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Higher algebraic K-theory for actions of diagonalizable groups

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Abstract. We study the K-theory of actions of diagonalizable group schemes on noetherian regular separated algebraic spaces: our main result shows how to reconstruct the K-theory ring of such an action from the K-theory rings of the loci where the stabilizers have constant dimension. We apply this to the calculation of the equivariant K-theory of toric varieties, and give conditions under which the Merkurjev spectral sequence degenerates, so that the equivariant K-theory ring determines the ordinary K-theory ring. We also prove a very refined localization theorem for actions of this type.

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Introduction

Fix a base noetherian separated connected scheme S, and let G be a diagonalizable group scheme of finite type over S (see [SGA3, Exposé VII]); recall that this means that G is the product of finitely many multiplicative

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Mathematics Subject Classification (2000): 19E08, 14L30 groups $\mathbb{G}_{\mathrm{m},S}$ and group schemes $\boldsymbol{\mu}_{n,S}$ of n^{th} roots of 1 for various values of n. Suppose that G acts on a separated noetherian regular algebraic space X over S.

If G acts on X with finite stabilizers, then [Ve-Vi] gives a decomposition theorem for the equivariant higher K-theory ring $\mathrm{K}_*(X, G)$; it says that, after inverting some primes, $\mathrm{K}_*(X, G)$ is a product of certain factor rings $\mathrm{K}_*(X^\sigma, G)_\sigma$ for each subgroup scheme $\sigma \subseteq G$ with $\sigma \simeq \boldsymbol{\mu}_n$ for some n and $X^\sigma \neq \emptyset$ (the primes to be inverted are precisely the ones dividing the orders of the σ). A slightly weaker version of this theorem was given in [Toen]. From this one can prove analogous formulas assuming that the stabilizers are of constant dimension (Theorem 7.4).

This paper deals with the general case, when the dimensions of the stabilizers are allowed to jump. In this case one sees already in the simplest examples that $\mathrm{K}_*(X, G)$ will not decompose as a product, not even after tensoring with $\mathbb{Q}$; for example, if S is a field, G is a torus and X is a representation of G, then $\mathrm{K}_0(X, G)$ is the ring of representations $\mathrm{R}G$, which is a ring of Laurent polynomials over $\mathbb{Z}$.

However, we show that the ring $\mathrm{K}_*(X, G)$ has a canonical structure of fibered product. More precisely, for each integer s we consider the locus X_s of X where the stabilizers have dimension precisely equal to s; this is a locally closed regular subspace of X. For each s consider the normal bundle N_s of X_s in X, and the subspace $\mathrm{N}_{s,s-1}$ where the stabilizers have dimension precisely $s-1$. There is a pullback map $\mathrm{K}_*(X_s, G) \to \mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$; furthermore in Sect. 3 we define a specialization homomorphism $\mathrm{Sp}^{s-1}_{X,s} \colon \mathrm{K}_*(X_{s-1}, G) \to \mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$, via deformation to the normal bundle. Our first main result (Theorem 4.5) shows that these specializations homomorphisms are precisely what is needed to reconstruct the equivariant K-theory of X from the equivariant K-theory of the strata.

Theorem 1 (The theorem of reconstruction from the strata). *Let n be the dimension of G. The restriction homomorphisms*

$$\mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(X_s, G)$$

induce an isomorphism

$$\begin{aligned}\mathrm{K}_*(X, G) \simeq \mathrm{K}_*(X_n, G) &\underset{\mathrm{K}_*(\mathrm{N}_{n,n-1}, G)}{\times} \mathrm{K}_*(X_{n-1}, G) \underset{\mathrm{K}_*(\mathrm{N}_{n-1,n-2}, G)}{\times} \\ &\dots \underset{\mathrm{K}_*(\mathrm{N}_{2,1}, G)}{\times} \mathrm{K}_*(X_1, G) \underset{\mathrm{K}_*(\mathrm{N}_{1,0}, G)}{\times} \mathrm{K}_*(X_0, G).\end{aligned}$$

In other words: the restrictions $\mathrm{K}_*(X, G) \to \mathrm{K}_*(X_s, G)$ induce an injective homomorphism $\mathrm{K}_*(X, G) \to \prod_s \mathrm{K}_*(X_s, G)$, and an element $(\alpha_n, \dots, \alpha_0)$ of the product $\prod_s \mathrm{K}_*(X_s, G)$ is in the image of $\mathrm{K}_*(X, G)$ if and only if the pullback of $\alpha_s \in \mathrm{K}_*(X_s, G)$ to $\mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$ coincides with $\mathrm{Sp}^{s-1}_{X,s}(\alpha_{s-1}) \in \mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$ for all $s = 1, \dots, n$.

This theorem is a powerful tool in studying the K-theory of diagonalizable group actions. From it one gets easily a description of the higher equivariant K-theory of regular toric varieties (Theorem 6.2). This is analogous to the description of their equivariant Chow ring in [Bri97, Theorem 5.4].

One can put Theorem 1 above together with the main result of [Ve-Vi] to give a very refined description of $\mathrm{K}_*(X, G)$; this is Theorem 7.12, which can be considered the ultimate localization theorem for actions of diagonalizable groups. However, notice that it does not supersede Theorem 1, because Theorem 1 holds with integral coefficients, while for the formula of Theorem 7.12 to be correct we have to invert some primes.

Many results are known for equivariant intersection theory, or for equivariant cohomology; often one can use our theorem to prove their K-theoretic analogues. For example, consider the following theorem of Brion, inspired in turn by results in equivariant cohomology due to Atiyah ([Ati74]), Bredon ([Bre74]), Hsiang ([Hsi75]), Chang and Skjelbred ([Ch-Sk74]), Kirwan ([Kir84]), Goresky, Kottwitz and MacPherson ([G-K-MP98]); see also the very useful discussion in [Bri98].

Theorem ([Bri97, 3.2, 3.3]). *Suppose that X is a smooth projective algebraic variety over an algebraically closed field with an action of an algebraic torus G.*

(i) *The rational equivariant Chow ring $\mathrm{A}^*_G(X)_{\mathbb{Q}}$ is free as a module over $\mathrm{A}^*_G(\mathrm{pt})_{\mathbb{Q}}$.*
(ii) *The restriction homomorphism*

$$\mathrm{A}^*_G(X)_{\mathbb{Q}} \longrightarrow \mathrm{A}^*_G(X^G)_{\mathbb{Q}} = \mathrm{A}^*(X^G)_{\mathbb{Q}} \otimes \mathrm{A}^*_G(\mathrm{pt})_{\mathbb{Q}}$$

is injective, and its image is the intersection of all the images of the restriction homomorphisms $\mathrm{A}^*_G(X^T)_{\mathbb{Q}} \to \mathrm{A}^*_G(X^G)_{\mathbb{Q}}$, where T ranges over all the subtori of codimension 1.

From this one gets a very simple description of the rational equivariant Chow ring when the fixed point locus X^G is zero dimensional, and the fixed point set X^T is at most 1-dimensional for any subtorus $T \subseteq G$ of codimension 1 ([Bri97, Theorem 3.4]).

In this paper we prove a version of Brion's theorem for algebraic K-theory. Remarkably, it holds with integral coefficients: we do not need to tensor with $\mathbb{Q}$. This confirms the authors' impression that when it comes to torsion, K-theory tends to be better behaved than cohomology, or intersection theory.

The following is a particular case of Corollary 5.11; when G is a torus, it is an an analogue of part (2) of Brion's theorem.

Theorem 2. *Suppose that G is a diagonalizable group acting a smooth proper scheme X over a perfect field; denote by G_0 the toral component of G, that is, the largest subtorus contained in G.*

Then the restriction homomorphism $\mathrm{K}_*(X, G) \to \mathrm{K}_*(X^{G_0}, G)$ *is injective, and its image equals the intersection of all the images of the restriction homomorphisms* $\mathrm{K}_*(X^T, G) \to \mathrm{K}_*(X^{G_0}, G)$ *for all the subtori* $T \subseteq G$ *of codimension 1.*

From this one gets a very complete description of $\mathrm{K}_*(X, G)$ when G is a torus and X is smooth and proper over an algebraically closed field, in the "generic" situation when X contains only finitely many invariant points and finitely many invariant curves (Corollary 5.12).

We also analyze the case of smooth toric varieties in detail in Sect. 6.

The analogue of Theorem 2 should hold for the integral equivariant topological K-theory of a compact differentable manifold with the action of a compact torus. Some related topological results are contained in [Ro-Kn].

Description of contents. Section 1 contains the setup that will be used throughout this paper. The K-theory that we use is the one described in [Ve-Vi]: see the discussion in Subsect. 1.1.

Section 2 contains some preliminary technical results; the most substantial of these is a very general self-intersection formula, proved following closely the proof of Thomason of the analogous formula in the non-equivariant case ([Tho93, Théorème 3.1]). Here we also discuss the stratification by dimensions of stabilizers, which is our basic object of study.

In Sect. 3 we define various types of specializations to the normal bundle in equivariant K-theory. This is easy for K_0, but for the whole higher K-theory ring we do not know how to give a definition in general without using the language of spectra.

Section 4 contains the proof of Theorem 1.

Section 5 is dedicated to the analysis of the case when X is complete, or, more generally, admits enough limits (Definition 5.8). The condition that $\mathrm{K}_*(X, G)$ be free as a module over the representation ring $\mathrm{R}G$ is not adequate when working with integral coefficients: here we analyze a rather subtle condition on the $\mathrm{R}G$-module $\mathrm{K}_*(X, G)$ that ensures that the analogue of Brion's theorem above holds, then we show, using a Białynicki-Birula stratification, that this condition is in fact satisfied when X admits enough limits over a perfect field.

We also apply our machinery to show that the degeneracy of the Merkurjev spectral sequence in [Mer97], that he proves when X is smooth and projective, in fact happens for torus actions with enough limits.

Section 6 is dedicated to the K-theory of smooth toric varieties. For any smooth toric variety X for a torus T, we give two descriptions of $\mathrm{K}_*(X, T)$. First of all, we show how Theorem 4.5 in this case gives a simple description of it as a subring of a product of representation rings, analogous to the description of its equivariant Chow ring in [Bri97, Theorem 5.4]. Furthermore, we give a presentation of $\mathrm{K}_*(X, T)$ by generators and relations over the K-theory ring of the base field (Theorem 6.4), analogous to the classical Stanley–Reisner presentation for its equivariant cohomology first obtained in [B-DC-P90]. For K_0 the result is essentially stated in [Kly83].

In Sect. 7 we generalize the result of [Ve-Vi] by giving a formula that holds for all actions of diagonalizable groups on regular noetherian algebraic spaces, irrespective of the dimensions of the stabilizers (Theorem 7.12).

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1. Notations and conventions

Throughout the paper we fix a base scheme S, that is assumed to be connected, separated and noetherian.

We will denote by G a diagonalizable group scheme of finite type over S (see [SGA3]), except when otherwise mentioned. Its groups of characters is $\widehat{G} \stackrel{\text{def}}{=} \operatorname{Hom}_S(G, \mathbb{G}_{\mathrm{m},S})$; the contravariant functor from the category of diagonalizable groups schemes of finite type over S to the category of finitely generated abelian groups given by $G \mapsto \widehat{G}$ is an antiequivalence of categories. The ring of representations of G is, by definition, $\mathrm{R}G = \mathbb{Z}\widehat{G}$, and furthermore $G = \operatorname{Spec} \mathrm{R}G \times_{\operatorname{Spec}\mathbb{Z}} S$.

We will denote by G_0 the *toral part* of G, that is, the largest subtorus of G. The group of characters $\widehat{G}_0$ is the quotient of $\widehat{G}$ by its torsion subgroup.

A G-space will always be a regular separated noetherian algebraic space over S over which G acts; sometimes we will talk about a *regular* G-space, for emphasis.

We notice explicitly that if $S' \to S$ is a morphism of schemes, with S' connected, then every diagonalizable subgroup scheme of $G \times_S S'$ is obtained by base change from a unique diagonalizable subgroup scheme of G. This will be used as follows: if $p : \operatorname{Spec}\Omega \to X$ is a geometric point, then we will refer to its stabilizer, which is a priori a subgroup scheme of $G \times_S \operatorname{Spec} \Omega$, as a subgroup scheme of G.

If $Y \hookrightarrow X$ is a regular embedding, we denote by $\mathrm{N}_Y X$ the normal bundle.

1.1. Equivariant K-theory. In this subsection G will be a group scheme over S that is flat, affine and of finite type. We use the same K-theoretic setup as in [Ve-Vi], that uses the language of [Th-Tr90]. The following is a slight extension of [Ve-Vi, Theorem 6.4].

Proposition 1.1. *Let G be flat affine separated group scheme of finite type over S, acting over a noetherian regular separated scheme X over S. Consider the following complicial bi-Waldhausen categories:*

(i) *the category* $\mathrm{W}_1(X, G)$ *of complexes of quasicoherent G-equivariant* $\mathcal{O}_X$*-modules with bounded coherent cohomology;*
(ii) *the category* $\mathrm{W}_2(X, G)$ *of bounded complexes of coherent G-equivariant* $\mathcal{O}_X$*-modules;*
(iii) *the category* $\mathrm{W}_3(X, G)$ *of complexes of flat quasicoherent G-equivariant* $\mathcal{O}_X$*-modules with bounded coherent cohomology, and*
(iv) *the category* $\mathrm{W}_4(X, G)$ *of bounded above complexes of G-equivariant quasi-coherent flat* $\mathcal{O}_X$*-Modules with bounded coherent cohomology.*

Then the inclusions

$$\mathrm{W}_2(X, G) \subseteq \mathrm{W}_1(X, G) \quad \textit{and} \quad \mathrm{W}_4(X, G) \subseteq \mathrm{W}_3(X, G) \subseteq \mathrm{W}_1(X, G)$$

induce isomorphisms on the corresponding Waldhausen K-theories. Furthermore the K-theory of any of the categories above coincides with the Quillen K-theory $\mathrm{K}'_*(X, G)$ *of the category of G-equivariant coherent* $\mathcal{O}_X$*-modules.*

Proof. For the first three categories, and the Quillen K-theory, the statement is precisely [Ve-Vi, Theorem 6.4].

Let us check that the inclusion $\mathrm{W}_4(X, G) \subseteq \mathrm{W}_1(X, G)$ induces an isomorphism in K-theory. By [Ve-Vi, Proposition 6.2], which shows that hypothesis 1.9.5.1 is satisfied, we can apply [Th-Tr90, Lemma 1.9.5], in the situation where $\mathcal{A}$ is the category of G-equivariant quasicoherent $\mathcal{O}_X$-Modules, $\mathcal{C}$ the category of cohomologically bounded complexes in $\mathcal{A}$, $\mathcal{D}$ the category of G-equivariant quasicoherent flat $\mathcal{O}_X$-Modules, $F : \mathcal{D} \hookrightarrow \mathcal{A}$ is the natural inclusion. In particular, any complex in $\mathrm{W}_1(X, G)$ receives a quasi-isomorphism from a complex in $\mathrm{W}_4(X, G)$. That is, [Th-Tr90, 1.9.7.1], applied to the inclusion $\mathrm{W}_4(X, G) \hookrightarrow \mathrm{W}_1(X, G)$, is satisfied; since the other hypothesis 1.9.7.0 of [Th-Tr90, 1.9.7] is obviously satisfied, we conclude by [Th-Tr90, Theorem 1.9.8]. □

We will denote by $\mathbb{K}(X, G)$ the Waldhausen K-theory spectrum and by $\mathrm{K}_*(X, G)$ the Waldhausen K-theory group of any of the categories above. As observed in [Ve-Vi, p. 39], it follows from results of Thomason that

$\mathrm{K}_*(-, G)$ is a covariant functor for proper maps of noetherian regular separated G-algebraic spaces over S; furthermore, each $\mathrm{K}_*(X, G)$ has a natural structure of a graded ring, and each equivariant morphism $f: X \to Y$ of noetherian regular separated G-algebraic spaces over S induces a pullback $f^*\colon \mathrm{K}_*(Y, G) \to \mathrm{K}_*(X, G)$, making $\mathrm{K}_*(-, G)$ into a contravariant functor from the category of noetherian regular separated G-algebraic spaces over S to graded-commutative rings. Furthermore, if $i: Y \hookrightarrow X$ is a closed embedding of noetherian regular separated G-algebraic spaces and $j: X \setminus Y \hookrightarrow X$ is the open embedding, then $\mathbb{K}(X \setminus Y, G)$ is the cone of the pushforward map $i_*\colon \mathbb{K}(Y, G) \to \mathbb{K}(X, G)$ ([Tho87, Theorem 2.7]), so there is an exact localization sequence

$$\cdots \longrightarrow \mathrm{K}_n(Y, G) \xrightarrow{i_*} \mathrm{K}_n(X, G) \xrightarrow{j^*} \mathrm{K}_n(X \setminus Y, G) \xrightarrow{\partial} \mathrm{K}_{n-1}(Y, G) \longrightarrow \cdots$$

Furthermore, if $\pi: E \to X$ is a G-equivariant vector bundle, the pullback

$$\pi^*\colon \mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(E, G)$$

is an isomorphism ([Tho87, Theorem 4.1]).

2. Preliminary results

2.1. The self-intersection formula. Here we generalize Thomason's self-intersection formula ([Tho93, Théorème 3.1]) to the equivariant case.

Theorem 2.1 (The self-intersection formula). *Suppose that a flat group scheme G separated and of finite type over S acts over a noetherian regular separated algebraic space X. Let $i: Z \hookrightarrow X$ be a regular G-invariant closed subspace of X. Then*

$$i^* i_*\colon \mathbb{K}(Z, G) \longrightarrow \mathbb{K}(Z, G),$$

coincides up to homotopy with the cup product

$$\lambda_{-1}\left(\mathrm{N}_Z^{\vee} X\right) \smile (-)\colon \mathbb{K}(Z, G) \longrightarrow \mathbb{K}(Z, G),$$

where $\mathrm{N}_Z^{\vee} X$ is the conormal sheaf of Z in X.

In particular, we have the equality

$$i^* i_* = \lambda_{-1}\left(\mathrm{N}_Z^{\vee} X\right) \smile (-)\colon \mathrm{K}_*(Z, G) \longrightarrow \mathrm{K}_*(Z, G).$$

Proof. The proof follows closely Thomason's proof of [Tho93, Théorème 3.1], therefore we will only indicate the changes we need for that proof to adapt to our situation.

Let us denote by $\mathrm{W}'(Z, G)$ the Waldhausen category consisting of pairs $(E^\bullet, \lambda : L^{\bullet\bullet} \to i_* E^\bullet)$ where $E^\bullet$ is a bounded above complex of

G-equivariant quasi-coherent flat $\mathcal{O}_Z$-Modules with bounded coherent cohomology, $L^{\bullet\bullet}$ is a bicomplex of G-equivariant quasi-coherent flat $\mathcal{O}_X$-Modules with bounded coherent total cohomology such that $L^{ij} = 0$ for $j \leq 0$, any i and also $L^{ij} = 0$ for $i > N$, for some integer N, any i; finally $\lambda : L^{\bullet\bullet} \to i_*E^\bullet$ is an exact augmentation of the bicomplex $L^{\bullet\bullet}$. In particular, for any i, the horizontal complex $L^{i\bullet}$ is a flat resolution of i_*E^i. The morphisms, cofibrations and weak equivalences in $\mathrm{W}'(Z, G)$ are as in [Tho93, 3.3, p. 209]. Thomason [Tho93, 3.3] shows that the forgetful functor $(E^\bullet, \lambda : L^{\bullet\bullet} \to i_*E^\bullet) \mapsto E^\bullet$ from $\mathrm{W}'(Z, G)$ to the category $\mathrm{W}_4(Z, G)$ of bounded above complexes of G-equivariant quasi-coherent flat $\mathcal{O}_Z$-Modules with bounded coherent cohomology induces a homotopy equivalence between the associated Waldhausen K-theory spectra. In other words, by Proposition 1.1, we can (and will) use $\mathrm{W}'(Z, G)$ as a "model" for $\mathbb{K}(Z, G)$.

With these choices, the morphism of spectra $i^*i_* : \mathbb{K}(Z, G) \longrightarrow \mathbb{K}(Z, G)$ can be represented by the exact functor $\mathrm{W}'(Z, G) \longrightarrow \mathrm{W}_1(Z, G)$ which sends $(E^\bullet, \lambda : L^{\bullet\bullet} \to i_*E^\bullet)$ to the total complex of the bicomplex $i^*(L^{\bullet\bullet})$.

The rest of the proof is exactly the same as in [Tho93, 3.3, pp. 210–212]. One first consider functors $T_k : \mathrm{W}'(Z, G) \to \mathrm{W}_1(Z, G)$ sending an object $(E^\bullet, \lambda : L^{\bullet\bullet} \to i_*E^\bullet)$ to the total complex of the bicomplex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \operatorname{im} \partial_h^{i,-k-1} \arrow[r] \arrow[d] & i^*L^{i,-k} \arrow[r] \arrow[d] & \cdots \arrow[r] & i^*L^{i,-1} \arrow[r] \arrow[d] & i^*L^{i,0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im} \partial_h^{i+1,-k-1} \arrow[r] \arrow[d] & i^*L^{i+1,-k} \arrow[r] \arrow[d] & \cdots \arrow[r] & i^*L^{i+1,-1} \arrow[r] \arrow[d] & i^*L^{i+1,0} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & & \vdots & \vdots &
\end{tikzcd}$$

which results from truncating all the horizontal complexes of $i^*L^{\bullet\bullet}$ at the k-th level.

The functors T_k are zero for $k < 0$ and come naturally equipped with functorial epimorphisms $T_k \twoheadrightarrow T_{k-1}$ whose kernel h_k has the property that $h_k(E^\bullet, \lambda : L^{\bullet\bullet} \to i_*E^\bullet)$ is quasi isomorphic to $\Lambda^k(\mathrm{N}_Z^\vee X) \otimes_{\mathcal{O}_Z} i^*E^\bullet[k]$ ([Tho93, 3.4.4], essentially because each horizontal complex in $L^{i\bullet}$ is a flat resolution of i_*E^i. Therefore, by induction on $k \geq -1$, starting from $T_{-1} = 0$, each T_k has values in $\mathrm{W}_1(Z, G)$ and preserves quasi-isomorphisms. Moreover, the arguments in [Tho93, 3.4, pp. 211–212], show that T_k actually preserves cofibrations and pushouts along cofibrations; hence each $T_k : \mathrm{W}'(Z, G) \longrightarrow \mathrm{W}_1(Z, G)$ is an exact functor of Waldhausen categories.

As in [Tho93, 3.4, p. 212], the quasi-isomorphism

$$h_k(E^\bullet, \lambda : L^{\bullet\bullet} \to i_* E^\bullet) \simeq \Lambda^k \left(\mathrm{N}_Z^\vee X\right) \otimes_{\mathcal{O}_Z} i^* E^\bullet[k]$$

shows that the canonical truncation morphism $i^*(L^{\bullet\bullet}) \to T_d(E^\bullet, \lambda : L^{\bullet\bullet} \to i_* E^\bullet)$, d being the codimension of Z in X, is a quasi-isomorphism, i.e. the morphism of spectra $i^* i_* : \mathbb{K}(Z, G) \longrightarrow \mathbb{K}(Z, G)$ can also be represented by the exact functor $T_d : \mathrm{W}'(Z, G) \longrightarrow \mathrm{W}_1(Z, G)$. Now, the Additivity Theorem ([Th-Tr90, 1.7.3 and 1.7.4]) shows that the canonical exact sequences of functors $h_k \hookrightarrow T_k \twoheadrightarrow T_{k-1}$ yield up-to-homotopy equalities $T_k = T_{k-1} + h_k$ between the induced map of spectra. And finally, recalling that a shift $[k]$ induces multiplication by $(-1)^k$ at the level of spectra, by induction on $k \geq -1$ we get equalities up to homotopy

$$\begin{aligned} i^* i_* &= T_d(-) \\ &= \sum_k h_k(-) \\ &= \sum_k \left[\Lambda^k \left(\mathrm{N}_Z^\vee X\right)\right] \otimes_{\mathcal{O}_Z} i^*(-)[k] \\ &= \lambda_{-1}\left(\mathrm{N}_Z^\vee X\right) \smile (-) \end{aligned}$$

of morphisms of spectra $\mathbb{K}(Z, G) \to \mathbb{K}(Z, G)$ ([Tho93, p. 212]). □

2.2. Stratification by dimensions of stabilizers. Let G be a diagonalizable group scheme of finite type acting on X as usual. Consider the group scheme $H \to X$ of stabilizers of the action. Since for a point $x \in X$ the dimension of the fiber H_x equals its dimension at the point $\gamma(x)$, where $\gamma : X \to H$ is the unit section, it follows from Chevalley's theorem ([EGAIV, 13.1.3]) that there is an open subset $X_{\leq s}$ where the fibers of H have dimension at most s. We will use also $X_{<s}$ with a similar meaning. We denote by X_s the locally closed subset $X_{\leq s} \setminus X_{<s}$; we will think of it as a subspace of X with the reduced scheme structure. Finally, we call N_s the normal bundle of X_s in X, and N_s^0 the complement of the 0-section in N_s. Notice that G acts on N_s, so we may consider the subscheme $(\mathrm{N}_s)_{<s} \subseteq \mathrm{N}_s$.

Proposition 2.2. *Let s be a nonzero integer.*

(i) *There exists a finite number of s-dimensional subtori $T_1, \ldots, T_r$ in G such that X_s is the disjoint union of the $X_{\leq s}^{T_j}$.*
(ii) *X_s is a regular locally closed subspace of X.*
(iii) $\mathrm{N}_s^0 = (\mathrm{N}_s)_{<s}$.

Proof. To prove part (i) we may restrict the action of G to its toral component. By Thomason's generic slice theorem ([Tho86a, Proposition 4.10]) there are only finitely many possible diagonalizable subgroup schemes of G that appear as stabilizers of a geometric point of X. Then we can take the T_j to be the toral components of the s-dimensional stabilizers.

Parts (ii) and (iii) follow from (i) and [Tho92, Proposition 3.1]. □

3. Specializations

In this section G will be a flat, affine and separated group scheme of finite type over S, acting on a noetherian regular separated algebraic space Y over S.

Definition 3.1. *A G-invariant morphism $Y \to \mathbb{P}^1_S$ is* regular at infinity *if the inverse image Y_∞ of the section at infinity in $\mathbb{P}^1_S$ is a regular effective Cartier divisor on Y.*

Theorem 3.2. *Let $\pi\colon Y \to \mathbb{P}^1_S$ be a G-invariant morphism over S that is regular at infinity. Denote by $i_\infty\colon Y_\infty \hookrightarrow Y$ the inclusion of the fiber at infinity, $j_\infty\colon Y \setminus Y_\infty \hookrightarrow Y$ the inclusion of the complement. Then there exists a specialization homomorphism of graded groups*

$$\mathrm{Sp}_Y\colon \mathrm{K}_*(Y \setminus Y_\infty, G) \longrightarrow \mathrm{K}_*(Y_\infty, G)$$

such that the composition

$$\mathrm{K}_*(Y, G) \xrightarrow{j_\infty^*} \mathrm{K}_*(Y \setminus Y_\infty) \xrightarrow{\mathrm{Sp}_Y} \mathrm{K}_*(Y_\infty, G)$$

coincides with $i_\infty^*\colon \mathrm{K}_*(Y, G) \to \mathrm{K}_*(Y_\infty, G)$.

Furthermore, if Y' is another noetherian separated regular algebraic space over S and $f\colon Y' \to Y$ is a G-equivariant morphism over S such that the composition $\pi f\colon Y' \to \mathbb{P}^1_S$ is regular at infinity, then the diagram

$$\begin{tikzcd}
\mathrm{K}_*(Y \setminus Y_\infty, G) \arrow[r, "\mathrm{Sp}_Y"] \arrow[d, "f^*"] & \mathrm{K}_*(Y_\infty, G) \arrow[d, "f_\infty^*"] \\
\mathrm{K}_*(Y' \setminus Y'_\infty, G) \arrow[r, "\mathrm{Sp}_{Y'}"] & \mathrm{K}_*(Y'_\infty, G)
\end{tikzcd}$$

commutes; here f_∞ is the restriction of f to $Y'_\infty \to Y_\infty$.

We refer to this last property as *the compatibility of specializations*.

Proof. Let us denote by $\mathbb{K}(X, G)$ the Quillen K-theory spectrum associated with the category of coherent equivariant G-sheaves on a noetherian separated algebraic space X. There is a homotopy equivalence

$$\mathrm{Cone}\big(\mathbb{K}(Y_\infty, G) \xrightarrow{i_{\infty *}} \mathbb{K}(Y, G)\big) \simeq \mathbb{K}(Y \setminus Y_\infty, G).$$

The commutative diagram

$$\begin{tikzcd}
\mathbb{K}(Y_\infty, G) \arrow[r, "i_{\infty *}"] \arrow[d, equal] & \mathbb{K}(Y, G) \arrow[d, "i_\infty^*"] \\
\mathbb{K}(Y_\infty, G) \arrow[r, "i_\infty^* i_{\infty *}"] & \mathbb{K}(Y_\infty, G)
\end{tikzcd}$$

induces a morphism of spectra

$$\begin{aligned} \mathbb{K}(Y \setminus Y_\infty, G) &\simeq \operatorname{Cone}\bigl(\mathbb{K}(Y_\infty, G) \xrightarrow{i_{\infty *}} \mathbb{K}(Y, G)\bigr) \\ &\longrightarrow \operatorname{Cone}\bigl(\mathbb{K}(Y_\infty, G) \xrightarrow{i_\infty^* i_{\infty *}} \mathbb{K}(Y_\infty, G)\bigr). \end{aligned} \tag{3.1}$$

By the self-intersection formula (Theorem 2.1) there is a homotopy

$$i_\infty^* i_{\infty *} \simeq \lambda_{-1}\left(\mathrm{N}_{Y_\infty}^\vee\right) \smile (-)\colon\ \mathbb{K}(Y_\infty, G) \to \mathbb{K}(Y_\infty, G);$$

on the other hand $\lambda_{-1}(N_{Y_\infty}^\vee) \smile (-)$ is homotopic to zero, because $\mathrm{N}_{Y_\infty}^\vee$ is trivial. So we have that

$$\operatorname{Cone}\bigl(\mathbb{K}(Y_\infty, G) \xrightarrow{i_\infty^* i_{\infty *}} \mathbb{K}(Y_\infty, G)\bigr) \simeq \mathbb{K}(Y_\infty, G)[1] \oplus \mathbb{K}(Y_\infty, G),$$

where $(-)[1]$ is the suspension of $(-)$. We define the specialization morphism of spectra

$$\mathcal{S}_Y\colon\ \mathbb{K}(Y \setminus Y_\infty, G) \longrightarrow \mathbb{K}(Y_\infty, G)$$

by composing the morphism (3.1) with the canonical projection

$$\mathbb{K}(Y_\infty, G)[1] \oplus \mathbb{K}(Y_\infty, G) \longrightarrow \mathbb{K}(Y_\infty, G).$$

Finally, Sp_Y is defined to be the homomorphism induced by $\mathcal{S}_Y$ on homotopy groups.

Let us check compatibility; it suffices to show that the diagram of spectra

$$\begin{tikzcd} \mathbb{K}(Y'_\infty, G) \arrow[r, "i'_{\infty *}"] \arrow[d, "f_\infty^*"] & \mathbb{K}(Y', G) \arrow[d, "f^*"] \\ \mathbb{K}(Y_\infty, G) \arrow[r, "i_{\infty *}"] & \mathbb{K}(Y, G) \end{tikzcd}$$

commutes up to homotopy. The essential point is that the diagram of algebraic spaces

$$\begin{tikzcd} Y'_\infty \arrow[r, "i'_\infty"] \arrow[d, "f_\infty"] & Y' \arrow[d, "f"] \\ Y_\infty \arrow[r, "i_\infty"] & Y \end{tikzcd}$$

is Tor-independent, that is, $\operatorname{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_{Y'}, \mathcal{O}_{Y_\infty}) = 0$ for all $i > 0$. Write $\mathrm{W}(Y', G)$ and $\mathrm{W}(Y'_\infty, G)$ for the Waldhausen categories of G-equivariant complexes of quasicoherent $\mathcal{O}_{Y'}$-modules and $\mathcal{O}_{Y'_\infty}$-modules with bounded coherent cohomology, while $\mathrm{W}(Y, G)$ and $\mathrm{W}(Y_\infty, G)$ will denote the Waldhausen categories of complexes of G-equivariant quasicoherent $\mathcal{O}_Y$-modules and $\mathcal{O}_{Y_\infty}$ bounded coherent cohomology, that are respectively degreewise

f^*-acyclic and degreewise f_∞^*-acyclic. By the Tor-independence of the diagram above, we have that $i_{\infty *}$ gives a functor $\mathrm{W}(Y_\infty, G) \to \mathrm{W}(Y, G)$, and the diagram

$$\begin{tikzcd} \mathrm{W}(Y_\infty, G) \arrow[r, "i_{\infty *}"] \arrow[d, "f_\infty^*"] & \mathrm{W}(Y, G) \arrow[d, "f^*"] \\ \mathrm{W}(Y'_\infty, G) \arrow[r, "i'_{\infty *}"] & \mathrm{W}(Y', G) \end{tikzcd}$$

commutes. By [Th-Tr90, 1.5.4] this concludes the proof of the theorem. □

Remark 3.3. For the projection $\mathrm{pr}_2\colon X \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S$ the specialization homomorphism

$$\mathrm{Sp}_{X\times_S\mathbb{P}^1_S}\colon \mathrm{K}_*\left(X \times_S \mathbb{A}^1_S, G\right) \longrightarrow \mathrm{K}_*(X, G)$$

coincides with the pullback $s_0^*\colon \mathrm{K}_*(X \times_S \mathbb{A}^1_S, G) \longrightarrow \mathrm{K}_*(X, G)$ via the zero-section $s_0\colon X \to X \times_S \mathbb{A}^1_S$.

In fact, the pullback $j_\infty^*\colon \mathrm{K}_*(X \times_S \mathbb{P}^1_S) \to \mathrm{K}_*(X \times_S \mathbb{A}^1_S)$ is surjective, and we have

$$\mathrm{Sp}_{X\times_S\mathbb{P}^1_S} \circ j_\infty^* = s_0^* \circ j_\infty^* = i_\infty^*\colon \mathrm{K}_*\left(X \times_S \mathbb{P}^1_S\right) \longrightarrow \mathrm{K}_*(X, G).$$

Remark 3.4. Since the restriction homomorphism $\mathrm{K}_0(Y, G) \to \mathrm{K}_0(Y \setminus Y_\infty, G)$ is a surjective ring homomorphism, and its composition with Sp_Y : $\mathrm{K}_0(Y \setminus Y_\infty, G) \to \mathrm{K}_0(Y_\infty, G)$ is a ring homomorphism, it follows that the specialization in degree 0 $\mathrm{Sp}_Y\colon \mathrm{K}_0(Y \setminus Y_\infty, G) \to \mathrm{K}_0(Y_\infty, G)$ is also a ring homomorphism. This should be true for the whole specialization homomorphism $\mathrm{Sp}_Y\colon \mathrm{K}_*(Y\setminus Y_\infty, G) \to \mathrm{K}_*(Y_\infty, G)$, but this is not obvious from the construction, and we do not know how to prove it.

Remark 3.5. From the proof of Theorem 3.2 we see that one can define a specialization homomorphism $\mathrm{K}_*(Y\setminus Z, G) \to \mathrm{K}_*(Z, G)$ if Z is a regular effective G-invariant divisor on Y whose normal sheaf is G-equivariantly trivial.

3.1. Specializations to the normal bundle. Let us go back to our standard situation, in which G is a diagonalizable group scheme of finite type acting on a regular separated noetherian algebraic space X. Fix a nonnegative integer s, and consider the closed immersion $X_s \hookrightarrow X_{\leq s}$; denote by N_s its normal bundle. Consider the deformation to the normal cone $\pi : \mathrm{M}_s \to \mathbb{P}^1_S$, the one denoted by $\mathrm{M}^0_{X_s} X_{\leq s}$ in [Ful93a, Chapter 5]. The morphism $\pi : \mathrm{M}_s \to \mathbb{P}^1_S$ is flat and G-invariant. Furthermore $\pi^{-1}(\mathbb{A}^1_S) = X_{\leq s} \times_S (\mathbb{A}^1_S)$, while the fiber at infinity of π is N_s. Consider the restriction $\pi^0\colon \mathrm{M}^0_s \to \mathbb{P}^1_S$ to the open subset $\mathrm{M}_s^0 = (\mathrm{M}_s)_{<s}$; then $(\pi^0)^{-1}(\mathbb{A}^1_S) = X_{<s} \times_S \mathbb{A}^1_S$, while the fiber at infinity of π^0 is $\mathrm{N}_s^0 = (\mathrm{N}_s)_{<s}$. We define a specialization homomorphism

$$\mathrm{Sp}_{X,s}\colon\ \mathrm{K}_*(X_{<s}, G) \longrightarrow \mathrm{K}_*\left(\mathrm{N}_s^0, G\right)$$

by composing the pullback

$$\mathrm{K}_*(X_{<s}, G) \longrightarrow \mathrm{K}_*\left(X_{<s} \times_S \mathbb{A}^1_S, G\right) = \mathrm{K}_*\left(\mathrm{M}_s^0 \setminus \mathrm{N}_s^0, G\right)$$

with the specialization homomorphism

$$\mathrm{Sp}_{\mathrm{M}_s^0}\colon\ \mathrm{K}_*\left(\mathrm{M}_s^0 \setminus \mathrm{N}_s^0, G\right) \longrightarrow \mathrm{K}_*\left(\mathrm{N}_s^0, G\right)$$

defined in the previous subsection.

We can also define more refined specializations.

Proposition 3.6. *Let $Y \to \mathbb{P}^1_S$ be regular at infinity.*

(i) *If $H \subseteq G$ is a diagonalizable subgroup scheme, then the restriction $Y^H \to \mathbb{P}^1$ is regular at infinity.*
(ii) *If s is a nonnegative integer, the restriction $Y_s \to \mathbb{P}^1_S$ is also regular at infinity.*

Proof. Part (i) and Proposition 2.2 (i) imply part (ii).

To prove (i), notice that, by [Tho92, Prop. 3.1], Y^H is regular, and so is Y^H_∞. Let f be the pullback to Y of a local equation for the section at infinity of $\mathbb{P}^1_S \to S$, and let p be a point of Y^H_∞. Since the conormal space to Y^H in Y has no nontrivial H-invariants, clearly the differential of f at p can not lie in this conormal space, hence f is not zero in any neighborhood of p in Y^H. This implies that Y^H_∞ is a regular Cartier divisor on Y^H, as claimed. □

If t is an integer with $t < s$, let us set $\mathrm{N}_{s,t} \overset{\text{def}}{=} (\mathrm{N}_s)_t$. We have that the restriction $(\mathrm{M}_s)_t \to \mathbb{P}^1_S$ is still regular at infinity, by Proposition 3.6 (ii); so we can also define a specialization homomorphism

$$\mathrm{Sp}^t_{X,s}\colon\ \mathrm{K}_*(X_t, G) \longrightarrow \mathrm{K}_*\big(\mathrm{N}_{s,t}, G\big)$$

by composing the pullback

$$\mathrm{K}_*(X_t, G) \longrightarrow \mathrm{K}_*\left(X_t \times_S \mathbb{A}^1_S, G\right) = \mathrm{K}_*\big(\big(\mathrm{M}_s^0\big)_t \setminus \mathrm{N}_{s,t}, G\big)$$

with the specialization homomorphism

$$\mathrm{Sp}_{(\mathrm{M}_s^0)_t}\colon\ \mathrm{K}_*\big(\big(\mathrm{M}_s^0\big)_t \setminus \mathrm{N}_{s,t}, G\big) \longrightarrow \mathrm{K}_*\big(\mathrm{N}_{s,t}, G\big).$$

The specializations above are compatible, in the following sense.

Proposition 3.7. *In the situation above, the diagram*

$$\begin{tikzcd} \mathrm{K}_*(X_{<s}, G) \arrow[r] \arrow[d, "\mathrm{Sp}_{X,s}"] & \mathrm{K}_*(X_t, G) \arrow[d, "\mathrm{Sp}^t_{X,s}"] \\ \mathrm{K}_*\left(\mathrm{N}^0_s, G\right) \arrow[r] & \mathrm{K}_*(\mathrm{N}_{s,t}, G) \end{tikzcd} ,$$

where the rows are restriction homomorphisms, commutes.

Proof. This follows immediately from the compatibility of specializations (Theorem 3.2). □

4. Reconstruction from the strata

4.1. K-rigidity.

Definition 4.1. *Let Y be a G-invariant regular locally closed subspace of X. We say that Y is* K-rigid *inside X if Y is regular and $\lambda_{-1}(\mathrm{N}_Y^\vee X)$ is not a zero-divisor in the ring* $\mathrm{K}_*(Y, G)$.

This condition may seem unlikely to ever be verified: in the non-equivariant case $\lambda_{-1}(\mathrm{N}_Y^\vee X)$ is always a nilpotent element, since it has rank zero over each component of X. However, in the equivariant case this is not necessary true. Here is the basic criterion that we will use use to check that a subspace is K-rigid.

Lemma 4.2. *Let Y be a G-space, E an equivariant vector bundle on Y. Suppose that there is a subtorus T of G acting trivially on Y, such that in the eigenspace decomposition of E with respect to T the subbundle corresponding to the trivial character is 0. Then $\lambda_{-1}(E)$ is not a zero-divisor in* $\mathrm{K}_*(Y, G)$.

Proof. Choose a splitting $G \simeq D \times T$; by [Tho86b, Lemme 5.6], we have

$$\mathrm{K}_*(Y, G) = \mathrm{K}_*(Y, D) \otimes \mathrm{R}T = \mathrm{K}_*(Y, D) \otimes \mathbb{Z}\left[t_1^{\pm 1}, \dots, t_n^{\pm 1}\right].$$

If $E = \bigoplus_{\chi \in \widehat{T}} E_\chi$ is the eigenspace decomposition of E, we have that $\lambda_{-1}(E)$ corresponds to the element $\prod_{\chi \in \widehat{T}} \lambda_{-1}(E_\chi \otimes \chi)$ of $\mathrm{K}_*(Y, D) \otimes \mathrm{R}T$, so it enough to show that $\lambda_{-1}(E_\chi \otimes \chi)$ is not a zero-divisor in $\mathrm{K}_*(Y, D) \otimes \mathrm{R}T$. But we can write

$$\lambda_{-1}(E_\chi \otimes \chi) = 1 + r_1\chi + r_2\chi^2 + \cdots + r_n\chi^n \in \mathrm{K}_*(Y, D) \otimes \mathrm{R}T,$$

where $r_n = (-1)^n[\det E_\chi]$ is a unit in $\mathrm{K}_*(Y, D)$.

Now we can apply the following elementary fact: suppose that A is a ring, $r_1, \dots, r_n$ central elements of A such that r_n is a unit, $\chi \in A[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ a monomial different from 1. Then the element $1 + r_1\chi + r_2\chi^2 + \cdots + r_n\chi^n$ is not a zero-divisor in $A[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. □

The next proposition is a K-theoretic variant of [Bri98, Proposition 3.2].

Proposition 4.3. *Let Y be a closed K-rigid subspace of X, and set $U = X \setminus Y$. Call $i\colon Y \hookrightarrow X$ and $j\colon U \hookrightarrow X$ the inclusions.*

(i) *The sequence*

$$0 \longrightarrow \mathrm{K}_*(Y, G) \xrightarrow{i_*} \mathrm{K}_*(X, G) \xrightarrow{j^*} \mathrm{K}_*(U, G) \longrightarrow 0$$

is exact.

(ii) *The two restriction maps*

$$i^*\colon \mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(Y, G) \quad \textit{and} \quad j^*\colon \mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(U, G)$$

induce a ring isomorphism

$$(i^*, j^*)\colon \mathrm{K}_*(X, G) \xrightarrow{\sim} \mathrm{K}_*(Y, G) \underset{\mathrm{K}_*(Y,G)/(\lambda_{-1}(\mathrm{N}_Y^\vee X))}{\times} \mathrm{K}_*(U, G)$$

where $\mathrm{N}_Y^\vee X$ is the conormal bundle of Y in X, the homomorphism

$$\mathrm{K}_*(Y, G) \longrightarrow \mathrm{K}_*(Y, G)/\big(\lambda_{-1}\big(\mathrm{N}_Y^\vee X\big)\big)$$

is the projection, while the homomorphism

$$\mathrm{K}_*(U, G) \simeq \mathrm{K}_*(X, G)/i_*\,\mathrm{K}_*(Y, G) \longrightarrow \mathrm{K}_*(Y, G)/\big(\lambda_{-1}\big(\mathrm{N}_Y^\vee X\big)\big)$$

is induced by $i^*\colon \mathrm{K}_*(X, G) \to \mathrm{K}_*(Y, G)$.

Proof. From the self-intersection formula (Theorem 2.1) we see that the composition $i^*i_*\colon \mathrm{K}_*(Y, G) \to \mathrm{K}_*(Y, G)$ is multiplication by $\lambda_{-1}(\mathrm{N}_Y^\vee X)$, so i_* is injective. We get part (i) from this and from the localization sequence.

Part (ii) follows easily from part (i), together with the following elementary fact.

Lemma 4.4. *Let A, B and C be rings, $f\colon B \to A$ and $g\colon B \to C$ ring homomorphisms. Suppose that there exist a homomorphism of abelian groups $\phi\colon A \to B$ such that:*

(i) *The sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{g} C \longrightarrow 0.$$

is exact;

(ii) *the composition $f \circ \phi\colon A \to A$ is the multiplication by a central element $a \in A$ which is not a zero divisor.*

Then f and g induce an isomorphism of rings

$$(f, g)\colon B \to A \underset{A/(a)}{\times} C,$$

where the homomorphism $A \to A/(a)$ is the projection, and the one $C \to A/(a)$ is induced by the isomorphism $C \simeq B/\operatorname{im}\phi$ and the ring homomorphism $f\colon B \to A$. □

4.2. The theorem of reconstruction from the strata. This section is entirely dedicated to the proof of our main theorem. Let us recall what it says. Let G act on X with the usual hypotheses. Consider the strata X_s defined in Subsect. 2.2, and the specialization homomorphisms

$$\mathrm{Sp}^t_{X,s}\colon \mathrm{K}_*(X_t, G) \longrightarrow \mathrm{K}_*\big(\mathrm{N}_{s,t}, G\big)$$

defined in Subsect. 3.1.

Theorem 4.5 (The theorem of reconstruction from the strata). *The homomorphism*

$$\mathrm{K}_*(X, G) \longrightarrow \prod_{s=0}^{n} \mathrm{K}_*(X_s, G)$$

obtained from the restrictions $\mathrm{K}_*(X, G) \to \mathrm{K}_*(X_s, G)$ *is injective. Its image consists of the sequences* $(\alpha_s) \in \prod_{s=0}^{n} \mathrm{K}_*(X_s, G)$ *with the property that for each* $s = 1, \dots, n$ *the pullback of* $\alpha \in \mathrm{K}_*(X_s, G)$ *to* $\mathrm{K}_*\big(\mathrm{N}_{s,s-1}, G\big)$ *coincides with* $\mathrm{Sp}^{s-1}_{X,s}(\alpha_{s-1}) \in \mathrm{K}_*\big(\mathrm{N}_{s,s-1}, G\big)$.

In other words, we can view $\mathrm{K}_*(X, G)$ as a fiber product

$$\begin{aligned}\mathrm{K}_*(X, G) \simeq \mathrm{K}_*(X_n, G) &\underset{\mathrm{K}_*(\mathrm{N}_{n,n-1},G)}{\times} \mathrm{K}_*(X_{n-1}, G) \underset{\mathrm{K}_*(\mathrm{N}_{n-1,n-2},G)}{\times} \\ &\dots \underset{\mathrm{K}_*(\mathrm{N}_{2,1},G)}{\times} \mathrm{K}_*(X_1, G) \underset{\mathrm{K}_*(\mathrm{N}_{1,0},G)}{\times} \mathrm{K}_*(X_0, G).\end{aligned}$$

Here is our starting point.

Proposition 4.6. *X_s is K-rigid in X.*

Proof. This follows from Proposition 2.2 (iii), and Lemma 4.2. □

So from Proposition 4.3 (ii) applied to the closed embedding $i_s\colon X_s \hookrightarrow X_{\le s}$, we get an isomorphism

$$\mathrm{K}_*(X_{\le s}, G) \simeq \mathrm{K}_*(X_s, G) \underset{\mathrm{K}_*(X_s,G)/(\lambda_{-1}(\mathrm{N}_s^\vee))}{\times} \mathrm{K}_*(X_{<s}, G).$$

We can improve on this.

Proposition 4.7. *The restrictions*

$$\mathrm{K}_*(X_{\le s}, G) \longrightarrow \mathrm{K}_*(X_s, G) \quad \textit{and} \quad \mathrm{K}_*(X_{\le s}, G) \longrightarrow \mathrm{K}_*(X_{<s}, G)$$

induce an isomorphism

$$\mathrm{K}_*(X_{\le s}, G) \xrightarrow{\sim} \mathrm{K}_*(X_s, G) \underset{\mathrm{K}_*(\mathrm{N}_s^0,G)}{\times} \mathrm{K}_*(X_{<s}, G),$$

where the homomorphism $\mathrm{K}_*(X_s, G) \to \mathrm{K}_*(\mathrm{N}_s^0, G)$ *is the pullback, while*

$$\mathrm{Sp}_{Y,s}\colon \mathrm{K}_*(X_{<s}, G) \to \mathrm{K}_*\left(\mathrm{N}_s^0, G\right)$$

is the specialization.

Proof. Let us start with a lemma.

Lemma 4.8. *The restriction homomorphism* $\mathrm{K}_*(X_s, G) \to \mathrm{K}_*(\mathrm{N}_s^0, G)$ *is surjective, and its kernel is the ideal* $\big(\lambda_{-1}(\mathrm{N}_s^\vee)\big) \subseteq \mathrm{K}_*(X_s, G)$.

Proof. Since the complement of the zero section $s_0 \colon X_s \hookrightarrow \mathrm{N}_s$ coincides with $(\mathrm{N}_s)_{<s}$ (Proposition 2.2 (iii)), we can apply Proposition 4.3 (i) to the normal bundle N_s, and conclude that there is an exact sequence

$$0 \longrightarrow \mathrm{K}_*(X_s, G) \xrightarrow{s_{0*}} \mathrm{K}_*(\mathrm{N}_s, G) \longrightarrow \mathrm{K}_*\left(\mathrm{N}_s^0, G\right) \longrightarrow 0.$$

Now, $s_0^* \colon \mathrm{K}_*(\mathrm{N}_s, G) \to \mathrm{K}_*(X_s, G)$ is an isomorphism, and the composition $s_0^* s_{0*} \colon \mathrm{K}_*(X_s, G) \to \mathrm{K}_*(X_s, G)$ is multiplication by $\lambda_{-1}(\mathrm{N}_s^\vee)$, because of the self-intersection formula 2.1, and this implies the thesis. $\square$

Therefore the restriction homomorphism $\mathrm{K}_*(X_s, G) \to \mathrm{K}_*(\mathrm{N}_s^0, G)$ induces an isomorphism of $\mathrm{K}_*(X_s, G)/\big(\lambda_{-1}(\mathrm{N}_s^\vee)\big)$ with $\mathrm{K}_*(\mathrm{N}_s^0, G)$; the proposition follows from this, and from Proposition 3.7. $\square$

Now we proceed by induction on the largest integer s such that $X_s \neq \emptyset$. If $s = 0$ there is nothing to prove. If $s > 0$, by induction hypothesis the homomorphism

$$\mathrm{K}_*(X_{<s}, G) \longrightarrow \mathrm{K}_*(X_{s-1}, G) \underset{\mathrm{K}_*(\mathrm{N}_{s,s-1}, G)}{\times} \cdots \underset{\mathrm{K}_*(\mathrm{N}_{1,0}, G)}{\times} \mathrm{K}_*(X_0, G)$$

induced by restrictions is an isomorphism; so from Proposition 4.7 we see that to prove Theorem 4.5 it is sufficient to show that if $\alpha_s \in \mathrm{K}_*(X_s, G)$, $\alpha_{<s} \in \mathrm{K}_*(X < s, G)$, α_{s-1} is the restriction of $\alpha_{<s}$ to $\mathrm{K}_*(X_{s-1}, G)$, α_s^0 is the pullback of α_s to $\mathrm{K}_*(\mathrm{N}_s^0, G)$ and $\alpha_{s,s-1}$ is the pullback of α_s to $\mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$, then $\mathrm{Sp}_{X,s}(\alpha_{<s}) = \alpha_s^0$ if and only if $\mathrm{Sp}_{X,s}^{s-1}(\alpha_{s-1}) = \alpha_{s,s-1}$. But the diagram

$$\begin{tikzcd}
\mathrm{K}_*(X_{<s}, G) \arrow[r, "\mathrm{Sp}_{X,s}"] \arrow[d] & \mathrm{K}_*\left(\mathrm{N}_s^0, G\right) \arrow[d] \\
\mathrm{K}_*(X_{s-1}, G) \arrow[r, "\mathrm{Sp}_{X,s}^{s-1}"] & \mathrm{K}_*(\mathrm{N}_{s,s-1}, G)
\end{tikzcd},$$

where the colums are restriction homomorphisms, is commutative (Proposition 3.7); hence it suffices to show that the restriction homomorphism

$$\mathrm{K}_*\left(\mathrm{N}_s^0, G\right) \longrightarrow \mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$$

is injective. To prove this we may suppose that the action of G on X_s is connected, that is, X_s is not a nontrivial disjoint union of open invariant subspaces. In this case the toral component of the isotropy group of a point of X_s is constant.

Set $E = \mathrm{N}_s$, and consider the eigenspace decomposition $E = \bigoplus_{\chi \in \widehat{T}} E_\chi$. We obtain a decomposition $E = \bigoplus_i E_i$ by grouping together E_χ and $E_{\chi'}$

when the characters χ and χ' are multiple of a common primitive character in $\widehat{T}$. Then clearly a geometric point of E is in E_{s-1} if and only if exactly one of its components according to the decomposition $E = \bigoplus_i E_i$ above is nonzero. In other words, $\mathrm{N}_{s,s-1}$ is the disjoint union $\coprod_i E_i^0$, where E_i^0 is embedded in E by setting all the other components equal to 0. The same argument as in the proof of Lemma 4.8 shows that the kernel of the pullback $\mathrm{K}_*(X_s, G) \to \mathrm{K}_*(E_i^0, G)$ is generated by $\lambda_{-1}E_i$, so the kernel of the pullback

$$\mathrm{K}_*(X_s, G) \longrightarrow \mathrm{K}_*(\mathrm{N}_{s,s-1}, G) = \bigoplus_i \mathrm{K}_*\left(E_i^0, G\right)$$

equals $\cap_i(\lambda_{-1}E_i)$; hence we need to show that $(\lambda_{-1}E) = \cap_i(\lambda_{-1}E_i)$.

This is done as follows. Choose a splitting $G = D \times T$: we have $\mathrm{K}_*(X_s, G) = \mathrm{K}_*(X_s, D) \otimes \mathrm{R}T$, as in the beginning of the proof of Lemma 4.2. First of all, we have $\lambda_{-1}E = \prod_i \lambda_{-1}E_i$. Furthermore, for each i we can choose a primitive character χ_i in $\widehat{T}$ such that all the characters which appear in the decomposition of E_i are multiples of χ_i; from this we see that $\lambda_{-1}E_i$ is of the form $\sum_{k=m_i}^{n_i} r_{i,k}\chi_i^k$, where m_i and n_i are (possibly negative) integers, $r_{i,k} \in \mathrm{K}_0(X_s, G)$, r_{i,m_i} and r_{i,n_i} are invertible. Then the conclusion of the proof follows from the following fact.

Lemma 4.9. *Let A be a ring, H a free finitely generated abelian group, $\chi_1, \dots, \chi_r$ linearly independent elements of H. Let $\gamma_1, \dots, \gamma_r$ be elements of the group ring AH of the form $\gamma_i = \sum_{k=m_i}^{n_i} r_{i,k}\chi_i^k$, where the $r_{i,k}$ are central elements of A such that r_{i,m_i} and r_{i,n_i} are invertible. Then we have an equality of ideals $(\gamma_1 \dots \gamma_r) = (\gamma_1) \cap \dots \cap (\gamma_r)$ in AH.*

Proof. By multiplying each γ_i by $r_{i,m_i}^{-1}\chi_i^{-m_i}$ we may assume that γ_i has the form $1 + a_{i,1}\chi_i + \dots + a_{i,s_i}\chi_i^{s_i}$ with $s_i \geq 0$ and a_{i,s_i} a central unit in A. We will show that for any $i \neq j$ the relation $\gamma_i \mid q\gamma_j$, $q \in AH$ implies $\gamma_i \mid q$; from this the thesis follows with a straightforward induction. We may assume that $r = 2$, $i = 1$ and $j = 2$.

Since χ_1, χ_2 are $\mathbb{Z}$-linearly independent elements of H, we may complete them to a maximal $\mathbb{Z}$-independent sequence $\chi_1, \dots, \chi_n$ of H; this sequence generates a subgroup $H' \subseteq H$ of finite index.

Suppose at first that $H' = H$, so that $AH = A\left[\chi_1^{\pm 1}, ..., \chi_n^{\pm 1}\right]$. Replacing A by $A\left[\chi_3^{\pm 1}, \dots, \chi_n^{\pm 1}\right]$, we may assume that $AH = A\left[\chi_1^{\pm 1}, \chi_2^{\pm 1}\right]$.

If $p\gamma_1 = q\gamma_2$, we can multiply this equality by a sufficiently high power of $\chi_1\chi_2$ and assume that p and q are polynomials in $A[\chi_1, \chi_2]$. Since γ_2 is a polynomial in $A[\chi_2]$ with central coefficients and invertible leading coefficient, the usual division algorithm allows us to write $p = s\gamma_2 + r \in A[\chi_1, \chi_2]$, where r is a polynomial whose degree in χ_2 is less than $s_2 = \deg_{\chi_2}\gamma_2$. By comparing the degrees in χ_2 in the equality $r\gamma_1 = (q - s\gamma_1)\gamma_2$ we see that $q - s\gamma_1$ must be zero, and this proves the result.

In the general case, choose representatives $u_1, \dots, u_r$ for the cosets of H' in H; then any element f of AH can be written uniquely as $\sum_{i=1}^r u_i f_i$

with $f_i \in AH'$. Then from the equality $\left(\sum_i u_i p_i\right)\gamma_1 = \sum_i u_i q_i$ we get $p_i\gamma_1 = q_i\gamma_2$ for all i, because γ_1 and γ_2 are in AH'; hence the thesis follows from the previous case. $\square$

5. Actions with enough limits

Let us start with some preliminaries in commutative algebra.

5.1. Sufficiently deep modules. Let A be a finitely generated flat Cohen–Macaulay $\mathbb{Z}$-algebra, such that each of the fibers of the morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ has pure dimension n. If V is a closed subset of $\operatorname{Spec} A$, we define the *fiber dimension* of V to be the largest of the dimensions of the fibers of V over $\operatorname{Spec} \mathbb{Z}$, and its *fiber codimension* to be n minus its fiber dimension. We say that V has *pure fiber dimension* if all the fibers of V have constant dimension at all points of V (of course some of the fibers may be empty).

The fiber dimension and codimension of an ideal in A will be the fiber dimension and codimension of the corresponding closed subset of $\operatorname{Spec} A$.

Definition 5.1. *Let M be an A-module. Then we say that M is* sufficiently deep *if the following two conditions are satisfied.*

(i) *All associated primes of M have fiber codimension 0.*
(ii) $\operatorname{Ext}^1_A(A/\mathfrak{p}, M) = 0$ *for all primes $\mathfrak{p}$ in A of fiber codimension at least 2.*

Here are the properties that we need.

Proposition 5.2.

(i) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, M' and M'' are sufficiently deep, then M is sufficiently deep.*
(ii) *Direct limits and direct sums of sufficiently deep modules are sufficiently deep.*
(iii) *If N is an abelian group, then $N \otimes_{\mathbb{Z}} A$ is sufficiently deep.*
(iv) *If M is a sufficiently deep A-module, then* $\operatorname{Ext}^1_A(N, M) = 0$ *for all A-modules N whose support has fiber codimension at least 2.*

Proof. Part (i) is obvious.

Part (ii) follows from the fact that A is noetherian, so formation of $\operatorname{Ext}^1_A(A/\mathfrak{p}, -)$ commutes with direct sums and direct limits.

Let us prove part (iii). From part (ii) we see that we may assume that N is cyclic. If $N = \mathbb{Z}$, then $M = A$, and the statement follows from the facts that A is Cohen–Macaulay, and that the height of a prime ideal is at least equal to its fiber codimension.

Assume that $N = \mathbb{Z}/m\mathbb{Z}$, so that $M = A/mA$. The associated primes of M are the generic points of the fibers of A over the primes dividing m, so condition 5.1 (i) is satisfied.

Take a prime $\mathfrak{p}$ of A of fiber codimension at least 2, and consider the exact sequence

$$0 = \operatorname{Ext}^1_A(A/\mathfrak{p}, A) \to \operatorname{Ext}^1_A(A/\mathfrak{p}, A/mA) \to \operatorname{Ext}^2_A(A/\mathfrak{p}, A) \xrightarrow{m} \operatorname{Ext}^2_A(A/\mathfrak{p}, A).$$

If the characteristic of $A/\mathfrak{p}$ is positive, then the height of $\mathfrak{p}$ is at least 3, so $\operatorname{Ext}^2_A(A/\mathfrak{p}, A) = 0$, because A is Cohen–Macaulay, and we are done. Otherwise, we have an exact sequence

$$0 \longrightarrow A/\mathfrak{p} \xrightarrow{m} A/\mathfrak{p} \longrightarrow A/((m) + \mathfrak{p}) \longrightarrow 0;$$

but the height of $(m) + \mathfrak{p}$ is at least 3, so $\operatorname{Ext}^2_A(A/((m) + \mathfrak{p}), A) = 0$. From this we deduce that multiplication by m is injective on $\operatorname{Ext}^2_A(A/\mathfrak{p}, A)$, and this concludes the proof of part (iii).

For part (iv), notice first of all that if N is a finitely generated A-module of fiber codimension at least 2 then we can filter N with successive quotients of type $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime of fiber codimension at least 2, so $\operatorname{Ext}^1_A(N, M) = 0$.

If N is not finitely generated and $0 \to M \to E \to N \to 0$ is an exact sequence of A-modules, N' is a finitely generated submodule of N, and E' is the pullback of E to N', then the sequence $0 \to M \to E' \to N'$ splits; but because of part (i) of the definition we have $\operatorname{Hom}_A(N', M) = 0$, hence there is a unique copy of N' inside E'. Hence there is a unique copy of N inside E, and the sequence splits. This completes the proof of the proposition. □

5.2. Sufficiently deep actions. Let G be a diagonalizable group scheme of finite type over S; all the actions will be upon noetherian separated regular algebraic spaces, as in our setup. The ring of representations $\mathrm{R}G = \mathbb{Z}\widehat{G}$ is a finitely generated flat Cohen–Macaulay $\mathbb{Z}$-algebra, and each of the fibers of the morphism $\operatorname{Spec} \mathrm{R}G \to \operatorname{Spec} \mathbb{Z}$ has pure dimension equal to the dimension of G.

Definition 5.3. *We say that the action of G on X is* sufficiently deep *when the* $\mathrm{R}G$*-module* $\mathrm{K}_*(X, G)$ *is sufficiently deep.*

Theorem 5.4. *Suppose that a diagonalizable group scheme of finite type G acts on a noetherian regular separated algebraic space X, and that the action is sufficiently deep. Then the restriction homomorphism* $\mathrm{K}_*(X, G) \to \mathrm{K}_*(X^{G_0}, G)$ *is injective, and its image is the intersection of the images of the restriction homomorphisms* $\mathrm{K}_*(X^T, G) \to \mathrm{K}_*(X^{G_0}, G)$*, where T ranges over all subtori of G of codimension 1.*

Proof. We need some preliminaries.

Lemma 5.5. *Suppose that G acts on X with stabilizers of constant dimension s. Then the support of* $\mathrm{K}_*(X, G)$ *as an* $\mathrm{R}G$*-module has pure fiber dimension s, and any associated prime of* $\mathrm{K}_*(X, G)$ *has pure fiber dimension s.*

Proof. Suppose first of all that s is 0. Then it follows easily from Thomason's localization theorem ([Tho92]) that the support of $\mathrm{K}_*(X, G)$ has fiber dimension 0, and from this we see that every associated prime must have fiber dimension 0.

In the general case, we may assume that the action is connected (that is, X is not a nontrivial disjoint union of open invariant subspaces); then there will be a splitting $G = H \times_S T$, where H is a diagonalizable group scheme of finite type acting on X with finite stabilizers, and T is a totally split torus that acts trivially on X. In this case

$$\mathrm{K}_*(X, G) = \mathrm{K}_*(X, H) \otimes_{\mathbb{Z}} RT = \mathrm{K}_*(X, H) \otimes_{\mathrm{R}H} \mathrm{R}G.$$

The proof is concluded by applying the following lemma.

Lemma 5.6. *Let A be a flat Cohen–Macaulay $\mathbb{Z}$-algebra of finite type, $A \to B$ a smooth homomorphism of finite type with fibers of pure dimension s. Suppose that M is an A-module whose support has fiber dimension 0. Then $M \otimes_A B$ has support of pure dimension s, and each of its associated primes has fiber dimension s.*

Proof. Since tensor product commutes with taking direct limits and B is flat over A, we may assume that M is of finite type over A. By an obvious filtration argument, we may assume that M is of the form $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of fiber dimension 0. In this case the only associated primes of $M \otimes_A B$ are the generic components of the fiber of Spec B over $\mathfrak{p}$, and this proves the result. □

Lemma 5.7. *Suppose that X and Y are algebraic spaces on which G acts with stabilizers of constant dimension respectively s and t. If N is an $\mathrm{R}G$-submodule of $\mathrm{K}_*(Y, G)$ and $t < s$, then there is no nontrivial homomorphism of $\mathrm{R}G$-modules from N to $\mathrm{K}_*(X, G)$.*

Proof. Given such a nontrivial homomorphism $N \to \mathrm{K}_*(X, G)$, call I its image. The support of I has fiber dimension at most t, so there is an associated prime of fiber dimension at most t in $\mathrm{K}_*(X, G)$, contradicting Lemma 5.5. □

Now we prove Theorem 5.4. Let n be the dimension of G, so that $X_n = X^{G_0}$. First of all, let us show that the natural projection

$$\mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(X_n, G) \underset{\mathrm{K}_*(\mathrm{N}_{n,n-1}, G)}{\times} \mathrm{K}_*(X_{n-1}, G)$$

is an isomorphism. This will be achieved by showing that for all s with $0 \le s \le n-1$ the natural projection $\mathrm{K}_*(X, G) \longrightarrow P_s$ is an isomorphism, where we have set

$$P_s = \mathrm{K}_*(X_n, G) \underset{\mathrm{K}_*(\mathrm{N}_{n,n-1}, G)}{\times} \mathrm{K}_*(X_{n-1}, G) \underset{\mathrm{K}_*(\mathrm{N}_{n-1,n-2}, G)}{\times} \cdots \underset{\mathrm{K}_*(\mathrm{N}_{s+1,s}, G)}{\times} \mathrm{K}_*(X_s, G).$$

For $s = 0$ this is our main Theorem 4.5, so we proceed by induction. If $s < n-1$ and the projection above is an isomorphism, we have an exact sequence

$$0 \longrightarrow \mathrm{K}_*(X, G) \longrightarrow P_{s+1} \times \mathrm{K}_*(X_s, G) \longrightarrow \mathrm{K}_*(\mathrm{N}_{s+1,s}, G),$$

where the last arrow is the difference of the composition of the projection $P_{s+1} \to \mathrm{K}_*(X_{s+1}, G)$ with the pullback $\mathrm{K}_*(X_{s+1}, G) \to \mathrm{K}_*(\mathrm{N}_{s+1,s}, G)$, and of the specialization homomorphism $\mathrm{K}_*(X_s, G) \to \mathrm{K}_*(\mathrm{N}_{s+1,s}, G)$. If we call N the image of this difference, we have an exact sequence of $\mathrm{R}G$-modules

$$0 \longrightarrow \mathrm{K}_*(X, G) \longrightarrow P_{s+1} \times \mathrm{K}_*(X_s, G) \longrightarrow N \to 0,$$

and the support of N is of fiber dimension at most $s \leq n-2$ by Lemma 5.5, hence it is of fiber codimension at least 2. It follows from the fact that $\mathrm{K}_*(X, G)$ is sufficiently deep and from Proposition 5.2 (iv) that this sequence splits. From the fact that $\mathrm{K}_*(X, G)$ has only associated primes of fiber dimension 0 we see that the pullback map $\mathrm{K}_*(X_s, G) \to N$ must be injective, and from Lemma 5.7 that a copy of N living inside $P_{s+1} \times \mathrm{K}_*(X_s, G)$ must in fact be contained in $\mathrm{K}_*(X_s, G)$; this implies that the projection $\mathrm{K}_*(X, G) \to P_{s+1}$ is an isomorphism.

So the projection

$$\mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(X_n, G) \times_{\mathrm{K}_*(\mathrm{N}_{n,n-1}, G)} \mathrm{K}_*(X_{n-1}, G)$$

is an isomorphism. Then the kernel of the specialization homomorphism from $\mathrm{K}_*(X_{n-1}, G)$ to $\mathrm{K}_*(\mathrm{N}_{n,n-1}, G)$ maps injectively in $\mathrm{K}_*(X, G)$, so it must be 0, again because $\mathrm{K}_*(X, G)$ has only associated primes of fiber dimension 0. Furthermore X_{n-1} is the disjoint union of the X^T_{n-1} when T ranges over all finite subtori of G of codimension 1, and similarly for $\mathrm{N}_{n,n-1}$. On the other hand, because of our main theorem applied to the action of G on X^T, we have the natural isomorphism $\mathrm{K}_*(X^T, G) \to \mathrm{K}_*(X^{G_0}, G) \times_{\mathrm{K}_*(\mathrm{N}^T_{n,n-1}, G)} \mathrm{K}_*(X^T_{n-1}, G)$, and this completes the proof of Theorem 5.4. □

5.3. Actions with enough limits are sufficiently deep. For the rest of this section S will be the spectrum of a field k, G is a diagonalizable group scheme of finite type acting on a smooth separated scheme X of finite type over k; call M the group of one-parameter subgroups $\mathbb{G}_{\mathrm{m},k} \to G$ of G. There is a natural Zariski topology on $M \simeq \mathbb{Z}^n$ in which the closed subsets are the loci of zeros of sets of polynomials in the symmetric algebra $\mathrm{Sym}^\bullet_{\mathbb{Z}} M^\vee$; we refer to this as the *Zariski topology on* M.

We will denote, as usual, by G_0 the toral component of G. If n is the dimension of G, then $X_n = X^{G_0}$. Furthermore, if we choose a splitting $G \simeq G_0 \times G/G_0$ we obtain an isomorphism of rings

$$\mathrm{K}_*(X^{G_0}, G) \simeq \mathrm{K}_*\left(X^{G_0}, G/G_0\right) \otimes \mathrm{R}G_0$$

([Tho86b, Lemme 5.6]).

Definition 5.8. *Suppose that k is algebraically closed. Consider a one parameter subgroup $H = \mathbb{G}_{\mathrm{m},k} \to G$, with the corresponding action of $\mathbb{G}_{\mathrm{m},k}$ on X. We say that this one parameter subgroup* admits limits *if for every closed point $x \in X$, the morphism $\mathbb{G}_{\mathrm{m},k} \to X$ which sends $t \in G$ to tx extends to a morphism $\mathbb{A}^1 \to X$. The image of* $0 \in \mathbb{A}^1(k)$ *in X is called* the limit of x for the one parameter subgroup H.

We say that the action of G on X admits enough limits *if the one parameter subgroups of G which admit limits form a Zariski-dense subset of M.*

If k is not algebraically closed, then we say that the action admits enough limits if the action obtained after base change to the algebraic closure of k does.

Remark 5.9. One can show that the locus of 1-parameter subgroups of G admitting limits is defined by linear inequalities, so the definition can be stated in more down to earth terms (we are grateful to the referee for pointing this out).

The notion of action with enough limits is a weakening of the notion of *filtrable* action due to Brion. More precisely, an action has enough limits if it satisfies condition (i) in [Bri97, Definition 3.2]; there is also a condition (ii) on closures of strata.

The main case when the action admits enough limits is when X is complete; in this case of course every one-parameter subgroup admits limits. Another case is when the action of $G_0 = \mathbb{G}_{\mathrm{m},k}^n$ on X extends to an action of the multiplicative monoid $\mathbb{A}^n$. Also, we give a characterization of toric varieties with enough limits in Proposition 6.7.

Theorem 5.10. *Suppose that a diagonalizable group scheme of finite type G over a perfect field k acts on a smooth separated scheme of finite type X over k. If the action of G admits enough limits, then it is sufficiently deep.*

By putting this together with Theorem 5.4 we get the following.

Corollary 5.11. *Suppose that a diagonalizable group scheme of finite type G over a perfect field k acts on a smooth separated scheme of finite type X over k. If the action of G admits enough limits, then the restriction homomorphism*

$$\mathrm{K}_*(X, G) \longrightarrow \mathrm{K}_*(X^{G_0}, G)$$

is injective, and its image is the intersection of the images of the restriction homomorphisms $\mathrm{K}_*(X^T, G) \to \mathrm{K}_*(X^{G_0}, G)$, *where T ranges over all subtori of G of codimension 1.*

For example, consider the following situation, completely analogous to the one considered in [Bri98, Corollary 7] and in [G-K-MP98]. Let G be an n-dimensional torus acting on a smooth complete variety X over an algebraically closed field k. Assume that the fixed point set $X^{G_0} = X_n$ is zero-dimensional, while X_{n-1} is 1-dimensional. Set $X^{G_0} = \{x_1, \dots, x_t\}$, and call $P_1, \dots, P_r$ the closures in X of the connected components of X_{n-1}. Then each P_j is isomorphic to $\mathbb{P}^1$, and contains precisely two of the fixed points, say x_i and $x_{i'}$. Call D_j the kernel of the action of G on P_j; then the image of the restriction homomorphism

$$\mathrm{K}_*(P_j, G) \to \mathrm{K}_*(x_i, G) \times \mathrm{K}_*(x_{i'}, G) = \mathrm{K}_*(k) \otimes \mathrm{R}G \times \mathrm{K}_*(k) \otimes \mathrm{R}G$$

consists of the pairs of elements

$$(\alpha, \beta) \in \mathrm{K}_*(k) \otimes \mathrm{R}G \times \mathrm{K}_*(k) \otimes \mathrm{R}G$$

whose images in $\mathrm{K}_*(k) \otimes \mathrm{R}D_j$ coincide (this follows immediately from Theorem 4.5). From this and from Corollary 5.11 we get the following.

Corollary 5.12. *In the situation above, the restriction map*

$$\mathrm{K}_*(X, G) \longrightarrow \prod_{i=1}^{t} \mathrm{K}_*(x_i, G) = \prod_{i=1}^{n} \mathrm{K}_*(k) \otimes \mathrm{R}G$$

is injective. Its image consist of all elements (α_i) *such that if* x_i *and* $x_{i'}$ *are contained in some* P_j*, then the restrictions of* α_i *and* $\alpha_{i'}$ *to* $\mathrm{K}_*(k) \otimes \mathrm{R}D_j$ *coincide.*

Theorem 5.10 is proved in the next subsection.

5.4. Białynicki-Birula stratifications. Let us prove Theorem 5.10: like in [Bri97], the idea is to use a Białynicki-Birula stratification. We will prove the following.

Proposition 5.13. *Suppose that a diagonalizable group scheme of finite type G over a perfect field k acts with enough limits on a smooth separated scheme of finite type X over k. Then the* $\mathrm{R}G$*-module* $\mathrm{K}_*(X, G)$ *is obtained by taking finitely many successive extensions of* $\mathrm{R}G$*-modules of the form* $N \otimes_{\mathbb{Z}} \mathrm{R}G$*, where N is an abelian group.*

Theorem 5.10 follows from this, in view of Proposition 5.2, parts (i) and (iii). Let us prove the proposition.

First of all, let us assume that k is algebraically closed. We will only consider closed points, and write X for $X(k)$.

It is a standard fact that the one-parameter subgroups $H = \mathbb{G}_{\mathrm{m},k} \to G_0$ with the property that $X^{G_0} = X^H$ form a nonempty Zariski open subset of M, so we can choose one with this property that admits enough limits. There is a (discontinuous) function $X \to X^{G_0}$ sending each point to its limit. Let $T_1, \dots, T_s$ be the connected components of X^{G_0}; call E_i the inverse image of T_i in X, and $\pi_j \colon E_j \to T_j$ the restriction of the limit function. The following is a fundamental result of Białynicki-Birula.

Theorem 5.14 (Białynicki-Birula). *In the situation above:*

(i) *The E_j are smooth locally closed G-invariant subvarieties of X.*
(ii) *The functions $\pi_j\colon E_j \to T_j$ are G-invariant morphisms.*
(iii) *For each j there is a representation V_j of H and an open cover $\{U_\alpha\}$ of T_j, together with equivariant isomorphisms $\pi_j^{-1}(U_\alpha) \simeq U_\alpha \times V_j$, such that the restriction $\pi_j\colon \pi_j^{-1}(U_\alpha) \to U_\alpha$ corresponds to the projection $U_\alpha \times V_j \to U_\alpha$.*
(iv) *If x is a point of T_j, then the normal space to E_j in X at x is the sum of the negative eigenspaces in the tangent space to X at x under the action of H.*

Of course in part (iii) we may take V_j to be the normal bundle to E_j in X at any point of T_j.

This theorem is proved in [Bia73]; we should notice that the condition that X is covered by open invariant quasiaffine subsets is always verified, thanks to a result of Sumihiro ([Sum75]).

Now we remove the hypothesis that k is algebraically closed: here is the variant of Białynicki-Birula's theorem that we need.

Theorem 5.15. *Suppose that a diagonalizable group scheme of finite type G over a perfect field k acts with enough limits on a smooth separated scheme of finite type X over k. Let $Y_1, \dots, Y_r$ be the connected components of X^{G_0}; there exists a stratification $X_1, \dots, X_r$ of X in locally closed G-invariant smooth subvarieties, together with G-equivariant morphisms $\rho_i\colon X_i \to Y_i$, such that:*

(i) *X_i contains Y_i for all i, and the restriction of ρ_i to Y_i is the identity.*
(ii) *If U is an open affine subset of Y_i and N_U is the restriction of the normal bundle $\mathrm{N}_{Y_i}X_i$ to U, then there is a G-equivariant isomorphism $\rho_i^{-1}(U) \simeq \mathrm{N}_U$ of schemes over U.*
(iii) *In the eigenspace decomposition of the restriction of $\mathrm{N}_{X_i}X$ to Y_i, the subbundle corresponding to the trivial character of G_0 is 0.*

Proof. Let $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and call Γ the Galois group of $\overline{k}$ over k. Choose a one parameter subgroup $H = \mathbb{G}_{\mathrm{m},k} \to G$ as before. Let $T_1, \dots, T_s$ be the connected components of $\overline{X}^{G_0}$, $\pi_j\colon E_j \to T_j$ as in Białynicki-Birula's theorem. The Y_i correspond to the orbits of the action of Γ on $\{T_1, \dots, T_s\}$; obviously Γ also permutes the E_j, so we let $X_1, \dots, X_r$ be the smooth subvarieties of X corresponding to the orbits of Γ on $\{E_1, \dots, E_s\}$. The $\pi_j\colon E_j \to T_j$ descend to morphisms $X_i \to Y_i$. Properties (i) and (iii) are obviously satisfied, because they are satisfied after passing to $\overline{k}$.

We have to prove (ii). Let E be the inverse image of U in X_i, I the ideal of U in the algebra $k[E]$. Because U is affine, I/I^2 is a projective $k[U]$-module, and G is diagonalizable, the projection $I \to I/I^2$ has a $k[U]$-linear and G-equivariant section $I/I^2 \to I$. This induces a G-equivariant morphism of U-schemes $E \to \mathrm{N}_U$, sending U to the 0-section, whose differential at the zero section is the identity (notice that N_U is also the restriction to U of the relative tangent bundle $\mathrm{T}_{X/U}$). We want to show that this is an isomorphism; it is enough to check that this is true on the fibers, so, let V be one of the fibers of N_U on some point $p \in U$. According to part (ii) of the theorem of Białynicki-Birula, the fiber of X on p is H-equivariantly isomorphic to V; hence an application of the following elementary lemma concludes the proof of Theorem 5.15.

Lemma 5.16. *Suppose that $\mathbb{G}_{\mathrm{m},k}$ acts linearly with positive weights on a finite dimensional vector space V over a field k. If $f : V \to V$ is an equivariant polynomial map whose differential at the origin is an isomorphism, then f is also an isomorphism.*

Proof. First all, notice that because of the positivity of the weights, we have $f(0) = 0$. By composing f with the inverse of the differential of f at the origin, we may assume that this differential of f is the identity. Consider the eigenspace decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where $\mathbb{G}_{\mathrm{m},k}$ acts on V_i with a character $t \mapsto t^{m_i}$, and $0 < m_1 < m_2 < \cdots < m_r$. Choose a basis of eigenvectors of V; we will use groups of coordinates $x_1, \ldots, x_r$, where x_i represents the group of elements of the dual basis corresponding to basis elements in V_i, so that the action of $\mathbb{G}_{\mathrm{m},k}$ is described by $t \cdot (x_1, \ldots, x_r) = (t^{m_1}x_1, \ldots, t^{m_r}x_r)$. Then it is a simple matter to verify that f is given by a formula of the type

$$\begin{aligned} &f(x_1, \ldots, x_r) \\ &\qquad = (x_1, x_2 + f_2(x_1), x_3 + f_3(x_1, x_2), \ldots, x_r + f_r(x_1, \ldots, x_{r-1})) \end{aligned}$$

and that every polynomial map of this form is an isomorphism. □

Now let us show that Theorem 5.15 implies Proposition 5.13. First of all, Theorem 5.15 (ii) and a standard argument with the localization sequence imply that the pullback map $\mathrm{K}_*(Y_i) \otimes_{\mathbb{Z}} \mathrm{R}G = \mathrm{K}_*(Y_i, G) \to K(X_i, G)$ is an isomorphism.

Now, let us order the strata $X_1, \ldots, X_r$ by decreasing dimension, and let us set $U_i = X_1 \cup \ldots \cup X_i$. Clearly X_i is closed in U_i. We claim that X_i is K-rigid in U_i for all i. In fact, it is enough to show that the restriction of $\lambda_{-1}(\mathrm{N}^\vee_{X_i} X)$ to Y_i is not a zero-divisor, and this follows from Lemma 4.2 and Theorem 5.15 (iii).

Then by Proposition 4.3 (i) we have an exact sequence

$$0 \longrightarrow \mathrm{K}_*(X_i, G) \longrightarrow \mathrm{K}_*(U_i, G) \longrightarrow \mathrm{K}_*(U_{i-1}, G) \longrightarrow 0;$$

so each $\mathrm{K}_*(U_i, G)$ is obtained by finitely many successive extensions of $\mathrm{R}G$-modules of the form $N \otimes_{\mathbb{Z}} \mathrm{R}G$, and $X = U_r$. This concludes the proof of Proposition 5.13, and of Theorem 5.10.

5.5. Comparison with ordinary K-theory for torus actions with enough limits. Assume that T is a totally split torus over a perfect field k, acting on a separated scheme X of finite type over k. We write T instead of G for conformity with the standard notation.

The following is a consequence of Proposition 5.13.

Corollary 5.17. *If X is smooth and the action has enough limits, we have*

$$\operatorname{Tor}_p^{\mathrm{R}T}\big(\mathrm{K}_*(X,T),\mathbb{Z}\big) = 0 \textit{ for all } p > 0.$$

The interest of this comes from the following result of Merkurjev.

Theorem 5.18 ([Mer97, Theorem 4.3]). *There is a homology spectral sequence*

$$E^2_{pq} = \operatorname{Tor}_p^{\mathrm{R}T}\big(\mathbb{Z}, \mathrm{K}_q(X,T)\big) \Longrightarrow \mathrm{K}_{p+q}(X)$$

such that the edge homomorphisms

$$\mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_q(X,T) \longrightarrow \mathrm{K}_q(X)$$

are induced by the forgetful homomorphism $\mathrm{K}_*(X,T) \to \mathrm{K}_*(X)$.

In particular the ring homomorphism $\mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_0(X,T) \to \mathrm{K}_0(X)$ is an isomorphism.

Furthemore, if X is smooth and projective we have $E^2_{pq} = 0$ for all $p > 0$, so the homomorphism $\mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_*(X,T) \to \mathrm{K}_*(X)$ is an isomorphism.

More generally, Merkurjev produces his spectral sequence for actions of reductive groups whose fundamental group is torsion-free.

From Corollary 5.17 we get the following extension of Merkurjev's degeneracy result.

Theorem 5.19. *Suppose that T is a totally split torus over a perfect field k, acting with enough limits on a smooth scheme separated and of finite type over k. Then the forgertful homomorphism $\mathrm{K}_*(X,T) \to \mathrm{K}_*(X)$ induces an isomorphism*

$$\mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_*(X,T) \xrightarrow{\sim} \mathrm{K}_*(X).$$

6. The K-theory of smooth toric varieties

Our reference for the theory of toric varieties will be [Ful93b].

In this section we take T to be a totally split torus over a fixed field k,

$$N = \operatorname{Hom}(\mathbb{G}_{\mathrm{m},k}, T) = \widehat{T}^{\vee}$$

its lattice of one-parameter subgroups, Δ a fan in $N \otimes \mathbb{R}$, $X = X(\Delta)$ the associated toric variety. We will always assume that X is smooth; this is equivalent to saying that every cone in Δ is generated by a subset of a basis of N.

We will give two different descriptions of the equivariant K-theory ring of X, one as a subring of a product of representation rings, and the second by generators and relations, analogously to what has been done for equivariant cohomology in [B-DC-P90].

6.1. The equivariant K-theory ring as a subring of a product of rings of representations. There is one orbit O_σ of T on X for each cone $\sigma \in \Delta$, containing a canonical rational point $x_\sigma \in O_\sigma(k)$. The dimension of O_σ is the codimension $\operatorname{codim}\sigma \overset{\text{def}}{=} \dim T - \dim\sigma$, and the stabilizer of any of its geometric points is the subtorus $T_\sigma \subseteq T$ whose group of one-parameter subgroups is precisely the subgroup $\langle\sigma\rangle = \sigma + (-\sigma) \subseteq N$; the dimension of T_σ is equal to the dimension of σ (see [Ful93b, 3.1]). Hence X_s is the disjoint union of the orbits $O_\sigma = T/T_\sigma$ with $\dim\sigma = s$.

Given a cone σ in $N \otimes \mathbb{R}$, we denote by $\partial\sigma$ the union of all of its faces of codimension 1.

Since

$$\mathrm{K}_*(O_\sigma, T) = \mathrm{K}_*(T/T_\sigma, T) = \mathrm{K}_*(\operatorname{Spec} k, T_\sigma) = \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$$

we have that

$$\prod_s \mathrm{K}_*(X_s, T) = \prod_{\sigma\in\Delta} \mathrm{K}_*(O_\sigma, T) \ = \prod_{\sigma\in\Delta} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma.$$

Lemma 6.1. *Fix an positive integer s. Then there is a canonical isomorphism*

$$\mathrm{N}_{s,s-1} \simeq \coprod_{\substack{\sigma\in\Delta\\ \dim\sigma=s}} \coprod_{\tau\in\partial\sigma} O_\tau.$$

Furthermore, for each pair σ, τ such that σ has dimension s, τ has dimension $s-1$, and τ is a face of σ, the composition of the specialization homomorphism

$$\begin{aligned}\mathrm{Sp}^{s-1}_{X,s}\colon\ \mathrm{K}_*(X_{s-1}, T) &= \prod_{\substack{\tau\in\Delta\\ \dim\tau=s-1}} \mathrm{K}_*(O_\tau, T)\\ &\longrightarrow \mathrm{K}_*(\mathrm{N}_{s,s-1}, T) = \prod_{\substack{\sigma\in\Delta\\ \dim\sigma=s}} \prod_{\tau\in\partial\sigma} \mathrm{K}_*(O_\tau, T),\end{aligned}$$

with the projection

$$\mathrm{pr}_{\sigma,\tau}\colon \prod_{\substack{\sigma\in\Delta\\ \dim\sigma=s}} \prod_{\tau\in\partial\sigma} \mathrm{K}_*(O_\tau, T) \longrightarrow \mathrm{K}_*(O_\tau, T)$$

is the projection

$$\mathrm{pr}_\tau\colon \prod_{\substack{\tau\in\Delta\\ \dim\tau=s-1}} \mathrm{K}_*(O_\tau, T) \longrightarrow \mathrm{K}_*(O_\tau, T).$$

Proof. We use the same notation as in [Ful93b]; in particular, for each cone σ of the fan of X, we denote by U_σ the corresponding affine open subscheme of X.

First of all, assume that the fan Δ consists of all the faces of an s-dimensional cone σ. Call B a part of a basis of N that spans σ: we have an action of T_σ on the k-vector space V_σ generated by B, by letting each 1-parameter subgroup $\mathbb{G}_{\mathrm{m}} \to G$ in B act by multiplication on the corresponding line in V_σ, and an equivariant embedding $T_\sigma \subseteq V_\sigma$. Then $X = U_\sigma$ is T-equivariantly isomorphic to the T-equivariant vector bundle

$$T \times^{T_\sigma} V_\sigma = (T \times V_\sigma)/T_\sigma \longrightarrow O_\sigma = T/T_\sigma$$

in such a way that the zero section corresponds to $O_\sigma \subseteq U_\sigma$. Since $X_s = O_\sigma$ and U_σ is a vector bundle over O_σ, we get a canonical isomorphism $U_\sigma \simeq \mathrm{N}_s$, and from this a canonical isomorphism

$$\mathrm{N}_{s,s-1} \simeq (U_\sigma)_{s-1} = \coprod_{\tau \in \partial\sigma} O_\tau.$$

It follows that the deformation to the normal bundle of O_σ in U_σ is also isomorphic to the product $U_\sigma \times_k \mathbb{P}^1$, and from this we get the second part of the statement.

In the general case we have $X_s = \coprod_{\dim\sigma=s} O_\sigma$, and if σ is a cone of dimension s in Δ, the intersection of X_s with U_σ is precisely O_σ. From this we get the first part of the statement in general.

The second part follows by applying the compatibility of specializations to the morphism of deformation to the normal bundle induced by the equivariant morphism $\coprod_{\dim\sigma=s} U_\sigma \to X_s$. $\square$

Using this lemma together with Theorem 4.5 we get that $\mathrm{K}_*(X, T)$ is the subring of

$$\prod_{\sigma\in\Delta} \mathrm{K}_*(O_\sigma, T) = \prod_{\sigma\in\Delta} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$$

consisting of elements (a_σ) with the property that the restriction of $a_\sigma \in \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$ to $\mathrm{K}_*(k) \otimes \mathrm{R}T_\tau$ coincides with $a_\tau \in \mathrm{K}_*(k) \otimes \mathrm{R}T_\tau$ every time τ is a face of codimension 1 in σ. Since every face of a cone is contained in a face of codimension 1, this can also be described as the subring of $\prod_\sigma \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$ consisting of elements (a_σ) with the property that the restriction of $a_\sigma \in \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$ to $\mathrm{K}_*(k) \otimes \mathrm{R}T_\tau$ coincides with $a_\tau \in \mathrm{K}_*(k) \otimes \mathrm{R}T_\tau$ every time τ is a face of σ. But every cone in Δ is contained in a maximal cone in Δ, so we get the following description of the equivariant K-theory of a smooth toric variety.

Theorem 6.2. *If $X(\Delta)$ is a smooth toric variety associated with a fan Δ in $N \otimes \mathbb{R}$, there is an injective homomorphism of $\mathrm{R}T$-algebras*

$$\mathrm{K}_*(X(\Delta), T) \hookrightarrow \prod_{\sigma\in\Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma,$$

where $\Delta_{\max}$ is the set of maximal cones in Δ.

An element $(a_\sigma) \in \prod_\sigma \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$ *is in the image of this homomorphism if and only if for any two maximal cones* σ_1 *and* σ_2*, the restrictions of* a_{σ_1} *and* a_{σ_2} *to* $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1 \cap \sigma_2}$ *coincide.*

This description of the equivariant K-theory ring of X is analogous to the description of its equivariant cohomology in [B-DC-P90], and of its equivariant Chow ring in [Bri97, Theorem 5.4].

6.2. The multiplicative Stanley–Reisner presentation. From the description above it is easy to get a presentation of the equivariant K-theory ring of the smooth toric variety $X(\Delta)$, analogous to the Stanley–Reisner presentation of its equivariant cohomology ring obtained in [B-DC-P90]. Denote by Δ_1 the subset of Δ consisting of 1-dimensional cones. We will use the following notation: if $\sigma \in \Delta_{\max}$, call $N_\sigma \subseteq N$ the group of 1-parameter subgroups of T_σ, so that $\widehat{T}_\sigma = (N_\sigma)^\vee$. We will use multiplicative notation for $\widehat{T}_\sigma$. Furthermore, for any $\rho \in \Delta_1$ we call $v_\rho \in N$ the generator for the monoid $\rho \cap N$.

For each $\rho \in \Delta_1$ we define an element u_ρ of the product $\prod_{\sigma \in \Delta_{\max}} \widehat{T}_\sigma$, as follows. If ρ is not a face of σ, we set $(u_\rho)_\sigma = 1$. If $\rho \in \Delta_1$ is a face of $\sigma \in \Delta_{\max}$ and $\{\rho = \rho_1, \rho_2 \dots, \rho_t\}$ is the set of 1-dimensional faces of σ, then, since the variety $X(\Delta)$ is smooth, we have that $v_{\rho_1}, \dots, v_{\rho_t}$ form a basis for N_σ. If $v_{\rho_1}^\vee, \dots, v_{\rho_t}^\vee$ is the dual basis in $\widehat{T}_\sigma = N_\sigma^\vee$, we set $u_\rho = v_{\rho_1}^\vee$.

Denote by $V_\Delta \subseteq \prod_{\sigma \in \Delta_{\max}} \widehat{T}_\sigma$ the subgroup consisting of the elements (x_σ) with the property that for all σ_1, σ_2 in $\Delta_{\max}$ the restrictions of $x_{\sigma_1} \in \widehat{T}_{\sigma_1}$ and $x_{\sigma_2} \in \widehat{T}_{\sigma_2}$ to $\widehat{T}_{\sigma_1 \cap \sigma_2}$ coincide. Then we have the following fact.

Proposition 6.3. *The elements* u_ρ *form a basis of* V_Δ.

The proof is straightforward.

We have the inclusions

$$\prod_{\sigma \in \Delta_{\max}} \widehat{T}_\sigma \subseteq \prod_{\sigma \in \Delta_{\max}} \mathrm{R}T_\sigma \subseteq \prod_{\sigma \in \Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma;$$

because of the description of the ring $\mathrm{K}_*\big(X(\Delta), T\big)$ as a subring of $\prod_{\sigma \in \Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$ given in Theorem 6.2, we see that we can consider the u_ρ as elements of $\mathrm{K}_*(X(\Delta), T)$.

There are some obvious relations that the u_ρ satisfy in

$$\mathrm{K}_*(X(\Delta), T) \subseteq \prod_{\sigma \in \Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma.$$

Suppose that S is a subset of Δ_1 not contained in any maximal cone of Δ. Then for all σ in $\Delta_{\max}$ there will be some $\rho \in S$ such that $(u_\rho)_\sigma = 1$ in $\widehat{T}_\sigma$; hence we have the relation

$$\prod_{\rho \in S} (u_\rho - 1) = 0 \quad \text{in} \quad \mathrm{K}_*(X(\Delta), T) \subseteq \prod_{\sigma \in \Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma.$$

From this we get a homomorphism of $\mathrm{K}_*(k)$-algebras

$$\frac{\mathrm{K}_*(k)\big[x_\rho^{\pm 1}\big]}{\big(\prod_{\rho\in S}(x_\rho - 1)\big)} \longrightarrow \mathrm{K}_*(X(\Delta), T) \tag{6.1}$$

where the x_ρ are indeterminates, where ρ varies over Δ_1, and S over the subsets of Δ_1 whose elements are not all contained in a maximal cone in Δ, by sending each x_ρ to u_ρ.

Theorem 6.4. *Suppose that $X(\Delta)$ is a smooth toric variety associated with a fan Δ in $N \otimes \mathbb{R}$. Then the homomorphism (6.1) above is an isomorphism.*

Proof. First of all, let us show that the u_ρ and their inverses generate

$$\mathrm{K}_*(X(\Delta), T) \subseteq \prod_{\sigma\in\Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma.$$

Set $\Delta_{\max} = \{\sigma_1, \dots, \sigma_r\}$, and let α be an element of $\mathrm{K}_*(X(\Delta), T)$; we want to show that α can be expressed as a Laurent polynomial in the x_ρ evaluated in the u_ρ. The ring $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1}$ is a ring of Laurent polynomials in the images of the u_ρ with $\rho \subseteq \sigma_1$, so we can find a Laurent polynomial $p_1(x_\rho)$, in which only the x_ρ with $\rho \subseteq \sigma_1$ appear, such that the image of $p_1(u_\rho)$ in $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1}$ equals the image of α in the same ring. By subtracting $p_1(u_\rho)$, we see that we may assume that the projection of α into $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1}$ is zero.

Now, let us repeat the procedure for the maximal cone σ_2: find a polynomial $p_1(x_\rho)$, in which only the x_ρ with $\rho \subseteq \sigma_2$ appear, such that the image of $p_2(u_\rho)$ in $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_2}$ equals the image of α in the same ring. The key point is that the restriction of α to $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1\cap\sigma_2}$ is zero, so in fact $p_2(x_\rho)$ can only contain the variables x_ρ with ρ not in σ_1. Hence the restriction of $p_2(x_\rho)$ to $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1}$ is also zero, and after having subtracted $p_2(x_\rho)$ from α we may assume that the restriction of α to both $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1}$ and $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_2}$ is zero. We can continue this process for the remaining cones $\sigma_3, \dots, \sigma_r$; at the end all the projections will be zero, and therefore α will be zero too.

Now we have to show that the kernel of the homomorphism

$$\mathrm{K}_*(k)\big[x_\rho^{\pm 1}\big] \longrightarrow \prod_{\sigma\in\Delta_{\max}} \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma \tag{6.2}$$

sending each x_ρ to u_ρ equals the ideal $\big(\prod_{\rho\in S}(x_\rho - 1)\big)$, where S varies over all subsets of Δ_1 not contained in any maximal cone. The kernel of the projection

$$\mathrm{K}_*(k)\big[x_\rho^{\pm 1}\big] \longrightarrow \mathrm{K}_*(k) \otimes \mathrm{R}T_\sigma$$

equals the ideal I_σ generated by the $x_\rho - 1$, where ρ varies over the set of 1-dimensional cones in Δ_1 not contained in σ, hence the kernel of the homomorphism 6.2 is the intersection of the I_σ. The result is then a consequence of the following lemma.

Lemma 6.5. *Let R be a (not necessarily commutative) ring, $\{x_\rho\}_{\rho\in E}$ a finite set of central indeterminates; consider the ring of Laurent polynomials $R\big[x_\rho^{\pm 1}\big]$. Let $A_1, \dots, A_r$ be subsets of E; for each $j = 1, \dots, r$ call I_j the ideal of $R\big[x_\rho^{\pm 1}\big]$ generated by the elements $x_\rho - 1$ with $\rho \in A_j$.*

Then the intersection $I_1 \cap \cdots \cap I_r$ is the ideal of $R\big[x_\rho^{\pm 1}\big]$ generated by the elements $\prod_{\rho\in S}(x_\rho - 1)$, where S varies over all subsets of E that meet each A_j.

Remark 6.6. When each A_j contains a single element this is a particular case of Lemma 4.9. The obvious common generalization should also hold.

Proof. We proceed by induction on r; the case $r = 1$ is clear. In general, take $p \in I_1 \cap \cdots \cap I_r \subseteq I_2 \cap \cdots \cap I_r$; by induction hypothesis, we can write

$$p = \sum_S \Big(\prod_{\rho\in S} (x_\rho - 1) \Big) q_S,$$

where S varies over all subsets of E whose intersection with each of the $A_2, \dots, A_r$ is not empty. We can split the sum as

$$p = \sum_{S\cap A_1 \neq \emptyset} \Big(\prod_{\rho\in S} (x_\rho - 1) \Big) q_S + \sum_{S\cap A_1 = \emptyset} \Big(\prod_{\rho\in S} (x_\rho - 1) \Big) q_S;$$

the first summand is in $I_1 \cap \cdots \cap I_r$ and is of the desired form, so we may subtract it from p and suppose that p is of the type

$$p = \sum_{S\cap A_1 = \emptyset} \Big(\prod_{\rho\in S} (x_\rho - 1) \Big) q_S.$$

Now, consider the ring $R\big[x_\rho^{\pm 1}\big]_{\rho\notin A_1}$ of Laurent polynomials not involving the variables in A_1; it is a subring of $R\big[x_\rho^{\pm 1}\big]$, and there is also a retraction

$$\pi \colon R\big[x_\rho^{\pm 1}\big] \longrightarrow R\big[x_\rho^{\pm 1}\big]_{\rho\notin A_1},$$

sending each x_ρ with $\rho \in A_1$ to 1, whose kernel is precisely I_1. The elements $\prod_{\rho\in S}(x_\rho - 1)$ are in $R\big[x_\rho^{\pm 1}\big]_{\rho\notin A_1}$, and

$$\pi p = \sum_{S\cap A_1 = \emptyset} \Big(\prod_{\rho\in S} (x_\rho - 1) \Big) \pi q_S = 0 \in R\big[x_\rho^{\pm 1}\big]_{\rho\notin A_1}$$

so we can write

$$p = \sum_{S\cap A_1 = \emptyset} \Big(\prod_{\rho\in S} (x_\rho - 1) \Big) (q_S - \pi q_S).$$

Then we write each $q_S - \pi q_S \in I_1$ as a linear combination of the polynomials $x_\rho - 1$ with $\rho \in A_1$; this concludes the proofs of the lemma and of the theorem. $\square$

6.3. Ordinary K-theory of smooth toric varieties. From Merkurjev's theorem (5.18) we get that $\mathrm{K}_0(X) = \mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_0(X, T)$. If the Merkurjev spectral sequence degenerates then we also have $\mathrm{K}_*(X) = \mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_*(X, T)$; this gives a way to compute the whole K-theory ring of X.

In general, the spectral sequence will not degenerate, and the ring $\mathrm{K}_*(X)$ tends to be rather complicated (for example, when $X = T$). When the toric variety has enough limits we can apply Theorem 5.19. Using the description of closures of orbits given in [Ful93b, 3.1], one shows that given a point $x \in X(\overline{k})$ lying in an orbit O_τ, and a one-parameter subgroup corresponding to an element $v \in N$, then the point has a limit under the one parameter subgroup if and only if v lies in the subset

$$\bigcup_{\sigma \in \operatorname{Star} \tau} (\sigma + \langle \tau \rangle) \subseteq N \otimes \mathbb{R},$$

where $\langle \tau \rangle$ denotes the subvector space $\tau + (-\tau) \subseteq N \otimes \mathbb{R}$, and $\operatorname{Star} \tau$ is the set of cones in Δ containing τ as a face. From this we obtain the following.

Proposition 6.7. *X has enough limits if and only if the subset*

$$\bigcap_{\tau \in \Delta} \bigcup_{\sigma \in \operatorname{Star} \tau} (\sigma + \langle \tau \rangle) \subseteq N \otimes \mathbb{R}$$

has nonempty interior.

Remark 6.8. If $X = X(\Delta)$ is a smooth toric variety with enough limits, the K-theory ring of X can be described in a slightly more efficient fashion: there is an injective homomorphism of $\mathrm{R}T$-algebras

$$\mathrm{K}_*(X, T) \hookrightarrow \prod_{\substack{\sigma \in \Delta \\ \dim \sigma = \dim T}} \mathrm{K}_*(k) \otimes \mathrm{R}T,$$

and an element $(a_\sigma) \in \prod_\sigma \mathrm{K}_*(k) \otimes \mathrm{R}T$ is in the image of this homomorphism if and only if for any two *adjacent* maximal cones σ_1 and σ_2, the restrictions of a_{σ_1} and a_{σ_2} to $\mathrm{K}_*(k) \otimes \mathrm{R}T_{\sigma_1 \cap \sigma_2}$ coincide.

Theorem 6.9. *If X is a smooth smooth toric variety with enough limits, then $\mathrm{K}_0(X, T)$ is a projective module over $\mathrm{R}T$ of rank equal to the number of maximal cones in its fan; furthermore the natural ring homomorphism $\mathrm{K}_*(k) \otimes \mathrm{K}_0(X, T) \to \mathrm{K}_*(X, T)$ is an isomorphism.*

In particular we have

$$\operatorname{Tor}_p^{\mathrm{R}T}(\mathrm{K}_*(X, T), \mathbb{Z}) = 0 \text{ for all } p > 0:$$

so from Merkurjev's theorem (5.18) we get the following.

Corollary 6.10. *Let X be a smooth toric variety with enough limits.*

(i) *The natural homomorphism of rings*

$$\mathbb{Z} \otimes_{\mathrm{R}T} \mathrm{K}_*(X, T) \longrightarrow \mathrm{K}_*(X)$$

is an isomorphism of $\mathrm{R}T$*-algebras.*

(ii) $\mathrm{K}_0(X)$ *is a free abelian group of rank equal to the number of maximal cones in* Δ.

(iii) *The natural homomorphism* $\mathrm{K}_*(k) \otimes \mathrm{K}_0(X) \to \mathrm{K}_*(X)$ *is an isomorphism.*

Remark 6.11. Suppose that the base field is the field $\mathbb{C}$ of complex numbers. The Merkurjev spectral sequence is an analogue of the Eilenberg–Moore spectral sequence ([Ei-Mo62])

$$E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(BT,\mathbb{Z})}\bigl(H_T^*(X, \mathbb{Z}), \mathbb{Z}\bigr) \Longrightarrow \mathrm{H}^{p+q}(X, \mathbb{Z}).$$

Then [B-B-F-K02] contains a description of the fans of the simplicial toric varieties for which this spectral sequence degenerates after tensoring with $\mathbb{Q}$. Presumably there should be a similar description for the case considered here.

7. The refined decomposition theorem

The main result of [Ve-Vi] shows that if G is an algebraic group acting with finite stabilizers on a noetherian regular algebraic space X over a field, the equivariant K-theory ring of X, after inverting certain primes, splits as a direct product of rings related with the K-theory of certain fixed points subsets. For actions of diagonalizable groups it is not hard to extend this decomposition to the case when the stabilizers have constant dimension.

So, in the general case when we do not assume anything about the dimension of the stabilizers, this theorem gives a description of the K-theory of each stratum X_s; it should clearly be possible to mix this with Theorem 4.5 to give a result that expresses $\mathrm{K}_*(X, G)$, after inverting certain primes, as a fibered product. This is carried out in this section.

The material in this section is organized as follows. We first extend the main result of [Ve-Vi] over an arbitrary noetherian separated base scheme S (Theorem 7.1) by giving a decomposition theorem in the case of an action with finite stabilizers of a diagonalizable group scheme G of finite type over S on a noetherian regular separated algebraic space X over S. Next, we deduce from this a decomposition theorem in the case where the action of G on X has stabilizers of fixed constant dimension (Theorem 7.4). Finally, we combine the analysis carried out in Sect. 4, and culminating with Theorem 4.5, together with Theorem 7.4 to prove a general decomposition theorem (Theorem 7.12) where no restriction is imposed on the stabilizers.

7.1. Actions with finite stabilizers. Here we recall the main result of [Ve-Vi] for actions of diagonalizable groups, extending it over any noetherian separated base scheme S.

Suppose that G is a diagonalizable group scheme of finite type acting with finite stabilizers on a noetherian regular separated algebraic space over a noetherian separated scheme S. A diagonalizable group scheme of finite type σ over S is called *dual cyclic* if its Cartier dual is finite cyclic, that is, if σ is isomorphic to a group scheme of the form $\boldsymbol{\mu}_{n,S}$ for some positive integer n.

A subgroup scheme $\sigma \subseteq G$ is called *essential* if it is dual cyclic, and $X^\sigma \neq \emptyset$.

There are only finitely many essential subgroups of G; we will fix a positive integer N which is divisible by the least common multiple of their orders.

Suppose that σ is a dual cyclic group of order n. The ring of representations $\mathrm{R}\sigma$ is of the form $\mathbb{Z}[t]/(t^n - 1)$, where t corresponds to a generator of the group of characters $\widehat{\sigma}$. Denote by $\widetilde{\mathrm{R}}\sigma$ the quotient of $\mathrm{R}\sigma$ corresponding to the quotient

$$\mathrm{R}\sigma = \mathbb{Z}[t]/(t^n - 1) \twoheadrightarrow \mathbb{Z}[t]/\bigl(\Phi_n(t)\bigr),$$

where Φ_n is the n^{th} cyclotomic polynomial. This quotient $\widetilde{\mathrm{R}}\sigma$ is independent of the choice of a generator for $\widehat{\sigma}$. We have a canonical homomorphism $\mathrm{R}G \to \mathrm{R}\sigma \twoheadrightarrow \widetilde{\mathrm{R}}\sigma$ induced by the embedding $\sigma \subseteq G$.

We also define a multiplicative system

$$\mathrm{S}_\sigma \subseteq \mathrm{R}G$$

as follows: an element of $\mathrm{R}G$ is in S_σ if its image in $\widetilde{\mathrm{R}}\sigma$ is a power of N.

For any $\mathrm{R}G$-module M, we define the *σ-localization M_σ* of M to be $\mathrm{S}_\sigma^{-1} M$. Consider the σ-localization

$$\mathrm{K}_*(X^\sigma, G)_\sigma = \mathrm{S}_\sigma^{-1}\,\mathrm{K}_*(X^\sigma, G)$$

of the $\mathrm{R}G$-algebra $\mathrm{K}_*(X^\sigma, G)$. The tensor product $\mathrm{K}_*(X^\sigma, G)_\sigma \otimes \mathbb{Q}$ is the localization

$$\bigl(\mathrm{K}_*(X^\sigma, G) \otimes \mathbb{Q}\bigr)_{\mathfrak{m}_\sigma}$$

of the $\mathrm{R}\sigma$-algebra $\mathrm{K}_*(X^\sigma, G) \otimes \mathbb{Q}$ at the maximal ideal

$$\mathfrak{m}_\sigma = \ker(\mathrm{R}\sigma \otimes \mathbb{Q} \twoheadrightarrow \widetilde{\mathrm{R}}\sigma \otimes \mathbb{Q}).$$

We are particularly interested in the σ-localization when σ is the trivial subgroup of G; in this case we denote it by $\mathrm{K}_*(X, G)_{\mathrm{geom}}$, and call it *the geometric equivariant K-theory of X*. The localization homomorphism

$$\mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \longrightarrow \mathrm{K}_*(X, G)_{\mathrm{geom}}$$

is surjective, and its kernel can be described as follows. Consider the kernel $\mathfrak{p} = \ker \mathrm{rk}$ of the localized rank homomorphism

$$\mathrm{rk}\colon \mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \longrightarrow \mathbb{Z}[1/N];$$

then the power $\mathfrak{p}^k$ is independent of k if k is large, and this power coincides with the kernel of the localization homomorphism.

For each essential subgroup $\sigma \subseteq G$, consider the compositions

$$\mathrm{loc}_\sigma\colon \mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \longrightarrow \mathrm{K}_*(X^\sigma, G) \otimes \mathbb{Z}[1/N] \longrightarrow \mathrm{K}_*(X^\sigma, G)_\sigma,$$

where the first arrow is a restriction homomorphism, and the second one is the localization.

There is also a homomorphism of $\mathrm{R}G$-algebras $\mathrm{K}_*(X^\sigma, G) \to \mathrm{K}_*(X^\sigma, G)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma$, defined as the composition

$$\begin{aligned}\mathrm{K}_*(X^\sigma, G) &\longrightarrow \mathrm{K}_*(X^\sigma, G \times \sigma) \simeq \mathrm{K}_*(X^\sigma, G) \otimes \mathrm{R}\sigma\\ &\longrightarrow \mathrm{K}_*(X^\sigma, G)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma,\end{aligned}$$

where the first morphism is induced by the multiplication $G \times \sigma \to G$, the second one is a natural isomorphism coming from the fact that σ acts trivially on X^σ ([Ve-Vi, Lemma 2.7]), and the third one is obtained from the localization homomorphism $\mathrm{K}_*(X^\sigma, G) \to \mathrm{K}_*(X^\sigma, G)_{\mathrm{geom}}$ and the projection $\mathrm{R}G \to \widetilde{\mathrm{R}}\sigma$. Then the homomorphism $\mathrm{K}_*(X^\sigma, G) \to \mathrm{K}_*(X^\sigma, G)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma$ factors through $\mathrm{K}_*(X^\sigma, G)_\sigma$ ([Ve-Vi, Lemma 2.8]), inducing a homomorphism

$$\theta_\sigma\colon \mathrm{K}_*(X^\sigma, G)_\sigma \longrightarrow \mathrm{K}_*(X^\sigma, G)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma.$$

Theorem 7.1.

(i) *There are finitely many essential subgroup schemes in G, and the homomorphism*

$$\prod_\sigma \mathrm{loc}_\sigma\colon \mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \longrightarrow \prod_\sigma \mathrm{K}_*(X^\sigma, G)_\sigma,$$

where the product runs over all the essential subgroup schemes of G, is an isomorphism.

(ii) *The homomorphism*

$$\theta_\sigma\colon \mathrm{K}_*(X^\sigma, G)_\sigma \longrightarrow \mathrm{K}_*(X^\sigma, G)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma$$

is an isomorphism of $\mathrm{R}G$*-algebras.*

Proof. If the base scheme S is the spectrum of a field, this is a particular case of the main theorem of [Ve-Vi].

If G is a torus, the proof of this statement given in [Ve-Vi] goes through without changes, because it only relies on Thomason's generic slice theorem for torus actions ([Tho86a, Proposition 4.10]).

In the general case, choose an embedding $G \hookrightarrow T$ into some totally split torus T over S, and consider the quotient space

$$Y \stackrel{\text{def}}{=} X \times^G T = (X \times T)/G$$

by the customary diagonal action of G; this exists as an algebraic space thanks to a result of Artin ([La-MB00, Corollaire 10.4]). The same argument as in the beginning of Sect. 5.1 of [Ve-Vi] shows that Y is separated.

Now observe that if $\sigma \subseteq T$ is an essential subgroup relative to the action of T on Y, we have $Y^\sigma = \emptyset$ unless $\sigma \subseteq G$ is an essential subgroup; hence the least common multiples of the orders of all essential subgroups are the same for the action of G on X and the action of T on Y.

Also, if $\sigma \subseteq G$ is an essential subgroup we have $Y^\sigma = X^\sigma \times^G T$, and therefore, by Morita equivalence ([Tho87, Proposition 6.2]), we get an isomorphism

$$\mathrm{K}_*(Y^\sigma, T) \simeq \mathrm{K}_*(X^\sigma, G)$$

which is an isomorphism of $\mathrm{R}T$-algebras, if we view $\mathrm{K}_*(X^\sigma, G)$ as an $\mathrm{R}T$-algebra via the restriction homomorphism $\mathrm{R}T \to \mathrm{R}G$.

Moreover, $\mathrm{S}_\sigma^T \subseteq \mathrm{R}(T)$ is exactly the preimage of $\mathrm{S}_\sigma^G \subseteq \mathrm{R}(G)$ under the natural surjection $\mathrm{R}T \to \mathrm{R}G$; therefore we have compatible σ-localized Morita isomorphisms

$$\left(\mathrm{S}_\sigma^T\right)^{-1} \mathrm{K}_*(Y^\sigma, T) \simeq \left(\mathrm{S}_\sigma^G\right)^{-1} \mathrm{K}_*(X^\sigma, G)$$

and (for σ equal to the trivial subgroup)

$$\mathrm{K}_*(Y^\sigma, T)_{\text{geom}} \simeq \mathrm{K}_*(X^\sigma, G)_{\text{geom}};$$

hence the theorem for the action of G on X follows from the theorem for the action of T on Y. □

7.2. Actions with stabilizers of constant dimension. From the theorem on actions with finite stabilizers we can easily get a decomposition result when we assume that the stabilizers have constant dimension. Assume that G is a diagonalizable group scheme of finite type over S, acting on a noetherian regular separated algebraic space X over S with stabilizers of constant dimension equal to s.

Definition 7.2. *A diagonalizable subgroup scheme $\sigma \subseteq G$ is* dual semi-cyclic *if σ/σ_0 is dual cyclic, where σ_0 is the toral component of σ.*

The order *of a dual semi-cyclic group σ is by definition equal the order of σ/σ_0.*

Equivalently, $\sigma \subseteq G$ is dual semicyclic if it is isomorphic to $\mathbb{G}_{\mathrm{m},S}^r \times \boldsymbol{\mu}_{n,S}$ for some $r \geq 0$ and $n > 0$.

Definition 7.3. *A subgroup scheme $\sigma \subseteq G$ is called* essential *if it is dual semicyclic and s-dimensional, and $X^\sigma \neq \emptyset$.*

There are finitely many subtori $T_j \subseteq G$ of dimension s in G with $X^{T_j} \neq \emptyset$, and X is the disjoint union of the X^{T_j}. The toral part of an essential subgroup of G coincides with one of the T_j; hence there are only finitely many essential subgroups of G.

We fix a positive integer N which is divisible by the least common multiple of the orders of the essential subgroups of G.

For each dual semicyclic subgroup $\sigma \subseteq G$, we define a multiplicative system

$$\mathrm{S}_\sigma \stackrel{\mathrm{def}}{=} \mathrm{S}_{\sigma/\sigma_0} \subseteq \mathrm{R}(G/\sigma_0) \subseteq \mathrm{R}G.$$

as the set of those elements of $\mathrm{R}(G/\sigma_0)$ whose image in $\widetilde{\mathrm{R}}(\sigma/\sigma_0)$ is a power of N. If M is a module over $\mathrm{R}G$, we define, as before, the σ-localization of M to be $M_\sigma = \mathrm{S}_\sigma^{-1} M$.

If $\sigma \subseteq G$ is an essential subgroup, we can choose a splitting $G \simeq (G/\sigma_0) \times \sigma_0$; according to [Tho86b, Lemme 5.6] this splitting induces an isomorphism

$$\mathrm{K}_*(X^\sigma, G) \simeq \mathrm{K}_*(X^\sigma, G/\sigma_0) \otimes \mathrm{R}\sigma_0$$

and also an isomorphism of σ-localizations

$$\mathrm{K}_*(X^\sigma, G)_\sigma \simeq \mathrm{K}_*(X^\sigma, G/\sigma_0)_{\sigma/\sigma_0} \otimes \mathrm{R}\sigma_0.$$

Fix one of the T_j, and choose a splitting $G \simeq G/T_j \times T_j$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_*(X^{T_j}, G) \otimes \mathbb{Z}[1/N] \arrow[r] \arrow[d, "\sim"] & \prod_{\substack{\sigma \text{ essential} \\ \sigma_0 = T_j}} \mathrm{K}_*(X^\sigma, G)_\sigma \arrow[d, "\sim"] \\
\mathrm{K}_*\left(X^{T_j}, G/T_j\right) \otimes \mathrm{R}T_j \otimes \mathbb{Z}[1/N] \arrow[r, "\sim"] & \prod_{\substack{\sigma \text{ essential} \\ \sigma_0 = T_j}} \mathrm{K}_*(X^\sigma, G/\sigma_0)_{\sigma/\sigma_0} \otimes \mathrm{R}\sigma_0
\end{tikzcd}$$

where the two columns are isomorphisms induced by the choice of a splitting $G \simeq G/T_j \times T_j$, and the rows are induced by composing the restriction homomorphism from X^{T_j} to X^σ with the localization homomorphism. The bottom row is in an isomorphism because of Theorem 7.1 (i).

Since the product of the restriction homomorphisms

$$\mathrm{K}_*(X, G) \longrightarrow \prod_j \mathrm{K}_*(X^{T_j}, G)$$

is an isomorphism, we obtain the following generalization of Theorem 7.1.

Theorem 7.4. *Suppose that a diagonalizable group scheme G of finite type over S acts with stabilizers of constant dimension on a noetherian regular separated algebraic space X over S.*

(i) *There are finitely many essential subgroup schemes in G, and the homomorphism*

$$\prod_{\sigma} \mathrm{loc}_\sigma \colon \mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \longrightarrow \prod_{\sigma} \mathrm{K}_*(X^\sigma, G)_\sigma,$$

where the product runs over all the essential subgroup schemes of G, is an isomorphism.

(ii) *For any essential subgroup scheme $\sigma \subseteq G$, a choice of a splitting $G \simeq (G/\sigma_0) \times \sigma_0$ gives an isomorphism*

$$\mathrm{K}_*(X^\sigma, G)_\sigma \longrightarrow \mathrm{K}_*(X^\sigma, G/\sigma_0)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma \otimes \mathrm{R}\sigma_0.$$

Remark 7.5. If $s = 0$, then $\sigma_0 = 1$ for each essential subgroup $\sigma \subseteq G$, so there is a unique splitting $G \simeq (G/\sigma_0) \times \sigma_0$, and the isomorphism in (ii) is canonical.

7.3. More specializations. For the refined decomposition theorem we need more specialization homomorphisms. Let a diagonalizable group scheme G of finite type over S act on a noetherian regular separated algebrac space X over S, with no restriction on the dimensions of the stabilizers.

Notation 7.6. Given a diagonalizable subgroup scheme $\sigma \subseteq G$, we set

$$X^{(\sigma)} = X^\sigma \cap X_{\leq \dim \sigma}.$$

Equivalently, $X^{(\sigma)} = (X_{\dim \sigma})^\sigma$.

Obviously $X^{(\sigma)}$ is a locally closed regular subspace of X.

Definition 7.7. *Let σ and τ be two diagonalizable subgroup schemes of G. We say that τ is* subordinate to σ, *and we write $\tau \prec \sigma$, if τ is contained in σ, and the induced morphism $\tau \to \sigma/\sigma_0$ is surjective.*

Suppose that σ and τ are diagonalizable subgroup schemes of G of dimension s and t respectively, and that τ is subordinate to σ. Consider the deformation to the normal cone $\mathrm{M}_s \to \mathbb{P}^1_S$ of X_s in $X_{\leq s}$, considered in Subsect. 3.1. By Proposition 3.6, the restriction $\mathrm{M}_s^{(\tau)} \to \mathbb{P}^1_S$ is regular at infinity, so we can define a specialization homomorphism

$$\mathrm{K}_*(X^{(\tau)}, G) \longrightarrow \mathrm{K}_*\left(\mathrm{N}_s^{(\tau)}, G\right).$$

Denote by N_σ the restriction of N_s to $X^{(\sigma)}$. We define the specialization homomorphism

$$\mathrm{Sp}^\tau_{X,\sigma} \colon \mathrm{K}_*(X^{(\tau)}, G) \longrightarrow \mathrm{K}_*\left(\mathrm{N}_\sigma^{(\tau)}, G\right)$$

as the composition of the homomorphism $\mathrm{K}_*(X^{(\tau)}, G) \to \mathrm{K}_*(\mathrm{N}_s^{(\tau)}, G)$ above with the restriction homomorphism $\mathrm{K}_*(\mathrm{N}_s^{(\tau)}, G) \to \mathrm{K}_*(\mathrm{N}_\sigma^{(\tau)}, G)$.

We also denote by

$$\mathrm{Sp}^{\tau}_{X,\sigma}\colon \mathrm{K}_*(X^{(\tau)}, G)_\tau \longrightarrow \mathrm{K}_*\left(\mathrm{N}_\sigma^{(\tau)}, G\right)_\tau$$

the τ-localization of this specialization homomorphism.

Remark 7.8. Since τ is subordinate to σ, it is easy to see that $\mathrm{N}_\sigma^{(\tau)}$ is a union of connected components of $\mathrm{N}_s^{(\tau)}$.

7.4. The general case. The hypotheses are the same as in the previous subsection: G is a diagonalizable group scheme of finite type over S, acting on a noetherian regular separated algebraic space X over S.

Definition 7.9. *An* essential subgroup of G *is a dual semicyclic subgroup scheme $\sigma \subseteq G$ such that $X^{(\sigma)} \neq \emptyset$.*

A semicyclic subgroup scheme of G is essential if and only if it is essential for the action of G on X_s for some s; hence there are only finitely many essential subgroups of G. We will fix a positive integer N that is divisible by the orders of all the essential subgroups of G.

If σ is a dual semicyclic subgroup of G, we define the multiplicative system $\mathrm{S}_\sigma \subseteq \mathrm{R}(G/\sigma_0) \subseteq \mathrm{R}G$ as before, as the subset of $\mathrm{R}(G/\sigma_0) \subseteq \mathrm{R}G$ consisting of elements whose image in $\widetilde{\mathrm{R}}(\sigma/\sigma_0)$ is a power of N. Also,

$$\mathrm{K}_*(X^{(\sigma)}, G)_\sigma \overset{\text{def}}{=} \mathrm{S}_\sigma^{-1}\,\mathrm{K}_*(X^{(\sigma)}, G),$$

as before.

Proposition 7.10. *Let σ and τ be two semi-cyclic subgroups of G. If $\tau \prec \sigma$, then $\mathrm{S}_\sigma \subseteq \mathrm{S}_\tau$.*

Proof. Consider the commutative diagram of group schemes

$$\begin{tikzcd} G/\tau_0 \arrow[r, two heads] & G/\sigma_0 \\ \tau/\tau_0 \arrow[r, two heads] \arrow[u, tail] & \sigma/\sigma_0 \arrow[u, tail] \end{tikzcd}\ ;$$

by taking representation rings we get a commutative diagram of rings

$$\begin{tikzcd} \mathrm{R}(G/\tau_0) \arrow[d, two heads] & \mathrm{R}(G/\sigma_0) \arrow[l, tail] \arrow[d, two heads] \\ \mathrm{R}(\tau/\tau_0) \arrow[d, two heads] & \mathrm{R}(\sigma/\sigma_0) \arrow[l, tail] \arrow[d, two heads] \\ \widetilde{\mathrm{R}}(\tau/\tau_0) & \widetilde{\mathrm{R}}(\sigma/\sigma_0) \arrow[l, dotted, tail] \end{tikzcd}$$

(without the dotted arrow). But it is easy to see that in fact the composition $\mathrm{R}(\sigma/\sigma_0) \to \mathrm{R}(\tau/\tau_0) \to \widetilde{\mathrm{R}}(\tau/\tau_0)$ factors through $\widetilde{\mathrm{R}}(\sigma/\sigma_0)$, so in fact the dotted arrows exists; and this proves the thesis. □

Now, consider the restriction of the projection $\pi_{\sigma,\tau}\colon \mathrm{N}_\sigma^{(\tau)} \to X^{(\sigma)}$, where σ and τ are dual semicyclic subgroups of G, and τ is subordinate to σ. Because of Proposition 7.10, we can consider the composition of the pullback $\mathrm{K}_*(X^{(\sigma)}, G)_\sigma \to \mathrm{K}_*(\mathrm{N}_\sigma^{(\tau)}, G)_\sigma$ with the natural homomorphism $\mathrm{K}_*(\mathrm{N}_\sigma^{(\tau)}, G)_\sigma \to \mathrm{K}_*(\mathrm{N}_\sigma^{(\tau)}, G)_\tau$ coming from the inclusion $\mathrm{S}_\sigma \subseteq \mathrm{S}_\tau$; we denote this homomorphism by

$$\pi_{\sigma,\tau}^*\colon\ \mathrm{K}_*(X^{(\sigma)}, G)_\sigma \longrightarrow \mathrm{K}_*\left(\mathrm{N}_\sigma^{(\tau)}, G\right)_\tau .$$

Definition 7.11. *Suppose that σ and τ are dual semicyclic subgroups of G and that τ is subordinate to σ. Two elements $a_\sigma \in \mathrm{K}_*(X^{(\sigma)}, G)_\sigma$ and $a_\tau \in \mathrm{K}_*(X_\tau, G)_\tau$ are* compatible *if*

$$\pi_{\sigma,\tau}^* a_\sigma = \mathrm{Sp}_{X,\sigma}^{\tau}\, a_\tau \in \mathrm{K}_*\left(\mathrm{N}_\sigma^{(\tau)}, G\right)_\tau .$$

For each essential dual semicyclic subgroup $\sigma \subseteq G$ we denote by

$$\mathrm{loc}_\sigma\colon\ \mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \to \mathrm{K}_*(X^{(\sigma)}, G)_\sigma$$

the composition of the restriction homomorphism

$$\mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \to \mathrm{K}_*(X^{(\sigma)}, G) \otimes \mathbb{Z}[1/N]$$

with the localization homomorphism

$$\mathrm{K}_*(X^{(\sigma)}, G) \otimes \mathbb{Z}[1/N] \to \mathrm{K}_*(X^{(\sigma)}, G)_\sigma .$$

The following is the main result of this section.

Theorem 7.12. *The ring homomorphism*

$$\prod_\sigma \mathrm{loc}_\sigma\colon\ \mathrm{K}_*(X, G) \otimes \mathbb{Z}[1/N] \longrightarrow \prod_{\substack{\sigma\subseteq G\\ \sigma \text{ essential}}} \mathrm{K}_*(X^{(\sigma)}, G)_\sigma$$

is injective. Its image consists of the elements (a_σ) of $\prod_\sigma \mathrm{K}_*(X^{(\sigma)}, G)_\sigma$ with the property that if σ and τ are essential, $\tau \prec \sigma$, and $\dim\sigma = \dim\tau + 1$, *then a_τ and a_σ are compatible.*

Notice that in the particular case that the action has stabilizers of constant dimension, all essential subgroups of G have the same dimension, and this reduces to Theorem 7.4.

Also, if σ is an essential subgroup of G then $X^{(\sigma)} = X^\sigma_{\dim\sigma}$, so it follows from Theorem 7.4 (ii) that a splitting $G \simeq (G/\sigma_0)\times\sigma_0$ gives an isomorphism of rings

$$\mathrm{K}_*(X^{(\sigma)}, G)_\sigma \simeq \mathrm{K}_*(X^{(\sigma)}, G/\sigma_0)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}\sigma \otimes \mathrm{R}\sigma_0 .$$

However, this isomorphism is not canonical in general, as it depends on the choice of a splitting.

Proof. To simplify the notation, we will implicitly assume that everything has been tensored with $\mathbb{Z}[1/N]$.

We apply Theorem 4.5 together with Theorem 7.4. According to Theorem 4.5 we have an injection $\mathrm{K}_*(X, G) \hookrightarrow \prod_s \mathrm{K}_*(X_s, G)$ whose image is the subring of sequences $(\alpha_s) \in \prod_{s=0}^n \mathrm{K}_*(X_s, G)$ with the property that for each $s = 1, \dots, n$ the pullback of $\alpha_s \in \mathrm{K}_*(X_s, G)$ to $\mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$ coincides with $\mathrm{Sp}_{X,s}^{s-1}(\alpha_{s-1}) \in \mathrm{K}_*(\mathrm{N}_{s,s-1}, G)$. Moreover, by Theorem 7.4, we can decompose further each $\mathrm{K}_*(X_s, G)$ as $\prod_\sigma \mathrm{K}_*(X^{(\sigma)}, G)_\sigma$, where σ varies in the (finite) set of essential subgroups of G of dimension s. By compatibility of specializations, for any $s \geq 0$, the following diagram is commutative

$$\begin{tikzcd}
\mathrm{K}_*(X_{s-1}, G) \arrow[r, "\sim"] \arrow[d, "\prod_\tau \mathrm{Sp}^{s-1}_{X,s}"'] & \prod\limits_{\substack{\tau \text{ essential} \\ \dim\tau = s-1}} \mathrm{K}_*(X^{(\tau)}, G)_\tau \arrow[d, "\mathrm{Sp}^\tau_{X,s}"] \arrow[dr, "\prod_{\tau,\sigma} \mathrm{Sp}^\tau_{X,\sigma}"] & \\
\mathrm{K}_*(\mathrm{N}_{s,s-1}, G) \arrow[r, "\sim"] & \prod\limits_{\substack{\tau \text{ essential} \\ \dim\tau = s-1}} \mathrm{K}_*(\mathrm{N}^{(\tau)}, G)_\tau \arrow[r, "\phi"] & \prod\limits_{\substack{\tau \text{ essential} \\ \dim\tau = s-1}} \prod\limits_{\substack{\sigma \succ \tau \\ \dim\sigma = s}} \mathrm{K}_*\left(\mathrm{N}^{(\tau)}_\sigma, G\right)
\end{tikzcd}$$

where ϕ is induced by the obvious pullbacks. On the other hand, the following diagram commutes by definition of $\pi^*_{\sigma,\tau}$

$$\begin{tikzcd}
\mathrm{K}_*(X_s, G) \arrow[r, "\sim"] \arrow[d] & \prod\limits_{\substack{\sigma \text{ essential} \\ \dim\sigma = s}} \mathrm{K}_*(X^{(\sigma)}, G)_\sigma \arrow[dr, "\prod_{\sigma,\tau} \pi^*_{\sigma,\tau}"] & \\
\mathrm{K}_*(\mathrm{N}_{s,s-1}, G) \arrow[r, "\sim"] & \prod\limits_{\substack{\tau \text{ essential} \\ \dim\tau = s-1}} \mathrm{K}_*(\mathrm{N}^{(\tau)}, G)_\tau \arrow[r, "\phi"] & \prod\limits_{\substack{\tau \text{ essential} \\ \dim\tau = s-1}} \prod\limits_{\substack{\sigma \succ \tau \\ \dim\sigma = s}} \mathrm{K}_*\left(\mathrm{N}^{(\tau)}_\sigma, G\right) .
\end{tikzcd}$$

Then the theorem will immediately follow if we show that ϕ is an isomorphism. This is true because of the following lemma.

Lemma 7.13. *Fix an essential subgroup τ of dimension $s - 1$. Then for any $\sigma \succ \tau$ with $\dim \sigma = s$, the scheme $\mathrm{N}^{(\tau)}_\sigma$ is open in $\mathrm{N}^{(\tau)}_s$; furthermore, $\mathrm{N}^{(\tau)}_s$ is the disjoint union of the $\mathrm{N}^{(\tau)}_\sigma$ for all essential σ with $\sigma \succ \tau$, $\dim \sigma = s$.*

Proof. We will show that $\mathrm{N}^{(\tau)}_s$ is the disjoint union of the $\mathrm{N}^{(\tau)}_\sigma$; since each $\mathrm{N}^{(\tau)}_\sigma$ is closed in $\mathrm{N}^{(\tau)}_s$, and there are only finitely many possible σ, it follows that each $\mathrm{N}^{(\tau)}_\sigma$ is also open in $\mathrm{N}^{(\tau)}_s$.

Let us first observe that if σ and σ' are essential subgroups in G of dimension s to which τ is subordinate, and $\mathrm{N}^{(\tau)}_\sigma \bigcap \mathrm{N}^{(\tau)}_{\sigma'} \neq \emptyset$, then $X^{(\sigma)} \bigcap X^{(\sigma')} \neq \emptyset$, therefore σ_0 is equal to σ'_0. But this implies that $\sigma = \sigma'$, since σ and σ' are both equal to the inverse image in G of the image of $\tau \to G/\sigma_0 = G/\sigma'_0$.

According to Proposition 2.2 (i), if $T_1, \dots, T_r$ are the essential s-dimensional subtori of G, then N_s is the disjoint union of the N_{T_j}. Clearly, if τ_0 is not contained in T_j, then $\mathrm{N}^{(\tau)}_{T_j}$ is empty, so $\mathrm{N}^{(\tau)}_s$ is the disjoint union of the $\mathrm{N}_T^{(\tau)}$, where T ranges over the essential s-dimensional subtori of G with $\tau_0 \subseteq T$. But there is a bijective correspondence between s-dimensional dual semi-cyclic subgroups $\sigma \subseteq G$ with $\sigma \succ \tau$ and s-dimensional subtori of G with $\tau_0 \subseteq T$: in one direction we associate with each σ its toral part σ_0, in the other we associate with each T the subgroup scheme $\tau + T \subseteq G$.

The proof is concluded by noticing that if σ and τ are as above, with $\sigma_0 = T$, then $\mathrm{N}_T^{(\tau)} = \mathrm{N}_\sigma^{(\tau)}$. □

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