The Chow ring of the classifying stack of PGL_2 is generated by Chern classes of the adjoint representation

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Abstract

Following the recent equivariant theory for Chow groups ([EG]), we prove the statement in the title. This is a new proof of a result already proved in [Pa]; there is some evidence that our method can be extended to the case of PGL_3 (at least over a field of characteristic $\neq 2,3$).

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In this paper the base field will be **C** but the results hold true over any field k of characteristic $\neq 2$. We freely use the functorial point of view for schemes and group schemes (e.g. [DG]) to be able to express maps, actions etc. as sending "elements to elements". Throughout we follow the notations of [Fu] and [EG].

We will prove the following result

Theorem 1 *The Chow ring with integer coefficients of the classifying stack of $PGL_{2,\mathbf{C}}$ is generated (as a ring) by $c_2(\mathrm{sl}_2)$ and $c_3(\mathrm{sl}_2)$.*

Remark 1 *Note that $c_1(\mathrm{sl}_2) = 0$ since $\det \mathrm{sl}_2$ is the trivial PGL_2-representation. Moreover, sl_2 is an autodual[1] PGL_2-representation so $2c_3(\mathrm{sl}_2) = 0$.*

R. Pandharipande [Pa] has proved that actually $A^*(BPGL_{2,\mathbf{C}}) \simeq \mathbf{Z}[c_2, c_3]/(2c_3)$; his proof makes use of the isomorphism $PGL_{2,\mathbf{C}} \simeq SO_{3,\mathbf{C}}$ therefore seems to be hardly generalizable to the computation of $A^*(BPGL_{3,\mathbf{C}})$. The present method of proof has some chances to carry over to that case.

Perhaps it is also worth mentioning that for any n, the classifying stack $BPGL_{n+1}$ is 1-isomorphic to the stack $\mathcal{BS}_n$ of Brauer-Severi schemes of type $\mathbf{P}^n$ (i.e. of twisted forms of $\mathbf{P}^n$ for the étale or fppf topology: [Mi], [DG] or [Sr]); therefore theorem1 gives generators for $A^*(\mathcal{BS}_1)$ too.

Proof. First of all, by self-intersection formula ([Fu], p.103) we have a ring isomorphism

$$A^*_{PGL_2}(\mathrm{sl}_2 \setminus \{0\}) \simeq \frac{A^*_{PGL_2}(\mathrm{sl}_2)}{c_{top}(\mathrm{sl}_2)} \simeq \frac{A^*(BPGL_2)}{c_3}.$$

We will use the stratification method to determine generators for $A^*_{PGL_2}(\mathrm{sl}_2 \setminus \{0\})$.

Let U be the open subset of $\mathrm{sl}_2 \setminus \{0\}$ consisting of matrices with distinct (hence opposite) eigenvalues and Z be the closed complement. There is a short exact sequence of graded groups

$$A^{PGL_2}_*(Z) \xrightarrow{\iota} A^{PGL_2}_*(\mathrm{sl}_2 \setminus \{0\}) \longrightarrow A^{PGL_2}_*(U) \to 0. \qquad (1)$$

Z is an orbit of PGL_2 (of Jordan canonical form $\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$ hence $A^*_{PGL_2}(Z) \simeq A^*(B(PGL_2)_Z)$ where $(PGL_2)_Z$ is "the" stabilizer of Z. $(PGL_2)_Z$ is unipotent so its classifying stack has trivial Chow ring

$$A^*_{PGL_2}(Z) \simeq [Z]_{PGL_2} \cdot \mathbf{Z} \quad (\text{concentrated in degree zero}).$$

We now claim that ι is the zero map. To prove this, we must show that $[Z]_{PGL_2}$ is zero in $A^*_{PGL_2}(\mathrm{sl}_2 \setminus \{0\})$. Consider the flat PGL_2- equivariant

[1] For any semisimple algebraic group the adjoint representation is self dual because of the nondegenerateness of the Killing form.

morphism $\det : \mathrm{sl}_2\backslash\{0\} \to \mathbf{A}^1$ ($\mathbf{A}^1$ as a trivial PGL_2-scheme); $\det^*([\{0\}]) = \left[\det^{-1}(\{0\})\right]_{PGL_2} = [Z]_{PGL_2}$ in $A^*_{PGL_2}(\mathrm{sl}_2\backslash\{0\})$. But $[\{0\}] = 0$ in $A^*_{PGL_2}(\mathbf{A}^1)$ (for example, because $\{0\}$ is the zero scheme of the invariant section x of the equivariantly trivial line bundle on $\mathbf{A}^1$). So $\iota = 0$.

(1) then tells us that $A^*_{PGL_2}(\mathrm{sl}_2\backslash\{0\}) \simeq A^*_{PGL_2}(U)$. So we are left to find generators of the ring $A^*_{PGL_2}(U)$.

Any $M \in U$ has stabilizer canonically isomorphic to the maximal torus

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha \in \mathbf{G}_m \right\} \simeq \mathbf{G}_m \subset PGL_2.$$

Considering T acting on the right of PGL_2 we have

$$U \simeq \left(Diag^*_{\mathrm{sl}_2} \times \frac{PGL_2}{T} \right) / S_2$$

where $Diag^*_{\mathrm{sl}_2}$ is the subset of sl_2 consisting of diagonal matrices with distinct eigenvalues (so $Diag^*_{\mathrm{sl}_2} \simeq \mathbf{A}^1\backslash\{0\}$) and S_2 left-acts on $Diag^*_{\mathrm{sl}_2}$ by permuting the diagonal entries and on $\frac{PGL_2}{T}$ as multiplication by $\underline{\sigma} \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the right ($\underline{\sigma}^2 = 1$). This isomorphism is PGL_2-equivariant with PGL_2 acting on the left of $\left(Diag^*_{\mathrm{sl}_2} \times \frac{PGL_2}{T} \right) / S_2$ as

$$g \cdot [diag(\lambda, \mu), [g']] \doteq [diag(\lambda, \mu), [gg']] .$$

Since S_2 acts freely on $Diag^*_{\mathrm{sl}_2} \times \frac{PGL_2}{T}$ and its action commutes with that of PGL_2, we have $A^*_{PGL_2}\left(\left(Diag^*_{\mathrm{sl}_2} \times \frac{PGL_2}{T} \right) / S_2 \right) \simeq A^*_{PGL_2 \times S_2}\left(Diag^*_{\mathrm{sl}_2} \times \frac{PGL_2}{T} \right)$. This is a general fact. Let G, H be algebraic groups having commuting actions on a scheme X and suppose G acts freely; if $(\widetilde{V}, \widetilde{U})$ is a "good couple" for (H, i) i.e. $\widetilde{U}$ is an open subset in the H-representation $\widetilde{V}$ with $\operatorname{co dim}(\widetilde{V}\backslash\widetilde{U}) > i$, then we have

$$\begin{aligned}
A^i_H(X / G) &\simeq && \text{(definition of equivariant Chow groups)} \\
&\simeq A^i\left(\left(\widetilde{U} \times \frac{X}{G} \right) / H \right) \simeq && \text{(the two actions commute)} \\
&\simeq A^i\left(\left(\widetilde{U} \times X \right) / G \times H \right) \simeq && (G \times H \text{ acts freely and [EG], Prop. 8)} \\
&\simeq A^i_{G\times H}\left(X \times \widetilde{U} \right).
\end{aligned}$$

By the equivariant version of the fundamental exact sequence ([Fu], Prop. 1.8) for Chow groups, we have

$$A^i_{G\times H}\left(X\times\widetilde{U}\right)\simeq A^i_{G\times H}\left(X\times\widetilde{V}\right)$$

for $i-co\dim(\widetilde{V}\backslash\widetilde{U})<0$; for any $G\times H$-representation E we have a ring isomorphism $A^*_{G\times H}(X)\simeq A^*_{G\times H}(X\times E)$, so we conclude that

$$A^i_H(X/G)\simeq A^i_{G\times H}(X).$$

Lemma 2 *Let $\Gamma_2\simeq S_2\ltimes T$ be the normalizer in PGL_2 of the maximal torus $T=\mathbf{G}_m$; Γ_2 left-acts on $Diag^*_{\mathrm{sl}_2}$ by $(\sigma,t)\cdot diag\,(\lambda,\mu)\doteq diag\,(\mu,\lambda)$ (i.e. T acts trivially). There is a canonical ring isomorphism $A^*_{PGL_2\times S_2}\left(Diag^*_{\mathrm{sl}_2}\times\frac{PGL_2}{T}\right)\simeq A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right)$.*

Proof. Let U_0 be a free open subset of a representation V_0 of $PGL_2\times S_2$ with complement of sufficiently high codimension. Since

$$1\to T\to\Gamma_2\to S_2\to 1$$

is exact, we have

$$\begin{aligned}\frac{Diag^*_{\mathrm{sl}_2}\times\frac{PGL_2}{T}\times U_0}{PGL_2\times S_2} &\simeq \left(\frac{Diag^*_{\mathrm{sl}_2}\times\frac{PGL_2}{T}\times U_0}{PGL_2}\right)\Big/ S_2\simeq\\ &\simeq \left(\frac{Diag^*_{\mathrm{sl}_2}\times U_0}{T}\right)\Big/ S_2\simeq\frac{Diag^*_{\mathrm{sl}_2}\times U_0}{\Gamma_2}\end{aligned}$$

where T acts on U_0 via $T\hookrightarrow PGL_2\hookrightarrow PGL_2\times S_2$ and Γ_2 via $\Gamma_2\hookrightarrow PGL_2\times S_2$. □

If $G=H\ltimes N$ and X is a G-scheme, then H acts on the stack $[X/N]$ hence on the Chow ring $A^*_N(X)$; this can also be seen concretely on approximations (for details see the forthcoming [V]).

In our case $G=\Gamma_2$ and we have $A^*_T\left(Diag^*_{\mathrm{sl}_2}\right)\simeq A^*(BT)$ since T acts trivially on $Diag^*_{\mathrm{sl}_2}$ which is open in $\mathbf{A}^1$.

Lemma 3 *There is a canonical ring isomorphism $A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right)\simeq A^*(BT)^{S_2}$.*

Assuming the lemma for a moment we can conclude the proof of Theorem 1. In fact S_2 acts on $A^*(BT) \simeq \mathbf{Z}[t]$ as $\sigma \cdot t = -t$ and (as it will be shown in the proof of the lemma) $c_2 \mapsto -t^2$ via the isomorphism[2]

$$A^*_{PGL_2}(U) \simeq A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right) \simeq A^*(BT)^{S_2};$$

therefore c_2 is a generator of the ring $A^*_{PGL_2}(U) \simeq A^*_{PGL_2}(\mathrm{sl}_2 \setminus \{0\})$ as we wanted.

Proof. (of Lemma 3)
The (split) exact sequence

$$1 \to T \xrightarrow{\iota} \Gamma_2 \xrightarrow{\pi} S_2 \to 1$$

induces pull-back morphisms

$$A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right) \xrightarrow{f} A^*_T\left(Diag^*_{\mathrm{sl}_2}\right)^{S_2} \hookrightarrow A^*_T\left(Diag^*_{\mathrm{sl}_2}\right)$$
$$A^*(B\Gamma_2) \xrightarrow{h} A^*(BT)^{S_2} \hookrightarrow A^*(BT)$$
$$A^*(BS_2) \xrightarrow{l} A^*(B\Gamma_2).$$

Consider the commutative diagram

$$\begin{array}{ccccc} & & A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right) & \xrightarrow{f} & A^*_T\left(Diag^*_{\mathrm{sl}_2}\right)^{S_2} \\ & & g\uparrow & & \uparrow_{iso} \\ & & A^*(B\Gamma_2) & \xrightarrow{h} & A^*(BT)^{S_2} \\ & & l\uparrow & & \\ A^+(BS_2) & \overset{i}{\hookrightarrow} & A^*(BS_2) & & \end{array} \qquad (2)$$

where g is induced via pull-back by the (Γ_2-equivariant) structural morphism $Diag^*_{\mathrm{sl}_2} \to Spec\mathbf{C}$. Note that g is surjective because $Diag^*_{\mathrm{sl}_2}$ is open in $\mathbf{A}^1$. Since the diagram.

$$\begin{array}{ccc} A^*_{S_2}\left(Diag^*_{\mathrm{sl}_2}\right) & \longrightarrow & A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right) \\ \uparrow & & g\uparrow \\ A^+_{S_2}\left(Diag^*_{\mathrm{sl}_2}\right) & & A^*(B\Gamma_2) \\ \uparrow & & l\uparrow \\ A^+(BS_2) & \overset{i}{\hookrightarrow} & A^*(BS_2) \end{array}$$

[2] We often denote by the same letter a Chern class and any of its pull-backs. In this case c_2 denotes both the second Chern class of sl_2 in $A^*(BPGL_2)$ and its pull-back to $A^*_{PGL_2}(U)$.

(where $A^*_{S_2}\left(Diag^*_{\mathrm{sl}_2}\right) \to A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right)$ is induced by $\Gamma_2 \twoheadrightarrow S_2$ and $A^+_{S_2}\left(Diag^*_{\mathrm{sl}_2}\right) \to A^+(BS_2)$ by $Diag^*_{\mathrm{sl}_2} \to Spec\mathbf{C}$) is commutative and $Diag^*_{\mathrm{sl}_2} \,/\, S_2$ is isomorphic to an open subset of $\mathbf{A}^1_{\mathbf{C}}$ (by the Symmetric Functions' Theorem), we have that $g \circ l$ is zero in positive degrees. Moreover f is obviously an isomorphism in degree zero because $Diag^*_{\mathrm{sl}_2}$ is irreducible. An easy diagram chasing on (2) shows then that f is an isomorphism if:

(a) h is surjective and

(b) $(\ker h)^+$ is generated by the image of $l \circ i$.

To prove (a) consider the Γ_2-representation $V = \mathbf{C}^2$ with $(\sigma, t) \cdot (x, y) \doteq (ty, t^{-1}x)$ and denote by λ_1, λ_2 its Chern classes in $A^*(B\Gamma_2)$. If $\iota : T \hookrightarrow \Gamma_2$, the induced T-representation $V_{(\iota)}$ has weights $(1, -1)$ so $c_2\left(V_{(\iota)}\right) = -t^2$ in $A^*(BT) \simeq \mathbf{Z}[t]$. Therefore $h(\lambda_2) = -t^2$ and since S_2 acts on $A^*(BT)$ as $\sigma \cdot t = -t$, h is surjective. So (a) is proved.

If we show that:

(b_1) λ_1 "comes" from $A^*(BS_2)$ and

(b_2) λ_1, λ_2 generate $A^*(B\Gamma_2)$ as a ring,

then (b) follows.

(b_1) is easy. Take $W = \mathbf{C}^2$ the S_2-representation with $\sigma \cdot (x, y) \doteq (y, x)$. If $\pi : \Gamma_2 \twoheadrightarrow S_2$ we have an induced Γ_2-representation $W_{(\pi)}$ and by definition $l(c_1(W)) = c_1\left(W_{(\pi)}\right)$. But

$$c_1\left(W_{(\pi)}\right) = c_1\left(\det W_{(\pi)}\right) = c_1(\det V) = c_1(V) \equiv \lambda_1$$

since $\det W_{(\pi)} \simeq \det V$ as Γ_2-representations. (b_1) is proved.

(b_2) is less straightforward. Consider the induced action of Γ_2 on $\mathbf{P}(V) \simeq \mathbf{P}^1$; by the equivariant projective bundle theorem ([EG] 3.3)

$$A^*_{\Gamma_2}(\mathbf{P}(V)) \simeq A^*(B\Gamma_2)[\ell] \,/\, \left(\ell^2 + \lambda_1 \ell + \lambda_2\right)$$

where $\ell = c_1(\mathcal{O}(1)) \in A^*_{\Gamma_2}(\mathbf{P}(V))$. To obtain ($b_2$) it is then enough to show that $A^*_{\Gamma_2}(\mathbf{P}(V))$ is generated by ℓ, λ_1 and λ_2.

There are two Γ_2-orbits in $\mathbf{P}(V)$

$$U' \doteq \{[x, y] \mid x \neq 0,\ y \neq 0\} \underset{open}{\overset{i}{\hookrightarrow}} \mathbf{P}(V)$$

$$Z' \doteq \{[0, 1] \equiv 0,\ [1, 0] \equiv \infty\} \underset{closed}{\overset{j}{\hookrightarrow}} \mathbf{P}(V)$$

with stabilizers $\Gamma_{2,U'} \simeq S_2 \times \mu_2$ (= stabilizer of $[1,1]$), $\Gamma_{2,Z'} \simeq T$. Therefore we have an exact sequence of graded groups

$$A_*^{\Gamma_2}(Z') \xrightarrow{j_*} A_*^{\Gamma_2}(\mathbf{P}(V)) \xrightarrow{i^*} A_*^{\Gamma_2}(U) \to 0 \tag{3}$$

and ring isomorphisms

$$A_{\Gamma_2}^*(Z') \simeq A^*(BT) \simeq \mathbf{Z}[t], \quad A_{\Gamma_2}^*(U') \simeq A^*(B(S_2 \times \mu_2)).$$

By [To] p.19, we have a Künneth isomorphism

$$A^*(B(S_2 \times \mu_2)) \simeq A^*(B\mu_2) \otimes_{\mathbf{Z}} A^*(BS_2) \simeq \frac{Z[\alpha]}{(2\alpha)} \otimes_{\mathbf{Z}} \frac{Z[\beta]}{(2\beta)}$$

where $\alpha = c_1(E)$ (resp. $\beta = c_1(W)$) with $E = \mathbf{C}$ with μ_2-action $\epsilon \cdot x = \epsilon x$ (resp. with W as above).

Since the following diagram is commutative ($\psi : \mathbf{P}(V) \to Spec\mathbf{C}$)

$$\begin{array}{ccc} A_{\Gamma_2}^*(\mathbf{P}(V)) & \xrightarrow{i^*} & A_{\Gamma_2}^*(U') \simeq A^*(B\mu_2) \otimes_{\mathbf{Z}} A^*(BS_2) \\ \psi^* \uparrow & & \uparrow 1 \otimes id \\ A^*(B\Gamma_2) & \xleftarrow{l} & A^*(BS_2) \end{array}$$

and we saw that $l(c_1(W)) = c_1(V)$, we have $i^*(\lambda_1) = \beta$. Moreover $i^*(\mathcal{O}(1))$ is isomorphic to the pull-back via $U' \to Spec\mathbf{C}$ of the Γ_2-representation E (S_2 acting trivially) viewed as a Γ_2-equivariant vector bundle on $Spec\mathbf{C}$. Then $i^*(\ell) = \alpha$.

Now let's find generators for the ideal[3] $im(j_*) \subset A_{\Gamma_2}^*(\mathbf{P}(V))$. By projection formula, the ideal $im(j_*)$ is generated by the image via j_* of any set of generators of $A_*^{\Gamma_2}(Z')$ as an $A_{\Gamma_2}^*(\mathbf{P}(V))$-module. Commutativity of the (Γ_2-equivariant) diagram

$$\begin{array}{ccc} Z' & \xrightarrow{j} & \mathbf{P}(V) \\ \varphi \searrow & & \swarrow \psi \\ & Spec\mathbf{C} & \end{array}$$

shows that $j^* \circ \psi^* = \varphi^*$; so $\varphi^*(\theta) \cdot \xi = \xi \cdot_j \psi^*(\theta)$, $\theta \in A^*(B\Gamma_2)$, $\xi \in A_*^{\Gamma_2}(Z')$, where the product in l.h.s. is the ring product in $A_{\Gamma_2}^*(Z')$ while on the r.h.s.

[3] $im(j_*)$ is an ideal by projection formula.

the product is in the $A^*_{\Gamma_2}(\mathbf{P}(V))$-module $A^{\Gamma_2}_*(Z')$. It is easy to check that $\varphi^*(\lambda_2) = -t^2$, therefore $\{1,t\}$ generates $A^{\Gamma_2}_*(Z')$ as an $A^*_{\Gamma_2}(\mathbf{P}(V))$-module and the ideal $im(j_*)$ is generated by $j_*(1)$ and $j_*(t)$. Let's compute these two push-forwards.

We have[4]

$$A^*_{\Gamma_2}(Z' = \{0,\infty\}) \ni 1 = [Z(xy)]$$

where $xy \in \Gamma(\mathbf{P}(V), \mathcal{O}(2))$ is Γ_2-invariant and regular; hence ([Fu] Example 3.2.16, p.61) $j_*(1) = c_1(\mathcal{O}(2)) = 2\ell$.

To compute $j_*(t)$ consider the following, general, transfer construction for Chow groups[5]. Let

$$1 \to H \xrightarrow{\phi} G \longrightarrow F \to 1$$

be an exact sequence of algebraic groups over a field k with F finite. If X is an algebraic smooth G-scheme then $p_1 : X \times F \longrightarrow X$ is proper G-equivariant and there is an equivariant push-forward

$$p_{1*} : A^G_*(X \times F) \to A^G_*(X).$$

If U is an open subset in a G-representation, with complement of sufficiently high codimension then we have

$$\begin{aligned} \frac{(X \times F) \times U}{G} &\simeq \left(\frac{(X \times F) \times U}{H}\right) / F \simeq \\ &\simeq \left(\frac{X \times U}{H} \times F\right) / F \simeq \frac{X \times U}{H} \end{aligned} \tag{4}$$

hence $A^*_G(X \times F) \simeq A^*_H(X)$ and p_{1*} induces a transfer morphism

$$tsf_X : A^H_*(X) \to A^G_*(X).$$

Remark 2 *Note that if $E \to X$ is a G-equivariant vector bundle and $E_{(\phi)}$ the induced H-equivariant vector bundle, $E_{(\phi)U} = \left(E_{(\phi)} \times U\right)/H$ corresponds to $(p_1^* E)_U = (p_1^* E \times U)/G$ via the isomorphism (4).*

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[4] $Z(s)$ denotes the zero-scheme of a section s.

[5] I learned it from Angelo Vistoli; it is completely similar to the well-known construction for ordinary group cohomology.

If $f : X \to Y$ is a flat G-equivariant morphism then there is a commutative diagram

$$\begin{array}{ccc} A_G^*(Y) & \xrightarrow{f^*} & A_G^*(X) \\ tsf_Y \uparrow & & \uparrow tsf_X \\ A_H^*(Y) & \xrightarrow{f^*} & A_H^*(X) \end{array} \tag{5}$$

(but transfers are not ring morphisms).

Let us return to our proof. The relevance of transfer to us is in that $j_*(t) = tsf_{\mathbf{P}(V)}(t \cdot [\{0\}])$, where $tsf_{\mathbf{P}(V)} : A_*^T(\mathbf{P}(V)) \to A_*^{\Gamma_2}(\mathbf{P}(V))$. In fact, by commutativity of the Γ_2-equivariant diagram

$$\begin{array}{ccc} Z' = \{0, \infty\} & \overset{\delta}{\hookrightarrow} & \mathbf{P}(V) \times S_2 \\ j \searrow & & \swarrow p_1 \\ & \mathbf{P}(V) & \end{array}$$

where $\delta(0) = (0, 1)$ and $\delta(\infty) = (\infty, \sigma)$, we have $j_* = p_{1*} \circ \delta_*$. But

$$\delta_*(1 = [\{0, \infty\}]) = [\{0\}] \in A_{\Gamma_2}^*(\mathbf{P}(V) \times S_2),$$

and projection formula then yields $j_*(t) = tsf_{\mathbf{P}(V)}(t \cdot [\{0\}])$ (note that $\{0\}$ is a T-invariant subscheme of $\mathbf{P}(V)$).

Now, $\{0\} = Z(x)$ but $x \in \Gamma(\mathbf{P}(V), \mathcal{O}(1))$ is not a T-invariant section[6]; however it is semi-invariant ([SGA3] Exposé VI_B, p. 406) with character $\chi : T = \mathbf{G}_m \to \mathbf{G}_m : s \mapsto s$. Therefore if $L_{\chi^{-1}}$ denote the 1-dimensional T-representation induced by the character χ^{-1} and $p : \mathbf{P}(V) \to Spec\mathbf{C}$,

$$\hat{x} \doteq x \otimes 1 \in \Gamma(\mathbf{P}(V), \mathcal{O}(1) \otimes p^* L_{\chi^{-1}})$$

is a T-invariant regular section (note that $p^* L_{\chi^{-1}}$ is trivial though not equivariantly trivial) and obviously $Z\left(\hat{x}\right) = Z(x)$. Since $c_1(L_{\chi^{-1}}) = -c_1(L_\chi) = -t \in A^*(BT) \simeq \mathbf{Z}[t]$, we get

$$[\{0\}] = \left[Z\left(\hat{x}\right)\right] = c_1(\mathcal{O}(1) \otimes p^* L_{\chi^{-1}}) = c_1(\mathcal{O}(1)) - c_1(L_\chi) = \ell - t$$

[6]The coaction (see [GIT]) of $\Gamma(T, \mathcal{O}_T) = \mathbf{C}[s, s^{-1}]$ on $\Gamma(\mathbf{P}(V), \mathcal{O}(1))$ is in fact given by $x \mapsto x \otimes s$, $y \mapsto y \otimes s^{-1}$.

in $A_*^T(\mathbf{P}(V)) \simeq \mathbf{Z}[t][\ell] \,/\, (\ell^2 - t^2)$[7]. So

$$j_*(t) = tsf_{\mathbf{P}(V)}(t \cdot [\{0\}]) = tsf_{\mathbf{P}(V)}\left(t\ell - t^2\right). \tag{6}$$

We now use property (5); hence we preliminarily compute the images of $1, t, t^2$ via

$$tsf_{pt} : A_T^*(pt = Spec\mathbf{C}) \equiv A^*(BT) \simeq \mathbf{Z}[t] \to A_{\Gamma_2}^*(pt) \equiv A^*(B\Gamma_2).$$

By projection formula

$$tsf_{pt}(1 = [pt]) = p_{1*}([S_2]) = p_{1*}(p_1^*(1)) = 2$$

since p_1 is finite of degree 2. To compute $tsf_{pt}(t)$ we note that the inclusion $S_2 \overset{\upsilon}{\hookrightarrow} \Gamma_2$ induces a pull-back $\upsilon^* : A^*(B\Gamma_2) \to A^*(BS_2)$ which is an isomorphism in degree 1. In fact, by [EG] Th. 1, we have natural isomorphisms

$$\begin{aligned} A^1(B\Gamma_2) &\simeq Pic^{\Gamma_2}(pt) = \frac{\{1\text{-dim.l representations of } \Gamma_2\}}{\text{iso}}, \\ A^1(BS_2) &\simeq Pic^{S_2}(pt) = \frac{\{1\text{- dim.l representations of } S_2\}}{\text{iso}} \end{aligned}$$

and υ^* is defined at the level of representations by $[L] \mapsto \left[L_{(\upsilon)}\right]$. Since $\pi \circ \upsilon = id_{S_2}$, υ^* is surjective in all degrees and it is enough to prove its injectivity in degree 1: this follows immediately from the elementary observation that any representation $\Gamma_2 \to GL_1$ factors through $\pi : \Gamma_2 \to S_2$.

Now, if U is an open subset in a Γ_2-representation, with complement of sufficiently high codimension, we have a cartesian diagram

$$\begin{array}{ccc} U \simeq \frac{U \times S_2}{S_2} & \overset{\psi'}{\longrightarrow} & \frac{U \times S_2}{\Gamma_2} \simeq \frac{U}{T} \\ \varphi' \downarrow & \square & \varphi \downarrow \\ \frac{U}{S_2} & \overset{\psi}{\longrightarrow} & \frac{U}{\Gamma_2} \end{array}$$

where φ is proper and ψ is flat. Hence ([Fu], Prop.1.7)

$$\psi^* \circ \varphi_*(t) = \psi^* \circ tsf_{pt}(t) = \varphi'_* \circ \psi'^*(t)$$

[7] We denote by the same symbol ℓ the Chern class of $\mathcal{O}(1)$ both in $A_{\Gamma_2}^*(\mathbf{P}(V))$ and in $A_T^*(\mathbf{P}(V))$ since they correspond under pull-back.

but $\psi'^{*}(t) \in A^1(U) = 0$ so $\psi^* \circ tsf_{pt}(t) \equiv v^* \circ tsf_{pt}(t) = 0$ and $tsf_{pt}(t) = 0$, v^* being an isomorphism in degree 1.

$tsf_{pt}(t^2)$ can be computed using Remark 2 and projection formula

$$tsf_{pt}\left(t^2 = -c_2\left(V_{(\iota)}\right)\right) = -p_{1*}\left(c_2\left(p_1^* V\right)\right) = -2\lambda_2.$$

Now we can come back to formula (6) and use property (5) to get

$$j_*(t) = tsf_{\mathbf{P}(V)}(t \cdot [\{0\}]) = tsf_{\mathbf{P}(V)}\left(t\ell - t^2\right) = tsf_{\mathbf{P}(V)}(t\ell) + 2\lambda_2.$$

Again by Remark 2 and projection formula we get

$$tsf_{\mathbf{P}(V)}(t\ell) = c_1(\mathcal{O}(1)) \cdot tsf_{\mathbf{P}(V)}(t)$$

which is zero by property (5) because $tsf_{pt}(t) = 0$. Therefore we conclude that $j_*(t) = 2\lambda_2$.

Referring back to (3) we summarize our computations as

$$\begin{array}{rcl} j_*(1) & = & 2\ell, \ j_*(t) = 2\lambda_2 \\ i^*(\ell) & = & \alpha, \ i^*(\lambda_1) = \beta. \end{array}$$

By (3) this in particular implies that $A^*_{\Gamma_2}(\mathbf{P}(V))$ is generated by ℓ, λ_1 and λ_2 i.e. that (b_2) holds. So the proof of lemma 3 is complete.

We conclude by showing, as promised, that $c_2 \mapsto -t^2$ via the isomorphism

$$A^*_{PGL_2}(U) \simeq A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right) \simeq A^*(BT)^{S_2}.$$

To begin with, we observe that the following diagram is commutative

$$\begin{array}{ccccc} A^*_{PGL_2}(\mathrm{sl}_2 \backslash \{0\}) & \longrightarrow & A^*_{PGL_2}(U) & \rightleftarrows & A^*_{\Gamma_2}\left(Diag^*_{\mathrm{sl}_2}\right) \\ \uparrow & & & & \downarrow f \\ A^*(BPGL_2) & \underset{u}{\longrightarrow} & A^*(B\Gamma_2) & \underset{h}{\longrightarrow} & A^*(BT)^{S_2} \end{array}$$

and that the bottom row composition $h \circ u$ is induced via pull-back by the inclusion $\varkappa : T \hookrightarrow PGL_2$. Choosing the usual basis

$$\left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

of sl_2, an easy computation shows that $(\mathrm{sl}_2)_{(\varkappa)} \simeq \mathbf{C}_{(0)} \oplus \mathbf{C}_{(1)} \oplus \mathbf{C}_{(-1)}$ as T-representations (where $\mathbf{C}_{(w)}$ denotes the 1-dimensional representation of weight w). Hence $h \circ u(c_2) = c_2\left((\mathrm{sl}_2)_{(\varkappa)}\right) = -t^2$ as stated. □ ■

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