WHAT IS . . .

a Derived Stack?

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Derived stacks are the "spaces" studied in derived algebraic geometry, a relatively new theory in which algebraic geometry meets homotopy theory—or higher category theory, depending on one's taste. Just as a scheme is locally modeled on commutative rings, derived schemes or stacks are modeled on some kind of *derived commutative rings*, a homotopy version of commutative rings. In order to define derived stacks more precisely, it will be useful to reexamine briefly the functorial point of view in (underived) algebraic geometry.

Let k be a *base* commutative ring. In algebraic geometry, a k-scheme may be given at least two equivalent definitions. The first one is as a special kind of pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative k-algebras on X (this is the so-called *ringed space* approach). The second one is as a special kind of functor from the category $\mathbf{CommAlg}_k$ of commutative k-algebras to the category of sets (this is the *functor of points* approach). For example, the n-dimensional projective space $\mathbb{P}^n_k$ over k may be identified with the functor sending $A \in \mathbf{CommAlg}_k$ to the set of surjective maps of A-modules $A^{n+1} \to A$, modulo the equivalence relation generated by multiplication by units in A. In the following, we will concentrate on the functor of points description.

Prompted by the study of moduli problems (e.g., classifying families of elliptic curves or vector bundles on a given algebraic variety), algebraic geometers have long been led to enlarge the *target* category of the functor of points from sets to *groupoids* (i.e., categories whose morphisms are all invertible) in order to classify objects *together with* their isomorphisms instead of just objects *modulo* isomorphisms. These functors are called *stacks,* and aficionados of the WHAT IS column already have met this notion ("What is a stack?" by Dan Edidin, *Notices*, April 2003). More recently, *higher stacks* came into play; they arise naturally when one is interested in classifying geometric objects (say, over a given scheme) for which the natural notion of *equivalence* is broader than just isomorphisms. Example: perfect complexes over a given scheme with equivalences given by quasi-isomorphisms, i.e., maps inducing isomorphisms on cohomology. In such cases it is natural to enlarge the target category for the corresponding moduli functors to the category of *simplicial sets* or, equivalently, to the category of *topological spaces*. A stack may be viewed as a higher stack via the *nerve* construction: the nerve of a groupoid is the simplicial set whose n-th level is the set of n composable morphisms in the groupoid. This simplicial set has homotopy groups only in degrees ≤ 1, with π_1 roughly corresponding to automorphisms of a given object. General simplicial sets or topological spaces are needed in order to accomodate "higher autoequivalences" of the objects being classified.

Another example of a higher stack is given by iterating the so-called *classifying stack* construction. For a k-group scheme G, there is a stack $\mathrm{B}G \equiv K(G,1)$ classifying principal G-bundles; by taking the nerve, we may view $K(G,1)$ as a functor to simplicial sets. If G is *abelian*, this functor is equivalent to a functor to simplicial abelian groups, and the classifying stack construction may then be applied to any simplicial level again to get $\mathrm{BB}G \equiv K(G,2)$. And so on. For any $n \geq 2$, $K(G,n)$ is not a stack but a higher stack (classifying

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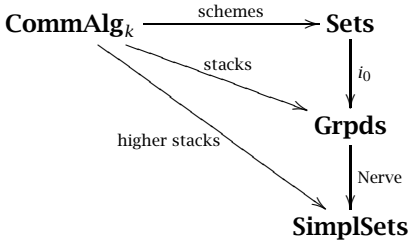
"higher" principal G-bundles); it is the algebro-geometric analog of the Eilenberg-Mac Lane space in topology, from which we borrowed the notation.

It is useful to draw a diagram summarizing the still underived situation we have just discussed.



Here i_0 is the functor identifying a set with the groupoid having that set of objects and only identities as arrows.

The main point of derived algebraic geometry *is to enlarge* (also) *the source category*, i.e., to replace commutative algebras with a more flexible notion of commutative rings serving as new or *derived rings*. Why? I could list here, among some of the actual historical motivations, the Kontsevich *hidden smoothness philosophy* and a geometrical definition of *universal elliptic cohomology* (aka *topological modular forms*). For expository reasons, I will concentrate instead on two more down-to-earth and classical instances that naturally lead to building a geometry based on these derived rings rather than on the usual commutative rings.

Derived Intersections

In algebraic geometry, the so-called *intersection multiplicities* are given by Serre's formula. Here is one form of it. Let X be an ambient complex smooth projective variety, and let Z, T be possibly singular subvarieties of X whose dimensions sum up to $\dim X$ and which intersect on a 0-dimensional locus. If $p \in Z \cap T$ is a point, its "weight" in the intersection, i.e., its intersection multiplicity, is given by

$$\mu_p(X; Z, T) = \sum_{i \geq 0} \dim_{\mathbb{C}} \operatorname{Tor}_i^{\mathcal{O}_{X,p}}(\mathcal{O}_{Z,p}, \mathcal{O}_{T,p}),$$

where $\mathcal{O}_{Y,p}$ denotes the local ring of a variety Y at $p \in Y$ and the Tor's are computed in the category of $\mathcal{O}_{X,p}$-modules. One can easily prove that the sum is finite and much less easily that it is nonnegative. But here we are interested in another aspect of this formula. In the lucky case of a flat intersection, i.e., when either $\mathcal{O}_{Z,p}$ or $\mathcal{O}_{T,p}$ are flat $\mathcal{O}_{X,p}$-modules, this formula tells us that the multiplicity is given by the dimension of the tensor product $\mathcal{O}_{Z,p} \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{T,p}$, which, for our purposes, has two peculiar features: it is a commutative ring, and it is the local ring of the scheme-theoretic intersection $Z \cap T$ at p. In other words, it carries a nice geometrical interpretation.

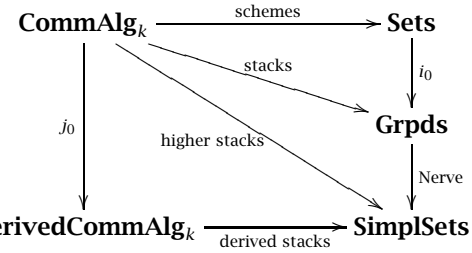
What about the general case? By definition, in order to compute such multiplicities, i.e., the Tor-groups, one has to resolve $\mathcal{O}_{Z,p}$ (or $\mathcal{O}_{T,p}$) via a complex of projective or flat $\mathcal{O}_{X,p}$-modules, tensor this resolution with $\mathcal{O}_{T,p}$ (or $\mathcal{O}_{Z,p}$), and compute the cohomology of the resulting complex. This is very much homological algebra, but where has the geometry gone? We started with three varieties and a point on each of them, and we ended up computing the cohomology of a rather geometrically obscure complex. Is there a way to reconcile the general case with the flat intersection case in a possibly wider geometrical picture? A possible answer is the following: we can still keep the two peculiar geometric features of the flat intersection case mentioned above, provided we are willing to contemplate a notion of commutative rings that is more general than the usual one. More precisely, one can first observe that it is possible to choose the resolution of, say, $\mathcal{O}_{Z,p}$ (as an $\mathcal{O}_{X,p}$-module) in such a way that it has the structure of a non-positively *commutative differential graded $\mathcal{O}_{X,p}$-algebra* with the differential increasing the degree (a *cdga*, for short) or equivalently of a simplicial commutative $\mathcal{O}_{X,p}$-algebra. This gives us an extension of the first feature of the flat intersection case. Then we can force the second feature by insisting that the tensor product of this cdga resolution with $\mathcal{O}_{T,p}$ does give the "scheme structure" of $Z \cap T$ (locally at the point p). Of course this is not a usual scheme structure, but rather a new kind of scheme-like structure which we will call a *derived structure*, the name coming from the fact that what we are computing is the derived tensor product $\mathcal{O}_{Z,p} \otimes^{\mathbb{L}}_{\mathcal{O}_{X,p}} \mathcal{O}_{T,p}$ (whose cohomology groups are the Tor-groups appearing in Serre's intersection formula). But we may, and therefore we do, view $\mathcal{O}_{Z,p} \otimes^{\mathbb{L}}_{\mathcal{O}_{X,p}} \mathcal{O}_{T,p}$ as a *derived* commutative ring, i.e., a cdga or a simplicial commutative algebra.

Deformation Theory

Another more or less classical topic in algebraic geometry that also leads to considering these kinds of derived rings is the theory of the *cotangent complex*. This object dates back to Quillen, Grothendieck, and Illusie (see L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Math. 239, Springer-Verlag, 1971) and is too technical to be carefully defined here. Let me just say that in the affine case $X = \operatorname{Spec} A$, for A a commutative k-algebra, it is defined as the classical module of Kähler differentials, but only after having replaced A itself by a resolution that is a simplicial commutative k-algebra that is free at each simplicial level (in characteristic 0, one might as well resolve using cdga's). Recall that a simplicial commutative algebra is a simplicial set that is at each level a commutative algebra and whose structural maps (face

and degeneracies) are required to be morphisms of algebras. Even with such a sketchy definition, it will not perhaps be too surprising that most topics in the deformation theory of schemes, and more generally of moduli stacks, are handled through this cotangent complex. Although this was confirmed and deepened several times in the course of the explosive development of moduli theory, this relationship with deformation theory was already clear to Quillen, Illusie, and Grothendieck. It was Grothendieck himself, back in 1968, who asked whether it was possible to give some geometrical interpretation of the cotangent complex, say of a scheme X (see p. 4 of *Catégories cofibrées additives et complexe cotangent relatif*, LNM 79, Springer-Verlag, 1968). Derived algebraic geometry offers a possible answer to this question: if one is willing to build algebraic geometry using derived rings, then in this new world the cotangent complex $\mathbb{L}_X$ of a scheme X *is* the actual cotangent space of X. As a consequence, in derived algebraic geometry the full cotangent complex (and not only some truncation thereof) has a deformation-theoretic interpretation: this might be considered as the main reason derived algebraic geometry is interesting for studying moduli problems.

The upshot of the previous discussion is that we are interested in enlarging the previous diagram to the following one:



Here **DerivedCommAlg**$_k$ is either the category of simplicial commutative k-algebras or, when the base ring k has characteristic 0, the category of *cdga*'s over k. We will freely pass from one description to the other whenever this is possible and will contribute to conciseness. The functor j_0 then sends a commutative algebra R to the corresponding constant simplicial commutative algebra (having R in each simplicial level and with identities as face and degeneracies) or to the cdga having R in degree 0, and 0 elsewhere.

An *equivalence* $f : A \to B$ between simplicial commutative algebras is a map inducing isomorphisms $\pi_i(A; a) \simeq \pi_i(A; f(a))$ between homotopy groups, for any base point $a \in A_0$ (A_0 being the 0-th level of the simplicial set A) and any $i \geq 0$. The corresponding notion for cdga's is given by *quasi-isomorphisms*, i.e., maps inducing isomorphisms on every cohomology group. Derived rings together with this notion of equivalences give rise to a *homotopy theory* (technically speaking, a Quillen model category structure); the same is true, with weak homotopy equivalences as equivalences, for the target category **SimplSets**. It is a basic rule of derived algebraic geometry that all constructions should be, in an appropriate sense, invariant under all these equivalences. This is exactly the point where derived algebraic geometry becomes a blend of algebraic geometry and homotopy theory, borrowing techniques and intuitions from both areas.

We are now ready to fulfill the aim of this column and give a definition. A *derived stack* over k is a functor **DerivedCommAlg**$_k \to$ **SimplSets** sending equivalences to weak homotopy equivalences and satisfying a *descent* or *gluing condition* with respect to some chosen "topology" on derived rings.

The descent condition mentioned here is just a derived (or homotopy) version of the usual sheaf condition with respect to an appropriate (i.e., invariant under equivalences) notion of *topology* on derived rings. Here is one example of such a topology, the so-called *strong étale topology*. We will assume we are working in characteristic 0 and are taking cdga's as our model for derived rings. A covering family for such a topology is a family $\{A \to B_i\}$ such that $\{H^0(A) \to H^0(B_i)\}$ is an étale covering family (in the sense of usual algebraic geometry) and the canonical maps $H^i(A) \otimes_{H^0(A)} H^0(B) \to H^i(B)$ are isomorphisms for any $i \geq 0$. A derived version of the Yoneda lemma gives us, for any derived ring A, a derived stack denoted as $\mathbb{R}\mathrm{Spec}\, A$ and called the *derived spectrum* of A. Any stack may be faithfully viewed as a derived stack, and conversely any derived stack $\mathcal{F}$ has a *truncation* $\mathrm{t}_0(\mathcal{F})$ that is a stack, e.g., $\mathrm{t}_0(\mathbb{R}\mathrm{Spec}\, A) \simeq \mathrm{Spec}(H^0(A))$. Passing to the truncation should be thought of as passing to the classical or underived part. And, intuitively speaking, $\mathcal{F}$ behaves much like a formal thickening of its truncation or as a scheme with respect to its reduced subscheme.

In the section "Deformation theory" I hinted that derived algebraic geometry might be a natural framework for deformation theory in algebraic geometry. Let me try to push this point further. The idea is that derived rings allow for more general deformation directions, but not so general that the usual geometrical intuition is completely lost.

Let $\mathrm{char}(k) = 0$ and $i \in \mathbb{N}$ and $k[i]$ be the k-dg-module having just k in degree $-i$. We can then consider the trivial square zero extension k-cdga

$$k[\varepsilon_i] := k \oplus k[i].$$

Note that $k[\varepsilon_i]$ is concentrated in nonpositive degrees (with a degree-increasing differential) and that its only nontrivial cohomology groups are concentrated in degrees 0 an $-i$, where they both equal k. $k[\varepsilon_i]$ is called the *derived ring of i-th order dual numbers* over k, and its derived

spectrum $\mathbf{D}_i := \mathbb{R}\mathrm{Spec}\, k[\varepsilon_i]$ is called the *derived i-th order infinitesimal disk* over k. Note that for odd i, $k[\varepsilon_i]$ is the free cdga on the k-dg module $k[i]$, i.e., on one generator in degree $-i$.

A useful intuitive way of thinking about $k[\varepsilon_i]$ is as the universal derived affine scheme carrying *generalized nilpotents* of order i. Here is one typical result explaining how derived stacks allow for a natural and geometrical reinterpretation of usual deformation theory.

Proposition. *Let X be a scheme over k and let $\mathbb{L}_X$ be its cotangent complex. If $x \in X(k)$ is a k-rational point in X, then for each $i \in \mathbb{N}$, there is a canonical group isomorphism*

$$\mathrm{Ext}^i_k(\mathbb{L}_{X,x}, k) \simeq \mathbb{R}\mathrm{Hom}_*(\mathbf{D}_i, (X,x)),$$

where $\mathbb{R}\mathrm{Hom}_*$ denotes the set of morphisms in the homotopy category of Spec k*-pointed derived stacks.*

In other words, for any $i \in \mathbb{N}$, the functor from Spec k-pointed schemes to abelian groups

$$\mathbf{Sch}_{*,k} \longrightarrow \mathbf{Ab} \ : (X,x) \longmapsto \mathrm{Ext}^i_k(\mathbb{L}_{X,x}, k),$$

while *not* co-representable in $\mathbf{Sch}_{*,k}$, is indeed co-represented by $\mathbf{D}_i$ in the larger category of pointed derived stacks. Therefore, the full cotangent complex has a moduli-theoretic interpretation in the world of derived algebraic geometry.

A *derived extension* of a stack F is a derived stack $\mathcal{F}$ together with an identification of F with the truncation $\mathrm{t}_0(\mathcal{F})$ of $\mathcal{F}$. Given a stack F, there is always a trivial derived extension (just viewing F itself as a derived stack), but in most cases there are other derived extensions. For example, the stack $\mathbf{Vect}_n(X)$ classifying rank n vector bundles over a smooth and proper scheme X has another *natural* and nontrivial derived extension $\mathbb{R}\mathbf{Vect}_n(X)$, obtained as the derived stack of maps from X to the classifying stack $\mathrm{B}GL_n$. One can prove that $\mathbb{R}\mathbf{Vect}_n(X)$ classifies a fairly natural derived version of rank n vector bundles on X. The choice of a derived extension of a given stack F endows F itself with important *additional* geometric structure. One interesting example of this further structure, still in the thread of deformation theory, arises when one starts with a Deligne-Mumford stack F (e.g., the stack of stable maps to a fixed smooth complex projective variety) and considers a derived Deligne-Mumford extension $\mathcal{F}$ of it. If this derived extension is *quasi-smooth* (i.e., its cotangent complex is of perfect amplitude $[-1,0]$), then the closed immersion $j : F \hookrightarrow \mathcal{F}$ induces a map of cotangent complexes $j^*\mathbb{L}_{\mathcal{F}} \to \mathbb{L}_F$ that is a $[-1,0]$-*perfect obstruction theory* in the sense of Behrend-Fantechi (*Invent. Math.* 128 (1997)). Moreover, it is true in all known cases, and expected to be true in general, that *any* such obstruction theory can be obtained as above from some derived extension. So, as already seen in the case of the cotangent complex of a scheme, obstruction theories are also classical ways of (partially) encoding derived geometric structures.

A Quick Guide to the Literature

The approach to derived algebraic geometry sketched above is contained essentially in [HAG-II], of which [Toën-2005] is a very readable overview. Due to some overlap in the topics, [V-2010] might be a useful complement to the present text. Another approach to derived geometry is in Jacob Lurie's book [H-Algebra], where the emphasis is on higher categorical aspects from the very beginning and whose spectrum of applications is broader.

References

[H-Algebra] J. Lurie, *Higher Algebra*, available at `http://www.math.harvard.edu/~lurie/`.

[Toën-2005] B. Toën, Higher and derived stacks: A global overview, pages 435–487 in *Algebraic Geometry–Seattle, 2005*, Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.

[HAG-II] B. Toën, G. Vezzosi, *Homotopical algebraic geometry. II: Geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.

[V-2010] G. Vezzosi, *A note on the cotangent complex in derived algebraic geometry*, arXiv:1008.0601.