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Infinitesimal Stokes' Formula for Higher-Order de Rham Complexes

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Abstract. A higher-order de Rham complex $\mathbf{dR}_\sigma$ [14] is associated with a commutative algebra A and a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots)$. It is called *regular* if σ is nondecreasing. We extend the algebraic definitions of the Lie derivative and interior product with respect to a derivation of A, to higher-order differential forms. These allow us to prove a generalization of the infinitesimal Stokes formula (also known as the Cartan homotopy formula) for higher regular de Rham complexes. In particular, this implies the homotopy invariance property of higher regular de Rham cohomologies for differentiable manifolds.

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Notations and Conventions

K: a commutative ring with unit;

A: a commutative, associative K-algebra with unit;

$\mathcal{M}(A)$ (resp. $\mathcal{M}(K)$): the category of A-modules (resp. of K-modules);

$\mathcal{BIM}(A)$: the category of (A, A)-bimodules, whose objects are *ordered couples* (P, P^+) of A-modules and whose morphisms are the usual morphisms of bimodules;

$[\mathbf{C}, \mathbf{C}]$: the category of functors $\mathbf{C} \to \mathbf{C}$, $\mathbf{C}$ being any category: its objects are functors $\mathbf{C} \to \mathbf{C}$ while its morphisms are natural transformations, also called functorial morphisms, between them;

If $\mathfrak{D}$ is a full subcategory of $\mathcal{M}(A)$, a functor T: $\mathfrak{D} \to \mathfrak{D}$ will be said *strictly representable* in $\mathfrak{D}$ if it exists an object τ in $\mathfrak{D}$ and a functorial isomorphism $T \simeq \hom_A(\tau, \cdot)$ in $[\mathfrak{D}, \mathfrak{D}]$.

If T_1 and T_2 are strictly representable functors $\mathfrak{D} \to \mathfrak{D}$ with representative objects τ_1 and τ_2, respectively, and φ: $T_1 \to T_2$ is a morphism in $[\mathfrak{D}, \mathfrak{D}]$ then its *dual representative* is the morphism $\varphi^\vee \doteq \varphi(\tau_1)(\mathrm{id}_{\tau_1}) \in \hom_{\mathfrak{D}}(\tau_2, \tau_1)$. When we write 'duality' we always mean this kind of duality.

A sequence $T_1 \to T_2 \to T_3$ of functors $T_i\colon \mathfrak{D} \to \mathfrak{D}$, $i = 1,2,3$, (and functorial morphisms) with $\mathfrak{D}$ an Abelian subcategory of $\mathcal{M}(A)$, will be said to be *exact in* $[\mathfrak{D}, \mathfrak{D}]$ if it is exact in $\mathfrak{D}$ when applied to any object of $\mathfrak{D}$.

If $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathrm{N}_+^n$ (resp., $(\sigma_1, \sigma_2, \ldots) \in \mathrm{N}_+^\infty \doteq \mathrm{inv\,lim}_{n>0} \mathrm{N}_+^n$) and $0 < r \leqslant n$ (resp., $r > 0$), then we define $\sigma(r) = (\sigma_1, \ldots, \sigma_r) \in \mathrm{N}_+^r$.

We will often write DO instead of 'differential operator'.

1. Introduction

Higher-order analogs of the standard de Rham complex were found in [14] as a result of a search for natural differential operators to deal with various kinds of prolongation procedures occurring in the formal theory of partial differential equations, differential geometry, etc. Some interesting applications of them are expected in secondary calculus ([16]). All these perspective applications requires the knowledge of higher de Rham cohomology, i.e. cohomology of higher analogs of the de Rham complex. It was conjectured in [14] that for smooth manifolds they are canonically isomorphic to the standard one. In this paper, we prove it for *regular* higher de Rham complexes. Recall that a higher-order de Rham complex is associated with a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots)$ and is called *regular* if σ is nondecreasing. The standard de Rham complex corresponds to the sequence $(1, 1, \ldots)$. This result is a consequence of the higher analog of the 'infinitesimal Stokes' formula', also known as the 'Cartan homotopy formula', which is, in fact, the main goal of this paper. The importance of such a formula is that it supplies the calculus of higher analogs of differential forms with necessary 'homotopy formulas'.

In [11] (see also [12]) we prove, in a much larger algebraic context, that under a natural smoothness hypothesis the higher de Rham cohomologies do not depend of σ for all, not necessarily regular, σ.

2. Higher-Order de Rham Complexes. Lie Derivative

We refer to [6] for all the basic definitions and notations regarding algebraic differential calculus. For any $k \geqslant 0$ and any couple (P, Q) of A-modules, we have the A-bimodule $\mathrm{Diff}_k^{(+)}(P, Q) = (\mathrm{Diff}_k(P, Q), \mathrm{Diff}_k^+(P, Q))$ of differential operators of order $\leqslant k$ from P to Q.

Remark 1. It is important here and in the following that the reader remember our *Notations and Conventions*: an object of $\mathcal{BIM}(A)$, though sometimes called simply an A-bimodule, is an *ordered* couple (P, P^+) of compatible A-module structures on the same underlying Abelian group. In particular, an A-bimodule in the standard sense gives rise to two different objects of $\mathcal{BIM}(A)$ depending

on the order of the corresponding modules. In many of the following definitions, this order will be crucial, therefore we have chosen to work with $\mathcal{BIM}(A)$.

As in [6] we put

$$\mathrm{Diff}_k(A, Q) \equiv \mathrm{Diff}_k Q \quad \text{and} \quad \mathrm{Diff}_k^+(A, Q) = \mathrm{Diff}_k^+ Q;$$

recall that by definition $\mathrm{Diff}_0^+ = \mathrm{Diff}_0 = \mathrm{Id}_{\mathcal{M}(A)}$ as functors from $\mathcal{M}(A)$ to itself.

$\mathrm{D}_k(Q) \doteq \{\Delta \in \mathrm{Diff}_k Q \mid \Delta(1) = 0\}$ is an A-submodule of $\mathrm{Diff}_k Q$ (but not of $\mathrm{Diff}_k^+ Q$) called module of kth order Q-valued derivations of the algebra A. This way we get a functor $\mathrm{D}_{(k)}\colon \mathcal{M}(A) \to \mathcal{M}(A)$ associating $\mathrm{D}_{(k)}(Q)$ with Q, together with the short exact sequence:

$$0 \longrightarrow \mathrm{D}_{(k)} \xrightarrow{i_k} \mathrm{Diff}_k \xrightarrow{p_k} \mathrm{Id}_{\mathcal{M}(A)} \longrightarrow 0 \tag{1}$$

in $[\mathcal{M}(A), \mathcal{M}(A)]$, where i_k is the obvious inclusion and p_k is defined by:

$$p_k(Q)\colon \mathrm{Diff}_k Q \longrightarrow Q\colon \Delta \mapsto \Delta(1), \quad \Delta \in \mathrm{Diff}_k Q$$

for any A-module Q. The functorial monomorphism $\mathrm{Id}_{\mathcal{M}(A)} \equiv \mathrm{Diff}_0 \hookrightarrow \mathrm{Diff}_k$ splits (1), so that $\mathrm{Diff}_k = \mathrm{D}_{(k)} \oplus \mathrm{Id}_{\mathcal{M}(A)}$. $\mathrm{D}_{(1)}(Q)$ coincides with the A-module $\mathrm{Der}_{A/K}(Q)$ of all (first-order) Q-valued K-linear derivations on A (see [2], for example).

If $k, l \geqslant 0$, we have a natural functorial morphism:

$$\begin{aligned}
&c_{k,l}\colon \mathrm{Diff}_k^+ \circ \mathrm{Diff}_l^+ \longrightarrow \mathrm{Diff}_{k+l}^+,\\
&[c_{k,l}(P)](\Delta)(a) \doteq [\Delta(a)](1),\\
&\Delta \in \mathrm{Diff}_k^+(\mathrm{Diff}_l^+(P)), \quad a \in A,
\end{aligned}$$

(P being an A-module) which is called the '*gluing*' morphism (see [6]).

Let P and P^+ be the left and right A-modules corresponding to an A-bimodule $P^{(+)} \equiv (P, P^+)$ (P and P^+ coincide as K-modules, hence as sets). Denote by $\mathrm{Diff}_k^\bullet(P^{(+)})$ (resp. $\mathrm{D}_{(k)}^\bullet(P^{(+)})$) the A-module which coincides with $\mathrm{Diff}_k(P^+)$ (resp. $\mathrm{D}_{(k)}(P^+)$) as K-module and inherits its A-module structure from P (and not from P^+). More precisely, an operator $\Delta \in \mathrm{Diff}_k^\bullet(P^{(+)})$ (resp., $\mathrm{D}_{(k)}^\bullet(P^{(+)})$) multiplied by $a \in A$ acts on A as

$$(a\Delta)(a') \doteq a\Delta(a'), \quad a' \in A$$

(where $(a, p) \mapsto ap$ is the multiplication in P) while the same map $\Delta\colon A \to P^+$ interpreted as an element of $\mathrm{Diff}_k(P^+)$ (resp., $\mathrm{D}_{(k)}(P^+)$) is multiplied by $a \in A$ as

$$(a\Delta)(a') \doteq a^+\Delta(a'), \quad a' \in A$$

(where $(a,p) \mapsto a^+p$ is the multiplication in P^+).

For an A-submodule $S \subset P$, we define the submodules

$$\mathrm{Diff}^\bullet_k(S \subset P^+) \doteq \{\Delta \in \mathrm{Diff}^\bullet_k(P^{(+)}) \mid \Delta(A) \subset S\} \underset{\mathcal{M}(A)}{\subset} \mathrm{Diff}^\bullet_k(P^{(+)}),$$

$$\mathrm{D}^\bullet_{(k)}(S \subset P^+) \doteq \{\Delta \in \mathrm{D}^\bullet_{(k)}(P^{(+)}) \mid \Delta(A) \subset S\} \underset{\mathcal{M}(A)}{\subset} \mathrm{D}^\bullet_{(k)}(P^{(+)}).$$

DEFINITION 1. Let $\sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathrm{N}^n_+$. The functor $\mathrm{D}_\sigma\colon \mathcal{M}(A) \to \mathcal{M}(A)$ is defined inductively by

$$\mathrm{D}_\sigma \doteq \mathrm{D}_{(\sigma_1)}, \quad n = 1,$$

$$\mathrm{D}_\sigma\colon P \longmapsto \mathrm{D}^\bullet_{(\sigma_1)}(\mathrm{D}_{(\sigma_2,\ldots,\sigma_n)}(P) \subset \mathrm{Diff}^+_{\sigma_2,\ldots,\sigma_n}(P)), \quad n > 1,$$

where to simplify notation we write (as in [6]) $\mathrm{Diff}^+_{\sigma_2,\ldots,\sigma_n}$ for $\mathrm{Diff}^+_{\sigma_2} \circ \cdots \circ \mathrm{Diff}^+_{\sigma_n}$.

Define

$$I_\sigma\colon \mathrm{D}_\sigma \longrightarrow \mathrm{D}^\bullet_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n}) \tag{2}$$

to be the natural inclusion and

$$\pi_\sigma\colon \mathrm{D}^\bullet_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n}) \longrightarrow \mathrm{D}_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}$$

to be the composition

$$\mathrm{D}^\bullet_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n}) \xrightarrow{I_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n})} \mathrm{D}^\bullet_{\sigma(n-2)}(\mathrm{Diff}^{(+)}_{\sigma_{n-1},\sigma_n}) \xrightarrow{c_{\sigma_{n-1},\sigma_n}} \mathrm{D}_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)},$$

where $c_{\sigma_{n-1},\sigma_n}$ is the 'gluing' morphism.

Then, for each $\sigma \in \mathrm{N}^n_+$ we have an exact sequence in $[\mathcal{M}(A), \mathcal{M}(A)]$:

$$0 \longrightarrow \mathrm{D}_{\sigma(n)} \xrightarrow{I_\sigma} \mathrm{D}^\bullet_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n}) \xrightarrow{\pi_\sigma} \mathrm{D}_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}. \tag{3}$$

DEFINITION 2. A full subcategory $\mathfrak{D} \subseteq \mathcal{M}(A)$ is *differentially closed* if

(i) $\mathfrak{D}$ is Abelian;

(ii) $\mathfrak{D}$ is closed under differential functors ($\mathrm{Diff}_n(P,\cdot), \mathrm{D}_\sigma, \mathrm{D}^\bullet_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n})$ etc., for any $\sigma \in \mathrm{N}^n_+$, any $n \in \mathrm{N}_+$ and any object P of $\mathfrak{D}$) and all these functors, when restricted to $\mathfrak{D}$, are strictly representable in $\mathfrak{D}$;

(iii) $A \in Ob(\mathfrak{D})$ and $\mathfrak{D}$ is closed under tensor product over A.

Remark 2. Condition (ii) is needed to have an ambient 'closed' with respect to functorial differential calculus; furthermore, as it will be clear in the following, since among the differential functors there are also compositions of 'elementary' ones (for example, $\mathrm{D}^\bullet_{(k)} \circ \mathrm{Diff}^+_l \equiv \mathrm{D}^\bullet_{(k)}(\mathrm{Diff}^+_l)$), we would like that representative objects of these nonelementary functors, if existing, could be expressed in terms of representative objects of the 'elementary' ones (for example, $\mathrm{D}_{(k)}$ and Diff_l). Condition (iii) makes it possible. This definition of differentially closed subcategory is the same as that in [6], except for the fact that we add (iii).

If $\mathfrak{D}$ is differentially closed then we denote by $\mathcal{J}^n_{\mathfrak{D}}(P)$, P being an object of $\mathfrak{D}$ (resp., by $\Lambda^\sigma_{\mathfrak{D}}$) the strict representative object of $\mathrm{Diff}_n(P,\cdot)_{|\mathfrak{D}}$ (resp., of $D_{\sigma(n)|\mathfrak{D}}$). $\mathcal{J}^n_{\mathfrak{D}}(P)$ is called the *n-jet module of* P in $\mathfrak{D}$ ([6]). Elements of $\Lambda^\sigma_{\mathfrak{D}}$ are called (*higher*) $\mathfrak{D}$*-differential forms of type* σ over the algebra A ([9, 14]). We also put formally $\Lambda^\emptyset_{\mathfrak{D}} \doteq A$.

The functorial exact sequence (3) in $[\mathfrak{D}, \mathfrak{D}]$, implies that $\mathcal{J}^{\sigma_n}_{\mathfrak{D}}(\Lambda^{\sigma(n-1)}_{\mathfrak{D}})$ (which is the strict representative object of $\mathrm{D}^\bullet_{\sigma(n-1)}(\mathrm{Diff}^+_{\sigma_n})_{|\mathfrak{D}}$, by condition (iii) above and Proposition 8 of [6]) fits into the exact sequence of representative objects

$$\Lambda^{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}_{\mathfrak{D}} \xrightarrow{\pi^\vee_\sigma} \mathcal{J}^{\sigma_n}_{\mathfrak{D}}(\Lambda^{\sigma(n-1)}_{\mathfrak{D}}) \xrightarrow{I^\vee_\sigma} \Lambda^\sigma_{\mathfrak{D}} \longrightarrow 0.$$

From it one gets the following inductive description of $\Lambda^\sigma_{\mathfrak{D}}$:

$$\Lambda^\sigma_{\mathfrak{D}} = \frac{\mathcal{J}^{\sigma_n}_{\mathfrak{D}}(\Lambda^{\sigma(n-1)}_{\mathfrak{D}})}{\pi^\vee_\sigma(\Lambda^{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}_{\mathfrak{D}})}. \tag{4}$$

$\mathfrak{D} = \mathcal{M}(A)$ is differentially closed ([17]).*

If $\mathfrak{D}$ is differentially closed and P is one of its objects, we denote by $j^{\mathfrak{D}}_k(P) \in \mathrm{Diff}_k(P, \mathcal{J}^k_{\mathfrak{D}}(P))$ the DO corresponding to the identity $\mathrm{id}_{\mathcal{J}^k_{\mathfrak{D}}(P)}$ under the canonical isomorphism

$$\hom_A(\mathcal{J}^k_{\mathfrak{D}}(P), \mathcal{J}^k_{\mathfrak{D}}(P)) \simeq \mathrm{Diff}_k(P, \mathcal{J}^k_{\mathfrak{D}}(P)).$$

Recall ([6]) that $\mathcal{J}^k_{\mathfrak{D}} \doteq \mathcal{J}^k_{\mathfrak{D}}(A)$ is a unitary commutative and associative K-algebra with multiplication

$$(a j^{\mathfrak{D}}_k(b)) \cdot (a' j^{\mathfrak{D}}_k(b')) = a a' j^{\mathfrak{D}}_k(b b'), \quad a, a', b, b' \in A.$$

Multiplication $(a, \theta) \mapsto j^{\mathfrak{D}}_k(a) \cdot \theta$ supplies it with the 'right' A-module structure. The A-module so obtained is denoted by $\mathcal{J}^k_{\mathfrak{D},+}$. This way the ordered couple $(\mathcal{J}^k_{\mathfrak{D}}, \mathcal{J}^k_{\mathfrak{D},+})$ becomes an A-bimodule.

EXAMPLE 1. (i) If $\mathfrak{D} = \mathcal{M}(A)$ and $\sigma = (1, \ldots, 1)$ (n times), then $\Lambda^\sigma_{\mathfrak{D}}$ coincides with $\Omega^n_{A/K}$, the nth module of Kähler–de Rham forms of A/K ([3] and [8]);

(ii) If $K = \mathbf{R}$ (the field of real numbers) and $A = C^\infty(M; \mathbf{R})$ (the algebra of real valued smooth functions on a smooth manifold M), then a $C^\infty(M;\mathbf{R})$-module P is called *geometric* if each element of P is uniquely defined by its

* As a rule, when $\mathfrak{D} = \mathcal{M}(A)$, we omit reference to $\mathfrak{D}$ in writing representative objects.

values on the points of M, i.e. if $\bigcap_{x\in M}\mu_x P=(0)$, μ_x being the maximal ideal of functions vanishing at x. Denote by $\mathfrak{D}=\mathcal{M}(A)_{\mathrm{geom}}$ the full subcategory of $\mathcal{M}(A)$ consisting of all geometric A-modules. Then $\mathfrak{D}$ is differentially closed and if $\sigma=(1,\dots,1)$, then $\Lambda^{\sigma}_{\mathfrak{D}}$ is the $C^\infty(M;\mathbf{R})$-module on nth order differential forms on M. Note, however, that if $\sigma\in\mathbf{N}^n_+$ is arbitrary, we may have $\Lambda^{\sigma}_{\mathfrak{D}}\neq(0)$ even if $n>\dim M$;

(iii) If A is Noetherian, then $\mathfrak{D}=\mathcal{M}(A)_N$, the subcategory of Noetherian A-modules is differentially closed ([1]).

A differentially closed category $\mathfrak{D}\subseteq\mathcal{M}(A)$ is called *smooth* if $\Lambda^{(1)}_{\mathfrak{D}}$ is a projective A-module of finite type. If A is the affine algebra of a regular affine algebraic variety over a characteristic zero field, then $\mathfrak{D}=\mathcal{M}(A)$ is smooth (see [5] or [8]). If A is the algebra of Example 1(ii), then $\mathfrak{D}=\mathcal{M}(A)_{\mathrm{geom}}$ is smooth.

Denote the set of all $\mathbf{N}_+$-valued sequences $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_n,\dots)$ by $\mathbf{N}^\infty_+$ and put $\sigma(n)\doteq(\sigma_1,\sigma_2,\dots,\sigma_n)$ for a $\sigma\in\mathbf{N}^\infty_+$ and any $n>0$. The A-homomorphism $I^\vee_{\sigma(n+1)}\colon\mathcal{J}^{\sigma_{n+1}}_{\mathfrak{D}}(\Lambda^{\sigma(n)}_{\mathfrak{D}})\to\Lambda^{\sigma(n+1)}_{\mathfrak{D}}$ dual to the functorial morphism (2)

$$I_{\sigma(n+1)}\colon \mathrm{D}_{\sigma(n+1)}\longrightarrow \mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}^{(+)}_{\sigma_{n+1}})$$

allows us to introduce the higher-order exterior (de Rham) differential

$$d^{\mathfrak{D}}_{\sigma(n+1)}\doteq I^\vee_{\sigma(n+1)}\circ j^{\mathfrak{D}}_{\sigma(n+1)}(\Lambda^{\sigma(n)}_{\mathfrak{D}})\colon \Lambda^{\sigma(n)}_{\mathfrak{D}}\longrightarrow\Lambda^{\sigma(n+1)}_{\mathfrak{D}}.$$

This way a de Rham-like complex of differential operators $\mathbf{dR}_\sigma(\mathfrak{D})$ can be associated with any $\sigma\in\mathbf{N}^\infty_+$:

$$\mathbf{dR}_\sigma(\mathfrak{D})\colon\ 0\longrightarrow A\xrightarrow{d^{\mathfrak{D}}_{\sigma(1)}}\Lambda^{\sigma(1)}_{\mathfrak{D}}\longrightarrow\cdots\longrightarrow\Lambda^{\sigma(n)}_{\mathfrak{D}}\xrightarrow{d^{\mathfrak{D}}_{\sigma(n+1)}}\Lambda^{\sigma(n+1)}_{\mathfrak{D}}\longrightarrow\cdots. \tag{5}$$

The 'higher' differential $d^{\mathfrak{D}}_{\sigma(n+1)}$ is a differential operator of order $\leqslant\sigma_{n+1}$ and $\mathbf{dR}_\sigma(\mathfrak{D})$ is called the *higher de Rham complex of type σ in $\mathfrak{D}$*.

If $\sigma=(1,1,\dots,1,\dots)$, then complex (5) coincides with the canonical 'algebraic' de Rham complex in the situation of Example 1(i) and with the standard 'differential-geometric' de Rham complex in that of Example 1(ii). We emphasize that the complexes $\mathbf{dR}_\sigma(\mathfrak{D})$, $\mathfrak{D}$ being the category of geometric $C^\infty(M;\mathbf{R})$-modules, are *natural* in the category of smooth manifolds.

In [11] (see also [12] or [9]) it is proved that if $\mathfrak{D}$ is smooth then all the complexes $\mathbf{dR}_\sigma(\mathfrak{D})$ are *quasi-isomorphic*, i.e. have the same cohomology. This is the case for example of a regular affine algebraic variety over a field of characteristic zero with $\mathfrak{D}=\mathcal{M}(A)$, A being the corresponding affine algebra, or of a differentiable manifold M of finite dimension with $\mathfrak{D}=\mathcal{M}(A)_{\mathrm{geom}}$, $A=C^\infty(M;\mathbf{R})$.

The following two equivalent descriptions of differential operators between representative objects will be useful later.

We work in a fixed differentially closed subcategory $\mathfrak{D}$ of $\mathcal{M}(A)$. All representative objects will be understood in $\mathfrak{D}$. Let F_1 and F_2 be representative objects of differential functors $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Suppose that $\mathcal{F}_1$ has an associated functor $\mathcal{F}^{\bullet}_1$ (having as domain the subcategory of $\mathcal{BIM}(A)$ whose objects are couples of objects in $\mathfrak{D}$) such that $\mathcal{F}^{\bullet}_1(\mathrm{Diff}^{(+)}_k)$ is strictly representable by $\mathcal{J}^k(F_1)$: this is the case, for example, of $\mathcal{F}_1=D_{\sigma(n)}$ or Diff_l. Let

$$\Delta\colon F_1\longrightarrow F_2 \tag{6}$$

be a DO of order $\leqslant k$. Then, there exists a unique A-homomorphism ([6]: jet-associated to Δ)

$$f_\Delta\colon \mathcal{J}^k(F_1)\longrightarrow F_2 \tag{7}$$

which represents Δ by duality: $\Delta=f_\Delta\circ j_k(F_1)$. Since $\mathcal{J}^k(F_1)$ is the representative object of $\mathcal{F}^{\bullet}_1(\mathrm{Diff}^{(+)}_k)$, f_Δ defines a unique morphism in $[\mathfrak{D},\mathfrak{D}]$:

$$f^\Delta\colon \mathcal{F}_2\longrightarrow\mathcal{F}^{\bullet}_1(\mathrm{Diff}^{(+)}_k), \tag{8}$$

called *generator* (functorial) *morphism* of Δ.

Formulas (7) and (8) give two different descriptions of a DO between representative objects. Formula (8) allows one to identify it with a functorial morphism which, as a rule, may be established in a straightforward way and can be used, then, to define the corresponding natural DO (6). The following examples show this procedure at work in two canonical cases (we assume for simplicity $\mathfrak{D}=\mathcal{M}(A)$):

(i) *Higher de Rham differential* $\mathrm{d}_{\sigma(n)}$.

If $\mathcal{F}_2=\mathrm{D}_{\sigma(n)}$, $\mathcal{F}_1=\mathrm{D}_{\sigma(n-1)}$, $k=\sigma_n$ and for (8) the natural inclusion

$$\mathrm{D}_{\sigma(n)}\hookrightarrow\mathrm{D}^{\bullet}_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n})$$

is taken, then $\mathrm{d}_{\sigma(n)}\colon\Lambda^{\sigma(n-1)}\to\Lambda^{\sigma(n)}$ is the corresponding DO (6).

(ii) *'Absolute' jet-operator* j_k.

In this almost tautological case,

$$\mathcal{F}_1=\mathrm{hom}_A(A,\cdot)\equiv\mathrm{Diff}_0\quad\text{and}\quad\mathcal{F}_2=\mathrm{Diff}_k\equiv\mathrm{hom}^{\bullet}_A(A,\cdot)(\mathrm{Diff}^{(+)}_k);$$

if the identity map

$$\mathrm{Id}\colon\ \mathrm{Diff}_k\longrightarrow\mathrm{hom}^{\bullet}_A(A,\cdot)(\mathrm{Diff}^{(+)}_k)\equiv\mathrm{Diff}_k$$

is taken for (8), then (6) becomes $j_k\colon A\to\mathcal{J}^k$.

The modules $\Lambda^{\sigma(n)}_{\mathfrak{D}}$ are generated over A by elements $d_{\sigma(n)}(a_1 d_{\sigma(n-1)}(a_2\dots d_{\sigma(1)}(a_n)\dots))$, $a_1,a_2,\dots,a_n\in A$ (reference to $\mathfrak{D}$ will be omitted, unless necessary).

EXAMPLE 2. If $A = K[x_1, \dots, x_n]$, K being a commutative ring, and $q > 0$, then $\Lambda^{(q)}_{\mathcal{M}(A)} \simeq I/I^{q+1}$ is a free A-module on the set of monomials ($\mathbf{n} \doteq \{1, \dots, n\} \subset \mathbf{N}$):

$$\Big\{[\hat{d}(x_{i_1})], [\hat{d}(x_{j_1}) \cdot \hat{d}(x_{j_2})], \dots, [\hat{d}(x_{r_1}) \cdots \hat{d}(x_{r_q})] \mid i_1, \dots, r_q \in \mathbf{n},$$
$$j_1 \leqslant j_2, \dots, r_1 \leqslant r_2 \leqslant \dots \leqslant r_q\Big\},$$

where

$$\hat{d}: A \longrightarrow I : a \longmapsto 1 \otimes a - a \otimes 1$$

and $[\xi]$ denotes the class modulo I^{q+1} of an element ξ of I. Moreover, by putting

$$\varepsilon_{i_1} \doteq [\hat{d}(x_{i_1})], \dots, \varepsilon_{r_1,\dots,r_q} \doteq [\hat{d}(x_{r_1}) \cdots \hat{d}(x_{r_q})],$$

we have for any $f \in A$:

$$d_{(q)}(f) = \sum \nabla_{i_1}(f)\varepsilon_{i_1} + \dots + \sum \nabla_{r_1,\dots,r_q}(f)\varepsilon_{r_1,\dots,r_q}, \tag{9}$$

where the elements $\nabla_{i_1}(f), \dots, \nabla_{r_1,\dots,r_q}(f) \in A$ are defined by the following identity:

$$f(x_1 + t_1, \dots, x_n + t_n) - f(x_1, \dots, x_n)$$
$$= \sum \nabla_{i_1}(f)t_{i_1} + \dots + \sum \nabla_{r_1,\dots,r_q}(f)t_{r_1} \cdots t_{r_q}$$

(for example, if $n = 1$, we have

$$\nabla_{\underbrace{1, \dots, 1}_{i \text{ times}}}(x^s) = \binom{s}{i} x^{s-i}, \quad \text{and} \quad \nabla_{\underbrace{1, \dots, 1}_{i \text{ times}}}$$

is a derivation of order $\leqslant i$). $\Lambda^{(q)}$ is also free over the set

$$\{d_{(q)}(x_{i_1}), d_{(q)}(x_{j_1}x_{j_2}), \dots, d_{(q)}(x_{r_1} \cdots x_{r_q}) \mid i_1, \dots, r_q \in \mathbf{n},$$
$$j_1 \leqslant j_2, \dots, r_1 \leqslant r_2 \leqslant \dots \leqslant r_q\}$$

and there is a more complicated formula analogous to (9).

If $\sigma, \tau \in \mathbf{N}^\infty_+$ with $\sigma \geqslant \tau$ (i.e. $\sigma_i \geqslant \tau_i$, $\forall i \geqslant 1$), then the sequence of monomorphisms $\mathrm{D}_{\tau(n)} \hookrightarrow \mathrm{D}_{\sigma(n)}$ in $[\mathfrak{D}, \mathfrak{D}]$ is defined easily due to the fact that obviously a DO of order $\leqslant k$ is also a DO of order $\leqslant k'$, $\forall k' \geqslant k$. This induces by duality a sequence of $\mathfrak{D}$-epimorphisms $\Lambda^{\sigma(n)}_{\mathfrak{D}} \to \Lambda^{\tau(n)}_{\mathfrak{D}}$, on representatives. All these epimorphisms commutes with higher de Rham differentials and, therefore, define a morphism of complexes

$$\mathbf{dR}_\sigma(\mathfrak{D}) \longrightarrow \mathbf{dR}_\tau(\mathfrak{D}) \tag{10}$$

(if $\sigma \geqslant \tau$). We may then consider the ($\mathfrak{D}$-epimorphic) inverse system $\{\mathbf{dR}_\sigma(\mathfrak{D})\}_{\sigma \in \mathbf{N}^\infty_+}$ and give the following:

DEFINITION 3. The infinitely prolonged (*or, simply,* infinite) de Rham complex of the K-algebra A in $\mathfrak{D}$, is defined as

$$\mathbf{dR}_\infty(\mathfrak{D}; A) \doteq \operatorname{inv} \lim_{\sigma \in \mathbf{N}^\infty_+} \mathbf{dR}_\sigma(\mathfrak{D}),$$
$$\mathbf{dR}_\infty(\mathfrak{D}; A)\colon\ 0 \longrightarrow A \xrightarrow{d_{(\infty)}} \Lambda^{(\infty)}_{\mathfrak{D}} \xrightarrow{d_{(\infty,\infty)}} \Lambda^{(\infty,\infty)_2}_{\mathfrak{D}} \longrightarrow \cdots$$
$$\longrightarrow \Lambda^{(\infty,\dots,\infty)_n}_{\mathfrak{D}} \longrightarrow \cdots, \tag{11}$$

where

$$\Lambda^{(\infty,\dots,\infty)_n}_{\mathfrak{D}} \doteq \operatorname{inv} \lim_{\sigma(n) \in \mathbf{N}^n_+} \Lambda^{\sigma(n)}_{\mathfrak{D}}, \quad \forall n > 0.$$

Now we generalize the *Lie derivative* to higher de Rham complexes.

In the sequel, $\mathfrak{D}$ will always denote a differentially closed subcategory of $\mathcal{M}(A)$.

Here on we will adopt the convention to write $\nabla(a_n)(a_{n-1}) \cdots (a_1)$ in place of the correct but cumbersome $(\cdots((\nabla(a_n))(a_{n-1}))\cdots)(a_1)$ for $\nabla \in \mathrm{Diff}^+_{\sigma_1,\dots,\sigma_n}(P)$, $a_1, a_2, \dots, a_n \in A$ and P an object of $\mathfrak{D}$.

DEFINITION 4. Let $X \in \mathrm{D}_{(1)}(A) \equiv \mathrm{D}(A)$, $\sigma \in \mathbf{N}^n_+$ and $n \geqslant 0$. The Lie derivative with respect to X

$$L_X\colon \Lambda^\sigma_{\mathfrak{D}} \longrightarrow \Lambda^\sigma_{\mathfrak{D}}$$

is defined to be the DO which corresponds to the functorial morphism (recall formula (8))

$$\varphi^{L_X}\colon \mathrm{D}_\sigma \longrightarrow \mathrm{D}^\bullet_\sigma(\mathrm{Diff}^{(+)}_1)$$

in $[\mathfrak{D}, \mathfrak{D}]$ given by

$$[\varphi^{L_X}(P)(\nabla)](a_n)(a_{n-1}) \cdots (a_1)(a_0)$$
$$\doteq a_0 \sum_{i=1}^{n} \nabla(a_n) \cdots (X(a_i)) \cdots (a_1) + X(a_0) \cdot \nabla(a_n)(a_{n-1}) \cdots (a_1),$$

where $\nabla \in \mathrm{D}_\sigma(P)$ and P is an object of $\mathfrak{D}$.

This is a DO of order $\leqslant 1$. Note that for $n = 0$ we have just $L_X = X$.

Let us note that L_X can be expressed in terms of generators as

$$L_X(a_0 d_{\sigma(n)}(a_1 d_{\sigma(n-1)}(a_2 \cdots d_{\sigma_1}(a_n) \cdots)))$$
$$= X(a_0) d_{\sigma(n)}(a_1 d_{\sigma(n-1)}(a_2 \cdots d_{\sigma_1}(a_n) \cdots)) +$$
$$+ \sum_{i=1}^{n} a_0 d_{\sigma(n)}(a_1 \cdots d_{\sigma(n-i+1)}(X(a_i) d_{\sigma(n-i)}(a_{i+1} \cdots d_{\sigma_1}(a_n) \cdots))). \tag{12}$$

The given definition is equivalent to the following inductive ones:

– *on functors:*

$$\varphi^{L_X}\colon \mathrm{D}_{(k)} \longrightarrow \mathrm{D}^{\bullet}_{(k)}(\mathrm{Diff}_l^{(+)})$$
$$[\varphi^{L_X}(P)](\nabla)(a_1)(a_0)$$
$$\doteq a_0(\nabla \circ X)(a_1) + X(a_0)\cdot\nabla(a_1),$$
$$\varphi^{L_X}\colon \mathrm{D}_\sigma \longrightarrow \mathrm{D}^{\bullet}_{\sigma}(\mathrm{Diff}_1^{(+)}),$$
$$[\varphi^{L_X}(P)(\nabla)](a_n)(a_{n-1})\cdots(a_1)(a_0)$$
$$\doteq a_0[[\varphi^{L_X}(P)\nabla(a_n)](a_{n-1})\cdots(a_1)(1) +$$
$$+(\nabla\circ X)(a_n)(a_{n-1})\cdots(a_1)] +$$
$$+X(a_0)\cdot\nabla(a_n)(a_{n-1})\cdots(a_1);$$

which implies on generators

$$L_X\colon \Lambda^\sigma \longrightarrow \Lambda^\sigma\colon a_0 d_{\sigma(n)}(\omega) \longmapsto X(a_0)d_{\sigma(n)}(\omega) + a_0 d_{\sigma(n)}(L_X(\omega)),$$
$$a_0\in A,\ \omega\in\Lambda^{\sigma(n-1)}.$$

– *on representatives:*

$$L_X \equiv X\colon A \longrightarrow A.$$

Supposing $L_X\colon \Lambda_{\mathfrak{D}}^{\sigma(r)} \to \Lambda_{\mathfrak{D}}^{\sigma(r)}$ to be already defined $\forall r < n$, it is possible to extend the operator L_X on $\mathcal{J}_{\mathfrak{D}}^{\sigma_n}(\Lambda_{\mathfrak{D}}^{\sigma(n-1)})$ by making use of the canonical isomorphism ([6] Prop. 8, or [7]):

$$\mathcal{J}_{\mathfrak{D}}^{\sigma_n}(\Lambda_{\mathfrak{D}}^{\sigma(n-1)}) \simeq \mathcal{J}_{\mathfrak{D}+}^{\sigma_n} \otimes_{\mathrm{in}} \Lambda_{\mathfrak{D}}^{\sigma(n-1)}$$

and putting

$$L_X\colon \mathcal{J}_{\mathfrak{D}+}^{\sigma_n} \otimes_{\mathrm{in}} \Lambda_{\mathfrak{D}}^{\sigma(n-1)} \longrightarrow \mathcal{J}_{\mathfrak{D}+}^{\sigma_n} \otimes_{\mathrm{in}} \Lambda_{\mathfrak{D}}^{\sigma(n-1)}$$
$$L_X(\xi\otimes\omega) \doteq L_X(\xi)\otimes\omega + \xi\otimes L_X(\omega),\quad \xi\in\mathcal{J}_{\mathfrak{D}+}^{\sigma_n},\ \omega\in\Lambda_{\mathfrak{D}}^{\sigma(n-1)}.$$

Then it is easy to verify that $L_X\circ\pi_\sigma^\vee = \pi_\sigma^\vee\circ L_X$ so that $\pi_\sigma^\vee(\Lambda_{\mathfrak{D}}^{(\sigma(n-2),\sigma_{n-1}+\sigma_n)})$ is L_X-stable. Remembering formula (4)

$$\Lambda_{\mathfrak{D}}^\sigma \equiv \Lambda_{\mathfrak{D}}^{\sigma(n)} \simeq \frac{\mathcal{J}_{\mathfrak{D}}^{\sigma_n}(\Lambda_{\mathfrak{D}}^{\sigma(n-1)})}{\pi_\sigma^\vee(\Lambda_{\mathfrak{D}}^{(\sigma(n-2),\sigma_{n-1}+\sigma_n)})},$$

one can therefore define $L_X\colon \Lambda_{\mathfrak{D}}^\sigma \to \Lambda_{\mathfrak{D}}^\sigma$ by passing to the quotient.

Remark 3. If $A = C^\infty(M;\mathbf{R})$ (M being a smooth manifold), $\mathfrak{D} = \mathcal{M}(A)_{\mathrm{geom.}}$ and $\sigma = (1,1,\ldots,1)$ the Lie derivative operator introduced above coincides with the classical Lie derivative of differential forms on M.

3. The Generalized Infinitesimal Stokes' Formula

In this section, we fix a differentially closed subcategory $\mathfrak{D}$ of $\mathcal{M}(A)$ and all representative objects will be taken in $\mathfrak{D}$. So we will omit the reference to it

in the notation. We call $\sigma\in\mathbf{N}_+^\infty$ *regular* if $\sigma_n \leqslant \sigma_{n+1}$, $\forall n>0$ as well as the complex $\mathbf{dR}_\sigma$ corresponding to σ. In this section we, first, define for regular $\mathbf{dR}_\sigma$'s the insertion operator (or 'interior product') associated to a 'vector field' $X\in\mathrm{D}_{(1)}(A)$ and then generalize infinitesimal Stokes' formula (or Cartan homotopy formula) to regular $\mathbf{dR}$-complexes. In what follows we still adopt the convention to write $\nabla(a_n)(a_{n-1})\cdots(a_1)$ instead of the correct but cumbersome $(\cdots((\nabla(a_n))(a_{n-1}))\cdots)(a_1)$ for $\nabla\in\mathrm{Diff}^+_{\sigma_1,\ldots,\sigma_n}(P)$, $a_1,a_2,\ldots,a_n\in A$ and P an object of $\mathfrak{D}$.

Associate to such a ∇ the operators ∇_r, $0\leqslant r\leqslant n$, defined by:

$$\nabla_r(a_n)\cdots(a_1)(a_0) \doteq \nabla(a_n)\cdots(a_r a_{r-1})\cdots(a_0),\quad 0<r\leqslant n$$
$$\nabla_0(a_n)\cdots(a_1)(a_0) \doteq a_0\cdot\nabla(a_n)\cdots(a_1),\quad a_0,\ldots,a_n\in A.$$

We have

$$[\delta_b(\nabla_r(a_n)\cdots(a_{r+1}))](a_r)\cdots(a_0)$$
$$=\nabla_r(a_n)\cdots(a_{r+1})(ba_r)(a_{r-1})\cdots(a_0),$$
$$-\nabla_r(a_n)\cdots(a_{r+1})(a_r)(ba_{r-1})\cdots(a_0)$$
$$=\nabla(a_n)\cdots(a_{r+1})(ba_r a_{r-1})\cdots(a_0) -$$
$$-\nabla(a_n)\cdots(a_{r+1})(a_r b a_{r-1})\cdots(a_0) = 0,$$

so ∇_r is a homomorphism with respect to a_r; moreover, $\forall k\geqslant 0$, we have

$$[\delta_{b_0,\ldots,b_k}(\nabla(a_n)\cdots(a_{n-j+1}))](a_{n-j})\cdots(a_0)$$
$$=[\delta_{b_0,\ldots,b_k}(\nabla(a_n)\cdots(a_{n-j+1}))](a_{n-j})\cdots(a_r a_{r-1})\cdots(a_0),$$

if $j\leqslant n-r-1$;

$$[\delta_{b_0,\ldots,b_k}(\nabla_r(a_n)\cdots(a_{n-j+1}))](a_{n-j})\cdots(a_0)$$
$$=[\delta_{b_0,\ldots,b_k}(\nabla(a_n)\cdots(a_{r+1}))](a_r a_{r-1})\cdots(a_0)$$

if $j=n-r+1$ and

$$[\delta_{b_0,\ldots,b_k}(\nabla_r(a_n)\cdots(a_{n-j+1}))](a_{n-j})\cdots(a_0)$$
$$=[\delta_{b_0,\ldots,b_k}(\nabla(a_n)\cdots(a_r a_{r-1})\cdots(a_{n-j+1}))](a_{n-j})\cdots(a_0) \tag{13}$$

if $j\geqslant n-r+2$. Therefore $\nabla_r\in\mathrm{Diff}^+_{\mu_1^{(r)},\mu_2^{(r)},\ldots,\mu_{n+1}^{(r)}}(P)$, where $\mu^{(r)}_{n-r+1}=0$ and

$$(\mu_1^{(r)},\ldots,\mu_{n-r}^{(r)},\mu_{n-r+2}^{(r)},\ldots,\mu_{n+1}^{(r)}) = \sigma(n).$$

The following definition supplies us with a useful tool to express the co-insertion morphism i^X.

DEFINITION 5. If $X\in\mathrm{D}_{(1)}(A)$, $l=0,1,\ldots,n$, $r=0,1,\ldots,l$ and $\nabla\in\mathrm{Diff}^+_{\sigma_1,\sigma_2,\ldots,\sigma_n}(P)$, we define $(\nabla,X)_{r,l}\in\mathrm{Diff}^+_{\nu_1^{(r,l)},\ldots,\nu_{n+1}^{(r,l)}}(P)$ as:

$$(\nabla,X)_{r,l}(a_n)\cdots(a_1)(a_0) \doteq ([\nabla(a_n)\cdots]_r\circ X)(a_l)\cdots(a_0),$$

$a_0,\ldots,a_n\in A.$

Remark 4. The positive integers $\nu_i^{(r,l)}$, $i = 1, \ldots, n+1$, can be described in terms of $\sigma(n)$ but we have no need of that in the sequel.

Now we pass to construct inductively the co-insertion morphism i^X.

Let P be an object of $\mathfrak{D}$, $\sigma \in \mathrm{N}_+^\infty$, $n > 0$, $\Delta \in \mathrm{D}_{\sigma(n)}(P)$, $p \in P$ and $a_0, \ldots, a_n \in A$. The induction starts from the definition of $i^X(P)\colon P \to \mathrm{D}_{(k)}(P)$, $k \geqslant 1$, as the composition of $i^X(P)\colon P \to \mathrm{D}(P)$ and the natural imbedding $\mathrm{D}(P) \hookrightarrow \mathrm{D}_{(k)}(P)$. The former of these morphisms is then given by

$$[i^X(P)(p)](a_0) \doteq X(a_0) \cdot p. \tag{14}$$

Assuming now that $i^X(P)$ is defined for all regular $\tau \in \mathrm{N}_+^{n-1}$, we define for a $\sigma \in \mathrm{N}_+^n$ the map

$$i^X(P)\colon \mathrm{D}_\sigma(P) \longrightarrow \mathrm{D}_{(\sigma_1,\ldots,\sigma_n,\sigma_n)}(P)$$

by posing

$$\begin{aligned}
&[i^X(P)(\Delta)](a_n)\cdots(a_1)(a_0)\\
&\doteq [i^X(P)(\Delta(a_n))](a_{n-1})\cdots(a_1)(a_0) +\\
&\quad + \left[\sum_{r=0}^{n}(-1)^r(\Delta_r \circ X)\right](a_n)\cdots(a_1)(a_0), \quad n \geqslant 1.
\end{aligned} \tag{15}$$

The following key assertion justifies the inductive procedure.

PROPOSITION 1. (i) *$P \mapsto i^X(P)$ is a well defined morphism in $[\mathfrak{D},\mathfrak{D}]$:*

$$i^X\colon \mathrm{D}_{\sigma(n)} \longrightarrow \mathrm{D}_{(\mu_1,\ldots,\mu_n,\mu_{n+1})},$$

where

$$\mu_{n+1} \doteq \sigma_n,\ \mu_n \doteq \max\{\sigma_n, \sigma_{n-1}\}, \ldots,\ \mu_2 \doteq \max\{\sigma_2, \sigma_1\},\ \mu_1 \doteq \sigma_1$$

(if $n = 1$, to avoid misunderstandings, we state explicitly that $i^X\colon \mathrm{D}_{(k)} \to \mathrm{D}_{(k,k)}$);

(ii) *the inductive definition* (15) *can be resolved by*

$$i^X(P)(\Delta) \doteq \sum_{l=0}^{n}\sum_{r=0}^{l}(-1)^r(\Delta, X)_{r,l}. \tag{16}$$

Proof. (ii) follows from the iteration of (15) taking into account Definition 5.

We prove (i) by induction on n, the length of σ. It is obvious for $n = 0$. So we suppose (i) to be true for $0 \leqslant k < n$ and pass to prove it for n. Below i^X stands for $i^X(P)$.

We begin by showing that $\forall a_n \in A$, $[i^X(\Delta)](a_n) \in \mathrm{D}_{(\mu_2,\ldots,\mu_{n+1})}(P)$. By (15):

$$[i^X(\Delta)](a_n) = i^X(\Delta(a_n)) + \sum_{r=0}^{n}(-1)^r\Delta_r(X(a_n)).$$

Since $\Delta(a_n) \in \mathrm{D}_{(\sigma_2,\ldots,\sigma_n)}(P)$,

$$i^X(\Delta(a_n)) \in \mathrm{D}_{(\sigma_2,\mu_3,\ldots,\mu_{n+1})}(P), \tag{17}$$

by the inductive hypothesis. Moreover $\nabla_r \in \mathrm{Diff}^+_{\mu_1^{(r)},\mu_2^{(r)},\ldots,\mu_{n+1}^{(r)}}(P)$ implies $\Delta_r \circ X \in \mathrm{Diff}^+_{\mu_1^{(r)}+1,\mu_2^{(r)},\ldots,\mu_{n+1}^{(r)}}(P)$, where

$$\mu_{n-r+1}^{(r)} = 0 \quad \text{and} \quad (\mu_1^{(r)}, \ldots, \mu_{n-r}^{(r)}, \mu_{n-r+2}^{(r)}, \ldots, \mu_{n+1}^{(r)}) = \sigma(n).$$

Hence, $\max_r\{\mu_i^{(r)}\} = \mu_i$, $\forall i = 0, \ldots, n$, so that the sum $[\sum_{r=0}^{n}(-1)^r(\Delta_r \circ X)]$ belongs to $\mathrm{Diff}^+_{\mu_1+1,\mu_2,\ldots,\mu_{n+1}}(P)$. Since $\Delta \in \mathrm{D}_{\sigma(n)}(P)$, we have for $a_j = 1$, $j = 1, \ldots, n-1$:

$$\begin{aligned}
&\left[\sum_{r=0}^{n}(-1)^r(\Delta_r \circ X)\right](a_n)\cdots(a_j = 1)\cdots(a_0)\\
&\quad = (-1)^j\Delta(X(a_n))(a_{n-1})\cdots(a_j)(a_{j-1})\cdots(a_0) +\\
&\quad\quad + (-1)^{j+1}\Delta(X(a_n))(a_{n-1})\cdots(a_j)(a_{j-1})\cdots(a_0)\\
&\quad = 0,
\end{aligned}$$

due to the fact that each term that appears in the sum with $r \neq j, j+1$ is zero. Similarly, we see that

$$\left[\sum_{r=0}^{n}(-1)^r(\Delta_r \circ X)\right](a_n)\cdots(a_1)(1) = 0,$$

so that

$$\left[\sum_{r=0}^{n}(-1)^r(\Delta_r \circ X)\right](a_n) \in \mathrm{D}_{(\mu_2,\ldots,\mu_{n+1})}(P). \tag{18}$$

Now it remains to note that $\mathrm{D}_{(\sigma_2,\mu_3,\ldots,\mu_{n+1})} \subset \mathrm{D}_{(\mu_2,\mu_3,\ldots,\mu_{n+1})}$ due to $\sigma_2 \leqslant \max\{\sigma_1, \sigma_2\} = \mu_2$.

To complete the proof it suffices to show that

(A) $i^X(\Delta) \in \mathrm{Diff}_{\sigma_1}(\mathrm{Diff}^+_{\mu_2,\ldots,\mu_{n+1}}(P))$;

(B) $[i^X(\Delta)](1) = 0$.

Assertion (B) follows from the fact that $X(1) = \Delta(1) = 0$. Hence, it remains to prove that $\delta_{b_0,\ldots,b_{\sigma_1}}[i^X(\Delta)] = 0$, $\forall b_0, \ldots, b_{\sigma_1} \in A$. A direct computation shows that for any $b \in A$ and any $\nabla \in \mathrm{Diff}^+_{\tau_1,\ldots,\tau_k}(P)$ the following equality holds

$$\begin{aligned}
\delta_b(i^X(\nabla)) &= i^X(\delta_b\nabla) + \sum_{r=0}^{n-1}(-1)^r[\delta_{X(b)}\nabla]_r +\\
&\quad + (-1)^n[\nabla_n \circ (\delta_b X) - [\delta_b\nabla]_n \circ X].
\end{aligned} \tag{19}$$

Now we are able to prove by induction on $s > 0$ that

$$\begin{aligned}
&\delta_{b_0,\dots,b_s}(i^X(\Delta))\\
&= i^X(\delta_{b_0,\dots,b_s}\Delta) + \sum_{r=0}^{n-1}(-1)^r \sum_{i=0}^{s}[\delta_{b_0,\dots,X(b_i),\dots b_s}\Delta]_r +\\
&\quad + (-1)^n[\Delta_n \circ (\delta_{b_0,\dots,b_s}X) - [\delta_{b_0,\dots,b_s}\Delta]_n \circ X].
\end{aligned} \tag{20}$$

So, assuming that (20) takes place for k, let us calculate $\delta_{b_0,\dots,b_{k+1}}(i^X(\Delta))$. By doing that we shall make use of the elementary formula $\delta_a(S \circ T) = (\delta_a S) \circ T + S \circ \delta_a T$ as well as the relation

$$\delta_a(\delta_b\Phi)_r = (\delta_{a,b}\Phi)_r, \quad \forall\Phi \in \mathrm{Diff}^+_{\tau_1,\dots,\tau_n}(P),$$

for any $(\tau_1, \dots, \tau_n) \in \mathrm{N}^n_+$ and $r \leqslant n-1$:

$$\begin{aligned}
&\delta_{b_0,\dots,b_{k+1}}(i^X(\Delta))\\
&= \delta_{b_{k+1}}[i^X(\delta_{b_0,\dots,b_k}\Delta)] + \sum_{r=0}^{n-1}(-1)^r \sum_{r=0}^{k}[\delta_{b_0,\dots,X(b_i),\dots,b_k,b_{k+1}}\Delta]_r +\\
&\quad + (-1)^n[(\delta_{b_{k+1}}(\Delta_n)) \circ (\delta_{b_0,\dots,b_k}X) + \Delta_n \circ (\delta_{b_0,\dots,b_{k+1}}X) -\\
&\quad - \delta_{b_{k+1}}(\delta_{b_0,\dots,b_k}\Delta)_n \circ X - (\delta_{b_0,\dots,b_k}\Delta)_n \circ \delta_{b_{k+1}}X].
\end{aligned}$$

Now applying (19) for $\nabla = \delta_{b_0,\dots,b_k}\Delta$ we have

$$\begin{aligned}
&\delta_{b_0,\dots,b_{k+1}}(i^X(\Delta))\\
&= i^X(\delta_{b_0,\dots,b_{k+1}}\Delta) + \sum_{r=0}^{n-1}(-1)^r[\delta_{b_0,\dots,b_k,X(b_{k+1})}\Delta]_r +\\
&\quad + (-1)^n[(\delta_{b_0,\dots,b_k}\Delta)_n \circ (\delta_{b_{k+1}}X) - [\delta_{b_0,\dots,b_{k+1}}\Delta]_n \circ X] +\\
&\quad + \sum_{r=0}^{n-1}(-1)^r \sum_{i=0}^{k}[\delta_{b_0,\dots,X(b_i),\dots,b_k,b_{k+1}}\Delta]_r +\\
&\quad + (-1)^n[(\delta_{b_{k+1}}(\Delta_n)) \circ (\delta_{b_0,\dots,b_k}X) + \Delta_n \circ (\delta_{b_0,\dots,b_{k+1}}X) -\\
&\quad - \delta_{b_{k+1}}(\delta_{b_0,\dots,b_k}\Delta)_n \circ X - (\delta_{b_0,\dots,b_k}\Delta)_n \circ \delta_{b_{k+1}}X]\\
&= i^X(\delta_{b_0,\dots,b_{k+1}}\Delta) + \sum_{r=0}^{n-1}(-1)^r[\delta_{b_0,\dots,b_k,X(b_{k+1})}\Delta]_r -\\
&\quad - (-1)^n[(\delta_{b_0,\dots,b_{k+1}}\Delta)_n \circ X] +\\
&\quad + \sum_{r=0}^{n-1}(-1)^r \sum_{i=0}^{k}[\delta_{b_0,\dots,X(b_i),\dots,b_k,b_{k+1}}\Delta]_r +\\
&\quad + (-1)^n[(\delta_{b_{k+1}}(\Delta_n)) \circ (\delta_{b_0,\dots,b_k}X) + \Delta_n \circ (\delta_{b_0,\dots,b_{k+1}}X) -\\
&\quad - \delta_{b_{k+1}}(\delta_{b_0,\dots,b_k}\Delta)_n \circ X].
\end{aligned}$$

This proves (20), remembering that ∇_n is a homomorphism in the first variable, $\forall\nabla \in \mathrm{Diff}_{\tau_1}(\mathrm{Diff}^+_{\tau_2,\dots,\tau_n})$, $\forall(\tau_1, \dots, \tau_n) \in \mathrm{N}^n_+$.

Formula (20) for $s = \sigma_1$ proves, obviously, (A).

From formula (16) it follows that $i^X(P)$ is an A-homomorphism. Its naturality with respect to P is evident. □

COROLLARY 2. *If $\sigma \in \mathrm{N}^\infty_+$ is regular, we may define the X-co-insertion (functorial) morphism:*

$$i^X\colon \mathrm{D}_{\sigma(n)} \longrightarrow \mathrm{D}_{\sigma(n+1)} \tag{21}$$

understood as the composition:

$$\mathrm{D}_{\sigma(n)} \longrightarrow \mathrm{D}_{(\sigma_1,\sigma_2,\dots,\sigma_n,\sigma_n)} \hookrightarrow \mathrm{D}_{\sigma(n+1)}.$$

Formula (16) together with Definition 5 implies the following explicit description of the co-insertion morphism:

$$\begin{aligned}
&[i^X(\Delta)](a_n)\cdots(a_1)(a_0)\\
&= X(a_0)\cdot\Delta(a_n)\cdots(a_1) + \sum_{l=1}^{n} a_0\Delta(a_n)\cdots(X(a_l))\cdots(a_1) +\\
&\quad + \sum_{l=1}^{n}(-1)^l\Delta(a_n)\cdots(a_{l+1})(X(a_l)a_{l-1})(a_{l-2})\cdots(a_0) +\\
&\quad + \sum_{l=2}^{n}\sum_{i=1}^{l-1}(-1)^i\Delta(a_n)\cdots(a_{l+1})(X(a_l))(a_{l-1})\cdots(a_ia_{i-1})(a_{i-2})\cdots(a_0).
\end{aligned} \tag{22}$$

Now we are ready to extend the definition of the standard insertion operation to regular higher-order differential forms:

DEFINITION 6. If $\sigma \in \mathrm{N}^\infty_+$ is regular and $X \in \mathrm{D}_{(1)}(A)$, we define the X-insertion homomorphism (or interior product by X):

$$i_X\colon \Lambda^{\sigma(n+1)} \longrightarrow \Lambda^{\sigma(n)}, \quad n \geqslant 0, \tag{23}$$

to be the dual-representative of i^X: $i_X \doteq (i^X)^\vee$.

Formula (22) leads now to the following explicit description of i_X in terms of generators:

$$\begin{aligned}
&i_X(d_{\sigma(n+1)}(a_0d_{\sigma(n)}(a_1d_{\sigma(n-1)}(a_2\cdots d_{\sigma(1)}(a_n)\cdots))))\\
&= X(a_0)\cdot d_{\sigma(n)}(a_1d_{\sigma(n-1)}(a_2\cdots d_{\sigma(1)}(a_n)\cdots)) +\\
&\quad + \sum_{l=1}^{n} a_0d_{\sigma(n)}(a_1\cdots d_{\sigma(n-l+1)}(X(a_l)\cdots d_{\sigma(1)}(a_n)\cdots)) +
\end{aligned}$$

$$+\sum_{l=1}^{n}(-1)^l d_{\sigma(n)}(a_0 d_{\sigma(n-1)}(a_1\cdots d_{\sigma(n-l+2)}(a_{l-2}d_{\sigma(n-l+1)}\times$$
$$\times(a_{l-1}X(a_l)\cdots d_{\sigma(1)}(a_n)\cdots)\cdots)))+$$
$$+\sum_{l=2}^{n}\sum_{i=1}^{l-1}(-1)^i d_{\sigma(n)}(a_0(\cdots d_{\sigma(n-i+2)}(a_{i-2}d_{\sigma(n-i+1)}\times$$
$$\times(a_i a_{i-1}\cdots d_{\sigma(n-l+1)}(X(a_l)\cdots d_{\sigma(1)}(a_n)\cdots))))). \tag{24}$$

To prove this formula it suffices to observe that the functorial isomorphism $\mathrm{D}_{\sigma(n)}\simeq\hom_A(\Lambda^{\sigma(n)},\cdot)$ is realized by means of the correspondence $\Delta\mapsto f^{\Delta}$ with

$$f^{\Delta}(a_0 d_{\sigma(n)}(a_1 d_{\sigma(n-1)}(a_2\cdots d_{\sigma(1)}(a_n)\cdots)))=a_0\Delta(a_n)\cdots(a_1).$$

If $A=C^\infty(M;\mathbf{R})$, M being a smooth manifold, $\mathfrak{D}=\mathcal{M}(A)_{\mathrm{geom.}}$ and $\sigma=(1,1,\ldots,1,\ldots)$, then i_X coincides with the standard differential geometric interior product with respect to the vector field X. In fact, formula (24) in this situation coincides with the corresponding one for the standard i_X.

If $\sigma,\tau\in\mathbf{N}_+^\infty$ are regular and $\sigma\geqslant\tau$ (i.e. $\sigma_i\geqslant\tau_i$, $\forall i\geqslant 1$), then $\forall n\geqslant 1$ the following diagram in $[\mathfrak{D},\mathfrak{D}]$ in which vertical arrows are natural inclusions is, obviously, commutative:

$$\begin{array}{ccc}\mathrm{D}_{\sigma(n)} & \xrightarrow{i^X} & \mathrm{D}_{\sigma(n+1)}\\ \uparrow & & \uparrow\\ \mathrm{D}_{\tau(n)} & \xrightarrow[i^X]{} & \mathrm{D}_{\tau(n+1)}\end{array}$$

By duality we get the commutative diagram of representative objects in $\mathfrak{D}$:

$$\begin{array}{ccc}\Lambda^{\sigma(n+1)} & \xrightarrow{i_X} & \Lambda^{\sigma(n)}\\ \downarrow & & \downarrow\\ \Lambda^{\tau(n+1)} & \xrightarrow[i_X]{} & \Lambda^{\tau(n)}\end{array}$$

with $\mathfrak{D}$-epimorphic vertical arrows.

Now we can extend the infinitesimal Stokes' formula to higher de Rham complexes:

THEOREM 3 (Generalized Infinitesimal Stokes' Formula). *If* $\mathbf{dR}_\sigma$ *is regular, then operators*

$$\Lambda^{\sigma(n-1)}\underset{i_X}{\overset{d_{\sigma(n)}}{\rightleftarrows}}\Lambda^{\sigma(n)}\underset{i_X}{\overset{d_{\sigma(n+1)}}{\rightleftarrows}}\Lambda^{\sigma(n+1)}$$

satisfy the formula

$$L_X=i_X\circ d_{\sigma(n+1)}+d_{\sigma(n)}\circ i_X. \tag{25}$$

Proof. Formula (24) allows us to compute the value of the right-hand side operator of (25) on a generator $a_0 d_{\sigma(n)}(a_1 d_{\sigma(n-1)}(a_2\cdots d_{\sigma(1)}(a_n)\cdots))$ of $\Lambda^{\sigma(n)}$ (we omit this straightforward but cumbersome computation) and to see that the result coincides with (12).

It is possible to prove (25) without resort to higher-differential forms directly. It suffices to show that

$$\varphi^{L_X}=\varphi^{d_{\sigma(n+1)}}\circ i^X+i^X\circ\varphi^{d_{\sigma(n)}} \tag{26}$$

by interpreting the ingredients in the following way:

$$\varphi^{L_X}\colon \mathrm{D}_{\sigma(n)}\longrightarrow \mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}_1^{(+)})\hookrightarrow \mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}^{(+)}_{\sigma_{n+1}}),$$

$$\varphi^{d_{\sigma(n+1)}}\circ i^X\colon \mathrm{D}_{\sigma(n)}\longrightarrow \mathrm{D}_{\sigma(n+1)}\longrightarrow \mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}^{(+)}_{\sigma_{n+1}}),$$

$$i^X\circ\varphi^{d_{\sigma(n)}}\colon \mathrm{D}_{\sigma(n)}\longrightarrow \mathrm{D}^{\bullet}_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n})\xrightarrow{i^{X\bullet}(\mathrm{Diff}^{+}_{\sigma_n})}\mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}^{(+)}_{\sigma_n})$$
$$\hookrightarrow \mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}^{(+)}_{\sigma_{n+1}}),$$

where the last inclusion is assured by regularity of σ and the morphism

$$i^{X\bullet}(\mathrm{Diff}^{+}_{\sigma_n})\colon \mathrm{D}^{\bullet}_{\sigma(n-1)}(\mathrm{Diff}^{+}_{\sigma_n})\longrightarrow \mathrm{D}^{\bullet}_{\sigma(n)}(\mathrm{Diff}^{+}_{\sigma_n})$$

is understood to be the morphism

$$i^X(\mathrm{Diff}^{+}_{\sigma_n})\colon \mathrm{D}_{\sigma(n-1)}(\mathrm{Diff}^{(+)}_{\sigma_n})\longrightarrow \mathrm{D}_{\sigma(n)}(\mathrm{Diff}^{(+)}_{\sigma_n})$$

in the left A-module structure. Formula (26) can be proved by a direct computation of its right-hand side by making use of (22). This computation, however, is essentially the same as that for (25). □

As a direct consequence of formula (25) we get the following corollary:

COROLLARY 4. *For any regular* $\sigma\in\mathbf{N}_+^n$, $L_X\circ d_\sigma=d_\sigma\circ L_X\colon \Lambda^{\sigma(n-1)}\to\Lambda^\sigma$.

Proof. Imbed σ in a regular $\bar\sigma\in\mathbf{N}_+^\infty$ and use formula (25) together with

$$d_{\bar\sigma(n+1)}\circ d_{\bar\sigma(n)}\equiv d_{\bar\sigma(n+1)}\circ d_\sigma=d_\sigma\circ d_{\sigma(n-1)}=0.$$ □

COROLLARY 5. *If* $A=C^\infty(M;\mathbf{R})$, *M being a differentiable manifold, and* $\mathfrak{D}$ *is the category of geometric A-modules, then for* σ *and* τ *regular the canonical epimorphisms (10) induce isomorphisms in cohomology. Therefore, all higher regular de Rham cohomologies coincide with the ordinary one.*

Proof. We prove this corollary by checking that the Eilenberg–Steenrod axioms for cohomological theories in the category of differentiable manifolds are satisfied by higher de Rham cohomologies.

Functoriality with respect to smooth maps between manifolds follows from the fact that any K-algebras morphism $F: A \to B$ induces morphisms $\Lambda^{\sigma(n)}_{A/K} \to \Lambda^{\sigma(n)}_{B/K}$ $\forall \sigma \in \mathrm{N}^{\infty}_{+}$, $\forall n \geqslant 0$, which commute with differentials, and that if F is the pull-back of a smooth map then the induced 'change-of-rings' functor $F^t: \mathcal{M}(B) \to \mathcal{M}(A)$ preserves the subcategories of geometric modules (see [7] Ch. I, §4).

If

$$M = \{\text{point}\}, \qquad \mathrm{D}_{\sigma(n)}(P) = (0), \quad \forall \sigma \in \mathrm{N}^{\infty}_{+}, \ \forall n > 0,$$

for any (geometric) A-module P: therefore the *Dimension Axiom* is verified.

Let $N \hookrightarrow M$ be a submanifold. The relative to N higher-differential forms are defined as those which vanishes on N. Since higher differentials commute with pullbacks, the relative-differential forms constitute a subcomplex of the 'absolute' one. Its cohomology is called relative cohomology (with respect to N) so that the relative cohomology groups satisfy the *Relative Cohomology Axiom* (also called *Exactness Axiom*).

Given the preceding definitions, the proof of the *Excision Axiom* is the same as in the case of ordinary de Rham cohomology.

Finally, verification of the *Homotopy Axiom* follows from the infinitesimal Stokes' formula (25) as in the case of ordinary de Rham cohomology: see, for example, [4], p. 178. □

In [12] and in a forthcoming article [11], it has been proved in a purely algebraic way the following more general result: if $\mathfrak{D}$ is a smooth differentially closed subcategory of $\mathcal{M}(A)$, then all the higher order de Rham complexes are quasi-isomorphic and in particular they are all quasi-isomorphic to the ordinary de Rham complex.

It is also worth mentioning that (25) is of great interest when dealing with the analog of the C-spectral sequence ([15]) built from the very beginning with higher or infinite de Rham complexes instead of the standard one (see [15], 12.2, p. 126).

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