_A, E^\vee)^\vee[-d].$$

With this notation we now have the following

Definition 8.4 *Let X be an $\mathcal{O}$-compact derived stack/k and let d be an integer. An* ***$\mathcal{O}$-orientation of X of degree*** *d is a morphism of cdga $[X] : C(X, \mathcal{O}) \to k[-d]$ such that for any $A \in dalg$ and any perfect complex on X_A, the natural map*

$$- \cup [X]_A : C(X_A, E) \to C(X_A, E^\vee)^\vee[-d].$$

is a quasi-isomorphism of A-dg modules.

Example 8.5 **(a)** Let X be a smooth proper Deligne-Mumford stack of dimension d over k, and let u be a Calabi-Yau structure on X. In other words u is an isomorphism between the structure sheaf of X and the canonical line bundle $\omega_X = \Omega_X^d$.

Since X is smooth and proper it is $\mathcal{O}$-compact when viewed as a derived stack/k. The isomorphism u composed with the trace map gives an isomorphism

$$H^d(X, \mathcal{O}) \xrightarrow{u} H^d(X, \omega_X) \xrightarrow{\mathrm{tr}} k.$$

This in turn gives a natural map of complexes

$$[X] : C(X, \mathcal{O}) \to k[-d]$$

which by Serre duality is an $\mathcal{O}$-orientation on X of degree d.

(b) Let M be an oriented connected and compact topological manifold of $\dim M = d$. Let $X = S(M)$ be the simplicial set of singular simplices in M viewed as a constant derived stack/k. The category $\mathbf{QCoh}(X)$ is naturally identified with the ∞-category of complexes of k-modules on M with locally constant cohomology sheaves. The perfect complexes on X correspond to complexes of k-modules on M that are locally quasi-isomorphic to finite complexes of constant sheaves of projective k-modules of finite type. I particular X is $\mathcal{O}$-compact. Furthermore, the fundamental class $[M] \in H_d(M,k)$ on M given by the orientation defines a morphism of complexes $[X] : C(M,k) = C(X,\mathcal{O}) \to k[-d]$. Finally Poincaré duality on M implies that $[X]$ is an $\mathcal{O}$-orientation on X of degree d.

Exercise 8.6 *Let $\mathbb{D} = \mathbf{Spec}\, k[t]/(t^2)$ be the spectrum of the dual numbers. Show that $\mathbb{D}$ admits a natural $\mathcal{O}$-orientation of degree* 0.

The main utility of $\mathcal{O}$-orientations is that they give natural integration maps on negative cyclic complexes and thus induce natural push-forward maps on shifted closed forms. Specifically let X be a derived $\mathcal{O}$-compact stack/k with an $\mathcal{O}$-orientation $[X] : C(X,\mathcal{O}) \to k[-d]$ of degree d. Suppose that X is a algebraic stack locally of finite presentation and let S be another algebraic derived stack locally of finite presentation. Then the $\mathcal{O}$-orientation on X induces a natural integration map of mixed graded complexes

$$\int_{[X]} : NC^w(X \times S) \to NC^w(S)[-d]$$

between the weighted negative cyclic complex on $X \times S$ and the weighted negative cyclic complex on X.

We will not spell out the definition of $\int_{[X]}$ but direct the reader to [PTVV, Section 2.1] which contains a detailed construction of this map. Here we only point out $\int_{[X]}$ induces a natural map between spaces of shifted closed forms. Indeed, recall (see Remark 7.4 (v)) that an n-shifted closed p-form α is nothing but a map $\alpha : k[n-p](p) \to NC^w$ of mixed graded complexes. Thus, given $\alpha \in \mathcal{A}^{p,cl}(X \times S; n)$ we can compose the map $\alpha : k[n-p](p) \to NC^w(X \times S)$ with $\int_{[X]}$ to obtan a map of mixed graded complexes

$$\begin{tikzcd} k[n-p](p) \arrow[r, "\alpha"] \arrow[rr, bend right, "\int_{[X]}\alpha"'] & NC^w(X \times S) \arrow[r, "\int_{[X]}"] & NC^w(S)[-d]. \end{tikzcd}$$

The assignment $\alpha \mapsto \int_{[X]} \alpha$ can therefore be viewed as a map of spaces of shifted closed p-forms:

$$\int_{[X]} : \mathcal{A}^{p,cl}(X \times S; n) \to \mathcal{A}^{p,cl}(S; n-d)$$

After these preliminaries we are now ready to state the algebraic version of the AKSZ formalism:

Theorem 8.7 (Theorem 5.2 in [PTVV]) *Let F be a derived algebraic stack equipped with an n-shifted symplectic form $\omega \in \mathrm{Symp}(F,n)$. Let X be an $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation $[X] : C(X,\mathcal{O}_X) \to k[-d]$ of degree d. Assume that the derived mapping stack $\mathbf{MAP}(X,F)$*

is itself a derived algebraic stack locally of finite presentation over k. Then, $\mathbf{MAP}(X, F)$ *carries a canonical $(n-d)$-shifted symplectic structure.*

The shifted symplectic form on $\mathbf{MAP}(X, F)$ is explicitly constructed from the symplectic form ω and the $\mathcal{O}$-orientation $[X]$. Indeed, consider the natural evaluation map

$$\begin{aligned} X \times \mathbf{MAP}(X, F) &\xrightarrow{\mathrm{ev}} F \\ (x, f) &\longmapsto f(x). \end{aligned}$$

Since $\omega \in \mathrm{Symp}(F, n) \to \mathcal{A}^{2,cl}(X; n)$ can be viewed as an n-shifted 2-form on F we can pull it back to $X \times \mathbf{MAP}(X, F)$ and integrate it against $[X]$ to obtain a closed $(n-d)$-shifted 2-form

$$\int_{[X]} \mathrm{ev}^* \omega \in \mathcal{A}^{2,cl}(\mathbf{MAP}(X, F); n-d).$$

It can be cheked directly [PTVV, Theorem 5.2] that the underlying shifted 2-form is non-degenerate, which shows that $\int_{[X]} \omega$ is symplectic.

Example 8.8 **(1)** Let $X/\mathbb{C}$ be a smooth and proper Calabi-Yau variety of dimension d and let G be complex reductive group. Then the derived moduli stack $\mathrm{Bun}_X(G) = \mathbf{MAP}(X, BG)$ of algebraic G-bundles on X is $(2-d)$-shifted symplectic.

(2) Let M be an oriented connected compact topological manifold of dimension d, and let G be a complex reductive group. If $X = S(M)$ is the simplicial set of singular simplices in M viewed as a constant derived stack, then the derived moduli of flat G-bundles on M, can be computed as the derived moduli $\mathrm{Bun}_X(G) = \mathbf{MAP}(X, BG)$ of algebraic G-bundles on X. Since X admits an $\mathcal{O}$-orientation of degree d, it again follows that $\mathrm{Bun}_X(G)$ is $(2-d)$-shifted symplectic.

Exercise 8.9 *(a) Suppose (M, ω) is an algebraic symplectic manifold over $\mathbb{C}$, and let $\mathbb{D} = \mathbf{Spec}\, \mathbb{C}[t]/(t^2)$ be the spectrum of the dual numbers. Then by the above theorem* $\mathbf{MAP}(\mathbb{D}, M)$ *is a symplectic manifold. But* $\mathbf{MAP}(\mathbb{D}, M)$ *is the total space of the tangent bundle T_M of M. Check that the symplectic structure on T_M is the pullback of the tautological symplectic structure on $T_M^\vee$ via the isomorphism $\omega^\flat : T_M \to T_M^\vee$.*

(b) Suppose (F, ω) is an n-shifted symplectic derived scheme over $\mathbb{C}$, and let $\mathbb{D} = \mathbf{Spec}\, \mathbb{C}[t]/(t^2)$ be the spectrum of the dual numbers. The theorem gives an n-shifted symplectic structure on the total space $\mathbb{T}_F = \mathbf{MAP}(\mathbb{D}, F)$ of the tangent stack of F. How does this symplectic structure depend on the key of ω?

Another source of examples of shifted symplectic structures is the derived intersection of Lagrangians. Before we can formulate the relevant construction we will need to discuss the notions of isotropic and Lagrangian structures on shifted symplectic geometry.

Definition 8.10 *Let (Y, ω) be a n-shifted symplectic derived stack/k and let $f : X \to Y$ be a map from a algebraic derived stack X locally of finite presentation/k.*

- *An* ***isotropic structure*** *on* $f : X \to Y$ *is a path* γ *in the space* $\mathcal{A}^{2,cl}(X;n)$ *connecting* $f^*\omega$ *to* 0
- *A* ***Lagrangian structure*** *on* $f : X \to Y$ *is an isotropic structure* γ *which is non-degenerate in the sense that the induced map* $\gamma^\flat : \mathbb{T}_f \to \mathbb{L}_X[n-1]$ *is an equivalence.*

Here $\mathbb{T}_f$ denotes the relative tangent complex of the map $f : X \to Y$, and the map $\gamma^\flat$ is constructed as follows. By definition γ gives a homotopy between the map $f^*(\omega^\flat) : f^*\mathbb{T}_Y \to f^*\mathbb{L}_Y[n]$ and the zero map of complexes. Pre-composing and post-composing $f^*(\omega^\flat)$ with the differential and the codifferential of f respectively we get a map of complexes

$$\begin{tikzcd} \mathbb{T}_X \arrow[r, "df^\vee \circ f^*(\omega^\flat) \circ df"] \arrow[d, "df"'] & \mathbb{L}_X[n] \\ f^*\mathbb{T}_Y \arrow[r, "f^*(\omega^\flat)"'] & f^*\mathbb{L}_Y[n] \arrow[u, "df^\vee"'] \end{tikzcd}$$

and hence pre-composing and post composing γ the differential and the codifferential of f we will get a homotopy between

$$df^\vee \circ f^*(\omega^\flat) \circ df : \mathbb{T}_X \to \mathbb{L}_X[n]$$

and the zero map of complexes.

If we write $\iota : \mathbb{T}_f \to \mathbb{T}_X$ for the natural map from the relative tangent complex of f to the tangent complex of X, then we will get a homotopy between the map

$$df^\vee \circ f^*(\omega^\flat) \circ df \circ \iota : \mathbb{T}_f \to \mathbb{L}_X[n]$$

and the zero map of complexes. On the other hand by definition we have an exact triangle of complexes

$$\mathbb{T}_f \xrightarrow{\iota} \mathbb{T}_X \xrightarrow{df} f^*\mathbb{T}_X \longrightarrow \mathbb{T}_f[1]$$

so we have an intrinsic homotopy between $df \circ \iota : \mathbb{T}_f \to f^*\mathbb{T}_Y$ and the zero map of complexes. In other words we get two homotopies between $df^\vee \circ f^*(\omega^\flat) \circ df \circ \iota$ and zero: one coming from the isotropic structure γ and the other coming from the definition of the relative tangent complex.

Composing these two homotopies we get a self-homotopy of the zero map of complexes $0 : \mathbb{T}_f \to \mathbb{L}_X[n]$, i.e. an element $\gamma^\flat$ in

$$\begin{aligned} \pi_1(\mathsf{Map}_{\mathbf{QCoh}(X)}(\mathbb{T}_f, \mathbb{L}_X[n]); 0) &= \pi_0(\mathsf{Map}_{\mathbf{QCoh}(X)}(\mathbb{T}_f, \mathbb{L}_X[n-1]); 0) \\ &= \mathrm{Hom}_{\mathbf{QCoh}(X)}(\mathbb{T}_f, \mathbb{L}_X[n-1]). \end{aligned}$$

Remark 8.11
- Any smooth Lagrangian $L \hookrightarrow (Y, \omega)$ where (Y, ω) is a smooth (0)-symplectic scheme has a natural Lagrangian structure in the derived sense. Moreover in this case the space of Lagrangian structures is contractible, so that this natural Lagrangian structure is essentially unique.
- For any n the point $\operatorname{Spec} k$ has a natural $(n+1)$-shifted symplectic form ω_{n+1}, namely the zero form. As first observed by D. Calaque, it is straightforward to check that the Lagrangian structures on the canonical map $X \to (\operatorname{Spec} k, \omega_{n+1})$ are the same thing as n-shifted symplectic structures on X.

The relevance of Lagrangian structures for constructing new shifted symplectic structures is captured by the following

Theorem 8.12 (Theorem 2.9 in [PTVV]) *Let (F, ω) be n-shifted symplectic derived stack, and let $L_i \to F$, $i = 1, 2$ be maps of derived stacks equipped with a Lagrangian structures. Then the derived intersection $L_1 \times^h_F L_2$ of L_1 and L_2 is canonically a $(n-1)$-shifted symplectic derived stack.*

Example 8.13 (1) An important special case of this construction is the (-1)-shifted symplectic structure on the derived critical locus of a function. Suppose $L/\mathbb{C}$ is a smooth variety and let $\mathsf{w} : L \to \mathbb{C}$ be a regular function. By definition the derived critical locus $\mathsf{Rctit}(\mathsf{w})$ of w is the derived zero scheme $\mathsf{Rzero}(d\mathsf{w})$ of the one form $d\mathsf{w} \in H^0(L, \Omega^1_L)$. Consider $T^\vee_L$ with its standard (0-shifted) symplectic structure. Since $d\mathsf{w}$ is closed, the map $d\mathsf{w} : L \to T^\vee_L$ is Lagrangian. But the zero section $o : L \to T^\vee_L$ is Lagrangian as well, and so

$$\mathsf{Rcrit}(\mathsf{w}) = L \underset{d\mathsf{w}, T^\vee_L, o}{\overset{h}{\times}} L$$

is equipped with a natural (-1)-shifted symplectic structure. In terms of the local description of Remark 7.12 the underlying (-1)-shifted 2-form is given by the pair$(d\mathsf{w}, \mathrm{id})$.
Variant: By the same construction the derived critical locus of a shifted function $\mathsf{w} : L \to \mathbb{A}^1[n]$ (i.e. an element $\mathsf{w} \in H^n(L, \mathcal{O})$ is equipped with a natural $(n-1)$-shifted symplectic structure.

Note that if w is locally constant, then $d\mathsf{w} = 0$ and so $\mathsf{Rcrit}(\mathsf{w}) = T^\vee_L[-1]$ is the (-1)-shifted cotangent stack with its canonical symplectic structure. By the same token if $\mathsf{w} \in H^n(L, \mathcal{O})$ is in the image of $H^n(L, \mathbb{C})$, then $\mathsf{Rcrit}(\mathsf{w}) = T^\vee_L[n-1]$ is the $(n-1)$-shifted cotangent stack with its canonical symplectic structure. In particular if L is smooth and projective over $\mathbb{C}$ the Hodge theorem implies that $H^n(L, \mathbb{C}) \to H^n(L, \mathcal{O})$ is surjective for all n and therefore the derived critical locus of any shifted function on L is a shifted cotangent bundle.

(2) Let (X, ω) be an algebraic symplectic manifold over $\mathbb{C}$ equipped with a Hamiltonian action of a complex reductive group G. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mu : X \to \mathfrak{g}^\vee$ be a G-equivariant moment map. Consider the stack quotient $[\mathfrak{g}^\vee/G]$ where G acts by the coadjoint action. Since $T^\vee_{BG} = [\mathfrak{g}[-1]/G]$ it follows that $[\mathfrak{g}^\vee/G] = T^\vee_{BG}[1]$ is a 1-shifted symplectic stack. The equivariant map μ induces a map of stacks

$$\mu : [X/G] \to [\mathfrak{g}^\vee/G]$$

and the moment map condition translates into the statement that the symplectic form ω induces a Lagrangian structure on this map.

Additionally, for any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^\vee$ the Kostant-Kirillov symplectic form on $\mathbb{O}$ induces a Lagrangian structure on the map of stacks

$$[\mathbb{O}/G] \to [\mathfrak{g}^\vee/G].$$

In particular the homotopy fiber product

$$[X/G] \underset{[\mathfrak{g}^\vee/G]}{\overset{h}{\times}} [\mathbb{O}/G]$$

will have a natural 0-shifted symplectic structure.

Note that this fiber product is simply the stack quotient $[R\mu^{-1}(\mathbb{O})/G]$, where $R\mu^{-1}(\mathbb{O}) = X \times^h_{\mathfrak{g}^\vee} \mathbb{O}$ is the derived preimage of the coadjoint orbit $\mathbb{O}$ under the moment map μ. Thus the 0-shifted symplectic structure on $[R\mu^{-1}(\mathbb{O})/G]$ can be viewed as an extension of the Marsden-Weinstein symplectic reduction: at the expense of adding a stacky and a derived structure on the reduction we obtain a symplectic structure that makes sense at all points along the preimage of the moment map. More details on the statements in this example can be found in [Ca-TQFT, Saf]

We conclude this section with a brief discussion of the local structure of shifted symplectic derived stacks. Recall that In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux-Weinstein theorem:** *a symplectic structure is locally (in the C^∞ or analytic setting) or formally (in the algebraic setting) isomorphic to the standard symplectic structure on a cotangent bundle.*

By analogy one might expect that in the derived stacky context shifted symplectic structures will be modelled locally by the standard symplectic structures on shifted cotangent bundles. This expectation is too naive.

Exercise 8.14 *Exhibit a derived critical locus (defined in Example 8.13 (1)) whose derived structure is not locally formal.*

However, as we already noted in Example 8.13, the shifted cotangent bundles are symplectically derived critical loci of shifted constant functions. It turns out that the derived critical loci of arbitrary shifted functions have enough flexibility to provide local models. This leads to a remarkable shifted version of the Darboux theorem:

Theorem 8.15 ([BBJ]) *Let X be a derived scheme, and let ω be an n-shifted symplectic structure on X, with $n < 0$. Then Zariski locally (X, ω) is isomorphic to $\big(\mathsf{Rcrit}(\mathsf{w}), \omega_{\mathsf{Rcrit}(\mathsf{w})}\big)$ for some shifted function $\mathsf{w} : M \to \mathbb{A}^1[n+1]$ on a derived scheme M.*

Remark 8.16 A more general result holds for locally finitely presented derived algebraic stacks (see [BBBJ]).

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