Infinitesimal and square-zero extensions of simplicial algebras

Notes for students

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October 2013

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Introduction

These notes were written to meet the requests of some students who pointed out that the exposition of the role of the cotangent complex in the Postnikov towers for simplicial commutative algebras in [HAG-II] was too terse and needed some kind of unzipping.
We took also the opportunity to enlarge a little bit the context, by introducing square-zero extensions and their relation with infinitesimal extensions (i.e. those coming from derivations). The idea is that infinitesimal extensions are captured by the cotangent complex, that square-zero extensions are special infinitesimal extensions, and that the Postnikov tower of a simplicial

commutative algebra is built out of square-zero extensions. We conclude the notes with two applications: we give connectivity estimates for the cotangent complex and we show how obstructions can be seen as deformations over simplicial rings.
All the material is well-known to experts but details might be useful to people meeting these topics for the first time. A similar path, in a broader and less elementary context, might be found in [HA, §8.3, §8.4].

Acknowledgments. It is a pleasure for both of us to thank all the participants to the Séminaire "Autour de la Géométrie Algébrique Dérivée 2013-2014" that took place at Paris 7 (Sophie Germain): Pieter Belmans, Brice Le Grignou, Valerio Melani, Marco Robalo, Yohann Ségalat, Pietro Vertechi. And to acknowledge the work done during the seminar (partially recorded at http://www.math.jussieu.fr/∼vezzosi/seminar/), and all the interesting questions raised, some of which were the motivation for the present notes.

Notational remarks. We denote by $\mathbf{sAlg}_k$ the model category of simplicial commutative k-algebras. All tensor products, unless differently stated, are implicitly derived.

1 Infinitesimal extensions

Infinitesimal extensions are defined by derived derivations:

Definition 1.1. Let $A \to B$ be a cofibrant A-algebra, M be a simplicial B-module and $\overline{d} \in \pi_0(\mathrm{Map}_{A/\mathbf{sAlg}_k/B}(B, B \oplus M[1]))$ be a derived derivation from B to $M[1]$, represented by a map $d\colon B \to B \oplus M[1]$ in $A/\mathbf{sAlg}_k/B$. If we denote by $\varphi_d : \mathbb{L}_{B/A} \to M[1]$ the map of B-modules corresponding to d, the *infinitesimal extension* $\psi_d : B \oplus_d M \to B$ *of* B *by* M *along* d is the map in $\mathrm{Ho}(A/\mathbf{sAlg}_k/B)$ defined by the following homotopy cartesian diagram in $A/\mathbf{sAlg}_k$

$$\begin{tikzcd} B \oplus_d M \arrow[r] \arrow[d, "\psi_d"'] & B \arrow[d, "0"] \\ B \arrow[r, "d"'] & B \oplus M[1] \end{tikzcd}$$

where 0 denotes the section corresponding to the trivial derived derivation $0\colon \mathbb{L}_{B/A} \to M[1]$.

The appearance of M (instead of any shift of it) in the notation $B \oplus_d M$ calls for an explanation.

Proposition 1.2. *If* $\psi_d : B \oplus_d M \to B$ *is an infinitesimal extension of* B *by* M *along* d*, then the homotopy fiber of* ψ_d *at* 0 *is isomorphic to* M *in* $\mathrm{Ho}(B\text{-}\mathbf{Mod})$.

Proof. Proposition A.5 shows that

$$\mathrm{hofib}\, \psi_d \simeq \mathrm{hofib}(0\colon B \to B \oplus M[1])$$

(where the fibres are taken over 0). In order to explicitly compute hofib(0) we observe that the square

$$\begin{tikzcd} B \arrow[r, "0"] \arrow[d] & B \oplus M[1] \arrow[d, "p"] \\ 0 \arrow[r] & M[1] \end{tikzcd}$$

is homotopy cartesian: in fact, p is a fibration (being surjective) and every object is fibrant, so that the statement follows from Corollary A.3 and from the fact that the previous diagram is obviously a strict pullback. As consequence, the outer rectangle in

$$\begin{tikzcd}
\operatorname{hofib} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
B \arrow[r, "0"] \arrow[d] & B \oplus M[1] \arrow[d] \\
0 \arrow[r] & M[1]
\end{tikzcd}$$

is a homotopy pullback, so that

$$\operatorname{hofib} \psi_d \simeq \operatorname{hofib} 0 \simeq \Omega(M[1])$$

Now, $M[1] = M \otimes_A A[S^1]$ is the suspension of M and $\Omega\Sigma(M) \simeq M$ by Corollary B.9. $\square$

2 Square-zero extensions

Definition 2.1. Let $n \geq 0$. Let $A \in \mathbf{sAlg}_k$, B_1 a cofibrant A-algebra, and $I \subseteq \pi_n(B_1)$ a sub-$\pi_0(B_1)$-module. A morphism of cofibrant A-algebras $\varphi\colon B_1 \to B_0$ in $A/\mathbf{sAlg}_k$ is a *A-square-zero extension by $I[n]$* if the following conditions are met

1. B_0 and B_1 are n-truncated;
2. φ is an $(n-1)$-equivalence of A-algebras;
3. For any n-truncated A-algebra E, the following diagram is homotopy cartesian

$$\begin{tikzcd}
\operatorname{Map}_{A/\mathbf{sAlg}_k}(B_0, E) \arrow[r, "\operatorname{Map}(\varphi, E)"] \arrow[d] & \operatorname{Map}_{A/\mathbf{sAlg}_k}(B_1, E) \arrow[d] \\
{[B_1, E]_{0,I}} \arrow[r] & {[B_1, E]}
\end{tikzcd}$$

   where $[B_1, E]$ denotes the set of homotopy classes of maps $B_1 \to E$, and $[B_1, E]_{0,I}$ the subset of $[B_1, E]$ consisting of those $[f]$ such that $\pi_n(f)$ is zero on I;

4. The canonical map $\pi_n(B_1) \to \pi_n(B_0)$ is surjective with kernel I, i.e. $\pi_n(B_1)/I \simeq \pi_n(B_0)$;
5. if $n = 0$ then $I^2 = 0$ (classical case).

Remark 2.2. 1. Equivalently, we can define $[B_1, E]_{0,I}$ as the following (homotopy) pullback in $\mathbf{sSet}$:

$$\begin{tikzcd}
{[B_1, E]_{0,I}} \arrow[r] \arrow[d] & {[B_1, E]} \arrow[d] \\
\operatorname{Hom}_{\pi_0(A)\text{-}\mathbf{Mod}}(\pi_n(B_1)/I, \pi_n(E)) \arrow[r, "\mathrm{can}"'] & \operatorname{Hom}_{\pi_0(A)\text{-}\mathbf{Mod}}(\pi_n(B_1), \pi_n(E))
\end{tikzcd}$$

In fact, inspection reveals that the above diagram is a strict pullback. It is a homotopy pullback because every object there is discrete, hence fibrant and, as consequence, the maps are fibrations.

2. For $n = 0$, and $A = k$ we get back the classical definition of square-zero extension of commutative k-algebras.

3. If $B_1 \to B_0$ is an $A-$square-zero extension by $I[n]$, then I is canonically a $\pi_0(B_0)$-module. This follows from $\pi_0(B_0) \simeq \pi_0(B_1)$, if $n > 0$, and is classical if $n = 0$ (since $I^2 = 0$).

Lemma 2.3. *If $\varphi\colon B_1 \to B_0$ is a square-zero extension by $I[n]$ in $A/\mathbf{sAlg}_k$, then* $\mathrm{hofib}\,\varphi$ *is a $K(I, n)$-space.*

Proof. We have by definition a fibre sequence

$$\mathrm{hofib}\,\varphi \to B_1 \xrightarrow{\varphi} B_0$$

in A-**Mod** (and therefore a fibre sequence of pointed simplicial sets). The long exact sequence of homotopy groups shows then that

$$\pi_m(\mathrm{hofib}\,\varphi) = 0$$

if $m > n$ or $m < n-1$. Moreover, for $m = n-1$ we have

$$\pi_n(B_1) \twoheadrightarrow \pi_n(B_0) \to \pi_{n-1}(\mathrm{hofib}\,\varphi) \to \pi_{n-1}(B_1) \xrightarrow{\sim} \pi_{n-1}(B_0)$$

so that $\pi_{n-1}(\mathrm{hofib}\,\varphi) = 0$. Finally, we have a short exact sequence

$$0 \to \pi_n(\mathrm{hofib}\,\varphi) \to \pi_n(B_1) \to \pi_n(B_0) \to 0$$

so that axiom 4. readily implies that

$$\pi_n(\mathrm{hofib}\,\varphi) \simeq I$$

completing the proof. $\square$

Proposition 2.4. *Let $n \geq 0$, $A \in \mathbf{sAlg}_k$, $\varphi : B_1 \to B_0$ and $\varphi' : B_1 \to B_0'$ in $A/\mathbf{sAlg}_k$ two A-square-zero extensions by $I[n]$ ($I \subseteq \pi_n(B_1)$ a fixed sub-$\pi_0(B_1)$-module). Then there is an isomorphism $B_0 \simeq B_0'$ in* $\mathrm{Ho}(B_1/\mathbf{sAlg}_k)$.

Proof. The mapping space axiom 3. tells us that the simplicial sets $\mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0, E)$ and $\mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0', E)$ are isomorphic in $\mathrm{Ho}(\mathbf{sSet})$, for any n-truncated $E \in A/\mathbf{sAlg}_k$. In particular, by taking $E = B_0'$, we get a map $u\colon B_0 \to B_0'$. Denote as $(A/\mathbf{sAlg}_k)_{\leq n}$ the left Bousfield localization of $A/\mathbf{sAlg}_k$ with respect to the single map $S := S^{n+1} \otimes A[T] \to A[T]$, and denote the left Quillen adjoint by $\tau_{\leq n} : A/\mathbf{sAlg}_k \to (A/\mathbf{sAlg}_k)_{\leq n}$. The fibrant objects in $(A/\mathbf{sAlg}_k)_{\leq n}$ are the S-local objects, i.e. n-truncated simplicial A-algebras. The homotopy category of $(A/\mathbf{sAlg}_k)_{\leq n}$ is identified as the full subcategory of $\mathrm{Ho}(A/\mathbf{sAlg}_k)$ consisting of n-truncated objects. Now, the mapping space axiom (3) implies that, for any S-local object $E \in \mathbf{sAlg}_k$, the map

$$u^* : \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0, E) \to \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0', E)$$

is an isomorphism in $\mathrm{Ho}(\mathbf{sSet})$, i.e. ([Hi], Prop. 3.5.3) $u\colon B_0 \to B_0'$ is an S-local equivalence. But both B_0 and B_0' are S-local objects (being n-truncated), so we conclude that in fact u is a weak equivalence in $A/\mathbf{sAlg}_k$ (an S-local equivalence between S-local objects is a weak equivalence: S-local Whitehead Theorem [Hi, Thm. 3.2.13]). How do we climb up to an equivalence of $B_1/\mathbf{sAlg}_k$? Simply observe that the isomorphism $\mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0, E) \simeq \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0', E)$ (in

Ho(**sSet**)), from which we deduced the map u, in fact commutes (up to homotopy) with the maps

$$\mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0, E) \xrightarrow{\mathrm{Map}(\varphi,E)} \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_1, E)$$

$$\mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0', E) \xrightarrow{\mathrm{Map}(\varphi',E)} \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_1, E)$$

Therefore, we may choose $u\colon B_0 \to B_0'$ as a map in $\mathrm{Ho}(B_1/\mathbf{sAlg}_k)$. □

Let $B_1 \to B_0$ be a square-zero extension by $I[n]$. We saw in Lemma 2.3 and in Proposition 2.4 that the sub-$\pi_0(B_0)$-module controls every information about the extension; in particular, the homotopy fiber is determined and there are no two different square-zero extensions associated to the same sub-$\pi_0(B_0)$-module. We are going now to show that every sub-$\pi_0(B_0)$-module determines a square-zero extension:

Proposition 2.5. *Let $n \geq 0$. Given a cofibrant and n-truncated $B_1 \in A/\mathbf{sAlg}_k$, and a sub-$\pi_0(B_1)$-module $I \subseteq \pi_n(B_1)$ (such that $I^2 = 0$ if $n = 0$), there exists a square zero extension $B_1 \to B_0$ by $I[n]$. Moreover any other such extension $B_1 \to B_0'$ is isomorphic to $B_1 \to B_0$ in $\mathrm{Ho}(B_1/\mathbf{sAlg}_k)$.*

Proof. The uniqueness statement is Proposition 2.4, so that we are left to prove the existence. The idea of the proof is to construct B_0 as "B_1/I" (i.e. to kill I inside B_1) and then to take the n-truncation as an A-algebra. To begin with, let us consider I as an A-module (via $A \to \pi_0(A) \to \pi_0(B_1)$); the category A-**Mod** being monoidal model we have a canonical identification

$$\mathrm{Hom}_{\mathrm{Ho}(A\text{-}\mathbf{Mod})}(I, \pi_n(B_1)) \simeq \mathrm{Hom}_{\mathrm{Ho}(A\text{-}\mathbf{Mod})}(I[n], B_1)$$

so that the inclusion $I \subseteq \pi_n(B_1)$ induces a map of A-modules $I[n] \to B_1$ (because in A-**Mod** every object is fibrant, hence maps in the homotopy category can be represented in A-**Mod**). At this point, we obtain by adjuntion an induced map of A-algebras

$$\mathrm{Sym}_A(I[n]) \to B_1$$

Define a new object $\widetilde{B_0}$ via the following pushout square in $A/\mathbf{sAlg}_k$:

$$\begin{tikzcd}
\mathrm{Sym}_A(I[n]) \arrow[r, "0"] \arrow[d] & A \arrow[d] \\
B_1 \arrow[r] & \widetilde{B_0}
\end{tikzcd}$$

where the map 0 is induced by the zero map of A-modules $I[n] \to A$. Finally, introduce $B_0 := \tau_{\leq n}\widetilde{B_0}$. B_0 comes equipped with a canonical map

$$\varphi\colon B_1 \to \widetilde{B_0} \to \tau_{\leq n}\widetilde{B_0} = B_0$$

We claim that φ is the square-zero extension by $I[n]$ we were looking for. Let us check that the conditions of Definition 2.1:

1. B_1 is n-truncated by hypothesis, while B_0 is n-truncated by construction;

2. in order to show that φ is an $(n-1)$-equivalence of A-algebras, and that the canonical map $\pi_n(B_0) \to \pi_n(B_1)$ induces an isomorphism $\pi_n(B_1)/I \simeq \pi_n(B_0)$, we use the spectral sequence of [Q, Theorem II.6.b]. Set first of all $R_* := \pi_*(\mathrm{Sym}_A(I[n]))$, so that the spectral sequence reads off as:

$$E^2_{pq} = \mathrm{Tor}_p^{R_*}(\pi_* B_1, \pi_* A)_q \Rightarrow \pi_{p+q}(\widetilde{B_0})$$

Let $C_{\bullet *} \to \pi_* B_1$ be a flat resolution of $\pi_* B_1$ as a graded$_*$ R_*-module, so that

$$\mathrm{Tor}_p^{R_*}(\pi_* B_1, \pi_* A)_q = H^p((C_{\bullet *} \otimes_{R_*} \pi_* A)_q)$$

Let us compute the degree q part of $C_{\bullet *} \otimes_{R_*} \pi_* A$:

$$(C_{\bullet *} \otimes_{R_*} \pi_* A)_q = \{x_{ij} \otimes y_k \mid x_{ij} \in C_{ij}, y_k \in \pi_k A, \, j+k=q\}$$

for $q \leq n$.

- If $q < n$, then $k < n$ and there are elements $\widetilde{y}_k \in R_k \simeq \pi_k A$ mapping to y_k, so that

  $$\{x_{ij} \otimes y_k \mid x_{ij} \in C_{ij}, \, y_k \in \pi_k A, \, j+k=q\} = \{\widetilde{y}_k x_{ij} \otimes 1 \mid x_{ij} \in C_{ij}, \, \tilde{y_k} \in R_k, \, j+k=q\}$$

  and therefore

  $$(C_{\bullet *} \otimes_{R_*} \pi_* A)_q \simeq C_{\bullet q}, \quad \text{for } 0 \leq q < n$$

- If $q = n$, for $j > 0$ (hence $k < n$) we still have

  $$x_{ij} \otimes y_k = \tilde{y_k} x_{i,0} \otimes 1$$

  while for $j = 0$, since $R_n \simeq \pi_n(A) \oplus I$, we get instead

  $$x_{i0} \otimes y_n = (y_n, 0) \cdot x_{i,0} \otimes 1 = (y_n, \xi) \cdot x_{i0} \otimes 1$$

  for any $\xi \in I$ (and $x_{i0} \in C_{i0}$, $y_n \in \pi_n A$). Therefore

  $$(C_{\bullet *} \otimes_{R_*} \pi_* A)_n \simeq C_{\bullet n}/I \cdot C_{\bullet 0},$$

  where $I \cdot C_{\bullet 0} := \{(0,\xi) \cdot x_{\bullet 0} \,|\, \xi \in I \subset R_n, \, x_{\bullet 0} \in C_{\bullet 0}\}$.

Therefore

$$\mathrm{Tor}_p^{R_*}(\pi_* B_1, \pi_* A)_q = \begin{cases} H^p(C_{\bullet q}) = \delta_{p0} \cdot \pi_q B_1 & \text{if } 0 \leq q < n \\ H^p(C_{\bullet n}/I \cdot C_{\bullet 0}) & \text{if } q = n \end{cases}$$

Let us compute $H^0(C_{\bullet n}/I \cdot C_{\bullet 0})$. Introduce first of all the ideal

$$J := I \oplus \bigoplus_{q > n} R_q$$

so that, given any graded R_*-module M_* we have

$$M_n / I \cdot M_0 \simeq (M_*/J \cdot M_*)_n \simeq (M_* \otimes_{R_*} R_*/J)_n$$

and now observe that we are given an exact sequence

$$C_{1,*} \to C_{0,*} \to \pi_*(B_1) \to 0$$

Tensoring with R_*/J preserves the right exactness, and taking the degree n part is obviously an exact functor, so that we obtain an exact sequence

$$C_{1,n}/I \cdot C_{1,0} \to C_{0,n}/I \cdot C_{0,0} \to \pi_n(C)/I \to 0$$

which readily implies that

$$H^0(C_{\bullet n}/I \cdot C_{\bullet 0}) \simeq \pi_n(B_1)/I$$

so the E^2 page of our homological spectral sequence is first quadrant and drawing it we obtain:

$$\begin{tikzcd}
q=n & \pi_n(B_1)/I \arrow[drr] & \bullet & \bullet & \bullet & \bullet & \bullet \\
q=n-1 & \pi_{n-1}(B_1)/I \arrow[drr] & 0 & 0 & 0 & 0 & 0 \\
q=n-2 & \pi_{n-2}(B_1)/I \arrow[drr] & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & {} & & & \\
\vdots & & & & & & \\
q=0 & \pi_0(B_1)/I & 0 & 0 & 0 & 0 & 0 \\
 & p=0 & p=1 & p=2 & p=3 & p=4 & \dots
\end{tikzcd}$$

Thus $E^\infty_{pq} = E^2_{pq}$ for $0 \leq p+q \leq n$, so that

$$\pi_i(\widetilde{B_1}) = \begin{cases} \pi_i(B_1) & \text{if } 0 \leq i < n \\ \pi_n(B_1)/I & \text{if } i = n \end{cases}$$

3. For any A-algebra E, the following diagram consists of homotopy cartesian squares

$$\begin{tikzcd}
\mathrm{Map}_{A/\mathbf{sAlg}_k}(\widetilde{B_0}, E) \arrow[r, "\mathrm{Map}(\varphi,E)"] \arrow[d] & \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_1, E) \arrow[d] \\
\mathrm{Map}_{A/\mathbf{sAlg}_k}(A, E) \arrow[r] \arrow[d] & \mathrm{Map}_{A/\mathbf{sAlg}_k}(\mathrm{Sym}_A(I[n]), E) \arrow[d] \\
{[A,E] \simeq [\tau_{\leq n}A \oplus I[n], E]_{0,\ \mathrm{I[n]}}} \arrow[r] & {[\tau_{\leq n}A \oplus I[n], E] \simeq [A \oplus I[n], E]}
\end{tikzcd}$$

(for the top square we use the homotopy pushout definition of $\widetilde{B_0}$ and the fact that $\mathrm{Map}_{A/\mathbf{sAlg}_k}(\widetilde{B_0}, E) \simeq \mathrm{Map}_{A/\mathbf{sAlg}_k}(B_0 := \tau_{\leq n}\widetilde{B_0}, E)$ since E is n-truncated; for the bottom square we use that $\tau_{\leq n}(\mathrm{Sym}_A(I[n])) \simeq \tau_{\leq n}(A \oplus I[n]) \simeq \tau_{\leq n}(A) \oplus I[n]$, and again the hypothesis that E is n-truncated). To conclude it just remains to remark that the diagram of sets

$$\begin{tikzcd} {[B_1, E]_{0,I}} \arrow[r] \arrow[d] & {[B_1, E]} \arrow[d] \\ {[A, E]} \arrow[r] & {[A \oplus I[n], E]} \end{tikzcd}$$

is cartesian.

□

The following result will be useful later

Lemma 2.6. *Let $n \geq 0$, and $\varphi : B_1 \to B_0$ in $A/\mathbf{sAlg}_k$ a square-zero extension by $I[n]$ ($I \subseteq \pi_n(B_1)$ a fixed sub-$\pi_0(B_1)$-module). Let $\widetilde{B_1}$ be defined by the following pushout square in $A/\mathbf{sAlg}_k$*

$$\begin{tikzcd} \mathrm{Sym}_A(I[n]) \arrow[r, "0"] \arrow[d] & A \arrow[d] \\ B_1 \arrow[r] & \widetilde{B_0} \end{tikzcd}$$

where the map 0 is induced by the zero map of ($\pi_0 A$ hence) A-modules $I[n] \to A$. Then

- *there is a canonical isomorphism $B_0 \simeq \tau_{\leq n}\widetilde{B_0}$ in $\mathrm{Ho}(B_1/\mathbf{sAlg}_k)$;*
- *there is a canonical isomorphism $\widetilde{B_0} \otimes_{B_1} B_0 \simeq \mathrm{Sym}_{B_0} I[n+1]$ in $\mathrm{Ho}(B_0/\mathbf{sAlg}_k)$.*

Proof. The proof of the first assert is part of the proof of Proposition 2.5. Let us prove the second part of the statement. We have the following ladder of homotopy pushouts in $\mathrm{Ho}(A/\mathbf{sAlg}_k)$:

$$\begin{tikzcd} \mathrm{Sym}_A(I[n]) \arrow[r, "0"] \arrow[d] & A \arrow[d] \\ B_1 \arrow[r] \arrow[d] & \widetilde{B_0} \arrow[dd] \\ \widetilde{B_0} \arrow[d] & \\ B_0 \arrow[r] & \widetilde{B_0} \otimes_{B_1} B_0 \end{tikzcd}$$

Now, by the upper homotopy cocartesian square, the composite $\mathrm{Sym}_A(I[n]) \to B_1 \to \widetilde{B_0}$ is isomorphic (in $\mathrm{Ho}(A/\mathbf{sAlg}_k)$) to the composite $\mathrm{Sym}_A(I[n]) \xrightarrow{0} A \longrightarrow \widetilde{B_0}$, so that the following square

$$\begin{tikzcd} \mathrm{Sym}_A(I[n]) \arrow[r, "0"] \arrow[d, "0"'] & A \arrow[dd] \\ A \arrow[d] & \\ B_0 \arrow[r] & \widetilde{B_0} \otimes_{B_1} B_0 \end{tikzcd}$$

is homotopy cocartesian as well. Therefore, if C is defined by the homotopy pushout

$$\begin{tikzcd} \mathrm{Sym}_A(I[n]) \arrow[r, "0"] \arrow[d, "0"'] & A \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

there is an induced homotopy pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B_0 \arrow[r] & \widetilde{B_0} \otimes_{B_1} B_0 \end{tikzcd}$$

Let us compute C. In order to do this, we observe that $\mathrm{Sym}_A : A\text{-}\mathbf{Mod} \to A/\mathbf{sAlg}_k$ is left Quillen, hence it commutes with homotopy pushouts; since $A \simeq \mathrm{Sym}_A(0)$, we get that $C \simeq \mathrm{Sym}_A(P)$, where P is defined by the homotopy pushout (in A-$\mathbf{Mod}$)

$$\begin{tikzcd} I[n] \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & P \end{tikzcd}$$

But, by definition of suspension functor in A-$\mathbf{Mod}$, we have then that $P \simeq \Sigma I[n] = I[n+1]$. Therefore $C \simeq \mathrm{Sym}_A I[n+1]$.

Now, coming back to the homotopy pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B_0 \arrow[r] & \widetilde{B_0} \otimes_{B_1} B_0 \end{tikzcd}$$

and recalling the base change property of the functor Sym_{-}, we conclude that there is a canonical isomorphism $\widetilde{B_0} \otimes_{B_1} B_0 \simeq \mathrm{Sym}_{B_0} I[n+1]$ in $\mathrm{Ho}(A/\mathbf{sAlg}_k)$. By tracing back the construction of this isomorphism, we see that it is indeed an isomorphism in $\mathrm{Ho}(B_0/\mathbf{sAlg}_k)$ (since the B_0-algebra structure comes in both cases from the bottom horizontal map of the pushout diagrams). $\square$

3 Any square-zero extension is an infinitesimal extension

Theorem 3.1. *Let $n \geq 0$, $A \in \mathbf{sAlg}_k$, and $u : B_1 \to B_0$ in $A/\mathbf{sAlg}_k$ a square-zero extension by $I[n]$ ($I \subseteq \pi_n(B_1)$ a fixed sub-$\pi_0(B_1)$-module). Then there exists a derived A-derivation d_u of B_0 into $I[n+1]$, and an isomorphism $B_0 \oplus_{d_u} I[n] \simeq B_1$ in $\mathrm{Ho}(A/\mathbf{sAlg}_k/B_0)$. Moreover, such a d_u is uniquely determined as a map in $\mathrm{Ho}(A/\mathbf{sAlg}_k/B_0)$.*

Proof. Throughout the proof, recall our standing convention $\otimes \equiv \otimes^{\mathbb{L}}$. Consider the fiber - cofiber sequence of A-modules

$$I[n] \longrightarrow B_1 \xrightarrow{u} B_0 \ .$$

It induces a fiber - cofiber sequence

$$B_1 \xrightarrow{u} B_0 \longrightarrow I[n+1] \ .$$

The idea is now to apply $(-) \otimes_{B_1} B_0$ to this sequence in order to obtain a split sequence; the one of the B_0-algebra structures on $B_0 \otimes_{B_1} B_0$ will induce the zero derivation while the other one will induce a derivation d_u such that $B_1 \simeq B_0 \times_{B_0 \oplus_{d_u} I[n]} B_0$. Let us work this idea out.

The sequence of B_0-modules

$$B_0 \simeq B_1 \otimes_{B_1} B_0 \xrightarrow{u \otimes \mathrm{id}} B_0 \otimes_{B_1} B_0 \longrightarrow B_0 \otimes_{B_1} I[n+1]$$

is clearly split by the product map $B_0 \otimes_{B_1} B_0 \to B_0$; therefore we get a canonical isomorphism

$$B_0 \otimes_{B_1} B_0 \simeq B_0 \oplus (B_0 \otimes_{B_1} I[n+1])$$

in the homotopy category of B_0-modules.
Let $\widetilde{B_0} := B_1 \otimes_{\mathrm{Sym}_A I[n]} A$, and let $\gamma \colon \tau_{\leq n}\widetilde{B_0} \to B_0$ be the isomorphism of B_1-algebras produced by Lemma 2.6. Introduce the morphism

$$t \colon \widetilde{B_0} \to \tau_{\leq n}\widetilde{B_0} \xrightarrow{\gamma} B_0$$

and consider the induced map

$$\theta := \mathrm{id} \otimes_{B_1} t \colon B_0 \otimes_{B_1} \widetilde{B_0} \longrightarrow B_0 \otimes_{B_1} B_0$$

which is a map of B_0-algebras, if we endow $B_0 \otimes_{B_1} B_0$ with the B_0-algebra structure given by

$$j_1 \colon B_0 \to B_0 \otimes_{B_1} B_0, \quad b \longmapsto b \otimes 1.$$

We claim that

$$\tau_{\leq n+1}\theta : \tau_{\leq n+1}(B_0 \otimes_{B_1} \widetilde{B_0}) \longrightarrow \tau_{\leq n+1}(B_0 \otimes_{B_1} B_0)$$

is an isomorphism in $\mathrm{Ho}(B_0/\mathbf{sAlg}_k)$. Let us prove this claim.

$\diamondsuit$ We compute how $\tau_{\leq n+1}\theta$ acts on homotopy groups. Let us first compute $\pi_i(B_0 \otimes_{B_1} I[n+1])$ by using the spectral sequence ([Q, II §6, Thm. 6.c])

$$\pi_p(\pi_q(B_0)[0] \otimes_{\pi_0 B_1} I[n+1]) \Rightarrow \pi_{p+q}(B_0 \otimes_{B_1} I[n+1]).$$

We have

$$\pi_p(\pi_q(B_0)[0] \otimes_{\pi_0 B_1} I[n+1]) = \begin{cases} \pi_q(B_0) \otimes_{\pi_0(B_1)} I & \text{if } p = n+1 \\ 0 & \text{if } p \neq n+1 \end{cases}$$

so the spectral sequence degenerates, and we have for $q = 0, p = n+1$

$$\pi_{n+1}(B_0 \otimes_{B_1} I[n+1]) = \pi_0(B_0) \otimes_{\pi_0(B_1)} I$$

Now, if $n = 0$ both B_0 and B_1 are discrete, $B_0 \simeq B_1/I$ as B_1-algebra and $I^2 = 0$, so that

$$\pi_0(B_0) \otimes_{\pi_0(B_1)} I \simeq I/I^2 \simeq I$$

If, instead, $n > 0$, then $\pi_0(B_1) \simeq \pi_0(B_0)$ and so

$$\pi_0(B_0) \otimes_{\pi_0(B_1)} I \simeq I$$

In conclusion we obtain

$$\pi_i(B_0 \otimes_{B_1} I[n+1]) = \begin{cases} 0 & \text{if } i < n+1 \\ I & \text{if } i = n+1 \\ \pi_q(B) \otimes I & \text{if } i = n+1+q,\ q > 0 \end{cases}$$

Since

$$B_0 \otimes_{B_1} B_0 \simeq B_0 \oplus (B_0 \otimes_{B_1} I[n+1]),$$

we conclude that

$$\pi_i(B_0 \otimes_{B_1} B_0) = \begin{cases} \pi_i(B_0) & \text{if } i < n+1 \\ \pi_{n+1}(B_0) \oplus I & \text{if } i = n+1 \\ \pi_i(B_0) \oplus (\pi_q(B) \otimes I) & \text{if } i = n+1+q,\ q > 0 \end{cases}$$

On the other hand, by Lemma 2.6,

$$B_0 \otimes_{B_1} \widetilde{B_0} \simeq \mathrm{Sym}_{B_0} I[n+1] = B_0 \oplus I[n+1] \oplus R$$

where R is $(n+1)$-connected (i.e. its π_i's vanish for $i \leq n+1$), so that there is an isomorphism

$$\tau_{\leq n+1}(B_0 \otimes_{B_1} \widetilde{B_0}) \simeq B_0 \oplus I[n+1]$$

in the homotopy category of B_0-algebras.

The reader may check that the following diagram is commutative

$$\begin{tikzcd} B_0 \otimes_{B_1} \widetilde{B_0} \arrow[rr, "\theta"] \arrow[dr] & & B_0 \otimes_{B_1} B_0 \arrow[dl] \\ & B_0 \oplus I[n+1] & \end{tikzcd}$$

This concluses our proof of the claim that $\tau_{\leq n+1}\theta$ is an equivalence. $\diamondsuit$

So we have proved that

$$\theta_{\leq n+1} := \tau_{\leq n+1}\theta \colon \tau_{\leq n+1}(B_0 \otimes_{B_1} \widetilde{B_0}) \simeq B_0 \oplus I[n+1] \longrightarrow \tau_{\leq n+1}(B_0 \otimes_{B_1} B_0)$$

is an isomorphism in $\mathrm{Ho}(B_0/\mathbf{sAlg}_k)$, and note that the B_0-algebra structure on the lhs is given by the map φ_0 corresponding to the zero derivation. Now we can use the other B_0-algebra structure

$$j_2 \colon B_0 \to B_0 \otimes_{B_1} B_0, \quad b \longmapsto 1 \otimes b,$$

to produce the derivation we are looking for. Let us define

$$\varphi_{d_u} \colon B_0 \simeq \tau_{\leq n+1} B_0 \xrightarrow{\tau_{\leq n+1} j_2} \tau_{\leq n+1}(B_0 \otimes_{B_1} B_0) \xrightarrow{\theta^{-1}_{\leq n+1}} B_0 \oplus I[n+1]$$

and observe that this is indeed a map in $\mathrm{Ho}(A/\mathbf{sAlg}_k/B)$, so it does correspond to a derived derivation $d_u : B_0 \to I[n+1]$ over A. Consider the corresponding infinitesimal extension defined by the homotopy pushout

$$\begin{tikzcd} B_0 \oplus_{d_u} I[n] \arrow[r, "\psi'"] \arrow[d, "\psi_{d_u}"'] & B_0 \arrow[d, "\varphi_0"] \\ B_0 \arrow[r, "\varphi_{d_u}"'] & B_0 \oplus I[n+1] \end{tikzcd}$$

and observe that, since the diagram

$$\begin{tikzcd} B_1 \arrow[r, "u"] & B_0 \arrow[r, shift left, "j_1"] \arrow[r, shift right, "j_2"'] & B_0 \otimes_{B_1} B_0 \end{tikzcd}$$

equalizes, the same is true for the diagram

$$\begin{tikzcd} B_1 \arrow[r, "u"] & B_0 \arrow[r, shift left, "j_1"] \arrow[r, shift right, "j_2"'] & B_0 \otimes_{B_1} B_0 \arrow[r] & \tau_{\leq n}(B_0 \otimes_{B_1} B_0) \simeq B_0 \oplus I[n+1] \end{tikzcd}$$

and therefore, by definition of φ_0 (induced by j_1) and φ_{d_u} (induced by j_2), the same is true for the diagram

$$\begin{tikzcd} B_1 \arrow[r, "u"] & B_0 \arrow[r, shift left, "\varphi_0"] \arrow[r, shift right, "\varphi_{d_u}"'] & B_0 \oplus I[n+1] \end{tikzcd} .$$

So, we have an induced map

$$\alpha : B_1 \to B_0 \oplus_{d_u} I[n]$$

in $\mathrm{Ho}(A/\mathbf{sAlg}_k/B_0)$ (where $B_0 \oplus_{d_u} I[n]$ is considered as an algebra over B_0 via the map ψ_{d_u}).

We are left to prove that α is an isomorphism. In order to do this, we will show that, in the following commutative diagram whose lines are fiber sequences, the map β is a weak equivalence:

$$\begin{tikzcd} \mathrm{hofib}(u) \arrow[r] \arrow[d, "\beta"] & B_1 \arrow[r, "u"] \arrow[d, "\alpha"] & B_0 \arrow[d, equal] \\ \mathrm{hofib}(\psi_{d_u}) \arrow[r] \arrow[d] & B_0 \oplus_{d_u} I[n] \arrow[r, "\psi_{d_u}"] \arrow[d, "\psi'"] & B_0 \arrow[d, "\varphi_{d_u}"] \\ \mathrm{hofib}(\varphi_0) \arrow[r, "\varphi_0"] & B_0 \arrow[r] & B_0 \oplus I[n+1] \end{tikzcd}$$

Proposition A.5 implies that the morphism

$$\mathrm{hofib}(\psi_{d_u}) \to \mathrm{hofib}(\varphi_0)$$

is a weak equivalence. Using the 2-out-of-3 property, it is sufficient to check that the composition

$$\mathrm{hofib}(u) \to \mathrm{hofib}(\psi_{d_u}) \to \mathrm{hofib}(\varphi_0)$$

is a weak equivalence. The definition of α implies $\psi' \circ \alpha = u$; moreover $\mathrm{hofib}(u)$ and $\mathrm{hofib}(\varphi_0)$ are (separately) isomorphic to $I[n]$. As consequence, it is sufficient to show that the left square in the following diagram

$$\begin{tikzcd} I[n] \arrow[r, "\gamma"] \arrow[d, equal] & B_1 \arrow[r, "u"] \arrow[d, "u"] & B_0 \arrow[d, "\varphi_{d_u}"] \\ I[n] \arrow[r, "\delta"] & B_0 \arrow[r, "\varphi_0"] & B_0 \oplus I[n+1] \end{tikzcd}$$

commutes *in the homotopy category*, where γ and δ denote the canonical morphisms

$$\gamma \colon I[n] \simeq \mathrm{hofib}(u) \to B_1, \qquad \delta \colon I[n] \simeq \mathrm{hofib}(\varphi_0) \to B_0$$

Recall from Proposition 1.2 that the morphism δ is obtained from the diagram

$$\begin{tikzcd}
I[n] \arrow[drr, bend left] \arrow[dr, dotted, "\delta"] \arrow[ddr, bend right, "0"'] & & \\
& B_0 \arrow[r] \arrow[d, "\varphi_0"'] & 0 \arrow[d] \\
& B_0 \oplus I[n+1] \arrow[r] & I[n+1]
\end{tikzcd}$$

so that $\varphi_0 \circ \delta \simeq 0 \simeq \varphi_0 \circ u \circ \gamma$. Since φ_0 is a section of the canonical projection $B_0 \oplus I[n+1] \to B_0$, it is in particular a split mono; as consequence, its image in the homotopy category is a (split) mono as well. We therefore get $\delta \simeq 0 \simeq u \circ \gamma$, completing the proof. □

4 Application to Postnikov towers

Proposition 4.1. *Let $n \geq 1$, and $C \in \mathbf{sAlg}_k$. Then the n-th stage $p_n : C_{\leq n} \to C_{\leq n-1}$ of the Postnikov tower is a $A = k$-square-zero extension by $\pi_n(C)[n]$.*

Proof. Let us check that the conditions of Definition 2.1 are met for $n \geq 1$ and $I = \pi_n C = \pi_n C_{\leq n}$.

1. Obviously $C_{\leq n}$ and $C_{\leq n-1}$ are n-truncated;
2. By definition of Postnikov tower, p_n is an $(n-1)$-equivalence of simplicial k-algebras;
3. Using Remark 2.2.1 we are reduced to show that for any n-truncated k-algebra E the following diagram is homotopy cartesian:

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{sAlg}_k}(C_{\leq n-1}, E) \arrow[r, "\mathrm{Map}(p_n, E)"] \arrow[d] & \mathrm{Map}_{\mathbf{sAlg}_k}(C_{\leq n}, E) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{k-\mathbf{Mod}}(\pi_n C, \pi_n E)
\end{tikzcd}$$

The idea is to kill π_n in $C_{\leq n}$ in order to obtain a better description of $C_{\leq n-1}$. In order to do so, consider the following homotopy pushout in $\mathbf{sAlg}_k$:

$$\begin{tikzcd}
\mathrm{Sym}_k(\pi_n(C)[n]) \arrow[r, "a"] \arrow[d, "b"'] & C_{\leq n} \arrow[d] \\
k \arrow[r] & D
\end{tikzcd}$$

where a is induced by the identity map $\pi_n C \to \pi_n C$ and b is induced by the zero map $\pi_n C \to k$ via the canonical identifications

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{sAlg}_k}(\mathrm{Sym}_k(\pi_n(C)[n]), E) &\cong \mathrm{Hom}_{k\text{-}\mathbf{Mod}}(\pi_n(C) \otimes_k k[S^n], E) \\
&\cong \mathrm{Hom}_{k\text{-}\mathbf{Mod}}(\pi_n(C), \mathrm{Map}(k[S^n], E)) \\
&\cong \mathrm{Hom}_{k\text{-}\mathbf{Mod}}(\pi_n(C), \pi_0 \mathrm{Map}(k[S^n], E)) \quad \text{(use Lemma B.1)} \\
&\cong \mathrm{Hom}_{k\text{-}\mathbf{Mod}}(\pi_n(C), \pi_n(E))
\end{aligned}$$

Assume for the moment that $\tau_{\leq n}D \simeq C_{\leq n-1}$ in $\mathrm{Ho}(C_{\leq n}/\mathbf{sAlg}_k)$; in this case, for any n-truncated object E in $\mathbf{sAlg}_k$, we get

$$\begin{aligned}\mathrm{Map}_{\mathbf{sAlg}_k}(C_{\leq n-1}, E) &\simeq \mathrm{Map}_{\mathbf{sAlg}_k}(\tau_{\leq n}D, E) \simeq \mathrm{Map}_{\mathbf{sAlg}_k}(D, E) \\ &\simeq \mathrm{Map}_{\mathbf{sAlg}_k}(C_{\leq n}, E) \times^h_{\mathrm{Map}_{\mathbf{sAlg}_k}(\mathrm{Sym}_k(\pi_n(C)[n]),E)} \mathrm{Map}_{\mathbf{sAlg}_k}(k, E)\end{aligned}$$

but

$$\mathrm{Map}_{\mathbf{sAlg}_k}(k, E) \simeq *$$

and

$$\mathrm{Map}_{\mathbf{sAlg}_k}(\mathrm{Sym}_k(\pi_n(C)[n]), E) \simeq \mathrm{Map}_{k-\mathbf{Mod}}(\pi_n(C)[n], E) \simeq \mathrm{Map}_{k-\mathbf{Mod}}(\pi_n C, \pi_n E)$$

Since $\pi_n(C)$ and $\pi_n(E)$ are discrete, it follows that $\mathrm{Map}_{k\text{-}\mathbf{Mod}}(\pi_n C, \pi_n E)$ is discrete as well, so that there is a weak equivalence:

$$\mathrm{Map}_{k\text{-}\mathbf{Mod}}(\pi_n C, \pi_n E) \simeq \pi_0 \mathrm{Map}_{k\text{-}\mathbf{Mod}}(\pi_n C, \pi_n E) = \mathrm{Hom}_{k\text{-}\mathbf{Mod}}(\pi_n C, \pi_n E)$$

completing the proof of this step.

4. The canonical map $\pi_n(C_{\leq n})/I = 0 \to \pi_n(C_{\leq n-1}) = 0$ is obviously an isomorphism.

Thus, we are left to show that there is a weak equivalence $\tau_{\leq n}D \simeq C_{\leq n-1}$ in $C_{\leq n}/\mathbf{sAlg}_k$. To prove this, we will be using the spectral sequence of [Q, Theorem II.6.b]. To begin with, set

$$R_* := \pi_*(\mathrm{Sym}_k(\pi_n(C)[n]))$$

so that the spectral sequence reads

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_p(\pi_*(C_{\leq n}), \pi_*(k))_q \Rightarrow \pi_{p+q}(D)$$

Choose a flat resolution $C_{\bullet *} \to \pi_*(C_{\leq n})$ as R_*-module so that

$$\mathrm{Tor}^{R_*}_p(\pi_*(C_{\leq n}), \pi_*(k))_q = H^p((C_{\bullet *} \otimes_{R_*} \pi_* k)_q)$$

Introduce the ideal

$$I := \bigoplus_{n \geq 1} R_n$$

Since

$$R_q = \begin{cases} k & \text{if } q = 0 \\ 0 & \text{if } 0 < q < n \\ \pi_n(C) & \text{if } q = n \end{cases}$$

it follows that

$$\pi_* k \simeq k \simeq R_*/I$$

and therefore

$$C_{\bullet *} \otimes_{R_*} k \simeq C_{\bullet *}/IC_{\bullet *}$$

In particular, being I a graded ideal, we get

$$(C_{\bullet *} \otimes_{R_*} k)_q \simeq C_{\bullet q}/J$$

where

$$J := \bigoplus_{i+j=q} I_i C_{\bullet j}$$

As consequence we see that

$$C_{\bullet q} \otimes_{R_*} k = \begin{cases} C_{\bullet q} & \text{if } q < n \\ C_{\bullet q}/\pi_n(C) & \text{if } q = n \end{cases}$$

Finally, this enables us to compute the second layer of the spectral sequence:

$$\mathrm{Tor}_p^{R_*}(\pi_*(C_{\leq n}), k)_q = \begin{cases} H^p(C_{\bullet q}) = \delta_{p0} \cdot \pi_q C_{\leq n} & \text{if } 0 \leq q < n \\ H^p(C_{\bullet n}/\pi_n(C_{\leq n})C_{\bullet 0}) & \text{if } q = n \end{cases}$$

In order to compute $H^0(C_{\bullet n}/\pi_n(C_{\leq n})C_{\bullet 0})$, we observe that by construction of $C_{\bullet *}$, the sequence of R_*-modules

$$C_{1*} \to C_{0*} \to \pi_*(C_{\leq n}) \to 0$$

is exact. Now, the functor $- \otimes_{R_*} R_*/I$ is right exact and the operation of taking the degree n of an R_*-modules defines obviously an exact functor

$$R_*\text{-}\mathbf{Mod} \to R_0\text{-}\mathbf{Mod}$$

Applying these two functors to the previous exact sequence yields the new sequence

$$C_{1n}/\pi_n(C_{\leq n})C_{1,0} \to C_{0n}/\pi_n(C_{\leq n})C_{0,0} \to \pi_n(C_{\leq n})/\pi_n(C) \to 0$$

which is still exact; in this way we obtain:

$$H^0(C_{\bullet n}/\pi_n(C_{\leq n})C_{\bullet 0}) \simeq \pi_n(C_{\leq n})/\pi_n(C) = 0$$

We finally get

$$\pi_q(D) = \begin{cases} \pi_q(C_{\leq n}) & \text{if } q < n \\ 0 & \text{if } q = n \end{cases}$$

Moreover, the map $C_{\leq n} \to D$ induces on the level of π_q the map

$$H^0(C_{\bullet q} \to C_{\bullet q} \otimes_{R_*} k)$$

which is the identity if $q < n$. This shows that $C_{\leq n} \to D$ is an $(n-1)$-equivalence.

At this point, consider the diagram

$$\begin{tikzcd}
\mathrm{Sym}_k(\pi_n(C)[n]) \arrow[r, "a"] \arrow[d, "b"'] & C_{\leq n} \arrow[d] \arrow[ddr, bend left, "p_n"] & \\
k \arrow[r] \arrow[drr, bend right, "\varphi"'] & D \arrow[dr, dotted] & \\
 & & C_{\leq n-1}
\end{tikzcd}$$

In order to prove the existence of the dotted map, we have to show that $p_n \circ a = \varphi \circ b$; by the universal property of the symmetric algebra, this is equivalent to show that the following square commutes:

$$\begin{tikzcd} \pi_n(C) \arrow[r, "\mathrm{id}"] \arrow[d] & \pi_n(C_{\leq n}) \arrow[d] \\ \pi_n(k) \arrow[r] & \pi_n(C_{\leq n-1}) \end{tikzcd}$$

and since $n \geq 1$ $\pi_n(k) = \pi_n(C_{\leq n-1}) = 0$, so that the last statement is trivially true.

The two-out-of-three property now shows that $D \to C_{\leq n-1}$ is an $(n-1)$-equivalence; since applying π_n we get $\pi_n(D) = \pi_n(C_{\leq n-1}) = 0$, it follows that the induced map

$$\tau_{\leq n} D \to C_{\leq n-1}$$

is an n-equivalence, hence an equivalence in $C_{\leq n}/\mathbf{sAlg}_k$ by the local Whitehead theorem. $\square$

At this point we easily recover the important [HAG-II, Lemma 2.2.1.1]:

Corollary 4.2. *Let $A \in \mathbf{sAlg}_k$ be a simplicial algebra. For every $n \geq 1$ there exists a unique (derived) derivation*

$$d_n \in \pi_0 \mathrm{Map}_{\mathbf{sAlg}_k/A_{\leq n-1}}(A_{\leq n-1}, A_{\leq n-1} \oplus \pi_n(A)[n+1])$$

such that the associated infinitesimal extension

$$A_{\leq n-1} \oplus_{d_n} \pi_n(A)[n] \to A_{\leq n-1}$$

is isomorphic in $\mathrm{Ho}(\mathbf{sAlg}_k/A_{n-1})$ *to*

$$A_{\leq n} \to A_{\leq n-1}$$

Proof. Proposition 4.1 implies that $A_{\leq n} \to A_{\leq n-1}$ is a square-zero extension, so that the result follows at once from Theorem 3.1. $\square$

Remark 4.3. In other words, Corollary 4.2 says that for every simplicial algebra A, the n-th stage $A_{\leq n}$ of its Postnikov decomposition is completely controlled by the $(n-1)$-th stage $A_{\leq n-1}$, the homotopy group $\pi_n(A)$ and an element of $\overline{k_n} \in [\mathbb{L}_{A_{\leq n-1}}, \pi_n(A)[n+1]]$ via the condition that the following is a homotopy pullback diagram:

$$\begin{tikzcd} A_{\leq n} \arrow[r, "p_n"] \arrow[d, "p_n"'] & A_{\leq n-1} \arrow[d, "0"] \\ A_{\leq n-1} \arrow[r, "k_n"] & A_{\leq n-1} \oplus \pi_n(A)[n+1] \end{tikzcd}$$

Such derived derivation $\overline{k_n}$ is called the n-th Postnikov invariant of A.

5 Connectivity estimates

Definition 5.1. Let $n \in \mathbb{N}$. A simplicial module M is said to be *n-connective* if $\pi_i M = 0$ for every $0 \leq i < n$. A map of simplicial modules $f \colon M \to N$ is said to be *n-connected* if $\mathrm{hofib}(f)$ is n-connective.

Proposition 5.2. *Let A be a simplicial k-algebra and let M a m-connective A-module.*

1. *if N is a n-connective A-module, then $M \otimes_A N$ is $(m+n)$-connective;*

2. *if $f \colon A \to B$ is a morphism of simplicial k-algebras such that $\pi_0(f)$ is an isomorphism, then the map $\varphi \colon M \to M \otimes_A B$ is a m-equivalence of simplicial A-modules.*

Proof. We use the spectral sequence of [Q, II §6, Thm 6.b]. Write

$$R_* := \pi_*(A)$$

so that the spectral sequence reads off as

$$\mathrm{Tor}_p^{R_*}(\pi_* M, \pi_* N)_q \Rightarrow \pi_{p+q}(M \otimes_A N)$$

We begin with the first statement. Choose a flat resolution $C_{\bullet *} \to \pi_* M$ and observe that we can in fact choose $C_{i,j} = 0$ for $j < m$ (just use the free resolutions given by the shifts of R_*). Then

$$\mathrm{Tor}_p^{R_*}(\pi_* M, \pi_* N)_q = H^p((C_{\bullet *} \otimes_{R_*} \pi_* N)_q)$$

and

$$(C_{\bullet *} \otimes_{R_*} \pi_* N)_q = \bigoplus_{i+j=q} C_{\bullet i} \otimes_{R_*} \pi_j N$$

Now, if $q \leq m + n - 1$ we have that necessarily $i < m$ or $j < n$, so that

$$(C_{\bullet *} \otimes_{R_*} \pi_* N)_q = 0$$

so that the spectral sequence degenerates yielding

$$\pi_p(M \otimes_A N) = 0$$

if $p \leq m + n - 1$.

We now turn to the second statement. Taking $N = B$ and $n = 0$ we see that $M \otimes_A B$ is m-connected, so that the map $\varphi \colon M \to M \otimes_A B$ is forcily an $(m-1)$-equivalence. We are left to compute $\pi_m(\varphi)$. However, the same computations as above show that $E^2_{0,m} = E^\infty_{0,m}$; since $\pi_0(A) \simeq \pi_0(B)$ it follows

$$(C_{\bullet *} \otimes_{R_*} \pi_* B)_m = C_{\bullet, m} \otimes_{\pi_0(A)} \pi_0 B \simeq C_{\bullet, m}$$

which implies

$$\pi_m(M \otimes_A B) \simeq H^0(C_{\bullet, m} \otimes_{R_*} \pi_0 B) \simeq H^0(C_{\bullet, m}) \simeq \pi_m M$$

Finally, we recall that the map $\pi_m(\varphi) \colon \pi_m M \to \pi_m(M \otimes_A B)$ can be computed as the 0-th homology of the canonical map

$$C_{\bullet, m} \to C_{\bullet, m} \otimes_{R_*} \pi_0 B$$

completing the proof. $\square$

Corollary 5.3. *Assume that k is of characteristic 0. Let $A \in \mathbf{sAlg}_k$ and $M \in A\text{-}\mathbf{Mod}$. If M is n-connective ($n > 0$), then $\mathrm{Sym}^p_A(M)$ is (pn)-connective.*

Proof. We may suppose that M is cofibrant, so that the derived tensor product and derived symmetric powers are the usual underived ones. Since k is of characteristic 0, the canonical map $r\colon M^{\otimes_A p} \to \mathrm{Sym}^p_A(M)$ has a right inverse (the antisymmetrization map) $i\colon \mathrm{Sym}^p_A(M) \to M^{\otimes_A p}$ (i.e. $r \circ i$ is the identity of $\mathrm{Sym}^p_A(M)$).

Now since M is n-connective, it follows from Prop. 5.2, that $M^{\otimes_A p}$ is pn-connective. But the composite

$$\pi_i(\mathrm{Sym}^p_A(M)) \xrightarrow{\pi_i(i)} \pi_i(M^{\otimes_A p}) \xrightarrow{\pi_i(r)} \pi_i(\mathrm{Sym}^p_A(M))$$

is the identity, and therefore $\pi_i(\mathrm{Sym}^p_A(M)) = 0$ whenever $\pi_i(M^{\otimes_A p}) = 0$. Hence $\mathrm{Sym}^p_A(M)$ has the same connectivity as $M^{\otimes_A p}$. $\square$

The proof of the following theorem is precisely the translation of the one given in [HA, Theorem 8.4.3.12]. However, the exposition given there is crystal-clear and we could not to improve it; as a consequence, we limit ourselves to sketch the outline of the proof.

Theorem 5.4. *Let $f\colon A \to B$ be a morphism in $\mathbf{sAlg}_k$ and $C_f := \mathrm{hocofib}(f) \in A\text{-}\mathbf{Mod}$ its homotopy cofiber. Then there exists a canonical map $\alpha : C_f \otimes_A B \to \mathbb{L}_f$ in $\mathrm{Ho}(B - Mod)$, and we have that α is $(2n+2)$-connected if f is n-connected ($n \in \mathbb{N}$).*

Proof. Let $\mathbb{L}(f)\colon \mathbb{L}_A \to \mathbb{L}_B$ be the canonical map induced by f, so that

$$\mathbb{L}_f \simeq \mathrm{hocofib}(\mathbb{L}(f))$$

We have a canonical map

$$\eta\colon \mathbb{L}_B \to \mathbb{L}_f$$

corresponding to a derived derivation

$$d_\eta\colon B \to B \oplus \mathbb{L}_f$$

Observe that $\eta \circ \mathbb{L}(f)$ is nullhomotopic; denote by φ_0^A the derivation associated to the null morphism $\mathbb{L}_A \to \mathbb{L}_f$; the equivalence of simplicial sets

$$\mathrm{Map}_{A\text{-}\mathbf{Mod}}(\mathbb{L}_A, \mathbb{L}_f) \simeq \mathrm{Map}_{\mathbf{sAlg}_k/A}(A, A \oplus \mathbb{L}_f)$$

implies that the associated derivations, $d_{\mathbb{L}(f)\circ\eta}$ and φ_0^A lie in the same path component, i.e. they are homotopic. Using the notations of Remark B.11 and Lemma B.13 we obtain

$$s(f, \mathrm{id}_{\mathbb{L}_f}) \circ d_{\eta\circ\mathbb{L}(f)} = d_\eta \circ f, \qquad \varphi_0^B \circ f = s(f, \mathrm{id}_{\mathbb{L}_f}) \circ \varphi_0^A$$

where φ_0^B denotes the derivation associated to the null map $\mathbb{L}_B \to \mathbb{L}_f$. It follows now

$$d_\eta \circ f = s(f, \mathrm{id}_{\mathbb{L}_f}) \circ d_{\eta\circ\mathbb{L}(f)} \simeq s(f, \mathrm{id}_{\mathbb{L}_f}) \circ \varphi_0^A = \varphi_0^B \circ f$$

Let at this point $\psi_\eta\colon B^\eta \to B$ be the induced infinitesimal extension, defined by the homotopy pullback

$$\begin{tikzcd} B^\eta \arrow[r, "\psi'"] \arrow[d, "\psi_\eta"'] & B \arrow[d, "\varphi_0^B"] \\ B \arrow[r, "d_\eta"'] & B \oplus \mathbb{L}_f \end{tikzcd}$$

Since A is cofibrant over k, Corollary A.4 can be used to deduce the existence of a map $f' \colon A \to B^\eta$ such that

$$f' \circ \psi_\eta \simeq f$$

We obtain in this way a canonical map of A-modules

$$\mathrm{hocofib}(f) \to \mathrm{hocofib}(\psi_\eta)$$

which corresponds, under adjunction, to a canonical map

$$\alpha_f \colon \mathrm{hocofib}(f) \otimes_A B \to \mathrm{hocofib}(\psi_\eta) \simeq \mathrm{hofib}(\psi_\eta)[1] \simeq \mathbb{L}_f$$

where the last isomorphism is due to Proposition 1.2. We are therefore left to show that α_f is $(2n+2)$-connected [1].

The proof proceeds now in several steps. The strategy is to describe the map f as a finite composition

$$f = f_{n+1} \circ \phi_{n+1} \circ \ldots \circ \phi_1$$

in such a way that $\alpha_{f_{n+1}}$ and α_{ϕ_i} are $(2n+2)$-connected for every i (plus some other conditions), and then deduce the property from stability properties of the connectivity of construction associating α_f to f. Having outlined the strategy, we prefer to begin with these stability properties:

1. assume that $h = gf$; if both f and g are $(n-1)$-connected and moreover both α_f and α_g are $(2n+2)$-connected, then α_h is $(2n+2)$-connected. This is (almost) straightforward after that one gives an appropriate estimate for the map $M \otimes_A N \to M \otimes_B N$, which can be found in [HA, Lemma 8.4.3.16], but which can also be obtained by the usual spectral sequence of [Q, II §6, Thm 6.b] by carefully choosing flat resolutions;

2. assume that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

is a pushout square. If α_f is $(2n+2)$-connected then $\alpha_{f'}$ is $(2n+2)$-connected. In fact, the naturality of the construction of α_f shows that we have a commutative diagram

$$\begin{tikzcd} \mathrm{hocofib}(f) \otimes_A B \arrow[r, "\alpha_f"] \arrow[d] & \mathbb{L}_f \arrow[d] \\ \mathrm{hocofib}(f') \otimes_{A'} B' \arrow[r, "\alpha_{f'}"] & \mathbb{L}_{f'} \end{tikzcd}$$

and now we have isomorphisms (cfr. [HAG-II, Proposition 1.2.1.6.(2)] for the first one):

$$\mathbb{L}_{f'} \simeq \mathbb{L}_f \otimes_B B', \qquad \mathrm{hocofib}(f) \otimes_A B \otimes_B B' \simeq \mathrm{hocofib}(f) \otimes_A B'$$

Moreover, the dual of Proposition A.5 implies $\mathrm{hocofib}(f) \simeq \mathrm{hocofib}(f')$; under this isomorphism we obtain

$$\mathrm{hocofib}(f) \otimes_A B' \simeq \mathrm{hocofib}(f') \otimes_{A'} B'$$

[1] The map α_f can be constructed also using a small generalization of the beginning of the proof of Theorem 3.1. We leave the details to the interested reader.

Since the functor $-\otimes_{A'} B'$ preserves cofiber sequences, it preserves fiber sequences as well, yielding

$$\mathrm{hofib}(\alpha_{f'}) \simeq \mathrm{hofib}(\alpha_f) \otimes_B B'$$

It is sufficient to apply now Proposition 5.2.(1) to deduce that if $\mathrm{hofib}(\alpha_f)$ is $(2n+2)$-connected then the same holds true for $\mathrm{hofib}(\alpha_{f'})$;

3. for every n-connected k-module M, the map $f\colon \mathrm{Sym}_k(M) \to k$ induced by the null map $M \to k$ is $(2n+2)$-connected. To prove this one first observe that there is a fiber sequence

$$M \to 0 \to \mathbb{L}_{k/\mathrm{Sym}_k(M)}$$

(this is essentially the formal computation that can be found in [HA, Proposition 8.4.3.14]), so that the codomain of α_f is $M[1]$. Next, we observe that

$$\mathrm{hofib}(f) \simeq \bigoplus_{i\geq 1} \mathrm{Sym}^i(M)$$

so that

$$\mathrm{hocofib}(f) \simeq \mathrm{hofib}(f)[1] \simeq \bigoplus_{i\geq 1} \mathrm{Sym}^i(M[1])$$

Finally, one checks directly that the composition

$$M[1] \simeq \mathrm{Sym}^1(M[1])[-1] \to \bigoplus_{i\geq 1} \mathrm{Sym}^i(M[1])[-1] \xrightarrow{\alpha_f} M[1]$$

is homotopic to the identity. This implies that

$$\mathrm{hofib}(\alpha_f) \simeq \bigoplus_{i\geq 2} \mathrm{Sym}^i(M[1])$$

Since $M[1]$ is $(n+1)$-connected, the result follows now from Corollary 5.3;

4. if f is $(2n+2)$-connected, then α_f is $(2n+2)$-connected. Indeed, it is sufficient to observe that both $B \otimes_A \mathrm{hocofib}(f)$ and $\mathbb{L}_f$ are $(2n+2)$-connective (the first thanks to Proposition 5.2.(1) and the second thanks to general properties of the cotangent complex - see for example [HA, Lemma 8.4.3.17]).

As a second step, we will need to produce a suitable factorization of the morphism $f\colon A \to B$. Let $M = \mathrm{hofib}(f)$. Then we have a natural map $\mathrm{Sym}_k(M) \to A$ induced by the universal property of the symmetric algebra, which enables us to form the homotopy pushout square

$$\begin{tikzcd} \mathrm{Sym}_k(M) \arrow[r, "\psi_1"] \arrow[d] & k \arrow[d] \\ A \arrow[r, "\phi_1"] & A_1 \end{tikzcd}$$

where ψ_1 is the map corresponding to the null morphism $M \to k$. This induces a morphism $f'\colon A' \to B$ such that $f_1 \circ \phi_1 \simeq f$; the 2-out-of-3 property of (local) weak equivalences readily implies that ϕ_1 is an $(n-1)$-equivalence. Then we claim that f_1 is $(n+1)$-connected. In fact, if

$$I := \bigoplus_{i\geq 1} \mathrm{Sym}^i(M)$$

denotes the homotopy fiber of ψ, we obtain (using the fact that $A \otimes_{\mathrm{Sym}_k(M)} -$ preserves cofiber sequences and hence fiber sequences) the following morphism of fiber sequences:

$$\begin{tikzcd} A \otimes_{\mathrm{Sym}_k(M)} I \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, equal] & A' \arrow[d, "f_1"] \\ M \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

which implies $\mathrm{hocofib}(f_1) \simeq \mathrm{hocofib}(g)[1]$. Observe now that the composition

$$M \simeq \mathrm{Sym}^1_k(M) \to I \to A \otimes_{\mathrm{Sym}_k(M)} I$$

is a section of g. It follows therefore that $\mathrm{hofib}(g) \simeq \mathrm{hocofib}(g)[-1]$ is a direct summand of $A \otimes_{\mathrm{Sym}_k(M)} I$. We therefore see that it is sufficient to show that this tensor product is n-connected. However, this follows at once from Proposition 5.2.(1) and Corollary 5.3.
We are finally ready to prove that α_f is always $(2n+2)$-connected. Using the second step we can write f as a composition

$$f = f_{n+1} \circ \phi_{n+1} \circ \ldots \circ \phi_1$$

where f_{n+1} is $(2n+2)$-connected. Using 4. and recalling that each of the maps ϕ_i is $(n-1)$-connected, we can use 1. to reduce ourselves to prove that α_{ϕ_i} is $(2n+2)$-connected for every i. However, this follows from 2. and 3. $\square$

Corollary 5.5. *Let $f : A \to B$ be a map in $\mathbf{sAlg}_k$, and $n \in \mathbb{N}$.*

1. *If f is n-connected, then $\mathbb{L}_f$ is $(n+1)$-connective.*
2. *If $\mathbb{L}_f$ is $(n+1)$-connective and $\pi_0(f)$ is an isomorphism, then f is n-connected.*

Proof. The first part is an immediate consequence of Theorem 5.4. In fact, using the notations of that theorem, if f is n-connected, then $C_f = \mathrm{hofib}(f)[1]$ is $(n+1)$-connective and $\alpha : C_f \otimes_A B \to \mathbb{L}_f$ is $(2n+2)$-connected; moreover, Proposition 5.2.(1) implies that $C_f \otimes_A B$ is at least $(n+1)$-connective; the long exact sequence associated to a cofiber sequence implies then that $\mathbb{L}_f$ is n-connective.

Conversely, assume that $\pi_0(f)$ is an isomorphism. We will show that if f is not n-connected, then $\mathbb{L}_f$ is not $(n+1)$-connective. We can assume that n is minimal with respect to this property, so that f is $(n-1)$-connected and $\pi_n C_f \neq 0$. Observe that $A \to B \to C_f$ is a fiber - cofiber sequence, and therefore $\pi_0(B) \to \pi_0(C_f)$ is surjective. If $\pi_0(f)$ is an isomorphism, we obtain $\pi_0(C_f) = 0$, so that $n \geq 1$.

Since C_f is n-connected, the map

$$\alpha \colon C_f \otimes_A B \to \mathbb{L}_f$$

is $(2n)$-connected. Since $2n > n$, it follows that

$$\pi_n(C_f \otimes_A B) \to \pi_n \mathbb{L}_f$$

is an isomorphism. Moreover, since $\pi_0(A) \simeq \pi_0(B)$ we can apply Proposition 5.2.(2) to conclude that the map

$$\pi_n C_f \to \pi_n(C_f \otimes_A B)$$

is an isomorphism as well. It follows that

$$\pi_n \mathbb{L}_f \simeq \pi_n C_f \neq 0$$

completing the proof. $\square$

Remark 5.6. 1. Note that the proof of Corollary 5.5 (1), shows a bit more than what is in the statement. In the fiber sequence

$$C_f \otimes_A B \xrightarrow{\alpha} \mathbb{L}_f \longrightarrow \mathrm{hocofib}(\alpha),$$

we know that hocofib(α) is $(2n+3)$-connective and that $C_f \otimes_A B$ is $(n+1)$-connective. Therefore we may also identify the first a priori non-zero homotopy group of $\mathbb{L}_f$:

$$\pi_{n+1}(C_f \otimes_A B) \simeq \pi_{n+1}(\mathbb{L}_f)$$

(since $2n+3 > n+2$ for $n \geq 0$). More generally, we have that the i-th homotopy groups of $C_f \otimes_A B$ and $\mathbb{L}_f$ are isomorphic for any $i < 2n+2$ (the interest of this remark grows linearly with n).

2. It follows from the previous corollary that the relative cotangent complex $\mathbb{L}_{\pi_0(A)/A}$ is 1-connective (i.e. $\pi_i \mathbb{L}_{\pi_0(A)/A} = 0$ for $i = 0, 1$). So the same is true for $\mathbb{L}_{\mathrm{t}(X)/X}$ where X is a Deligne-Mumford derived stack and $\mathrm{t}(X)$ its truncation.

Corollary 5.7. *For a morphism $f : A \to B$ in $\mathbf{sAlg}_k$ the following properties are equivalent*

1. *f is a weak equivalence*
2. *$\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is an isomorphism, and $\mathbb{L}_f \simeq 0$.*

Proof. (1) $\Rightarrow$ (2) is obvious. From Corollary 5.5, we get that f is n-connected for any $n \geq 0$, i.e. it is a weak equivalence. So (2) $\Rightarrow$ (1). $\square$

6 An exercise in derived deformation theory

We want to explain how derived deformation theory fills the gaps in classical deformation theory, by working out an explicit example of a very 'classical' deformation problem: the *infinitesimal deformations of a proper smooth scheme over $k = \mathbb{C}$.*

Since we work in characteristic zero, the reader might, in this §, switch from $\mathbf{sAlg}_k$ to $\mathbf{cdga}_{\mathbb{C}}^{\leq 0}$, if he wishes to.

Let us recall that the object of study of classical (formal) deformation theory are reduced functors

$$F \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Grpd} \hookrightarrow \mathbf{sSet}$$

(i.e. $F(\mathbb{C})$ is weakly contractible). Here, $\mathbf{Art}_{\mathbb{C}}$ denote the category of artinian $\mathbb{C}$-algebras with residue field isomorphic to $\mathbb{C}$. For example, if $F \colon \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Grpd}$ is a classical moduli problem and $\xi \in F(\mathbb{C})$ is a point, we can obtain a formal reduced functor by forming the homotopy pullback

$$\hat{F}_\xi := F \times_{F(\mathbb{C})} \xi$$

and then restricting it to $\mathbf{Art}_{\mathbf{C}}$; this is called the formal completion of F at ξ.

A classically well known moduli functor is given by

$$F \colon \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Grpd}$$

sending a $\mathbb{C}$-algebra R into the groupoid of proper smooth morphisms

$$Y \to \mathrm{Spec}(R)$$

and isomorphisms between them. In this case, if we fix a proper smooth scheme

$$\xi\colon X_0 \to \mathrm{Spec}(\mathbb{C})$$

the corresponding homotopy base change $\hat{F}_\xi$ is exactly the usual functor Def_{X_0}. The following properties are well known:

1. $\hat{F}_\xi(\mathbb{C}[t]/t^{n+1})$ is the groupoid of n-th order infinitesimal deformations of ξ;
2. if $\xi_1 \in \hat{F}_\xi(\mathbb{C}[\varepsilon])$ is a first order deformation of ξ, then $\mathrm{Aut}_{\hat{F}_\xi(\mathbb{C}[\varepsilon])}(\xi_1) \simeq H^0(X_0, T_{X_0})$;
3. $\pi_0(\hat{F}_\xi(\mathbb{C}[\varepsilon])) \simeq H^1(X_0, T_{X_0})$;
4. if ξ_1 is a first order deformation there exists an obstruction $\mathrm{obs}(\xi_1) \in H^2(X_0, T_{X_0})$ such that $\mathrm{obs}(\xi_1) = 0$ if and only if ξ_1 extends to a second order deformation.

The first three properties are really satisfactory, but not the fourth one. It raises two *questions* :

1. *how to interpret geometrically the entire* $H^2(X_0, T_{X_0})$ *?*
2. *how to identify intrinsically the space of all obstructions*[2] *inside* $H^2(X_0, T_{X_0})$ *?*

Derived deformation theory gives a more general perspective on the subject, and answers both questions. It allows a natural interpretation of $H^2(X_0, T_{X_0})$ as the group of derived deformations i.e. (isomorphism classes of) deformations over a specific non-classical ring, and it identifies, consequently, the obstructions space in a very natural way. Let's work these answers out.

Define

$$\underline{F}\colon \mathrm{sAlg}_{\mathbb{C}} \to \mathbf{sSet}$$

sending a simplicial algebra A to the nerve of the category of proper smooth maps of derived schemes

$$Y \to \mathbb{R}\mathrm{Spec}(A)$$

and equivalences between them. It is clear that $\underline{F}$ is a derived enhancement of F, and it can be shown that it preserves homotopy pullbacks. Introduce the full subcategory $\mathrm{sArt}_{\mathbb{C}}$ of $\mathrm{sAlg}_{\mathbb{C}}$ of simplicial $\mathbb{C}$-algebras A such that $\pi_0(A) \in \mathbf{Art}_{\mathbb{C}}$; if we fix

$$\xi \in \underline{F}(\mathbb{C}) = F(\mathbb{C})$$

then we can, as above, form the derived completion of $\underline{F}$ at ξ by taking the homotopy pullback:

$$\hat{\underline{F}}_\xi := \underline{F} \times^h_{\underline{F}(\mathbb{C})} \xi$$

The following proposition answers to Question 1 above, by saying that the entire $H^2(X_0, T_{X_0})$ can be interpreted as a space of derived deformations.

Proposition 6.1. $\pi_0(\hat{\underline{F}}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X_0, T_{X_0})$.

[2]It can happen that every obstruction is trivial and $H^2(X_0, T_{X_0}) \neq 0$. An example is given by a smooth projective surface $X_0 \subseteq \mathbb{P}^3_{\mathbb{C}}$ of degree ≥ 6.

Proof. First of all, $\underline{F}$ has a cotangent complex at ξ in the sense of [HAG-II, Definition 1.4.1.5] and it can be shown that

$$\mathbb{T}_{\underline{F},\xi} \simeq \mathbb{R}\Gamma(\mathbb{T}_{X_0}[1])$$

Using [HAG-II, Proposition 1.4.1.6] we obtain

$$\mathbb{L}_{\underline{F},\xi} \simeq \mathbb{T}^*_{\underline{F},\xi} \simeq \mathbb{R}\Gamma(X_0, \mathbb{L}_{X_0}[-1])$$

Therefore

$$\begin{aligned} \pi_0(\underline{\mathrm{Der}}_{\underline{F}}(\xi; \mathbb{C}[1])) &\simeq \pi_0(\mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{L}_{\underline{F},\xi}, \mathbb{C}[1])) \\ &\simeq \mathrm{Ext}^0(\mathbb{L}_{\underline{F},\xi}, \mathbb{C}[1]) \\ &= \mathrm{Ext}^1(\mathbb{L}_{\underline{F},\xi}, \mathbb{C}) \\ &= \mathrm{Ext}^0(\mathbb{L}_{\underline{F},\xi}[-1], \mathbb{C}) \\ &= \mathbb{T}_{\underline{F},\xi}[1] \simeq \mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0}[2]) \end{aligned}$$

since X_0 is smooth, we obtain $\mathbb{T}_{X_0} \simeq T_{X_0}$, so that

$$\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0}[2]) \simeq H^2(X_0, T_{X_0})$$

In conclusion

$$\begin{aligned} \pi_0(\hat{\underline{F}}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) &\simeq \pi_0(\mathrm{hofib}(\underline{F}(\mathbb{C} \oplus \mathbb{C}[1]) \to \underline{F}(\mathbb{C}), \xi)) \\ &\simeq \pi_0(\underline{\mathrm{Der}}_{\underline{F}}(\xi; \mathbb{C}[1])) \simeq H^2(X_0, T_{X_0}) \end{aligned}$$

□

Now that we have a derived deformation interpretation of $H^2(X_0, T_{X_0})$ at hand, we can proceed by answering Question 2 above. We begin by the following

Lemma 6.2. *Let*

$$I \longrightarrow A' \xrightarrow{f} A$$

be a square zero extension of (augmented) artinian $\mathbb{C}$-algebras (i.e. $I^2 = 0$). Then there exist a derivation $d\colon A \to A \oplus I[1]$ and a homotopy cartesian diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "\pi\circ d"] \\ \mathbb{C} \arrow[r] & \mathbb{C} \oplus I[1] \end{tikzcd}$$

where $\pi\colon A \oplus I[1] \to \mathbb{C} \oplus I[1]$ is the natural map induced by the augmentation $A \to \mathbb{C}$.

Proof. Use Theorem 3.1 to deduce the existence of a derivation $d\colon A \to A \oplus I[1]$ such that

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "\varphi_0"] \\ A \arrow[r, "d"] & A \oplus I[1] \end{tikzcd}$$

is a homotopy pullback. We are left to show that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi_0"'] & \mathbb{C} \arrow[d] \\ A \oplus I[1] \arrow[r] & \mathbb{C} \oplus I[1] \end{tikzcd}$$

is a homotopy pullback. However, the map $A \oplus I[1] \to C \oplus I[1]$ is a fibration, hence it is sufficient to show that it is a pullback, and this is straightforward verification. $\square$

If in particular we take the square-zero extension $A' = \mathbb{C}[s]/(s^3) \to \mathbb{A} = C[s]/(s^2)$, we obtain a homotopy pullback

$$\begin{tikzcd} \mathbb{C}[s]/(s^3) \arrow[r] \arrow[d] & \mathbb{C}[s]/(s^2) \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} \oplus \mathbb{C}[1] \end{tikzcd}$$

Using the fact that $\underline{\hat{F}_\xi}$ is reduced and preserves pullbacks, we obtain a fiber sequence of pointed simplicial sets

$$\underline{\hat{F}_\xi}(\mathbb{C}[s]/(s^3)) \to \underline{\hat{F}_\xi}(\mathbb{C}[s]/(s^2)) \to \underline{\hat{F}_\xi}(\mathbb{C} \oplus \mathbb{C}[1])$$

Then, Proposition 6.1 allows then to write the long exact sequence:

$$\pi_0(\underline{\hat{F}_\xi}(\mathbb{C}[s]/(s^3))) \to \pi_0(\underline{\hat{F}_\xi}(\mathbb{C}[s]/(s^2))) \to \pi_0(\underline{\hat{F}_\xi}(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X_0, T_{X_0})$$

of pointed sets (note that the middle and the rightmost ones are vector spaces). As a consequence, we see that a first order deformation extends to a second order deformation if and only if its image in $H^2(X_0, T_{X_0})$ vanishes. In other words, the space Obs of all obstructions is given by the image of the obstruction map

$$\mathrm{obs} : \pi_0(\underline{\hat{F}_\xi}(\mathbb{C}[s]/(s^2))) \to \pi_0(\underline{\hat{F}_\xi}(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X_0, T_{X_0}).$$

We have therefore answered Question 2, too.

Exercise. Extend the previous arguments to higher order infinitesimal deformations and obstructions.

A Homotopical nonsense

A.1 Homotopy pullbacks

The first technique we want to recall is how to compute homotopy pullbacks in a general model category. Recall first of all the following result:

Proposition A.1. *Let $\mathcal{M}$ be a right proper model category. If we have a diagram*

$$X \xrightarrow{g} Z \xleftarrow{h} Y$$

where at least one of g and h is a fibration, then the pullback $X \times_Z Y$ is naturally weakly equivalent to the homotopy pullback.

Proof. See [Hi, Corollary 13.3.8]. $\square$

We can obtain a similar result for general model categories adding the hypothesis that every object X, Y and Z is fibrant. To see this, recall first of all the following proposition:

Proposition A.2. *Let $\mathcal{M}$ be a model category and let*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r, "w"] & Y \end{tikzcd}$$

be a pullback. If p is a fibration and w is a weak equivalence between fibrant objects, then u is a weak equivalence.

Proof. There is a simple argument due to Reedy (cfr. [Hi, Proposition 13.1.2]), but there is also a more elaborate proof that avoid any lifting argument and therefore can be carried out in the more general context of categories of fibrant objects (see [GJ, Proposition II.8.5]). $\square$

Corollary A.3. *Let $\mathcal{M}$ be a model category. Suppose given a pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

where B, C and D are fibrant objects and p is a fibration. Then the square is a homotopy pullback.

Proof. The same proof of Proposition A.1 applies, because the only needed fact is the stability of weak equivalences under pullback by fibrations, and this is guaranteed by Proposition A.2. $\square$

We conclude describing the "universal homotopy mapping property" of the pullback that everyone could imagine (but for which we don't have any written reference):

Corollary A.4. *Let $\mathcal{M}$ be a model category and let*

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

be a homotopy pullback in $\mathcal{M}$. If X is a cofibrant object and $\alpha\colon X \to B$, $\beta\colon X \to C$ are morphisms such that $f \circ \alpha \simeq g \circ \beta$, then there is a map $\gamma\colon X \to A$ in the homotopy category of $\mathcal{M}$ such that $g' \circ \gamma \simeq \alpha$ and $f' \circ \gamma \simeq \beta$.

Proof. We can assume B, C and D to be fibrant and the maps f and g to be fibrations. In this case, use the cofibrancy of X to choose a cylinder object

$$X \sqcup X \xrightarrow{(i_0, i_1)} \mathrm{Cyl}(X) \xrightarrow{w} X$$

for X and a homotopy $H\colon \mathrm{Cyl}(X) \to D$ such that

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, bend left, "f\circ\alpha"] & \\ \mathrm{Cyl}(X) \arrow[r, "H"] & D \\ X \arrow[u, "i_1"] \arrow[ur, bend right, "g\circ\beta"'] & \end{tikzcd}$$

commutes. The liftings in the diagrams

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "i_1"'] & C \arrow[d, "g"] \\ \mathrm{Cyl}(X) \arrow[r, "H"'] \arrow[ur, dotted, "K_1"] & D \end{tikzcd} , \qquad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i_0"'] & B \arrow[d, "f"] \\ \mathrm{Cyl}(X) \arrow[r, "H"'] \arrow[ur, dotted, "K_2"] & D \end{tikzcd}$$

exist because i_0 and i_1 are trivial cofibrations, while f and g are fibrations by assumption. In particular we get

$$f \circ K_2 = H = g \circ K_1$$

which produces a unique map $\delta\colon X \times I \to A$. Set

$$\gamma := \delta \circ i_0$$

We therefore have

$$g' \circ \gamma = g' \circ \delta \circ i_0 = K_2 \circ i_0 = \alpha$$

and

$$f' \circ \gamma = f' \circ \delta \circ i_0 = K_1 \circ i_0 \simeq K_1 \circ i_1 = \beta$$

The uniqueness up to homotopy of γ is easily seen with a similar construction. □

A.2 Homotopy fibres

Proposition A.5. *Let $\mathcal{M}$ be a pointed model category and let*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "\beta"] & D \end{tikzcd}$$

be a given homotopy pullback. Then $\mathrm{hofib}\, f \simeq \mathrm{hofib}\, g$.

Proof. We can compute an explicit model for the homotopy pullback by replacing B, C and D by fibrant objects and the maps g and β by fibrations. This means that we can assume from the beginning that g and β are fibrations between fibrant objects. Then f is a fibration as well and $\mathrm{hofib}\, f$ is defined to be the homotopy pullback

$$\begin{tikzcd} \mathrm{hofib}\, f \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ * \arrow[r] & C \end{tikzcd}$$

Since C, $*$ and A are fibrant and f is a fibration it follows from Corollary A.3 that the (strict) pullback of the maps $* \to C \leftarrow A$ is an explicit model for the homotopy fiber. It follows that the outer rectangle in

$$\begin{tikzcd} \operatorname{hofib} f \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ * \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

is a pullback, hence (for the same reason as above) a homotopy pullback, showing that

$$\operatorname{hofib} f \simeq \operatorname{hofib} g$$

□

B Homotopy of Simplicial rings

B.1 Simplicial algebras

Throughout this section we will denote by k a fixed field and we will denote by $\mathbf{sAlg}_k$ the category of simplicial objects in $\mathbf{Alg}_k$. The canonical adjunction

$$\mathrm{Sym}_k\colon \mathbf{sSet} \leftrightarrows \mathbf{sAlg}_k \colon \mathcal{U}$$

where $\mathcal{U}$ is the obvious forgetful functor satisfies the hypothesis of the transfer principle, so that we can endow $\mathbf{sAlg}_k$ with a model structure where

1. a map $f\colon A \to B$ is a weak equivalence or a fibration if and only if the map $\mathcal{U}(f)$ is so;
2. a map $f\colon A \to B$ is a cofibration if and only if it has the left lifting property with respect to every trivial fibration.

We have a natural inclusion $i\colon \mathbf{Alg}_k \to \mathbf{sAlg}_k$ which defines a reflective subcategory. In fact one has the following:

Lemma B.1. *The functor $\pi_0\colon \mathbf{sAlg}_k \to \mathbf{Alg}_k$ is left adjoint to the inclusion functor i.*

Proof. Let A be any simplicial k-algebra and consider the k-algebra $\pi_0(A)$. We clearly have a morphism

$$\eta_A\colon A \to \pi_0(A)$$

defined by sending an n-simplex $a \in A_n$ into the path component of any of its vertices. The compatibility with the sum and the product is a natural consequence of the fact that the face maps of A are compatible with the algebra structure (i.e. $d_n\colon A_n \to A_{n-1}$ is a morphism of k-algebras).

If B is any discrete k-algebra and $\varphi\colon A \to B$ is any morphism we immediately obtain a morphism of k-algebras

$$\pi_0(\varphi)\colon \pi_0(A) \to \pi_0(B) = B$$

which moreover satisfies $\pi_0(\varphi) \circ \eta_A = \varphi$. The uniqueness of $\pi_0(\varphi)$ is clear, hence it follows that $\pi_0 \dashv i$ by the standard characterization of the adjuctions via the universal property of the unit. □

B.2 Modules over simplicial rings

Let $A \in \mathbf{sAlg}_k$ be a fixed simplicial k-algebra. The category of (simplicial) A-modules, denoted A-**Mod** inherits a model structure from $\mathbf{sAlg}_k$ using the classical result that can be found in [SS]. This category is naturally endowed with a forgetful functor

$$A\text{-}\mathbf{Mod} \to \mathbf{sSet}$$

which is right adjoint to

$$A[-]\colon \mathbf{sSet} \to A\text{-}\mathbf{Mod}$$

Definition B.2. Let A be a simplicial k-algebra. For every A-module M and any positive integer $n \geq 0$ set

$$M[n] := M \otimes_A A[S^n]$$

where S^n is a simplicial model for the n-sphere.

If M is an A-module, we can define its homotopy groups simply using the forgetful functor to **sSet**. With this definition one immediately obtains the following lemma:

Lemma B.3. *For any A-module M it holds*

$$\pi_n(N) \simeq \pi_0 \mathrm{Map}_{A\text{-}\mathbf{Mod}}(A[S^n], N)$$

Proof. One has to observe that setting $M \otimes K := M \otimes_A A[K]$ for any A-module M and any simplicial set K define a tensor over **sSet** which is in fact part of a simplicial model structure over A-**Mod** (see for example [Q, Chapter II.4]). It follows that

$$\mathrm{Map}_{A\text{-}\mathbf{Mod}}(A[S^n], N) \simeq \mathrm{Map}_{\mathbf{sSet}}(S^n, N)$$

and now the thesis follows by definition of $\pi_n(N)$. □

Since A-**Mod** is a pointed model category, it follows that we can define a suspension and a loop functor. More precisely, we consider the following definition:

Definition B.4. Let M be an A-module. The suspension of M is defined to be the homotopy pushout

$$\begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma(M) \end{tikzcd}$$

We define the loop functor in a similar way:

Definition B.5. Let M be an A-module. The loop of M is defined to be the homotopy pullback

$$\begin{tikzcd} \Omega(M) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & M \end{tikzcd}$$

With these definitions, we can prove that A-**Mod** is "almost stable", in the sense that Σ is not an equivalence, but $\Omega\Sigma(M) \simeq M$ for any simplicial module M. The result is essentially due to Quillen, see [Q, Proposition II.6.1]. We will need a preliminary result on the form of cofibrations of A-**Mod**.

Definition B.6. A map $f\colon M \to N$ in A-**Mod** is said to be *free* if there are subsets $C_q \subset N_q$ for each $q \in \mathbb{N}$ such that:

1. $\eta^* C_p \subseteq C_q$ whenever $\eta\colon \mathbf{q} \to \mathbf{p}$ is a surjective monotone map;
2. for every $q \in \mathbb{N}$ the map $(f_q, g_q)\colon M_q \oplus A[C_q] \to N_q$ is an isomorphism, where $g_q\colon A[C_q] \to N_q$ is the map induced by the inclusion $C_q \subseteq N_q$.

Remark B.7. A free morphism $f\colon M \to N$ in A-**Mod** is always degreewise injective. In fact, $M_q \to \oplus A[C_q]$ is injective, so that $f_q\colon M_q \to N_q$ is injective for each $q \in \mathbb{N}$.

Proposition B.8. *A morphism $f\colon M \to N$ in A-**Mod** is a cofibration if and only if it is a retract of a free map. In particular, every cofibration in A-**Mod** is degreewise injective.*

Proof. See [Q, Remark 4, page II.4.11] for a proof that every free map is a cofibration. The small object argument can be used to show that every map f admits a factorization as $f = pi$, where p is a trivial fibration and i is a free map. It follows that if f is a cofibration, then it is a retract of a free map. The second statement follows at once, since the retract of an injective map is still an injective map. □

Corollary B.9. *Let A be a simplicial k-algebra. Then for any A-module M there is a weak equivalence $\Omega\Sigma(M) \simeq M$.*

Proof. We have to show that if the square

$$\begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & N \end{tikzcd}$$

is a homotopy pushout, then it is also a homotopy pullback. First of all, we can suppose without loss of generality that M is a cofibrant object; next, we can replace the map $M \to 0$ with a cofibration $j\colon M \to D$ where D is weakly equivalent to 0. The dual of Corollary A.3 shows that the pushout

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d] & D \arrow[d, "p"] \\ 0 \arrow[r] & N' \end{tikzcd}$$

is an explicit model for the suspension of M. In other words, we have

$$\Sigma(M) \simeq N' := \mathrm{coker}(j)$$

Now, N' and D are fibrant objects and $p\colon D \to N'$ is a surjective map, hence it is a fibration. It follows again from Corollary A.3 that $\ker(p)$ is an explicit model for $\Omega(N')$. Since Proposition B.8 implies that j is injective, we see that

$$M \simeq \ker(j) \simeq \Omega\Sigma(M)$$

completing the proof. □

B.3 Derived derivations and cotangent complex

Recall the following definition of derived derivation:

Definition B.10. Let A be a simplicial k-algebra and let B be an A-algebra. An A-derivation of B with values in a B-module M is a section of $B \oplus M \to B$, where $B \oplus M$ is defined by performing the classical square-zero extension degreewise.

Remark B.11. Fix two simplicial k-algebras A and C. The previous definition gives rise to a bifunctor

$$s\colon A/\mathbf{sAlg}_k/B \times B\text{-}\mathbf{Mod} \to A/\mathbf{sAlg}_k/B$$

defined by

$$s\colon (A \to C \to B, M) \mapsto C \oplus M$$

where M is thought as C-module by forgetting along the given map $C \to B$.

We have also another functor

$$\pi\colon A/\mathbf{sAlg}_k/B \times B\text{-}\mathbf{Mod} \to A/\mathbf{sAlg}_k/B$$

defined simply by

$$\pi\colon (A \to C \to B, M) \mapsto A \to C \to B$$

Finally, we have a natural transformation $p\colon s \to \pi$ which assigns to the pair (C, M) in $A/\mathbf{sAlg}_k/B \times B\text{-}\mathbf{Mod}$ the natural projection

$$C \oplus M \to C$$

We will denote by abuse of notation this map p_C (instead of $p_{C,M}$). These are easy checks left to the reader.

The set of A-derivations of B into M is naturally endowed with a k-module structure, which allows to define a functor

$$\mathrm{Der}_A(B, -)\colon B\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$$

We can see this functor as the π_0 of another, much more interesting functor

$$\mathbb{D}er_A(B, -)\colon B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$$

defined by

$$\mathbb{D}er_A(B, M) := \mathrm{Map}_{A/\mathbf{sAlg}_k/B}(B, B \oplus M)$$

Lemma B.12. *The functor $\mathbb{D}er_A(B, -)$ is representable by a simplicial B-module $\mathbb{L}_{B/A}$. In particular, it is a left Quillen functor.*

Proof. Let $Q(B)$ be a cofibrant replacement for B in the model category $A/\mathbf{sAlg}_k$. Define

$$\mathbb{L}_{B/A} := \Omega^1_{Q(B)/A} \otimes_{Q(B)} B$$

where the construction of $\Omega^1_{Q(B)/A}$ is meant to be performed degreewise. It can be checked that $\mathbb{L}_{B/A}$ is the desired representative (see for example [HAG-II, Chapter I.1]).

The second part of the statement follows from the fact that $\mathrm{Map}_B(\mathbb{L}_{B/A}, -)$ is right adjoint to $- \otimes_B \mathbb{L}_{B/A}$ and the fact that $\mathrm{Map}_B(\mathbb{L}_{B/A}, -)$ respects fibrations and trivial fibrations (since it is defined as the internal hom of $\mathbf{sSet}$). □

Lemma B.13. *Let $f\colon A \to B$ be a morphism of C-algebras and let $g\colon M \to N$ be a morphism of B-module. Any commutative triangle of A-modules*

$$\begin{tikzcd} \mathbb{L}_{A/C} \arrow[r, "u"] \arrow[d, "\mathbb{L}(f)"'] & M \arrow[d, "g"] \\ \mathbb{L}_{B/C} \arrow[r, "v"'] & N \end{tikzcd}$$

gives rise to a commutative diagram of C-algebras as follows:

$$\begin{tikzcd} A \arrow[r, "d_u"] \arrow[d, "f"'] & A \oplus M \arrow[d, "s(f,g)"] \\ B \arrow[r, "d_v"] & B \oplus N \end{tikzcd}$$

where $s(f,g)$ denotes the bifunctor of Remark B.11 and d_u, d_v are the C-derivation induced by the universal property of the cotangent complexes $\mathbb{L}_{A/C}$ and $\mathbb{L}_{B/C}$.

Proof. Using the notations of Remark B.11, we see that d_u is a section of p_A, and moreover naturality yields

$$f \circ p_A = p_B \circ s(f,g)$$

Since p_A is an epimorphism, we conclude that the equality

$$s(f,g) \circ d_u = d_v \circ f$$

holds if and only if

$$s(f,g) \circ d_u \circ p_A = d_v \circ f \circ p_A$$

and now this follows from the already stated properties. □

References

[Hi] P. Hirschhorn, *Model categories and their localizations*, AMS Monographs

[HTT] J. Lurie, *Higher Algebra*, www.math.harvard.edu/ lurie/.

[HA] J. Lurie, *Higher Algebra*, http://www.math.harvard.edu/∼lurie/papers/higheralgebra.pdf

[Q] D. Quillen, *Homotopical algebra*, LNM 43, 1967.

[HAG-II] B. Toën, G. Vezzosi, *Homotopical algebraic geometry II: Geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.

[SS] S. Schwede, Shipley, *Algebras and modules in monoidal model categories*, Proceedings of London Mathematical Society (80), 2000

[GJ] P. Goerss, J.F. Jardine, *Sipmlicial Homotopy Theory*, Birkaüser, 1999