DUKE MATHEMATICAL JOURNAL
Vol. 113, No. 1, © 2002

HIGHER ALGEBRAIC K-THEORY OF GROUP ACTIONS WITH FINITE STABILIZERS

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Abstract

We prove a decomposition theorem for the equivariant K-theory of actions of affine group schemes G of finite type over a field on regular separated Noetherian algebraic spaces, under the hypothesis that the actions have finite geometric stabilizers and satisfy a rationality condition together with a technical condition that holds, for example, for G abelian or smooth. We reduce the problem to the case of a GL_n-action and finally to a split torus action.

Contents

1. Introduction

The purpose of this paper is to prove a decomposition theorem for the equivariant K-theory of actions of affine group schemes of finite type over a field on regular separated Noetherian algebraic spaces. Let X be a regular connected separated Noetherian scheme with an ample line bundle, and let $K_0(X)$ be its Grothendieck ring of

DUKE MATHEMATICAL JOURNAL
Vol. 113, No. 1, © 2002
Received 27 April 2000. Revision received 13 June 2001.
2000 *Mathematics Subject Classification*. Primary 19E08, 14L30; Secondary 14A20.
Authors' work partially supported by the University of Bologna funds for selected research topics.

vector bundles. Then the kernel of the rank morphism $K_0(X) \to \mathbb{Z}$ is nilpotent (see [SGA6, Exp. VI, Th. 6.9]), so the ring $K_0(X)$ is indecomposable and remains such after tensoring with any indecomposable $\mathbb{Z}$-algebra.

The situation is quite different when we consider the equivariant case. Let G be an algebraic group acting on a Noetherian separated regular scheme, or algebraic space, let X be over a field k, and consider the Grothendieck ring $K_0(X, G)$ of G-equivariant perfect complexes. This is the same as the Grothendieck group of G-equivariant coherent sheaves on X, and it coincides with the Grothendieck ring of G-equivariant vector bundles if all G-coherent sheaves are quotients of locally free coherent sheaves (which is the case, e.g., when G is finite or smooth and X is a scheme). Assume that the action of G on X is connected, that is, that there are no nontrivial invariant open and closed subschemes of X. Still, $K_0(X, G)$ usually decomposes, after inverting some primes; for example, if G is a finite group and $X = \operatorname{Spec} \mathbb{C}$, then $K_0(X, G)$ is the ring of complex representations of G, which becomes a product of fields after tensoring with $\mathbb{Q}$.

In [Vi2] the second author analyzes the case where the action of G on X has finite reduced geometric stabilizers. Consider the ring of representations $\mathrm{R}(G)$, and consider the kernel $\mathfrak{m}$ of the rank morphism $\mathrm{rk} : K_0(X, G) \to \mathbb{Z}$. Then $K_0(X, G)$ is an $\mathrm{R}(G)$-algebra; he shows that the localization morphism

$$K_0(X, G) \otimes \mathbb{Q} \longrightarrow K_0(X, G)_{\mathfrak{m}}$$

is surjective and that the kernel of the rank morphism $K_0(X, G)_{\mathfrak{m}} \otimes \mathbb{Q} \to \mathbb{Q}$ is nilpotent. Furthermore, he conjectures that $K_0(X, G) \otimes \mathbb{Q}$ splits as a product of the localization $K_0(X, G)_{\mathfrak{m}}$ and some other ring, and he formulates a conjecture about what the other factor ring should be when G is abelian and the field is algebraically closed of characteristic zero. The proofs of the results in [Vi2] depend on an equivariant Riemann-Roch theorem, whose proof was never published by the author; however, all of the results have been proved and generalized in [EG].

The case where G is a finite group is studied in [Vi1]. Assume that k contains all nth roots of 1, where n is the order of the group G. Then the author shows that, after inverting the order of G, the K-theory ring $K_*(X, G)$ of G-equivariant vector bundles on X (which is assumed to be a scheme in that paper) is canonically the product of a finite number of rings, expressible in terms of ordinary K-theory of appropriate subschemes of fixed points of X. Here $K_*(X, G) = \bigoplus_i K_i(X, G)$ is the graded higher K-theory ring. The precise formula is as follows.

Let σ be a cyclic subgroup of G whose order is prime to the characteristic of k; then the subscheme X^σ of fixed points of X under the actions of σ is also regular. The representation ring $\mathrm{R}(\sigma)$ is isomorphic to the ring $\mathbb{Z}[t]/(t^n - 1)$, where t is a generator of the group of characters $\hom(\sigma, k^*)$. We call $\widetilde{\mathrm{R}}(\sigma)$ the quotient of the ring $\mathrm{R}(\sigma)$ by the ideal generated by the element $\Phi_n(t)$, where Φ_n is the nth cyclotomic polynomial;

this is independent of t. The ring $\widetilde{\mathrm{R}}(\sigma)$ is isomorphic to the ring of integers in the nth cyclotomic field. Call $\mathrm{N}_G(\sigma)$ the normalizer of σ in G; the group $\mathrm{N}_G(\sigma)$ acts on the scheme X^σ and, by conjugation, on the group σ. Consider the induced actions of $\mathrm{N}_G(\sigma)$ on the K-theory ring $K_*(X^\sigma)$ and on the ring $\widetilde{\mathrm{R}}(\sigma)$.

Choose a set $\mathscr{C}(G)$ of representatives for the conjugacy classes of cyclic subgroups of G whose order is prime to the characteristic of the field. The statement of the main result of [Vi1] is as follows.

THEOREM
There is a canonical ring isomorphism

$$K_*(X, G) \otimes \mathbb{Z}[1/|G|] \simeq \prod_{\sigma \in \mathscr{C}(G)} \bigl(K_*(X^\sigma) \otimes \widetilde{\mathrm{R}}(\sigma)\bigr)^{\mathrm{N}_G(\sigma)} \otimes \mathbb{Z}[1/|G|].$$

In the present paper we generalize this decomposition to the case in which G is an algebraic group scheme of finite type over a field k, acting with finite geometric stabilizers on a Noetherian regular separated algebraic space X over k. Of course, we cannot expect a statement exactly like the one for finite groups, expressing equivariant K-theory simply in terms of ordinary K-theory of the fixed point sets. For example, when X is the Stiefel variety of r-frames in n-space, then the quotient of X by the natural free action of GL_r is the Grassmannian of r-planes in n-space, and $K_0(X, \mathrm{GL}_r) = K_0(X/\mathrm{GL}_r)$ is nontrivial, while $K_0(X) = \mathbb{Z}$.

Let X be a Noetherian regular algebraic space over k with an action of an affine group scheme G of finite type over k. We consider the Waldhausen K-theory group $K_*(X, G)$ of complexes of quasi-coherent G-equivariant sheaves on X with coherent bounded cohomology. This coincides on the one hand with the Waldhausen K-theory group $K_*(X, G)$ of the subcategory of complexes of quasi-coherent G-equivariant flat sheaves on X with coherent bounded cohomology (and hence has a natural ring structure given by the total tensor product) and on the other hand with the Quillen group $K'_*(X, G)$ of coherent equivariant sheaves on X; furthermore, if every coherent equivariant sheaf on X is the quotient of a locally free equivariant coherent sheaf, it also coincides with the Quillen group $K_*^{\text{naive}}(X, G)$ of coherent locally free equivariant sheaves on X. These K-theories and their relationships are discussed in the appendix.

Our result is as follows. First we have to see what plays the role of the cyclic subgroups of a finite group. This is easy; the group schemes whose rings of representations are of the form $\mathbb{Z}[t]/(t^n - 1)$ are not the cyclic groups, in general, but their Cartier duals, that is, the group schemes that are isomorphic to the group scheme μ_n of nth roots of 1 for some n. We call these group schemes *dual cyclic*. If σ is a dual cyclic group, we can define $\widetilde{\mathrm{R}}\sigma$ as before. A dual cyclic subgroup σ of G is called essential if $X^\sigma \neq \emptyset$. The correct substitute for the ordinary K-theory of the subspaces of invariants is the *geometric equivariant K-theory* $K_*(X, G)_{\mathrm{geom}}$, which is defined as follows. Call N the least common multiple of the orders of all the essential dual cyclic subgroups of G. Call S_1 the multiplicative subset of the ring $\mathrm{R}(G)$ consisting of elements whose virtual rank is a power of N; then $K_*(X, G)_{\mathrm{geom}}$ is the localization $S_1^{-1}K_*(X, G)$. Notice that $K_*(X, G)_{\mathrm{geom}} \otimes \mathbb{Q} = K_*(X, G)_{\mathfrak{m}}$, with the notation above. Moreover, if every coherent equivariant sheaf on X is the quotient of a locally free equivariant coherent sheaf, by [EG], we have an isomorphism of rings

$$K_0(X, G)_{\mathrm{geom}} \otimes \mathbb{Q} = A_G^*(X) \otimes \mathbb{Q},$$

where $A_G^*(X)$ denotes the direct sum of G-equivariant Chow groups of X.

We prove the following. Assume that the action of G on X is connected. Then the kernel of the rank morphism $K_0(X, G)_{\mathrm{geom}} \to \mathbb{Z}[1/N]$ is nilpotent (see Cor. 5.2). This is remarkable; we have made what might look like a small step toward making the equivariant K-theory ring indecomposable, and we immediately get an indecomposable ring. Indeed, $K_*(X, G)_{\mathrm{geom}}$ "feels like" the K-theory ring of a scheme; we want to think of $K_*(X, G)_{\mathrm{geom}}$ as what the K-theory of the quotient X/G should be, if X/G were smooth, after inverting N (see Conj. 5.8).

Furthermore, consider the centralizer $\mathrm{C}_G(\sigma)$ and the normalizer $\mathrm{N}_G(\sigma)$ of σ inside G. The quotient $\mathrm{w}_G(\sigma) = \mathrm{N}_G(\sigma)/\mathrm{C}_G(\sigma)$ is contained inside the group scheme of automorphisms of σ, which is a discrete group, so it is also a discrete group. It acts on $\widetilde{\mathrm{R}}(\sigma)$, by conjugation, and it also acts on the equivariant K-theory ring $K_*(X^\sigma, \mathrm{C}_G(\sigma))$ and on the geometric equivariant K-theory ring $K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}}$ (see Cor. 2.5).

We say that the action of G on X is sufficiently rational when the following two conditions are satisfied. Let $\overline{k}$ be the algebraic closure of k.

(1) Each essential dual cyclic subgroup $\sigma \subseteq G_{\overline{k}}$ is conjugate by an element of $G(\overline{k})$ to a dual cyclic subgroup of G.

(2) If two essential dual cyclic subgroups of G are conjugate by an element of $G(\overline{k})$, they are also conjugate by an element of $G(k)$.

Obviously, every action over an algebraically closed field is sufficiently rational. Furthermore, if G is GL_m, SL_m, Sp_m, or a totally split torus, then any action of G is sufficiently rational over an arbitrary base field (see Prop. 2.3). If G is a finite group, then the action is sufficiently rational when k contains all nth roots of 1, where n is the least common multiple of the orders of the cyclic subgroups of k of order prime to the characteristic, whose fixed point subscheme is nonempty. Denote by $\mathscr{C}(G)$ a set of representatives for essential dual cyclic subgroup schemes, under conjugation by elements of the group $G(k)$. Here is the statement of our result.

MAIN THEOREM

Let G be an affine group scheme of finite type over a field k, acting on a Noetherian separated regular algebraic space X. Assume the following three conditions.

(a) *The action has finite geometric stabilizers.*

(b) *The action is sufficiently rational.*

(c) *For any essential cyclic subgroup σ of G, the quotient $G/\mathrm{C}_G(\sigma)$ is smooth.*

Then $\mathscr{C}(G)$ is finite, and there is a canonical isomorphism of $\mathrm{R}(G)$*-algebras*

$$K_*(X, G) \otimes \mathbb{Z}[1/N] \simeq \prod_{\sigma \in \mathscr{C}(G)} \left(K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)\right)^{\mathrm{W}_G(\sigma)}.$$

Conditions (a) and (b) are clearly necessary for the theorem to hold. We are not sure about (c). It is rather mild, as it is satisfied, for example, when G is smooth (this is automatically true in characteristic zero) or when G is abelian. A weaker version of condition (c) is given in Section 5.2.

In the case when G is abelian over an algebraically closed field of characteristic zero, the main theorem implies [Vi2, Conj. 3.6]. When G is a finite group, and the base field contains enough roots of 1, as in the statement of Theorem 1, then the conditions of the main theorem are satisfied; since the natural maps $K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \to K_*(X^\sigma)^{\mathrm{C}_G(\sigma)}$ become isomorphisms after inverting the order of G (see Prop. 5.7), the main theorem implies [Vi1, Th. 1]. However, the proof of the main theorem here is completely different from [Vi1, proof of Th. 1].

As B. Toen pointed out to us, a weaker version of our main theorem (with $\mathbb{Q}$-coefficients and assuming G smooth, acting with finite reduced stabilizers) follows from his [To1, Th. 3.15]; the étale techniques he uses in proving this result make it impossible to avoid tensoring with $\mathbb{Q}$ (see also [To2]).

Here is an outline of the paper. First we define the homomorphism (see Sec. 2.2). Next, in Section 3, we prove the result when G is a totally split torus. Here the basic tool is the result of R. Thomason, which gives a generic description of the action of a torus on a Noetherian separated algebraic space, and we prove the result by Noetherian induction, using the localization sequence for the K-theory of equivariant coherent sheaves. As in [Vi1], the difficulty here is that the homomorphism is defined via pullbacks, and thus it does not commute with the pushforwards intervening in the localization sequence. This is solved by producing a different isomorphism between the two groups in question, using pushforwards instead of pullbacks, and then relating this to our map, via the self-intersection formula.

The next step is to prove the result in the case when $G = \mathrm{GL}_n$; here the key point is a result of A. Merkurjev which links the equivariant K-theory of a scheme with a GL_n-action to the equivariant K-theory of the action of a maximal torus. This is carried out in Section 4. Finally (see Sec. 5), we reduce the general result to the case of GL_n, by considering an embedding $G \subseteq \mathrm{GL}_n$, and the induced action of GL_n on $Y = \mathrm{GL}_n \times^G X$. It is at this point that condition (c) enters, allowing a clear description of Y^σ where σ is an essential dual cyclic subgroup of G (see Prop. 5.6).

2. General constructions

Notation. If S is a separated Noetherian scheme, X is a Noetherian separated S-algebraic space (which is most of the time assumed to be regular), and G is a flat affine group scheme over S acting on X, we denote by $K_*(X, G)$ (resp., $K'(X, G)$) the Waldhausen K-theory of the complicial bi-Waldhausen (see [ThTr]) category $\mathscr{W}_{1,X,G}$ of complexes of quasi-coherent G-equivariant $\mathscr{O}_X$-modules with bounded coherent cohomology (resp., the Quillen K-theory of G-equivariant coherent $\mathscr{O}_X$-modules). As shown in the appendix, if X is regular, $K_*(X, G)$ is isomorphic to $K'_*(X, G)$ and has a canonical graded ring structure. When X is regular, the isomorphism $K_*(X, G) \simeq K'_*(X, G)$ then allows us to switch between the two theories when needed.

2.1. Morphisms of actions and induced maps on K-theory

Let S be a scheme. By an *action* over S we mean a triple (X, G, ρ) where X is an S-algebraic space, G is a group scheme over S, and $\rho : G \times_S X \to X$ is an action of G on X over S. A morphism of actions

$$(f, \phi) : (X, G, \rho) \longrightarrow (X', G', \rho')$$

is a pair of S-morphisms $f : X \to X'$ and $\phi : G \to G'$, where ϕ is a morphism of S-group schemes, such that the following diagram commutes:

$$\begin{tikzcd} G \times_S X \arrow[r, "\rho"] \arrow[d, "\phi\times f"'] & X \arrow[d, "f"] \\ G' \times_S X' \arrow[r, "\rho'"'] & X' \end{tikzcd}$$

Equivalently, f is required to be G-equivariant with respect to the given G-action on X and the G-action on X' induced by composition with ϕ.

A morphism of actions $(f, \phi) : (X, G, \rho) \to (X', G', \rho')$ induces an exact functor $(f, \phi)^* : \mathscr{W}_{3,X',G'} \to \mathscr{W}_{3,X,G}$, where $\mathscr{W}_{3,Y,H}$ denotes the complicial bi-Waldhausen category of complexes of H-equivariant flat quasi-coherent modules with bounded coherent cohomology on the H-algebraic space Y (see appendix). Let $(\mathscr{E}^*, \varepsilon^*)$ be an object of $\mathscr{W}_{3,X',G'}$; that is, $\mathscr{E}^*$ is a complex of G'-equivariant flat quasi-coherent $\mathscr{O}_{X'}$-modules with bounded coherent cohomology, and for any i,

$$\varepsilon^i : \mathrm{pr}_2'^* \mathscr{E}^i \xrightarrow{\sim} \rho'^* \mathscr{E}^i$$

is an isomorphism satisfying the usual cocycle condition. Here $\mathrm{pr}_2' : G' \times_S X' \to X'$ denotes the obvious projection, and similarly for $\mathrm{pr}_2 : G \times_S X \to X$. Since

$$\rho^* f^* \mathscr{E}^* = (f\rho)^* \mathscr{E}^* = (\phi \times f)^* \rho'^* \mathscr{E}^*$$

and

$$\mathrm{pr}_2^* f^* \mathscr{E}^* = (\phi \times f)^* \mathrm{pr}_2'^* \mathscr{E}^*,$$

we define $(f, \phi)^*(\mathscr{E}^*, \varepsilon^*) \doteq (f^*\mathscr{E}^*, (\phi \times f)^*(\varepsilon^*)) \in \mathscr{W}_{3,X,G}$ (the cocycle condition for each $(\phi \times f)^*(\varepsilon^i)$ following from the same condition for ε^i); $(f, \phi)^*$ is then defined on morphisms in the only natural way. Now, $(f, \phi)^*$ is an exact functor and, if X and X' are regular so that the Waldhausen K-theory of $\mathscr{W}_{3,X,G}$ (resp., of $\mathscr{W}_{3,X',G'}$) coincides with $K_*(X, G)$ (resp., $K_*(X', G')$) (see appendix), it defines a ring morphism

$$(f, \phi)^* : K_*(X', G') \longrightarrow K_*(X, G).$$

A similar argument shows that if f is flat, (f, ϕ) induces a morphism

$$(f, \phi)^* : K_*'(X', G') \longrightarrow K_*'(X, G).$$

Example 2.1
Let G and H be group schemes over S, and let X be an S-algebraic space. Moreover, suppose that

(1) G and H act on X;

(2) G acts on H by S-group scheme automorphisms (i.e., it is given a morphism $G \to \mathrm{Aut}_{(\mathrm{GrSch})/S}(H)$ of group functors over S);

(3) the two preceding actions are compatible; that is, for any S-scheme T, any $g \in G(T)$, $h \in H(T)$, and $x \in X(T)$, we have

$$g \cdot (h \cdot x) = h^g \cdot (g \cdot x),$$

where $(g, h) \mapsto h^g$ denotes the action of $G(T)$ on $H(T)$.

If $g \in G(S)$ and if g_T denotes its image via $G(S) \to G(T)$, let us define a morphism of actions $(f_g, \phi_g) : (X, H) \to (X, H)$ as

$$f_g(T) : X(T) \longrightarrow X(T) : x \longmapsto g_T \cdot x,$$
$$\phi_g(T) : H(T) \longrightarrow H(T) : h \longmapsto h^{g_T}.$$

This is an isomorphism of actions and induces an action of the group $G(S)$ on $K_*'(X, H)$ and on $K_*(X, H)$. This applies, in particular, to the case where X is an algebraic space with a G action and $G \triangleright H$, G acting on H by conjugation.

2.2. The basic definitions and results

Let G be a linear algebraic k-group scheme G acting with finite geometric stabilizers on a regular Noetherian separated algebraic space X over k.

We denote by $\mathrm{R}(G)$ the representation ring of G.

A (Cartier) *dual cyclic subgroup* of G over k is a k-subgroup scheme $\sigma \subseteq G$ such that there exist an $n > 0$ and an isomorphism of k-groups $\sigma \simeq \mu_{n,k}$. If σ, ρ are dual cyclic subgroups of G and if L is an extension of k, we say that σ and ρ are conjugate over L if there exists $g \in G(L)$ such that $g\sigma_{(L)}g^{-1} = \rho_{(L)}$ (where $H_{(L)} \doteq H \times_{\operatorname{Spec} k} \operatorname{Spec} L$, for any k-group scheme H) as L-subgroup schemes of $G_{(L)}$.

A dual cyclic subgroup $\sigma \subseteq G$ is said to be *essential* if $X^\sigma \neq \emptyset$.

We say that the action of G on X is *sufficiently rational* if

(1) any two essential dual cyclic subgroups of G are conjugated over k if and only if they are conjugated over an algebraic closure $\overline{k}$ of k;

(2) any essential dual cyclic subgroup $\overline{\rho}$ of $G_{(\overline{k})}$ is conjugated over $\overline{k}$ to a dual cyclic subgroup of the form $\sigma_{(\overline{k})}$ where $\sigma \subseteq G$ is (essential) dual cyclic.

We denote by $\mathscr{C}(G)$ a set of representatives for essential dual cyclic subgroups of G with respect to the relation of conjugacy over k.

Remark 2.2

Note that if the action is sufficiently rational and if ρ, σ are essential dual cyclic subgroups of G which are conjugate over an algebraically closed extension Ω of k, then they are also conjugate over k.

PROPOSITION 2.3

Any action of $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$, *or of a split torus is sufficiently rational.*

Proof

If G is a split torus, condition (1) is clear because G is abelian, while it follows from the rigidity of diagonalizable groups that any subgroup scheme of $G_{\overline{k}}$ is in fact defined over k. Let $\sigma \subseteq \mathrm{GL}_m$ be a dual cyclic subgroup. Since σ is diagonalizable, we have an eigenspace decomposition

$$V = k^m = \bigoplus_{\chi \in \widehat{\sigma}} V_\chi^\sigma$$

such that the χ with $V_\chi \neq 0$ generate $\widehat{\sigma}$. Conversely, given a cyclic group C and a decomposition

$$V = \bigoplus_{\chi \in \widehat{C}} V_\chi$$

such that the χ with $V_\chi \neq 0$ generate $\widehat{C}$, there is a corresponding embedding of the Cartier dual σ of C into GL_n with $V_\chi = V_\chi^\sigma$ for each $\chi \in C = \widehat{\sigma}$. Now, if $\sigma \subseteq \mathrm{GL}_{m,\overline{k}}$ is a dual cyclic subgroup defined over $\overline{k}$, we can apply an element of $\mathrm{GL}_m(\overline{k})$ to make the V_χ defined over k, and then $g\sigma g^{-1}$ is defined over k. If $\sigma \subseteq \mathrm{GL}_m$ and $\tau \subseteq \mathrm{GL}_m$ are dual cyclic subgroups that are conjugate over $\overline{k}$, pick an element of $\mathrm{GL}_m(\overline{k})$ sending σ to τ. This induces an isomorphism $\phi : \sigma_{\overline{k}} \simeq \tau_{\overline{k}}$, which by rigidity is defined over k. Then if χ and χ' are characters that correspond under the isomorphism of $\widehat{\sigma}$ and $\widehat{\tau}$ induced by ϕ, then the dimension of V_χ^σ is equal to the dimension of $V_{\chi'}^\tau$, so we can find an element g of GL_m which carries each V_χ^σ onto the corresponding $V_{\chi'}^\tau$; conjugation by this element carries σ onto τ. For SL_m, the proof is very similar if we remark that to give a dual cyclic subgroup $\sigma \subseteq \mathrm{SL}_m \subseteq \mathrm{GL}_m$ corresponds to giving a decomposition

$$V = k^m = \bigoplus_{\chi \in \widehat{\sigma}} V_\chi^\sigma$$

such that the χ with $V_\chi^\sigma \neq 0$ generate $\widehat{\sigma}$, with the condition $\prod_{\chi \in \widehat{\sigma}} \chi^{\dim V_\chi^\sigma} = 1 \in \widehat{\sigma}$. For $\mathrm{Sp}_m \subseteq \mathrm{GL}_{2m}$, a dual cyclic subgroup $\sigma \subseteq \mathrm{Sp}_m$ gives a decomposition

$$V = k^{2m} = \bigoplus_{\chi \in \widehat{\sigma}} V_\chi^\sigma$$

such that the χ with $V_\chi^\sigma \neq 0$ generate $\widehat{\sigma}$, with the condition that for $v \in V_\chi^\sigma$ and $v' \in V_{\chi'}^\sigma$ the symplectic product of v and v' is always zero, unless $\chi\chi' = 1 \in \widehat{\sigma}$. Both conditions then follow rather easily from the fact that any two symplectic forms over a vector space are isomorphic. $\square$

Let $N_{(G,X)}$ denote the least common multiple of the orders of essential dual cyclic subgroups of G. Notice that $N_{(G,X)}$ is finite: since the action has finite stabilizers, the group scheme of stabilizers is quasi-finite over X; therefore the orders of the stabilizers of the geometric points of X are globally bounded.

We define $\Lambda_{(G,X)} \doteq \mathbb{Z}[1/N_{(G,X)}]$.

If $H \subseteq G$ is finite, we also write Λ_H for $\mathbb{Z}[1/|H|]$. Note that, if $\sigma \subseteq G$ is dual cyclic, then $\Lambda_\sigma = \Lambda_{(\sigma, \operatorname{Spec} k)}$, and if, moreover, σ is essential, $\Lambda_\sigma \subseteq \Lambda_{(G,X)}$.

If $H \subseteq G$ is a subgroup scheme and if A is a ring, we write $\mathrm{R}(H)_A$ for $\mathrm{R}(H) \otimes_{\mathbb{Z}} A$. We denote by $\mathrm{rk}_H : \mathrm{R}(H) \longrightarrow \mathbb{Z}$ and by $\mathrm{rk}_{H,\Lambda_{(G,X)}} : \mathrm{R}(H)_{\Lambda_{(G,X)}} \longrightarrow \Lambda_{(G,X)}$ the rank ring homomorphisms.

We let

$$K'_*(X,G)_{\Lambda_{(G,X)}} = K'_*(X,G) \otimes \Lambda_{G,X}$$

and

$$K_*(X,G)_{\Lambda_{(G,X)}} = K_*(X,G) \otimes \Lambda_{G,X}$$

(for the notation, see the beginning of this section). Recall that $K_*(X, G)_{\Lambda_{(G,X)}}$ is an $\mathrm{R}(G)$-algebra via the pullback $\mathrm{R}(G) \simeq K_0(\operatorname{Spec} k, G) \to K_0(X, G)$ and that $K_*(X, G) \simeq K'_*(X, G)$ since X is regular (see appendix).

If σ is a dual cyclic subgroup of G of order n, the choice of a generator t for the dual group $\widehat{\sigma} \doteq \mathrm{Hom}_{\mathrm{GrSch}_{/k}}(\sigma, \mathbb{G}_{m,k})$ determines an isomorphism

$$\mathrm{R}(\sigma) \simeq \frac{\mathbb{Z}[t]}{(t^n - 1)}.$$

Let p_σ be the canonical ring surjection

$$\frac{\mathbb{Z}[t]}{(t^n - 1)} \longrightarrow \prod_{d|n} \frac{\mathbb{Z}[t],}{(\Phi_d)},$$

and let $\widetilde{p_\sigma}$ be the induced surjection

$$\frac{\mathbb{Z}[t]}{(t^n - 1)} \longrightarrow \frac{\mathbb{Z}[t]}{(\Phi_n)},$$

where Φ_d is the dth cyclotomic polynomial. If $\mathfrak{m}_\sigma$ is the kernel of the composition

$$\mathrm{R}(\sigma) \simeq \frac{\mathbb{Z}[t]}{(t^n - 1)} \longrightarrow \frac{\mathbb{Z}[t]}{(\Phi_n)},$$

the quotient ring $\mathrm{R}(\sigma)/\mathfrak{m}_\sigma$ does not depend on the choice of the generator t for $\widehat{\sigma}$.

Notation. We denote by $\widetilde{\mathrm{R}}(\sigma)$ the quotient $\mathrm{R}(\sigma)/\mathfrak{m}_\sigma$. We remark that if σ is dual cyclic of order n and if t is a generator of $\widehat{\sigma}$, there are isomorphisms

$$\mathrm{R}(\sigma)_{\Lambda_\sigma} \simeq \frac{\Lambda_\sigma[t]}{(t^n - 1)} \simeq \prod_{d|n} \frac{\Lambda_\sigma[t]}{(\Phi_d)}. \tag{1}$$

Let $\widetilde{\pi_\sigma} : \mathrm{R}(G)_{\Lambda_{(G,X)}} \to \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{(G,X)}}$ be the canonical ring homomorphism. The *σ-localization* $K'_*(X, G)_\sigma$ of $K'_*(X, G)_{\Lambda_{(G,X)}}$ is the localization of the $\mathrm{R}(G)_{\Lambda_{(G,X)}}$-module $K'_*(X, G)_{\Lambda_{(G,X)}}$ at the multiplicative subset $S_\sigma \doteq \widetilde{\pi_\sigma}^{-1}(1)$. The σ-localization $K_*(X, G)_\sigma$ is defined in the same way. If $H \subseteq G$ is a subgroup scheme, we also write $\mathrm{R}(H)_\sigma$ for $S_\sigma^{-1}(\mathrm{R}(H)_{\Lambda_{(G,X)}})$.

If *σ is the trivial group*, we denote by $K'_*(X, G)_{\mathrm{geom}}$ the σ-localization of $K'_*(X, G)_{\Lambda_{(G,X)}}$ and call it the *geometric part* or *geometric localization* of $K'_*(X, G)_{\Lambda_{(G,X)}}$. Note that $\widetilde{\pi_1}$ coincides with the rank morphism $\mathrm{rk}_{G,\Lambda_{(G,X)}} : \mathrm{R}(G)_{\Lambda_{(G,X)}} \longrightarrow \Lambda_{(G,X)}$. We have the same definition for $K_*(X, G)_{\mathrm{geom}}$.

Let $\mathrm{N}_G(\sigma)$ (resp., $\mathrm{C}_G(\sigma) \subseteq \mathrm{N}_G(\sigma)$) be the normalizer (resp., the centralizer) of σ in G; since $\mathrm{Aut}(\sigma)$ is a finite constant group scheme,

$$\mathrm{W}_G(\sigma) \doteq \frac{\mathrm{N}_G(\sigma)}{\mathrm{C}_G(\sigma)}$$

is also a constant group scheme over k associated to a finite group $w_G(\sigma)$.

LEMMA 2.4
Let H be a k-linear algebraic group, let $\sigma \simeq \mu_{n,k}$ be a normal subgroup, and let Y be an algebraic space with an action of H/σ. Then there is a canonical action of $w_H(\sigma)$ on $K'_*(Y, \mathrm{C}_H(\sigma))$.

Proof
Let us first assume that the natural map

$$H(k) \longrightarrow w_H(\sigma) \tag{2}$$

is surjective (which is true, e.g., if k is algebraically closed). Since $\mathrm{C}_H(\sigma)(k)$ acts trivially on $K'_*(Y, \mathrm{C}_H(\sigma))$ and, by Example 2.1, $H(k)$ acts naturally on $K'_*(Y, \mathrm{C}_H(\sigma))$, we may use (2) to define the desired action. In general, (2) is not surjective and we proceed as follows. Suppose that we can find a closed immersion of k-linear algebraic groups $H \hookrightarrow H'$ such that

(i) σ is normal in H';
(ii) $H'/\mathrm{C}_{H'}(\sigma) \simeq W_H(\sigma)$;
(iii) $H'(k) \to w_H(\sigma)$ is surjective.

Consider the open and closed immersion $Y \times \mathrm{C}_H(\sigma) \hookrightarrow Y \times H$; this induces an open and closed immersion $Y \times^{\mathrm{C}_H(\sigma)} \mathrm{C}_{H'}(\sigma) \hookrightarrow Y \times^{\mathrm{C}_H(\sigma)} H'$ whose composition with the étale cover $Y \times^{\mathrm{C}_H(\sigma)} H' \to Y \times^H H'$ is easily checked (e.g., on geometric points) to be an isomorphism. Therefore,

$$K'_*\big(Y \times^H H', \mathrm{C}_{H'}(\sigma)\big) \simeq K'_*\big(Y \times^{\mathrm{C}_H(\sigma)} \mathrm{C}_{H'}(\sigma), \mathrm{C}_{H'}(\sigma)\big) \simeq K'_*\big(Y, \mathrm{C}_H(\sigma)\big),$$

where the last isomorphism is given by the Morita equivalence theorem (see [Th3, Prop. 6.2]). By (i) and (iii) we can apply the argument at the beginning of the proof and get an action of $w_H(\sigma)$ on $K'_*(Y \times^H H', \mathrm{C}_{H'}(\sigma))$ and therefore on $K'_*(Y, \mathrm{C}_H(\sigma))$, as desired. It is not difficult to check that this action does not depend on the chosen immersion $H \hookrightarrow H'$.

Finally, let us prove that there exists a closed immersion $H \hookrightarrow H'$ satisfying conditions (i) – (iii) above. First choose a closed immersion $j : H \hookrightarrow \mathrm{GL}_{n,k}$ for some n. Clearly,

$$H/\mathrm{C}_H(\sigma) \hookrightarrow \mathrm{GL}_{n,k}/\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma),$$

and, embedding σ in a maximal torus of $\mathrm{GL}_{n,k}$, it is easy to check that

$$\mathrm{GL}_{n,k}(k) \to \big(\mathrm{GL}_{n,k}/\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\big)(k)$$

is surjective. Now define H' as the inverse image of $H/\mathrm{C}_H(\sigma)$ in the normalizer $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)$. □

COROLLARY 2.5
There is a canonical action of $w_G(\sigma)$ on $K'_*(X^\sigma, \mathrm{C}_G(\sigma))$ which induces an action on $K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}}$.

Proof
Since $\mathrm{C}_G(\sigma) = \mathrm{C}_{N_G(\sigma)}(\sigma)$, Lemma 2.4, applied to $Y = X^\sigma$ (resp., $Y = \operatorname{Spec} k$) and $H = N_G(\sigma)$, yields an action of $w_G(\sigma)$ on $K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)$ $\big($resp., on $K_0(\operatorname{Spec} k, \mathrm{C}_G(\sigma)) = \mathrm{R}(\mathrm{C}_G(\sigma))\big)$. The multiplicative system $S_1 = \mathrm{rk}^{-1}(1)$ is preserved by this action so that there is an induced action on the ring $S_1^{-1}\mathrm{R}(\mathrm{C}_G(\sigma))$. The pullback

$$K_0\big(\operatorname{Spec} k, \mathrm{C}_G(\sigma)\big) \to K_0\big(X^\sigma, \mathrm{C}_G(\sigma)\big)$$

is $w_G(\sigma)$-equivariant, and then $w_G(\sigma)$ acts on $K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}}$. □

Remark 2.6
If Y is regular, Lemma 2.4 also gives an action of $w_H(\sigma)$ on $K_*(Y, \mathrm{C}_H(\sigma))$ since $K_*(Y, \mathrm{C}_H(\sigma)) \simeq K'_*(Y, \mathrm{C}_H(\sigma))$ (see appendix). In particular, since by [Th5, Prop. 3.1], X^σ is regular, Corollary 2.5 still holds for $K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}}$.

Note also that the embedding of k-group schemes $W_G(\sigma) \hookrightarrow \mathrm{Aut}_k(\sigma)$ induces, by Example 2.1, an action of $w_G(\sigma)$ on $K_0(\operatorname{Spec} k, \sigma) = \mathrm{R}(\sigma)$. The product in σ induces a morphism of k-groups,

$$\sigma \times \mathrm{C}_G(\sigma) \longrightarrow \mathrm{C}_G(\sigma),$$

which in its turn induces a morphism

$$m_\sigma : K_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big) \longrightarrow K_*\big(X^\sigma, \sigma \times \mathrm{C}_G(\sigma)\big).$$

LEMMA 2.7
If $H \subseteq G$ is a subgroup scheme and if σ is contained in the center of H, there is a canonical ring isomorphism

$$K_*(X^\sigma, \sigma \times H) \simeq K_*(X^\sigma, H) \otimes \mathrm{R}(\sigma).$$

Proof
Since σ acts trivially on X^σ, we have an equivalence (see [SGA3, Exp. I, par. 4.7.3])

$$(\sigma \times H) - \mathrm{Coh}_{X^\sigma} \simeq \bigoplus_{\widehat{\sigma}} (H - \mathrm{Coh}_{X^\sigma}) \tag{3}$$

(where $\widehat{\sigma}$ is the character group of σ) which induces an isomorphism

$$K'_*(X^\sigma, \sigma \times H) \simeq K'_*(X^\sigma, H) \otimes \mathrm{R}(\sigma).$$

We conclude since $K_*(Y,H) \simeq K'_*(Y,H)$ and $K_*(X^\sigma, \sigma \times H) \simeq K'_*(X^\sigma, \sigma \times H)$ (see appendix). $\square$

For any essential dual cyclic subgroup $\sigma \subseteq G$, let $\Lambda \doteq \Lambda_{(G,X)}$, and consider the composition

$$\begin{aligned} K_*(X,G)_\Lambda \to K_*\big(X, \mathrm{C}_G(\sigma)\big)_\Lambda \to K_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\Lambda \\ \xrightarrow{m_\sigma} K_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\Lambda \otimes_\Lambda \mathrm{R}(\sigma)_\Lambda \\ \longrightarrow K_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda, \end{aligned} \tag{4}$$

where the first map is induced by group restriction, the last one is the geometric localization map tensored with the projection $\mathrm{R}(\sigma)_\Lambda \to \widetilde{\mathrm{R}}(\sigma)_\Lambda$, and we have used Lemma 2.7 with $H = \mathrm{C}_G(\sigma)$; the second map is induced by restriction along $X^\sigma \hookrightarrow X$ which is a regular closed immersion (see [Th5, Prop. 3.1]) and therefore has finite Tor-dimension, so that the pullback on K-groups is well defined (see appendix). It is not difficult to show that the image of (4) is actually contained in the invariant submodule

$$\big(K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_G(\sigma)},$$

so that we get a map

$$\psi_{\sigma,X} : K_*(X,G)_\Lambda \longrightarrow \big(K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_G(\sigma)}.$$

Our basic map is

$$\begin{aligned} \Psi_{X,G} \doteq \prod_{\sigma \in \mathscr{C}(G)} \psi_{\sigma,X} : K_*(X,G)_\Lambda \\ \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \big(K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_G(\sigma)}. \end{aligned} \tag{5}$$

Note that $\Psi_{X,G}$ is a morphism of $\mathrm{R}(G)$-algebras as a composition of morphisms of $R(G)$-algebras.

The following technical lemma is used in Propositions 3.5 and 4.6.

LEMMA 2.8
Let G be a linear algebraic k-group acting with finite stabilizers on a Noetherian separated k-algebraic space X, and let $\Lambda \doteq \Lambda_{(G,X)}$. Let $H \subseteq G$ be a subgroup, and let σ be an essential dual cyclic subgroup contained in the center of H. Consider the composition

$$K'_*(Y^\sigma, H)_\Lambda \longrightarrow K'_*(Y^\sigma, H)_\Lambda \otimes_\Lambda \mathrm{R}(\sigma)_\Lambda \longrightarrow K'_*(Y^\sigma, H)_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda, \tag{6}$$

where the first morphism is induced by the product morphism $\sigma \times H \to H$ (recall Lem. 2.7) and the second is the tensor product of the geometric localization morphism with the projection $\mathrm{R}(\sigma)_\Lambda \to \widetilde{\mathrm{R}}(\sigma)_\Lambda$. Then (6) factors through $K'_*(Y^\sigma, H)_\Lambda \to K'_*(Y^\sigma, H)_\sigma$, yielding a morphism

$$\theta_{H,\sigma} : K'_*(Y^\sigma, H)_\sigma \longrightarrow K'_*(Y^\sigma, H)_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda. \tag{7}$$

Proof

Let S_1 (resp., S_σ) be the multiplicative subset in $\mathrm{R}(H)_\Lambda$ consisting of elements going to 1 via the homomorphism $\mathrm{rk}_{H,\Lambda} : \mathrm{R}(H)_\Lambda \to \Lambda$ (resp., $\mathrm{R}(H)_\Lambda \to \widetilde{\mathrm{R}}(\sigma)_\Lambda$). Observe that $K'_*(X^\sigma, H)_\Lambda \otimes_\Lambda \mathrm{R}(\sigma)_\Lambda$ (resp., $K'_*(X^\sigma, H)_{\mathrm{geom}} \otimes_\Lambda \widetilde{\mathrm{R}}(\sigma)_\Lambda$) is canonically an $\mathrm{R}(H)_\Lambda \otimes \mathrm{R}(\sigma)_\Lambda$-module (resp., an $S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$-module) and therefore an $\mathrm{R}(H)$-module via the coproduct ring morphism

$$\Delta_\sigma : \mathrm{R}(H)_\Lambda \longrightarrow \mathrm{R}(H)_\Lambda \otimes \mathrm{R}(\sigma)_\Lambda$$

(resp., via the ring morphism

$$f_\sigma : \mathrm{R}(H)_\Lambda \xrightarrow{\Delta_\sigma} \mathrm{R}(H)_\Lambda \otimes \mathrm{R}(\sigma)_\Lambda \longrightarrow S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big).$$

If we denote by A' the $\mathrm{R}(H)_\Lambda$-algebra $f_\sigma : \mathrm{R}(H)_\Lambda \longrightarrow S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$, it is enough to show that the localization homomorphism

$$l'_\sigma : A' \longrightarrow S_\sigma^{-1}(A')$$

is an isomorphism, because in this case the morphism (7) is induced by the S_σ-localization of (6). Let A denote the $\mathrm{R}(H)_\Lambda$-algebra

$$\lambda_1 \otimes 1 : \mathrm{R}(H)_\Lambda \longrightarrow S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda,$$

where $\lambda_1 : \mathrm{R}(H)_\Lambda \to S_1^{-1}\mathrm{R}(H)_\Lambda$ denotes the localization homomorphism. It is a well-known fact that there is an isomorphism of $\mathrm{R}(H)_\Lambda$-algebras $\varphi : A' \to A$; this is exactly the dual assertion to "the action $H \times \sigma \to \sigma$ is isomorphic to the projection on the second factor $H \times \sigma \to \sigma$." Therefore, we have a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\varphi"] \arrow[d, "l'_\sigma"'] & A \arrow[d, "l_\sigma"] \\ S_\sigma^{-1}A' \arrow[r, "S_\sigma^{-1}\varphi"'] & S_\sigma^{-1}A \end{tikzcd}$$

where l_σ denotes the localization homomorphism, and it is enough to prove that l_σ is an isomorphism. To see this, note that the ring $\widetilde{\mathrm{R}}(\sigma)_\Lambda$ is a free Λ-module of finite rank (equal to $\phi(|\sigma|)$, ϕ being the Euler function), and there is a norm homomorphism

$$\mathrm{N} : \widetilde{\mathrm{R}}(\sigma)_\Lambda \longrightarrow \Lambda$$

sending an element to the determinant of the Λ-endomorphism of $\widetilde{\mathrm{R}}(\sigma)_\Lambda$ induced by multiplication by this element; obviously, we have

$$\mathrm{N}^{-1}(\Lambda^*) = \big(\widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^*.$$

Analogously, there is a norm homomorphism

$$\mathrm{N}' : A' = S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \longrightarrow S_1^{-1}\mathrm{R}(H)_\Lambda,$$

and

$$\mathrm{N}^{-1}\big((S_1^{-1}\mathrm{R}(H)_\Lambda)^*\big) = \big(S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^*.$$

There is a commutative diagram

$$\begin{tikzcd}
S_1^{-1}\mathrm{R}(H)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \arrow[r, "\mathrm{N}'"] \arrow[d, "\mathrm{rk}_{H,\Lambda}\otimes\mathrm{id}"'] & S_1^{-1}\mathrm{R}(H)_\Lambda \arrow[d, "\mathrm{rk}_{H,\Lambda}"] \\
\widetilde{\mathrm{R}}(\sigma)_\Lambda \arrow[r, "\mathrm{N}"'] & \Lambda
\end{tikzcd}$$

and, by definition of S_1, we get $\mathrm{rk}_{H,\Lambda}^{-1}(\Lambda^*) = (S_1^{-1}\mathrm{R}(H)_\Lambda)^*$. Therefore, by definition of S_σ, $S_\sigma/1$ consist of units in A, and we conclude the proof of the lemma. □

The following lemma, which is an easy consequence of a result of Merkurjev, is the main tool in reducing the proof of the main theorem from $G = \mathrm{GL}_{n,k}$ to its maximal torus T.

LEMMA 2.9
Let X be a Noetherian separated algebraic space over k with an action of a split reductive group G over k such that $\pi_1(G)$ (see [Me, Par. 1.1]) is torsion free. Then if T denotes a maximal torus in G, the canonical morphism

$$K'_*(X, G) \otimes_{\mathrm{R}(G)} \mathrm{R}(T) \longrightarrow K'_*(X, T)$$

is an isomorphism.

Proof
Let $B \supseteq T$ be a Borel subgroup of G. Since $\mathrm{R}(B) \simeq \mathrm{R}(T)$ and $K'_*(X, B) \simeq K'_*(X, T)$ (see [Th4, proof of Th. 1.13, p. 594]), by [Me, Prop. 4.1], the canonical ring morphism

$$K'_*(X, G) \otimes_{\mathrm{R}(G)} \mathrm{R}(T) \longrightarrow K'_*(X, T)$$

is an isomorphism.

Since Merkurjev states his theorem for a scheme, we briefly indicate how it extends to a Noetherian separated algebraic space X over k. By [Th1, Lem. 4.3], there exists an open dense G-invariant separated subscheme $U \subset X$. Since Merkurjev's map commutes with localization, by the localization sequence and Noetherian induction it is enough to know the result for U. And this is given in [Me, Prop. 4.1]. Note that by [Me, Prop. 1.22], $\mathrm{R}(T)$ is flat (actually free) over $\mathrm{R}(G)$, and therefore the localization sequence remains exact after tensoring with $\mathrm{R}(T)$. $\square$

The following is [Vi1, Lem. 3.2]. It is used frequently in the rest of the paper, and it is stated here for the convenience of the reader.

LEMMA 2.10
Let W be a finite group acting on the left on a set $\mathscr{A}$, and let $\mathscr{B} \subseteq \mathscr{A}$ be a set of representatives for the orbits. Assume that W acts on the left on a product of abelian groups of the type $\prod_{\alpha \in \mathscr{A}} M_\alpha$ in such a way that

$$sM_\alpha = M_{s\alpha}$$

for any $s \in W$.

For each $\alpha \in \mathscr{B}$, let us denote by W_α the stabilizer of α in W. Then the canonical projection

$$\prod_{\alpha \in \mathscr{A}} M_\alpha \longrightarrow \prod_{\alpha \in \mathscr{B}} M_\alpha$$

induces an isomorphism

$$\Big(\prod_{\alpha \in \mathscr{A}} M_\alpha\Big)^W \longrightarrow \prod_{\alpha \in \mathscr{B}} (M_\alpha)^{W_\alpha}.$$

3. The main theorem: The split torus case

In this section, T is a split torus over k.

PROPOSITION 3.1
Let $T' \subset T$ be a closed subgroup scheme (diagonalizable, by [SGA3, Exp. IX, par. 8.1]), finite over k. Then the canonical morphism

$$\delta : \mathrm{R}(T')_{\Lambda_{T'}} \longrightarrow \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{T'}}$$

is a ring isomorphism.

Proof
Since both $\mathrm{R}(T')_{\Lambda_{T'}}$ and $\prod \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{T'}}$ are free $\Lambda_{T'}$-modules of finite rank, it is enough

to prove that, for any nonzero prime $p \nmid |T'|$, the induced morphism of $\mathbb{F}_p$-vector spaces

$$\mathrm{R}(T')_{\Lambda_{T'}} \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{T'}} \otimes_{\mathbb{Z}} \mathbb{F}_p \tag{8}$$

is an isomorphism. Now, for any finite abelian group A, we have an equality $|A| = \sum_{A \twoheadrightarrow C} \varphi(|C|)$, where φ denotes the Euler function, $|H|$ denotes the order of the group H, and the sum is extended to all cyclic quotients of A; applying this to the group of characters $\widehat{T'}$ (so that the corresponding cyclic quotients C are exactly the group of characters $\widehat{\sigma}$ for σ dual cyclic subgroups of T'), we see that the ranks of both sides in (8) coincide with $|T'|$, and it is then enough to prove that (8) is injective. Define a morphism

$$f : \prod_{\substack{\tau \text{ dual cyclic} \\ \tau \subseteq T'}} \widetilde{\mathrm{R}}(\tau)_{\Lambda_{T'}} \longrightarrow \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \mathrm{R}(\sigma)_{\Lambda_{T'}}$$

of $\mathrm{R}(T')_{\Lambda_{T'}}$-modules by requiring, for any dual cyclic subgroup $\sigma \subseteq T'$, the commutativity of the following diagram:

$$\begin{tikzcd}
\prod_{\substack{\tau \text{ dual cyclic} \\ \tau \subseteq T'}} \widetilde{\mathrm{R}}(\tau)_{\Lambda_{T'}} \arrow[r, "f"] \arrow[d, "\mathrm{Pr}_\sigma"'] & \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \mathrm{R}(\sigma)_{\Lambda_{T'}} \arrow[d, "\mathrm{pr}_\sigma"] \\
\prod_{\tau \subseteq \sigma} \widetilde{\mathrm{R}}(\tau)_{\Lambda_{T'}} & \mathrm{R}(\sigma)_{\Lambda_{T'}} \arrow[l, "\sim"', "\varphi"]
\end{tikzcd}$$

where Pr_σ and pr_σ are the obvious projections and φ is the isomorphism

$$\mathrm{R}(\sigma)_{\Lambda_{T'}} \xrightarrow{\prod_{\tau \subseteq \sigma} \mathrm{res}^{\sigma}_{\tau}} \prod_{\tau \subseteq \sigma} \mathrm{R}(\tau)_{\Lambda_{T'}} \xrightarrow{(\widetilde{\mathrm{pr}}_\tau)_\tau} \prod_{\tau \subseteq \sigma} \widetilde{\mathrm{R}}(\tau)_{\Lambda_{T'}}$$

induced by (1). Obviously, $f \circ \delta$ coincides with the map

$$\prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \mathrm{res}^{T'}_{\sigma} : \mathrm{R}(T')_{\Lambda_{T'}} \longrightarrow \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \mathrm{R}(\sigma)_{\Lambda_{T'}},$$

so we are reduced to proving that

$$\mathrm{R}(T')_{\Lambda_{T'}} \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \mathrm{R}(\sigma)_{\Lambda_{T'}} \otimes_{\mathbb{Z}} \mathbb{F}_p$$

is injective, that is, that if A is a finite abelian group and $p \nmid |A|$, then

$$\varphi : \mathbb{F}_p[A] \longrightarrow \prod_{C \in \{\text{cyclic quotients of } A\}} \mathbb{F}_p[C] \tag{9}$$

is injective. If $\widehat{A} = \mathrm{Hom}_{\mathrm{AbGrps}}(A, \mathbb{C}^*)$ denotes the complex characters group of A, then $\mathrm{R}(\widehat{A}) \simeq \mathbb{Z}[A]$ and

$$\varphi = \prod_{\widehat{C}} \mathrm{res}^{\widehat{A}}_{\widehat{C}} : \mathrm{R}(\widehat{A}) \longrightarrow \prod_{\widehat{C} \in \{\text{cyclic subgroups of } \widehat{A}\}} \mathrm{R}(\widehat{C}).$$

Since $p \nmid |A|$, it is enough to prove that if $\xi \in \mathrm{R}(\widehat{A}) \dot{\otimes}_{\mathbb{Z}} \mathbb{Z}[1/|A|]$ has image via

$$\mathrm{res}^{\widehat{A}}_{\widehat{C}} \otimes \mathbb{Z}[1/|A|] : \mathrm{R}(\widehat{A}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|A|] \longrightarrow \mathrm{R}(\widehat{C}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|A|]$$

contained in $p\big(\mathrm{R}(\widehat{C}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|A|]\big)$ for each cyclic $\widehat{C} \subseteq \widehat{A}$, then $\xi \in p\big(\mathrm{R}(\widehat{A}) \dot{\otimes}_{\mathbb{Z}} \mathbb{Z}[1/|A|]\big)$.

By [Se, p. 73], there exists $(\theta'_{\widehat{C}})_{\widehat{C}} \in \prod_{\widehat{C} \in \{\text{cyclic subgroups of} \widehat{A}\}} \mathrm{R}(\widehat{C}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|A|]$ such that

$$1 = \sum_{\widehat{C}} \big(\mathrm{ind}^{\widehat{C}}_{\widehat{A}} \otimes \mathbb{Z}[1/|A|]\big)(\theta'_{\widehat{C}});$$

therefore

$$\begin{aligned} \xi &= \sum_{\widehat{C}} \xi \big(\mathrm{ind}^{\widehat{C}}_{\widehat{A}} \otimes \mathbb{Z}[1/|A|]\big)(\theta'_{\widehat{C}}) \\ &= \sum_{\widehat{C}} \big(\mathrm{ind}^{\widehat{C}}_{\widehat{A}} \otimes \mathbb{Z}[1/|A|]\big)\big(\theta'_{\widehat{C}}(\mathrm{res}^{\widehat{A}}_{\widehat{C}} \otimes \mathbb{Z}[1/|A|])(\xi)\big) \end{aligned}$$

(by the projection formula), and we conclude the proof of the proposition. □

Remark 3.2
The proof of Proposition 3.1 is similar to [Vi1, proof of Prop. 1.5], which is, however, incomplete; that is why we have decided to give all the details here.

COROLLARY 3.3
We have the following.

(i) *If $\sigma \neq \sigma'$ are dual cyclic subgroups of T, we have $\widetilde{\mathrm{R}}(\sigma)_{\sigma'} = 0$ and $\widetilde{\mathrm{R}}(\sigma)_{\sigma} = \widetilde{\mathrm{R}}(\sigma)$.*

(ii) *If $T' \subset T$ is a closed subgroup scheme, finite over k, and if σ is a dual cyclic subgroup of T, we have $\mathrm{R}(T')_{\sigma} = 0$ if $\sigma \nsubseteq T'$.*

(iii) *If $T' \subset T$ is a closed subgroup scheme, finite over k, the canonical morphism of $\mathrm{R}(T)$-algebras*

$$\mathrm{R}(T')_{\Lambda_{T'}} \longrightarrow \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subseteq T'}} \mathrm{R}(T')_{\sigma}$$

is an isomorphism.

Proof

(i) Suppose $\sigma \neq \sigma'$, and let $T' \subset T$ be the closed subgroup scheme of T generated by σ and σ'. The obvious morphism $\pi : \mathrm{R}(T)_{\Lambda_{T'}} \to \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{T'}} \times \widetilde{\mathrm{R}}(\sigma')_{\Lambda_{T'}}$ factors through $\mathrm{R}(T')_{\Lambda_{T'}} \to \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{T'}} \times \widetilde{\mathrm{R}}(\sigma')_{\Lambda_{T'}}$, which is an epimorphism by Proposition 3.1. If $\xi \in \mathrm{R}(T)_{\Lambda_{T'}}$ with $\pi(\xi) = (0, 1) \otimes 1$, we have

$$\xi \in S_{\sigma'} \cap \ker\left(\mathrm{R}(T)_{\Lambda_{T'}} \to \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{T'}}\right).$$

Then $\widetilde{\mathrm{R}}(\sigma)_{\sigma'} = 0$. The second assertion is obvious.

(ii) and (iii) These follow immediately from (i) and Proposition 3.1. □

Now let X be a regular Noetherian separated k-algebraic space on which T acts with finite stabilizers, and let $\Lambda \doteq \Lambda_{(T,X)}$. Obviously, $\mathscr{C}(T)$ is just the set of essential dual cyclic subgroups of T since T is abelian.

PROPOSITION 3.4

We have the following.

(i) *If $j_\sigma : X^\sigma \hookrightarrow X$ denotes the inclusion, the pushforward $(j_\sigma)_*$ induces an isomorphism*

$$K'_*(X^\sigma, T)_\sigma \longrightarrow K'_*(X, T)_\sigma.$$

(ii) *The canonical localization morphism*

$$K'_*(X, T)_\Lambda \longrightarrow \prod_{\sigma \in \mathscr{C}(T)} K'_*(X, T)_\sigma$$

is an isomorphism, and the product on the left is finite.

Proof

(i) The proof is the same as that of [Th5, Th. 2.1], but we substitute Corollary 3.3(ii) for [Th5, Th. 2.1] since we use a localization different from Thomason's.

(ii) By the generic slice theorem for torus actions (see [Th1, Prop. 4.10]), there exist a T-invariant nonempty open subspace $U \subset X$, a closed (necessarily diagonalizable) subgroup T' of T, and a T -equivariant isomorphism

$$U \simeq T/T' \times (U/T) \simeq (U/T) \times^{T'} T.$$

Since U is nonempty and T acts on X with finite stabilizers, T' is finite over k and $K'_*(U, T) \simeq K'_*(U/T) \otimes_{\mathbb{Z}} \mathrm{R}(T')$, by Morita equivalence theorem (see [Th3, Prop. 6.2]) and [Th1, Lem. 5.6]. By Corollary 3.3(ii), the proposition for $X = U$ follows from Corollary 3.3(iii). By Noetherian induction and the localization sequence for K'-groups (see [Th3, Th. 2.7]), the statement for U implies the same for X.

Again using Noetherian induction, Thomason's generic slice theorem for torus actions, and (i), one similarly shows that the product $\prod_{\sigma \in \mathscr{C}(T)} K'_*(X, T)_\sigma$ is finite. □

By Proposition 3.4, there is an induced isomorphism (of $\mathrm{R}(T)$-modules, not a ring isomorphism due to the composition with pushforwards)

$$\prod_{\sigma \in \mathscr{C}(T)} K'_*(X^\sigma, T)_\sigma \longrightarrow K'_*(X, T)_\Lambda. \tag{10}$$

As shown in Lemma 2.8, the product morphism $\sigma \times T \to T$ induces a morphism

$$\theta_{T,\sigma} : K'_*(X^\sigma, T)_\sigma \longrightarrow K'_*(X^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda. \tag{11}$$

PROPOSITION 3.5
For any $\sigma \in \mathscr{C}(T)$, $\theta_{T,\sigma}$ is an isomorphism.

Proof
We write $\theta_{X,\sigma}$ for $\theta_{T,\sigma}$ in order to emphasize the dependence of the map on the space. We proceed by Noetherian induction on X^σ. Let $X' \subseteq X^\sigma$ be a T-invariant closed subspace, and let us suppose that (11) is an isomorphism with X replaced by any T-invariant proper closed subspace Z of X'. By Thomason's generic slice theorem for torus actions (see [Th1, Prop. 4.10]), there exist a T-invariant nonempty open subscheme $U \subset X'$, a (necessarily diagonalizable) subgroup T' of T, and a T-equivariant isomorphism

$$U^\sigma \equiv U \simeq T/T' \times (U/T) \simeq (U/T) \times^{T'} T.$$

Since U is nonempty and T acts on X with finite stabilizers, T' is finite over k and, obviously, $\Lambda_{T'} \subseteq \Lambda$. Let $Z^\sigma \equiv Z \doteq X' \smallsetminus U$. Since

$$\begin{tikzcd}
{} \arrow[r] & K'_*(Z^\sigma, T)_\sigma \arrow[r] \arrow[d, "\theta_{Z,\sigma}"'] & K'_*(X'^\sigma, T)_\sigma \arrow[r] \arrow[d, "\theta_{Y',\sigma}"'] & K'_*(U^\sigma, T)_\sigma \arrow[r] \arrow[d, "\theta_{U,\sigma}"'] & {} \\
{} \arrow[r] & K'_*(Z^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{R}(\sigma)_\Lambda \arrow[r] & K'_*(X'^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{R}(\sigma)_\Lambda \arrow[r] & K'_*(U^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{R}(\sigma)_\Lambda \arrow[r] & {}
\end{tikzcd}$$

is commutative, by the induction hypothesis and the five-lemma it is enough to show that $\theta_{U,\sigma}$ is an isomorphism. By Morita equivalence theorem (see [Th3, Prop. 6.2]) and [Th1, Lem. 5.6], $K'_*(U, T) \simeq K'_*(U/T) \otimes_{\mathbb{Z}} \mathrm{R}(T')$, so it is enough to prove that

$$\begin{aligned}
\theta_{\mathrm{Spec}\, k,\sigma} : K'_0(\mathrm{Spec}\, k, T')_\sigma = \mathrm{R}(T')_\sigma &\to K'_0(\mathrm{Spec}\, k, T')_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \\
&= \mathrm{R}(T')_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda
\end{aligned}$$

is an isomorphism. But this follows immediately from Corollary 3.3(i) and (iii). □

Combining Proposition 3.5 with (10), we get an isomorphism

$$\Phi_{X,T} : \prod_{\sigma \in \mathscr{C}(T)} K'_*(X^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \longrightarrow K'_*(X, T)_\Lambda. \tag{12}$$

The following lemma is a variant of [Th5, Lem. 3.2], which already proves it after tensoring with $\mathbb{Q}$.

LEMMA 3.6
Let X be a Noetherian regular separated algebraic space over k on which a split k-torus acts with finite stabilizers, and let $\sigma \in \mathscr{C}(T)$. Let X^σ denote the regular σ-fixed subscheme, let $j_\sigma : X^\sigma \hookrightarrow X$ be the regular closed immersion (see [Th5, Prop. 3.1]), and let $\mathscr{N}(j_\sigma)$ be the corresponding locally free conormal sheaf. Then, for any T-invariant algebraic subspace Y of X^σ, the cap-product

$$\lambda_{-1}\big(\mathscr{N}(j_\sigma)\big) \cap (-) : K'_*(Y, T)_\sigma \longrightarrow K'_*(Y, T)_\sigma$$

is an isomorphism.

Proof
We proceed by Noetherian induction on closed T-invariant subspaces Y of X^σ. The statement is trivial for $Y = \emptyset$, so let us suppose Y nonempty and

$$\lambda_{-1}\big(\mathscr{N}(j_\sigma)\big) \cap (-) : K'_*(Z, T)_\sigma \longrightarrow K'_*(Z, T)_\sigma$$

an isomorphism for any proper T-invariant closed subspace Z of Y. By Thomason's generic slice theorem for torus actions (see [Th1, Prop. 4.10]), there exist a T-invariant nonempty open subscheme $U \subset Y$, a closed (necessarily diagonalizable) subgroup T' of T, and a T -equivariant isomorphism

$$U^\sigma \equiv U \simeq T/T' \times (U/T).$$

Since U is nonempty and T acts on X with finite stabilizers, T' is finite over k. Using the localization sequence and the five-lemma, we reduce ourselves to showing that

$$\lambda_{-1}\big(\mathscr{N}(j_\sigma)\big) \cap (-) : K'_*(U, T)_\sigma \longrightarrow K'_*(U, T)_\sigma$$

is an isomorphism. For this, it is enough to show that (the restriction of) $\lambda_{-1}(\mathscr{N}(j_\sigma))$ is a unit in $K_0(U, T)_\sigma \simeq K_0(U/T)_\Lambda \otimes \mathrm{R}(T')_\sigma$ (see [Th3, Prop. 6.2]). Decomposing $\mathscr{N}(j_\sigma)$ according to the characters of T', we may write, shrinking U if necessary,

$$\mathscr{N}(j_\sigma) = \bigoplus_{\rho \in \widehat{T'}} \mathscr{O}_{U/T}^{r_\rho} \otimes \mathscr{L}_\rho,$$

where $\mathscr{L}_\rho$ is the line bundle attached to the T'-character ρ and $r_\rho \geq 0$, and therefore $\lambda_{-1}(\mathscr{N}(j_\sigma)) = \prod_{\rho \in \widehat{T'}} (1-\rho)^{r_\rho}$ in $K_0(U/T) \otimes \mathrm{R}(T')$. The localization map $\mathrm{R}(T')_\Lambda \to \mathrm{R}(T')_\sigma \simeq \widetilde{\mathrm{R}}(\sigma)_\Lambda$ coincides with the composition

$$\mathrm{R}(T')_\Lambda \xrightarrow{\pi_\sigma} \mathrm{R}(\sigma)_\Lambda \xrightarrow{p_\sigma} \widetilde{\mathrm{R}}(\sigma)_\Lambda$$

of the restriction to σ followed by the projection (see Cor. 3.3), and then

$$(\mathrm{id}_{K_0(U/T)_\Lambda} \otimes \pi_\sigma)\big(\mathscr{N}(j_\sigma)\big) = \bigoplus_{\chi\in\widehat{\sigma}\setminus\{0\}} \mathscr{O}_{U/T}^{r_\chi} \otimes \mathscr{L}_\chi,$$

in $K_0(U/T)_\Lambda \otimes \mathrm{R}(\sigma)_\Lambda$, where the summand omits the trivial character since the decomposition of $\mathscr{N}(j_\sigma)$ according to the characters of σ has vanishing fixed subsheaf $\mathscr{N}(j_\sigma)_0$ (see, e.g., [Th5, Prop. 3.1]). Therefore,

$$\lambda_{-1}\big((\mathrm{id}_{K_0(U/T)_\Lambda} \otimes \pi_\sigma)(\mathscr{N}(j_\sigma))\big) = \prod_{\chi\in\widehat{\sigma}\setminus\{0\}} (1-\chi)^{r_\chi},$$

and it is enough to show that the image of $1-\chi$ in $\widetilde{\mathrm{R}}(\sigma)_\Lambda$ via p_σ is a unit for any nontrivial character χ of σ. Now, the image of such a χ in

$$\widetilde{\mathrm{R}}(\sigma)_\Lambda \simeq \frac{\Lambda[t]}{(\Phi_{|\sigma|})}$$

($\Phi_{|\sigma|}$ being the $|\sigma|$th cyclotomic polynomial) is of the form $[t^l]$ for some $1 \le l < |\sigma|$, where $[-]$ denotes the class mod$\Phi_{|\sigma|}$; therefore the cokernel of the multiplication by $1-[t^l]$ in $\Lambda[t]/(\Phi_{|\sigma|})$ is

$$\frac{\Lambda[t]}{(\Phi_{|\sigma|}, 1-t^l)} = 0$$

since $\Phi_{|\sigma|}$ and $(1-t^l)$ are relatively prime in $\Lambda[t]$ for $1 \le l < |\sigma|$. Thus $1-[t^l]$ is a unit in $\Lambda[t]/(\Phi_{|\sigma|})$, and we conclude the proof of the lemma. □

We are now able to prove our main theorem for $G = T$.

THEOREM 3.7
If X is a regular Noetherian separated k-algebraic space, then

$$\Psi_{X,T} : K_*(X,T)_\Lambda \longrightarrow \prod_{\sigma\in\mathscr{C}(T)} K_*(X^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$$

is an isomorphism of $\mathrm{R}(T)$*-algebras.*

Proof
Recall (see appendix) that $K_*(X,T) \simeq K'_*(X,T)$ and $K_*(X^\sigma,T) \simeq K'_*(X^\sigma,T)$, since both X and X^σ are regular (see [Th5, Prop. 3.1]). Since $\Phi_{X,T}$ is an isomorphism of $\mathrm{R}(T)$-modules, it is enough to show that the composition $\Psi_{X,T} \circ \Phi_{X,T}$ is an isomorphism. A careful inspection of the definitions of $\Psi_{X,T}$ and $\Phi_{X,T}$ easily reduces the problem to proving that, for any $\sigma \in \mathscr{C}(T)$, the composition

$$K'_*(X^\sigma, T)_\sigma \xrightarrow{j_{\sigma *}} K'_*(X,T)_\sigma \xrightarrow{j_\sigma^*} K'_*(X^\sigma,T)_\sigma$$

is an isomorphism, $j_\sigma : X^\sigma \hookrightarrow X$ being the natural inclusion. Since j_σ is regular, there is a self-intersection formula

$$j_\sigma^* \circ j_{\sigma *}(-) = \lambda_{-1}\big(\mathscr{N}(j_\sigma)\big) \cap (-), \tag{13}$$

$\mathscr{N}(j_\sigma)$ being the conormal sheaf associated to j_σ, and we conclude by Lemma 3.6. To prove the self-intersection formula (13), we adapt [Th5, proof of Lem. 3.3]. First, by Proposition 3.4(i), $j_{\sigma *}$ is an isomorphism, so it is enough to prove that $j_{\sigma *} j_\sigma^* j_{\sigma *}(-) = j_{\sigma *}(\lambda_{-1}(\mathscr{N}(j_\sigma)) \cap (-))$. By the projection formula (see Prop. A.5), we have

$$\begin{aligned} j_{\sigma *} j_\sigma^* j_{\sigma *}(-) &= j_{\sigma *} j_\sigma^*(1) \cap j_{\sigma *}(-) = j_{\sigma *} j_\sigma^*(\mathscr{O}_X) \cap j_{\sigma *}(-) \\ &= j_{\sigma *}(\mathscr{O}_{X^\sigma}) \cap j_{\sigma *}(-) = j_{\sigma *}\big(j_\sigma^*(\mathscr{O}_{X^\sigma}) \cap (-)\big). \end{aligned}$$

Now, as explained in the appendix, to compute $j_\sigma^*(\mathscr{O}_{X^\sigma})$ we choose a complex F^* of flat quasi-coherent G-equivariant modules on X which is quasi-isomorphic to $\mathscr{O}_{X^\sigma}$, and then

$$j_\sigma^*(\mathscr{O}_{X^\sigma}) = [j_\sigma^*(F^*)] = [F^* \otimes \mathscr{O}_{X^\sigma}] = \sum_i (-1)^i [H^i(F^* \otimes \mathscr{O}_{X^\sigma})].$$

But F^* is a flat resolution of $\mathscr{O}_{X^\sigma}$, so $H^i(F^* \otimes \mathscr{O}_{X^\sigma}) = \mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_{X^\sigma}, \mathscr{O}_{X^\sigma}) \simeq \bigwedge^{\mathrm{i}} \mathscr{N}(j_\sigma)$, where the last isomorphism (see [SGA6, Exp. VII, par. 2.5]) is natural and hence T-equivariant. Therefore, $j_\sigma^*(\mathscr{O}_{X^\sigma}) = \lambda_{-1}(\mathscr{N}(j_\sigma))$, and we conclude the proof of the theorem. □

4. The main theorem: The case of $G = \mathrm{GL}_{n,k}$

In this section we use the result for $\Psi_{X,T}$ to deduce the same result for $\Psi_{X,\mathrm{GL}_{nk}}$.

THEOREM 4.1
Let X be a Noetherian regular separated algebraic space over a field k on which $G = \mathrm{GL}_{n,k}$ acts with finite stabilizers. Then the map defined in (5),

$$\begin{aligned} \Psi_{X,G} : K_*(X, G)_{\Lambda_{(G,X)}} \\ \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \Big(K_*(X^\sigma, C(\sigma))_{\mathrm{geom}} \otimes_{\Lambda_{(G,X)}} \widetilde{\mathrm{R}}(\sigma)_{\Lambda_{(G,X)}}\Big)^{w_G(\sigma)}, \end{aligned} \tag{14}$$

is an isomorphism of $\mathrm{R}(G)$*-algebras and the product on the right is finite.*

Throughout this section, entirely devoted to the proof of Theorem 4.1, we simply write G for $\mathrm{GL}_{n,k}$, Λ for $\Lambda_{(G,X)}$, and T for the maximal torus of diagonal matrices in $\mathrm{GL}_{n,k}$. First, let us observe that we can choose each $\sigma \in \mathscr{C}(G)$ contained in T. Moreover, $\Lambda_{(T,X)} = \Lambda_{(G,X)}$.

We need the following three preliminary lemmas (Lems. 4.2, 4.3, 4.4).

If $\sigma, \tau \subset T$ are dual cyclic subgroups, they are conjugate under the $G(k)$-action if and only if they are conjugated via an element in the Weyl group S_n. For any group scheme H with a dual cyclic subgroup $\sigma \subseteq H$, we denote by $\mathfrak{m}_\sigma^H$ the kernel of $\mathrm{R}(H)_\Lambda \to \widetilde{\mathrm{R}}(\sigma)_\Lambda$ and by $\mathrm{R}\widehat{(H)_{\Lambda,\sigma}}$ the completion of $\mathrm{R}(H)_\Lambda$ with respect to the ideal $\mathfrak{m}_\sigma^H$.

The following lemma is essentially a variant of Lemma 2.9 for σ-localizations.

LEMMA 4.2

Let $G = \mathrm{GL}_{n,k}$, let T be the maximal torus of G consisting of diagonal matrices, and let X be an algebraic space on which G acts with finite stabilizers.

(i) *For any essential dual cyclic subgroup $\sigma \subseteq T$, the morphisms*

$$\omega_{\sigma,\mathrm{geom}} : K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_{\mathrm{geom}} \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \longrightarrow K'_*(X^\sigma, T)_{\mathrm{geom}},$$
$$\omega_\sigma : K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \longrightarrow K'_*(X^\sigma, T)_\sigma$$

induced by $T \hookrightarrow \mathrm{C}_G(\sigma)$ are isomorphisms.

(ii) *For any essential dual cyclic subgroup $\sigma \subseteq T$,*

$$(\mathfrak{m}_\sigma^{\mathrm{C}_G(\sigma)})^N \cdot K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma = 0, \quad N \gg 0,$$

and the morphism induced by $T \hookrightarrow \mathrm{C}_G(\sigma)$,

$$\widehat{\omega_\sigma} : K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma \otimes_{\mathrm{R}\widehat{(\mathrm{C}_G(\sigma))_{\Lambda,\sigma}}} \mathrm{R}\widehat{(T)_{\Lambda,\sigma}} \longrightarrow K'_*(X^\sigma, T)_\sigma,$$

is an isomorphism.

Proof

(i) Since $\mathrm{C}_G(\sigma)$ is isomorphic to a product of general linear groups over k and since T is a maximal torus in $\mathrm{C}_G(\sigma)$, by Lemma 2.9 the canonical ring morphism

$$K'_*\big(X, \mathrm{C}_G(\sigma)\big) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))} \mathrm{R}(T) \longrightarrow K'_*(X, T) \tag{15}$$

is an isomorphism. If $H \subseteq G$ is a subgroup scheme, we denote by S_σ^H the multiplicative subset of $\mathrm{R}(H)_\Lambda$ consisting of the elements sent to 1 by the canonical ring homomorphism $\mathrm{R}(H)_\Lambda \to \widetilde{\mathrm{R}}(\sigma)_\Lambda$. By (15), ω_σ coincides with the composition

$$\begin{aligned}
&K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \\
&\quad \simeq K'_*(X^\sigma, T) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \big((S_\sigma^{\mathrm{C}_G(\sigma)})^{-1}\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda\big) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \\
&\quad \xrightarrow{\mathrm{id}\otimes\nu_\sigma} K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} (S_\sigma^T)^{-1}\mathrm{R}(T)_\Lambda \simeq K'_*(X^\sigma, T)_\sigma,
\end{aligned}$$

where

$$\nu_\sigma : (S_\sigma^{\mathrm{C}_G(\sigma)})^{-1}\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \to (S_\sigma^T)^{-1}\mathrm{R}(T)_\Lambda \tag{16}$$

is induced by $T \hookrightarrow \mathrm{C}_G(\sigma)$ and the last isomorphism follows from (15); the same is true for $\omega_{\sigma,\mathrm{geom}}$. Therefore, it is enough to prove that ν_σ and

$$\nu_{\sigma,\mathrm{geom}} : (S_1^{\mathrm{C}_G(\sigma)})^{-1}\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \to (S_1^T)^{-1}\mathrm{R}(T)_\Lambda$$

are isomorphisms; that is, if S_τ denotes the image of $S_1^{\mathrm{C}_G(\sigma)}$ via the restriction map

$$\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \longrightarrow \mathrm{R}(T)_\Lambda,$$

then $S_\tau^T/1$ consists of units in $(S_\tau)^{-1}\mathrm{R}(T)_\Lambda$ for $\tau = 1$ and $\tau = \sigma$. If Δ_σ denotes the Weyl group of $\mathrm{C}_G(\sigma)$, which is a product of symmetric groups, we have $\mathrm{R}(\mathrm{C}_G(\sigma)) \simeq \mathrm{R}(T)^{\Delta_\sigma}$ and therefore

$$(S_\tau^{\mathrm{C}_G(\sigma)})^{-1}\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \simeq \big((S_\tau)^{-1}\mathrm{R}(T)_\Lambda\big)^{\Delta_\sigma}$$

since $\mathrm{R}(T)$ is torsion free. Moreover, there is a commutative diagram

$$\begin{tikzcd}
(S_\tau^{\mathrm{C}_G(\sigma)})^{-1}\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \arrow[r, hook] \arrow[d, "\psi"'] & (S_\tau)^{-1}\mathrm{R}(T)_\Lambda \arrow[d] \arrow[dl, "\doteq\varphi"] \\
\widetilde{\mathrm{R}}(\tau)_\Lambda \simeq (S_\tau^T)^{-1}\mathrm{R}(\tau)_\Lambda & (S_\tau^T)^{-1}\mathrm{R}(T)_\Lambda \arrow[l, "(S_\tau^T)^{-1}\operatorname{res}_\tau^T"']
\end{tikzcd}$$

where ψ is induced by $\widetilde{\pi}_\tau$ and the isomorphism $\widetilde{\mathrm{R}}(\tau)_\Lambda \simeq (S_\tau^T)^{-1}\mathrm{R}(\tau)_\Lambda$ is obtained from Proposition 3.1 and Corollary 3.3. If we define the map

$$M : (S_\tau)^{-1}\mathrm{R}(T)_\Lambda \longrightarrow \big((S_\tau)^{-1}\mathrm{R}(T)_\Lambda\big)^{\Delta_\sigma},$$
$$\xi \longmapsto \prod_{g\in\Delta_\sigma} g\cdot\xi\ ,$$

it is easily checked that for $\xi \in (S_\tau)^{-1}\mathrm{R}(T)_\Lambda$, ξ is a unit if $M(\xi)$ is a unit, and that $\psi(M(\xi)) = 1$ implies that ξ is a unit in $((S_\tau)^{-1}\mathrm{R}(T)_\Lambda)^{\Delta_\sigma}$. But φ is Δ_σ-equivariant, and therefore $S_\tau^T/1$ consists of units in $(S_\tau)^{-1}\mathrm{R}(T)_\Lambda$ for $\tau = 1$ or σ.

(ii) Since $\mathrm{R}(\mathrm{C}_G(\sigma)) \to \mathrm{R}(T)$ is faithfully flat, by (i) it is enough to prove that

$$(\mathfrak{m}_\sigma^T)^N K'_*(X^\sigma, T)_\sigma = 0 \quad \text{for } N \gg 0. \tag{17}$$

But (17) can be proved using the same technique as in the proof of, for example, Proposition 3.5, that is, Noetherian induction together with Thomason's generic slice theorem for torus actions. The second part of (ii) follows, arguing as in (i), from the fact that (16) is an isomorphism since

$$K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \widehat{\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)}_{\Lambda,\sigma} \simeq K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma,$$
$$K'_*(X^\sigma, T)_\sigma \otimes_{\mathrm{R}(T)_\Lambda} \widehat{\mathrm{R}(T)}_{\Lambda,\sigma} \simeq K'_*(X^\sigma, T)_\sigma. \qquad \square$$

If $\sigma, \tau \subset T$ are dual cyclic subgroups conjugated under $G(k)$, they are conjugate through an element of the Weyl group S_n and we write $\tau \approx_{S_n} \sigma$; moreover, we have $\mathfrak{m}^G_\sigma = \mathfrak{m}^G_\tau$ because conjugation by an element in S_n (actually, by any element in $G(k)$) induces the identity morphism on K-theory and, in particular, on the representation ring. Then there are canonical maps

$$\widehat{\mathrm{R}(G)}_{\Lambda,\sigma} \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \longrightarrow \prod_{\tau \text{ dual cyclic}^{\tau \approx_{S_n} \sigma}} \widehat{\mathrm{R}(T)}_{\Lambda,\tau}, \tag{18}$$

$$\widehat{\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)}_{\Lambda,\sigma} \otimes_{R(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \longrightarrow \widehat{\mathrm{R}(T)}_{\Lambda,\sigma}. \tag{19}$$

LEMMA 4.3
Maps (18) and (19) are isomorphisms.

Proof
Since $\mathrm{R}(G) = \mathrm{R}(T)^{S_n} \to \mathrm{R}(T)$ is finite, the canonical map $\widehat{\mathrm{R}(G)}_{\Lambda,\sigma} \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \to \widehat{\mathrm{R}(T)_\Lambda}^{\mathfrak{m}^G_\sigma}$ (where $\widehat{\mathrm{R}(T)_\Lambda}^{\mathfrak{m}^G_\sigma}$ denotes the $\mathfrak{m}^G_\sigma$-adic completion of the $\mathrm{R}(G)_\Lambda$-module $\mathrm{R}(T)_\Lambda$) is an isomorphism. Moreover, $\mathrm{R}(G)_\Lambda = (\mathrm{R}(T)_\Lambda)^{S_n}$ implies that

$$\sqrt{\mathfrak{m}^G_\sigma \mathrm{R}(T)_\Lambda} = \bigcap_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \sqrt{\mathfrak{m}^T_\tau} = \bigcap_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \mathfrak{m}^T_\tau,$$

and, by Corollary 3.3(i), $\mathfrak{m}^T_\tau + \mathfrak{m}^T_{\tau'} = \mathrm{R}(T)_\Lambda$ if $\tau \neq \tau'$. By the Chinese remainder lemma, we conclude that (18) is an isomorphism.

Arguing in the same way, we get that the canonical map

$$\widehat{\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)}_{\Lambda,\sigma} \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \longrightarrow \prod_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{\Delta_\sigma} \sigma}} \widehat{\mathrm{R}(T)}_{\Lambda,\tau}$$

is an isomorphism, where $\Delta_\sigma = S_n \cap \mathrm{C}_G(\sigma)$ is the Weyl group of $\mathrm{C}_G(\sigma)$ with respect to T and we write $\tau \approx_{\Delta_\sigma} \sigma$ to denote that τ and σ are conjugate through an element of Δ_σ. But $\Delta_\sigma \subset \mathrm{C}_G(\sigma)$, so that $\tau \approx_{\Delta_\sigma} \sigma$ if and only if $\tau = \sigma$, and we conclude that (19) is an isomorphism. $\square$

LEMMA 4.4
For any essential dual cyclic subgroup $\sigma \subseteq G$, the canonical morphism

$$\mathrm{R}\widehat{(G)_{\Lambda,\sigma}} \longrightarrow \mathrm{R}\big(\widehat{\mathrm{C}_G(\sigma)\big)_{\Lambda,\sigma}}$$

is a finite étale Galois cover (see [SGA1, Exp. V]) with Galois group $w_G(\sigma)$.

Proof
Since $\mathrm{R}(T)$ is flat over $\mathrm{R}(G) = \mathrm{R}(T)^{S_n}$, we have

$$\begin{aligned}
\mathrm{R}\widehat{(G)_{\Lambda,\sigma}} &\simeq \mathrm{R}\widehat{(G)_{\Lambda,\sigma}} \otimes_{\mathrm{R}(G)_\Lambda} \big(\mathrm{R}(T)_\Lambda\big)^{S_n} \\
&\simeq \big(\widehat{\mathrm{R}(G)_{\Lambda,\sigma}} \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda\big)^{S_n} \\
&\simeq \Big(\prod_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \mathrm{R}\widehat{(T)_{\Lambda\cdot\tau}}\Big)^{S_n},
\end{aligned}$$

the last isomorphism being given in Lemma 4.3. By Lemma 2.10, we get

$$\widehat{\mathrm{R}(G)_\sigma} \simeq \big(\widehat{\mathrm{R}(T)_\sigma}\big)^{S_{n,\sigma}},$$

where S_n acts on the set of dual cyclic subgroups of T which are S_n-conjugated to σ and where $S_{n,\sigma}$ denotes the stabilizer of σ. Analogously, denoting by Δ_σ the Weyl group of $\mathrm{C}_G(\sigma)$, by Lemma 4.2(ii) we have

$$\begin{aligned}
\mathrm{R}\big(\widehat{\mathrm{C}_G(\sigma)\big)_{\Lambda,\sigma}} &\simeq \mathrm{R}\big(\widehat{\mathrm{C}_G(\sigma)\big)_{\Lambda,\sigma}} \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \big(\mathrm{R}(T)_\Lambda\big)^{\Delta_\sigma} \\
&\simeq \big(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))_{\Lambda,\sigma}} \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda\big)^{\Delta_\sigma} \\
&\simeq \big(\mathrm{R}\widehat{(T)_{\Lambda,\sigma}}\big)^{\Delta_\sigma},
\end{aligned}$$

where the last isomorphism is given by Lemma 4.3. From the exact sequence

$$1 \to \Delta_\sigma \longrightarrow S_{n,\sigma} \longrightarrow w_G(\sigma) \to 1,$$

we conclude that

$$\mathrm{R}\widehat{(G)_{\Lambda,\sigma}} \simeq \big(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))_{\Lambda,\sigma}}\big)^{w_G(\sigma)}. \tag{20}$$

By [SGA1, Prop. 2.6, Exp. V], it is now enough to prove that the stabilizers of geometric points (i.e., the inertia groups of points) of Spec $\big(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))_{\Lambda,\sigma}}\big)$ under the $w_G(\sigma)$-action are trivial.

First, let us observe that $\mathrm{Spec}(\widetilde{\mathrm{R}}(\sigma)_\Lambda)$ is a closed subscheme of $\mathrm{Spec}(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))_{\Lambda,\sigma}})$. This can be seen as follows. It is obviously enough to show that if s denotes the order of σ, the map

$$\pi_\sigma : \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \longrightarrow \mathrm{R}(\sigma)_\Lambda = \frac{\Lambda[t]}{(t^s - 1)}$$

is surjective. First, consider the case where σ is contained in the center of G.

Since $\mathrm{R}(\sigma)_\Lambda$ is of finite type over Λ, we show that for any prime* $p \nmid s$ the induced map

$$\pi_{\sigma,p} : \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes \mathbb{F}_p \longrightarrow \mathrm{R}(\sigma)_\Lambda \otimes \mathbb{F}_p$$

is surjective. Note that if E denotes the standard n-dimensional representation of G, π_σ sends $\bigwedge^r E$ to $\binom{n}{r}t^r$. If $p \nmid n$, then $\pi_{\sigma,p}$ is surjective (in fact, $\pi_\sigma(E) = nt$ and n is invertible in $\mathbb{F}_p$). If $p \mid n$, let us write $n = qm$, with $q = p^i$ and $p \nmid m$. Since $(s, q) = 1$, t^q is a ring generator of $\mathrm{R}(\sigma)_\Lambda$, and to prove $\pi_{\sigma,p}$ is injective, it is enough to show that $p \nmid \binom{n}{q}$. This is elementary since the binomial expansion of

$$(1 + X)^n = (1 + X^q)^m$$

in $\mathbb{F}_p[X]$ yields $\binom{n}{q} = m$ in $\mathbb{F}_p$. For a general $\sigma \subseteq T$, let $\mathrm{C}_G(\sigma) = \prod_{i=1}^{l} \mathrm{GL}_{d_i,k}$, where $\sum d_i = n$, and let σ_i denote the image of σ in $\mathrm{GL}_{d_i,k}$, $i = 1, \dots, l$. Since $\sigma \subseteq \prod_{i=1}^{l} \sigma_i$ is an inclusion of diagonalizable groups, the induced map

$$\mathrm{R}\Big(\prod_{i=1}^{l} \sigma_i\Big) = \bigotimes_{i=1}^{l} \mathrm{R}(\sigma_i) \longrightarrow \mathrm{R}(\sigma)$$

is surjective (e.g., see [SGA3, Vol. II]). But $\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \to \mathrm{R}(\sigma)_\Lambda$ factors as

$$\mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda = \bigotimes_{i=1}^{l} \mathrm{R}(\mathrm{GL}_{d_i,k})_\Lambda \longrightarrow \bigotimes_{i=1}^{l} \mathrm{R}(\sigma_i)_\Lambda \longrightarrow \mathrm{R}(\sigma)_\Lambda,$$

and also the first map is surjective (by the previous case, since σ_i is contained in the center of $\mathrm{GL}_{d_i,k}$ and $|\sigma_i|$ divides $|\sigma|$). This proves that $\mathrm{Spec}(\widetilde{\mathrm{R}}(\sigma)_\Lambda)$ is a closed subscheme of $\mathrm{Spec}(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))}_{\Lambda,\sigma})$. Since $\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))}_{\Lambda,\sigma}$ is the completion of $\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda$ along the ideal

$$\ker\big(\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \to \widetilde{\mathrm{R}}(\sigma)_\Lambda\big),$$

any nonempty closed subscheme of $\mathrm{Spec}\big(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))}_{\Lambda,\sigma}\big)$ meets the closed subscheme $\mathrm{Spec}(\widetilde{\mathrm{R}}(\sigma)_\Lambda)$. To prove that $w_G(\sigma)$ acts freely on the geometric points of $\mathrm{Spec}\big(\mathrm{R}(\widehat{\mathrm{C}_G(\sigma))}_{\Lambda,\sigma}\big)$, it is then enough to show that it acts freely on the geometric points of $\mathrm{Spec}(\widetilde{\mathrm{R}}(\sigma)_\Lambda)$.

Actually, more is true: the map $q : \mathrm{Spec}(\widetilde{\mathrm{R}}(\sigma)_\Lambda) \to \mathrm{Spec}(\Lambda)$ is a $(\mathbb{Z}/s\mathbb{Z})^*$-torsor†. In fact, if $\mathrm{Spec}(\Omega) \to \mathrm{Spec}(\Lambda)$ is a geometric point, the corresponding geometric fiber of q is isomorphic to the spectrum of

$$\prod_{\alpha_i \in \widetilde{\mu}_s(\Omega)} \frac{\Omega[t]}{(t - \alpha_i)} \simeq \prod_{\alpha_i \in \widetilde{\mu}_s(\Omega)} \Omega$$

*Recall that σ is essential; hence s is invertible in Λ.
†Recall that the constant group scheme associated to $(\mathbb{Z}/s\mathbb{Z})^*$ is isomorphic to $\mathrm{Aut}_k(\sigma)$.

and $(\mathbb{Z}/s\mathbb{Z})^*$ acts by permutation on the primitive roots $\widetilde{\mu}_s(\Omega)$, by $\alpha \mapsto \alpha^k$, $(k, s) = 1$. In particular, the action of the subgroup $w_G(\sigma) \subset (\mathbb{Z}/s\mathbb{Z})^*$ on $\operatorname{Spec}(\widetilde{\mathrm{R}}(\sigma)_\Lambda)$ is free. □

PROPOSITION 4.5
The canonical morphism

$$K'_*(X, G)_\Lambda \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \left(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\right)^{w_G(\sigma)}$$

is an isomorphism.

Proof
By Lemma 2.9, the canonical ring morphism

$$K'_*(X, G) \otimes_{\mathrm{R}(G)} \mathrm{R}(T) \longrightarrow K'_*(X, T)$$

is an isomorphism. Since $\mathrm{R}(G) \to \mathrm{R}(T)$ is faithfully flat, it is enough to show that

$$\begin{aligned} K'_*(X, T)_\Lambda \simeq K'_*(X, G)_\Lambda \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \\ \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \left(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\right)^{w_G(\sigma)} \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \end{aligned}$$

is an isomorphism. By Proposition 3.4(ii), we are left to prove that

$$\prod_{\sigma \in \mathscr{C}(G)} \left(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\right)^{w_G(\sigma)} \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \simeq \prod_{\substack{\sigma \text{ dual cyclic} \\ \sigma \subset T}} K'_*(X, T)_\sigma. \tag{21}$$

For any $\tau \in \mathscr{C}(G)$ ($\tau \subseteq T$, as usual), we have

$$\begin{aligned} &K'_*\big(X^\tau, \mathrm{C}_G(\tau)\big)_\tau \otimes_{\mathrm{R}\widehat{(G)_{\Lambda,\tau}}} \mathrm{R}\widehat{(T)_{\Lambda,\tau}} \\ &\quad\simeq \left(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau \otimes_{\mathrm{R}\widehat{(\mathrm{C}_G(\tau))_{\Lambda,\tau}}} \mathrm{R}\widehat{(\mathrm{C}_G(\tau))_{\Lambda,\tau}}\right) \otimes_{\mathrm{R}\widehat{(G)_{\Lambda,\tau}}} \mathrm{R}\widehat{(T)_{\Lambda,\tau}} \\ &\quad\simeq \left(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau \otimes_{R\widehat{(G)_{\Lambda,\tau}}} \mathrm{R}\widehat{(\mathrm{C}_G(\tau))_{\Lambda,\tau}}\right) \otimes_{\mathrm{R}\widehat{(\mathrm{C}_G(\tau))_{\Lambda,\tau}}} \mathrm{R}\widehat{(T)_{\Lambda,\tau}}. \end{aligned}$$

By Lemma 4.4, for any $\mathrm{R}\widehat{(G)_{\Lambda,\tau}}$-module M, we have

$$M \otimes_{\mathrm{R}\widehat{(G)_{\Lambda,\tau}}} \mathrm{R}\widehat{\big(\mathrm{C}_G(\tau)\big)}_{\Lambda,\tau} \simeq w_G(\tau) \times M$$

since a torsor is trivial when base changed along itself. Therefore,

$$\begin{aligned} &K'_*\big(X^\tau, \mathrm{C}_G(\tau)\big)_\tau \otimes_{\mathrm{R}\widehat{(G)_{\Lambda,\tau}}} \mathrm{R}\widehat{(T)_{\Lambda,\tau}} \\ &\qquad\simeq w_G(\tau) \times \left(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau \otimes_{\mathrm{R}\widehat{(\mathrm{C}_G(\tau))_{\Lambda,\tau}}} \mathrm{R}\widehat{(T)_{\Lambda,\tau}}\right) \end{aligned} \tag{22}$$

with $w_G(\tau)$ acting on left-hand side by left multiplication on $w_G(\tau)$. Applying Lemma 4.2(ii) to the left-hand side, we get

$$K'_*\big(X^\tau, \mathrm{C}_G(\tau)\big)_\tau \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\tau}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}} \simeq w_G(\tau) \times K'_*(X^\tau, T)_\tau,$$

and taking invariants with respect to $w_G(\tau)$,

$$\big(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\tau}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}}\big)^{w_G(\tau)} \simeq K'_*(X^\tau, T)_\tau. \tag{23}$$

Comparing (21) to (23), we are reduced to proving that for any $\sigma \in \mathscr{C}(G)$ there is an isomorphism

$$\begin{aligned}\big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} &\otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \\ &\simeq \prod_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \big(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\tau}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}}\big)^{w_G(\tau)}.\end{aligned}$$

Since $\widehat{\mathrm{R}(T)_{\Lambda,\tau}}$ is flat over $\widehat{\mathrm{R}(G)_{\Lambda,\tau}}$ and $w_G(\tau)$ acts trivially on it, we have (see [SGA1])

$$\begin{aligned}\big(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau &\otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\tau}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}}\big)^{w_G(\tau)} \\ &\simeq \big(K'_*(X^\tau, \mathrm{C}_G(\tau))_{\Lambda,\tau}\big)^{w_G(\tau)} \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\tau}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}}.\end{aligned}$$

By Lemma 4.3, we have isomorphisms

$$\begin{aligned}\big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} &\otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \\ &\simeq \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\sigma}}} \widehat{\mathrm{R}(G)_{\Lambda,\sigma}} \otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \\ &\simeq \prod_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\sigma}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}}.\end{aligned}$$

(Recall that $\widehat{\mathrm{R}(G)_{\Lambda,\sigma}} = \widehat{\mathrm{R}(G)_{\Lambda,\tau}}$ for any $\tau \approx_{S_n} \sigma$, since $\mathfrak{m}_\sigma^G = \mathfrak{m}_\tau^G$.) For each τ, choosing an element $g \in S_n$ such that $g\sigma g^{-1} = \tau$ determines an isomorphism $K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma \simeq K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau$ whose restriction to invariants

$$\big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} \simeq \big(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau\big)^{w_G(\tau)}$$

is independent of the choice of g. Therefore, we have a canonical isomorphism

$$\begin{aligned}\big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} &\otimes_{\mathrm{R}(G)_\Lambda} \mathrm{R}(T)_\Lambda \\ &\simeq \prod_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_\sigma\big)^{w_G(\sigma)} \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\sigma}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}} \\ &\simeq \prod_{\substack{\tau \text{ dual cyclic} \\ \tau \approx_{S_n} \sigma}} \big(K'_*(X^\tau, \mathrm{C}_G(\tau))_\tau\big)^{w_G(\tau)} \otimes_{\widehat{\mathrm{R}(G)_{\Lambda,\tau}}} \widehat{\mathrm{R}(T)_{\Lambda,\tau}},\end{aligned}$$

as desired. □

Since $K_*(X,G) \simeq K'_*(X,G)$ and $K_*(X^\sigma, \mathrm{C}_G(\sigma)) \simeq K'_*(X^\sigma, \mathrm{C}_G(\sigma))$, comparing Proposition 4.5 with (14), we see that the proof of Theorem 4.1 can be completed by the following.

PROPOSITION 4.6
For any $\sigma \in \mathscr{C}(G)$, the morphism given by Lemma 2.8 and induced by the product $\mathrm{C}_G(\sigma) \times \sigma \to \mathrm{C}_G(\sigma)$,

$$\theta_{\mathrm{C}_G(\sigma),\sigma} : K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma \longrightarrow K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda,$$

is an isomorphism.

Proof
To simplify the notation, we write θ_σ for $\theta_{\mathrm{C}_G(\sigma),\sigma}$. As usual, we may suppose σ contained in T. Since $\mathrm{C}_G(\sigma)$ is isomorphic to a product of general linear groups over k, we can take T as its maximal torus, and by Lemma 2.9, the canonical ring morphism

$$K'_*\big(X, \mathrm{C}_G(\sigma)\big) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))} \mathrm{R}(T) \longrightarrow K'_*(X,T)$$

is an isomorphism. Moreover, $\mathrm{R}(\mathrm{C}_G(\sigma)) \to \mathrm{R}(T)$ being faithfully flat, it is enough to prove that $\theta_\sigma \otimes \mathrm{id}_{\mathrm{R}(T)}$ is an isomorphism. To prove this, let us consider the commutative diagram

$$\begin{array}{ccc}
K'_*\big(X^\sigma, \mathrm{C}_G(\sigma)\big)_\sigma \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda & \xrightarrow{\theta_\sigma \otimes \mathrm{id}} & \big(K'_*(X, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \\
\Big\downarrow{\scriptstyle \omega_\sigma} & & \Big\downarrow{\scriptstyle \widetilde{\gamma_\sigma}} \\
K'_*(X^\sigma, T)_\sigma & \xrightarrow[\theta_{T,\sigma}]{} & K'_*(X^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda
\end{array}$$

where

- $K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$ is an $\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda$-module via the coproduct ring morphism $\Delta_{\mathrm{C}_G(\sigma)} : \mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \to \mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$ (induced by the product $\mathrm{C}_G(\sigma) \times \sigma \to \mathrm{C}_G(\sigma)$);
- ω_σ is the canonical map induced by the inclusion $T \hookrightarrow \mathrm{C}_G(\sigma)$ and is an isomorphism by Lemma 4.2;
- $\theta_{T,\sigma}$ is an isomorphism as shown in the proof of Theorem 3.7;
- $\widetilde{\gamma_\sigma}$ sends $(x \otimes u) \otimes t$ to $(\Delta_T(t) \cdot x_{|T}) \otimes u$, for $x \in K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}}$, $u \in \widetilde{\mathrm{R}}(\sigma)_\Lambda$, $t \in \mathrm{R}(T)_\Lambda$, $\Delta_T : \mathrm{R}(T)_\Lambda \to \mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$ being the coproduct induced by the product $T \times \sigma \to T$.

So we are left to prove that $\widetilde{\gamma_\sigma}$ is an isomorphism.

First, let us observe that if R is a ring, $A \to A'$ is a ring morphism, and M is an A-module, there is a natural isomorphism

$$\begin{aligned}(M \otimes_{\mathbb{Z}} R) \otimes_{A \otimes_{\mathbb{Z}} B} (A' \otimes_{\mathbb{Z}} R) &\longrightarrow (M \otimes_A A') \otimes_{\mathbb{Z}} R,\\ (m \otimes r_1) \otimes (a' \otimes r_2) &\longmapsto (m \otimes a') \otimes r_1 r_2.\end{aligned}$$

Applying this to $M = K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}}$, $R = \widetilde{\mathrm{R}}(\sigma)_\Lambda$, $A = \mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda$, $A' = \mathrm{R}(T)_\Lambda$ and using Lemma 4.2, we get a canonical isomorphism

$$\begin{aligned}&\mathrm{f} : K'_*(X^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\\ &\longrightarrow \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big) \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda} \big(\mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)',\end{aligned} \tag{24}$$

where we have denoted by $(\mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda)'$ the $\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$-algebra

$$\mathrm{res} \otimes \mathrm{id} : \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \longrightarrow \mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda.$$

It is an elementary fact that there are mutually inverse isomorphisms $\alpha_{\mathrm{C}_G(\sigma)}$, $\beta_{\mathrm{C}_G(\sigma)}$, and α_T, β_T fitting into the commutative diagrams

$$\begin{tikzcd} & \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \arrow[dd, "\alpha_{C_G(\sigma)}"] \\ \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \arrow[ur, "\Delta_{C_G(\sigma)}"] \arrow[dr, "\mathrm{id}\otimes 1"'] & \\ & \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \end{tikzcd} \tag{25}$$

$$\begin{tikzcd} & \mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \arrow[dd, "\alpha_T"] \\ \mathrm{R}(T)_\Lambda \arrow[ur, "\Delta_T"] \arrow[dr, "\mathrm{id}\otimes 1"'] & \\ & \mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \end{tikzcd} \tag{26}$$

and compatible with restriction maps induced by $T \hookrightarrow \mathrm{C}_G(\sigma)$. This is exactly the dual assertion to the general fact that "an action $H \times Y \to Y$ is isomorphic over X to the projection on the second factor $\mathrm{pr}_2 : H \times Y \to Y$," for any group scheme H and any algebraic space Y. From (25) we get an isomorphism

$$\widetilde{\alpha} : \big(\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)' \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \longrightarrow \mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda,$$

where $\big(\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)'$ denotes the $\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda$-algebra

$$\Delta_{\mathrm{C}_G(\sigma)} : \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \to \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda.$$

Therefore, if we denote by $(\mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda)''$ the $\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$-algebra

$$(\mathrm{res} \otimes \mathrm{id}) \circ \alpha_T : \mathrm{R}\big(\mathrm{C}_G(\sigma)\big)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \longrightarrow \mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda,$$

the composition

$$\begin{aligned}
&\big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)' \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda \\
&\quad = \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big) \\
&\qquad\qquad \bigotimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda} \Big(\big(\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)' \otimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda} \mathrm{R}(T)_\Lambda\Big) \\
&\quad \xrightarrow{\mathrm{id} \otimes \widetilde{\alpha}} \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big) \\
&\qquad\qquad \bigotimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda} \big(\mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)'' \\
&\quad \xrightarrow{\mathrm{id} \otimes \beta_T} \big(K'_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big) \\
&\qquad\qquad \bigotimes_{\mathrm{R}(\mathrm{C}_G(\sigma))_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda} \big(\mathrm{R}(T)_\Lambda \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)' \\
&\quad \simeq K'_*(X^\sigma, T)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda
\end{aligned}$$

is an isomorphism and it can be easily checked to coincide with $\widetilde{\gamma_\sigma}$. □

5. The main theorem: The general case

In this section, we use Theorem 4.1 to deduce the same result for the action of a linear algebraic k-group G, having finite stabilizers, on a regular separated Noetherian k-algebraic space X. We write Λ for $\Lambda_{(G,X)}$.

We start with a general fact.

PROPOSITION 5.1
Let X be a regular Noetherian separated k-algebraic space on which a linear algebraic k-group G acts with finite stabilizers. Then there exists an integer $N > 0$ such that if $a_1, \ldots, a_N \in K_0(X, G)_{\mathrm{geom}}$ have rank zero on each connected component of X, then the multiplication by $\prod_{i=1}^N a_i$ on $K'_*(X, G)_{\mathrm{geom}}$ is zero.

In particular, we have the following.

COROLLARY 5.2
Let X be a regular Noetherian separated k-algebraic space with a connected action of a linear algebraic k-group G having finite stabilizers. Then the geometric localization

$$\mathrm{rk}_{0,\mathrm{geom}} : K_0(X, G)_{\mathrm{geom}} \longrightarrow \Lambda$$

of the rank morphism has a nilpotent kernel.

Proof of Proposition 5.1
Let us choose a closed immersion $G \hookrightarrow \mathrm{GL}_{n,k}$ (for some $n > 0$). By Morita equivalence,

$$K'_*(X, G) \simeq K'_*(X \times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k})$$

and

$$K_0(X, G) \simeq K_0(X \times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k}).$$

Moreover, $\Lambda_{(X\times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k})} = \Lambda$. Let $\xi = x/s \in K'_*(X, G)_{\mathrm{geom}}$ with $x \in K'_*(X, G)_\Lambda$ and $s \in \mathrm{rk}^{-1}(1)$ where $\mathrm{rk} : \mathrm{R}(G) \to \Lambda$ is the rank morphism, and let $a_i = \alpha_i/s_i$ with $\alpha_i \in K_0(X, G)_\Lambda$ and $s_i \in \mathrm{rk}^{-1}(1)$ for $i = 1, \ldots, N$. Let us consider the elements $x/1$ in $K'_*(X \times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k})_{\mathrm{geom}}$, and $\alpha_i/1$ in $K_0(X \times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k})_{\mathrm{geom}}$ for $i = 1, \ldots, N$. Since the canonical homomorphism

$$K'_*(X \times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k})_{\mathrm{geom}} \longrightarrow K'_*(X, G)_{\mathrm{geom}}$$

is a morphism of modules over the ring morphism

$$K_0(X \times^G \mathrm{GL}_{n,k}, \mathrm{GL}_{n,k})_{\mathrm{geom}} \longrightarrow K_0(X, G)_{\mathrm{geom}},$$

if the theorem holds for $G = \mathrm{GL}_{n,k}$ and if N is the corresponding integer, the product $\prod_i \alpha_i/1$ in $K_0(X, G)_{\mathrm{geom}}$ annihilates $x/1 \in K'_*(X, G)_{\mathrm{geom}}$ and a fortiori $\prod_i a_i$ annihilates ξ in $K'_*(X, G)_{\mathrm{geom}}$. So, we may assume $G = \mathrm{GL}_{n,k}$. Let T be the maximal torus of diagonal matrices in G. By Lemma 4.2(i) with $\sigma = 1$, there are isomorphisms

$$\begin{aligned}\omega_{1,\mathrm{geom}} : K_0(X, \mathrm{GL}_{n,k})_{\mathrm{geom}} \otimes_{\mathrm{R}(\mathrm{GL}_{n,k})_\Lambda} \mathrm{R}(T)_\Lambda &\simeq K_0(X, T)_{\mathrm{geom}},\\ K'_*(X, \mathrm{GL}_{n,k})_{\mathrm{geom}} \otimes_{\mathrm{R}(\mathrm{GL}_{n,k})_\Lambda} \mathrm{R}(T)_\Lambda &\simeq K'_*(X, T)_{\mathrm{geom}}.\end{aligned}$$

Since $\mathrm{R}(\mathrm{GL}_{n,k}) \to \mathrm{R}(T)$ is faithfully flat and the diagram

$$\begin{tikzcd} K_0(X, \mathrm{GL}_{n,k})_{\mathrm{geom}} \otimes_{\mathrm{R}(\mathrm{GL}_{n,k})_\Lambda} \mathrm{R}(T)_\Lambda \arrow[r, "\mathrm{rk}_{0,\mathrm{geom}} \otimes \mathrm{id}"] \arrow[d, "\omega_{1,\mathrm{geom}}"'] & \Lambda \otimes_{\mathrm{R}(\mathrm{GL}_{n,k})} \mathrm{R}(T)_\Lambda \arrow[d, "\mathrm{id}_\Lambda \otimes \mathrm{rk}_T"] \\ K_0(X, T)_{\mathrm{geom}} \arrow[r, "\mathrm{rk}_{0,\mathrm{geom}}"'] & \Lambda \end{tikzcd}$$

commutes, we reduce ourselves to proving the proposition for $G = T$, a split torus.

To handle this case, we proceed by Noetherian induction on X. By [Th1, Prop. 4.10], there exist a T-invariant nonempty open subscheme $j : U \hookrightarrow X$, a closed diagonalizable subgroup T' of T, and a T-equivariant isomorphism

$$U \simeq T/T' \times (U/T).$$

Since U is nonempty and T acts on X with finite stabilizers, T' is finite over k and $K'_*(U, T) \simeq K'_*(U/T) \otimes_{R(T)} R(T')$, by Morita equivalence theorem (see [Th3, Prop. 6.2]). Let $i : Z \hookrightarrow X$ be the closed complement of U in X, and let N' be an integer satisfying the proposition for both Z and U. Consider the geometric localization sequence

$$K'_*(Z, T)_{\text{geom}} \xrightarrow{i_*} K'_*(X, T)_{\text{geom}} \xrightarrow{j^*} K'_*(U, T)_{\text{geom}},$$

and let $\xi \in K'_*(X, T)_{\text{geom}}$. Let $a_1, \ldots, a_{2N'} \in K_0(X, T)_{\text{geom}}$. By our choice of N',

$$j^*(a_{N'+1} \cdot \ldots \cdot a_{2N'} \cup \xi) = 0;$$

thus $a_{N'+1} \cdot \ldots \cdot a_{2N'} \cap \xi = i_*(\eta)$ for some η in $K'_*(Z, T)_{\text{geom}}$. By the projection formula, we get

$$a_1 \cdot \ldots \cdot a_{2N'} \cup \xi = i_*\big(i^*(a_1) \cdot \ldots \cdot i^*(a_{N'}) \cup \eta\big),$$

which is zero by our choice of N' and by the fact that rank morphisms commute with pullbacks. Thus, $N \doteq 2N'$ satisfies our proposition. □

Remark 5.3

By Corollary 5.2, $K_*(X, G)_{\text{geom}}$ is isomorphic to the localization of $K_*(X, G)_\Lambda$ at the multiplicative subset $(\text{rk}_0)^{-1}(1)$, where $\text{rk}_0 : K_0(X, G)_\Lambda \to \Lambda$ is the rank morphism. Therefore, if X is regular, $K_*(X, G)_{\text{geom}}$ depends only on the quotient stack $[X/G]$ (see [LMB]) and not on its presentation as a quotient.

The main theorem of this paper is the following.

THEOREM 5.4

Let X be a Noetherian regular separated algebraic space over a field k, and let G be a linear algebraic k-group with a sufficiently rational action on X having finite stabilizers. Suppose, moreover, that for any essential dual cyclic k-subgroup scheme $\sigma \subseteq G$, the quotient algebraic space $G/\mathrm{C}_G(\sigma)$ is smooth over k (which is the case if, e.g., G is smooth or abelian). Then $\mathscr{C}(G)$ is finite, and the map defined in (5),

$$\Psi_{X,G} : K_*(X, G)_\Lambda \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \big(K_*(X^\sigma, \mathrm{C}(\sigma))_{\text{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_G(\sigma)},$$

is an isomorphism of $\mathrm{R}(G)$*-algebras.*

Remark 5.5

In Section 5.1 we also give less restrictive hypotheses on G under which Theorem 5.4 still holds.

Note also that if X has the "G-equivariant resolution property" (i.e., any G-equivariant coherent sheaf is the G-equivariant epimorphic image of a G-equivariant locally free coherent sheaf), then in Theorem 5.4 one can replace our K_* with Quillen K-theory of G-equivariant locally free coherent sheaves. This happens, for example, if X is a scheme and G is smooth or finite (see [Th3]).

5.1. *Proof of Theorem 5.4*

Let us choose, for some n, a closed immersion $G \hookrightarrow \mathrm{GL}_{n,k}$ and consider the algebraic space quotient

$$Y \doteq \mathrm{GL}_{n,k} \times^G X.$$

We claim that if the theorem holds for Y with the induced $\mathrm{GL}_{n,k}$-action, then it holds for X with the given G-action. First, let us note that Y is separated. The action map ψ : $G \times (\mathrm{GL}_{n,k} \times X) \to (\mathrm{GL}_{n,k} \times X) \times (\mathrm{GL}_{n,k} \times X)$ is proper (hence a closed immersion) since its composition with the separated map

$$p_{123} : \mathrm{GL}_{n,k} \times X \times \mathrm{GL}_{n,k} \times X \to \mathrm{GL}_{n,k} \times X \times \mathrm{GL}_{n,k}$$

(here we use that X is separated) is just $\mathrm{id}_X \times a$, where a is the action map of G on $\mathrm{GL}_{n,k}$; hence it is proper (see [EGAI, Rem. 5.1.7], which obviously carries over to algebraic spaces). Let $P \doteq X \times \mathrm{GL}_{n,k}$. In the Cartesian diagram

$$\begin{array}{ccc} P \times_Y P & \xrightarrow{j} & P \times P \\ \downarrow & & \downarrow{\scriptstyle \pi \times \pi} \\ Y & \xrightarrow[\Delta_Y]{} & Y \times Y \end{array}$$

$P \times_Y P \simeq G \times P$ since $\pi : P \to Y$ is a G-torsor, and π is faithfully flat; therefore, Δ_Y is a closed immersion; that is, Y is separated.

Note that $\Lambda_{(Y,\mathrm{GL}_{n,k})} = \Lambda$.

Consider the morphism defined in (5),

$$\Psi_{X,G} : K_*(X,G)_\Lambda \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \left(K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\right)^{w_G(\sigma)}. \tag{27}$$

By Theorem 4.1, the map

$$\Psi_{Y,\mathrm{GL}_{n,k}} : K_*(Y, \mathrm{GL}_{n,k})_\Lambda \\ \longrightarrow \prod_{\rho \in \mathscr{C}(\mathrm{GL}_{n,k})} \left(K_*(Y^\rho, \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\rho)_\Lambda\right)^{w_{\mathrm{GL}_{n,k}}(\rho)}$$

is an isomorphism, and by the Morita equivalence theorem (see [Th3, Prop. 6.2]), $K_*(Y, \mathrm{GL}_{n,k})_\Lambda \simeq K_*(X, G)_\Lambda$. We prove the theorem by constructing an isomorphism

$$\prod_{\rho \in \mathscr{C}(\mathrm{GL}_{n,k})} \big(K_*(Y^\rho, \mathrm{C}(\rho))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\rho)\big)^{w_{\mathrm{GL}_{n,k}}(\rho)} \longrightarrow \prod_{\sigma \in \mathscr{C}(G)} \big(K_*(X^\sigma, \mathrm{C}_G(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_G(\sigma)} \tag{28}$$

commuting with the Ψ's and Morita isomorphisms.

Let $\alpha : \mathscr{C}(G) \to \mathscr{C}(\mathrm{GL}_{n,k})$ be the natural map. If $Y^\rho \neq \emptyset$, there exists a dual cyclic subgroup $\sigma \subseteq G$, $\mathrm{GL}_{n,k}$-conjugate to ρ (and $X^\sigma \neq \emptyset$); therefore, $Y^\rho = \emptyset$ unless $\rho \in \mathrm{im}(\alpha)$, and we may restrict the first product in (28) to those ρ in the image of α and suppose $\mathrm{im}(\alpha) \subseteq \mathscr{C}(G)$ as well. The following proposition describes the Y^ρ's that appear.

PROPOSITION 5.6

Let X be a Noetherian regular separated algebraic space over a field k, and let G be a linear algebraic k-group with a sufficiently rational action on X having finite stabilizers. Suppose, moreover, that for any essential dual cyclic k-subgroup scheme $\sigma \subseteq G$, the quotient algebraic space $G/\mathrm{C}_G(\sigma)$ is smooth over k. Let $G \hookrightarrow \mathrm{GL}_{n,k}$ be a closed embedding, let $\rho \in \mathrm{im}(\alpha)$ be an essential dual cyclic subgroup, and let $Y \doteq \mathrm{GL}_{n,k} \times^G X$ be the algebraic space quotient for the left diagonal action of G. If $\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho) \subseteq \mathscr{C}(G)$ denotes the fiber $\alpha^{-1}(\rho)$, then

(i) *choosing for each $\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$ an element $u_{\rho,\sigma} \in \mathrm{GL}_{n,k}(k)$ such that $u_{\rho,\sigma}\sigma u_{\rho,\sigma}^{-1} = \rho$ (in the obvious functor-theoretic sense) determines a unique isomorphism of algebraic spaces over k,*

$$j_\rho : \coprod_{\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)} \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma \longrightarrow Y^\rho;$$

(ii) *$\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$ is finite.*

Proof

Part (ii) follows from (i) since Y^ρ is quasi-compact. The proof of (i) requires several steps.

(a) *Definition of j_ρ.* If $\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$, let $\mathscr{N}_\sigma$ be the presheaf on the category $\mathrm{Sch}_{/k}$ of k-schemes which associates to $T \to \operatorname{Spec} k$ the set

$$\mathscr{N}_\sigma(T) \doteq \frac{\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)(T) \times X^\sigma(T)}{\mathrm{N}_G(\sigma)(T)};$$

since $\mathrm{N}_G(\sigma)$ acts freely on $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times X^\sigma$ (on the left), the flat sheaf associated to $\mathscr{N}_\sigma$ is $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma$. Let $\widehat{Y}^\rho$ be the presheaf on $\mathrm{Sch}_{/k}$ which associates to $T \to \operatorname{Spec} k$ the set

$$\widehat{Y}^\rho(T) \doteq \Big\{ [A, x] \in \frac{\mathrm{GL}_{n,k}(T) \times X(T)}{G(T)} \,\Big|\, \forall T' \to T, \forall r \in \rho(T'), [r A_{T'}, x_{T'}] = [A_{T'}, x_{T'}] \Big\};$$

the flat sheaf associated to $\widehat{Y}^\rho$ is Y^ρ (e.g., see [DG, Chap. II, §1, n. 3]). If $u_{\rho,\sigma} \in \mathrm{GL}_{n,k}(k)$ is such that $u_{\rho,\sigma}\sigma u_{\rho,\sigma}^{-1} = \rho$ (in the obvious functor-theoretic sense), the presheaf map

$$\widehat{j}_{\rho,\sigma} : \mathscr{N}_\sigma \longrightarrow \widehat{Y}^\rho,$$
$$\widehat{j}_{\rho,\sigma}(T) : \mathscr{N}_\sigma(T) \ni [B, x] \longrightarrow [u_{\rho,\sigma} B, x] \in \widehat{Y}^\rho(T)$$

is easily checked to be well defined. Let $j_{\rho,\sigma} : \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma \to Y^\rho$ denote the associated sheaf map, and define $j_\rho \doteq \coprod_{\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)} j_{\rho,\sigma}$.

(b) *The map j_ρ induces a bijection on geometric points.* This is an elementary check. Let $\xi \in Y^\rho(\Omega)$ be a geometric point. Then there exist an fppf cover $T_0 \to \operatorname{Spec}\Omega$ and an element $[A, x] \in \widehat{Y}^\rho(T_0)$ representing ξ. Therefore, for each $T \to T_0$ and each $r \in \rho(T)$ there exists $g \in G(T)$ such that

$$r A_T g^{-1} = A_T,$$
$$g x_T = x_T.$$

Then $A^{-1}\rho A$ defines (functorially over T_0) a dual cyclic subgroup scheme σ_0' of $G_{(T_0)}$ over T_0. Since σ_0' is isomorphic to some μ_{n,T_0}, it descends to a dual cyclic subgroup σ' of G over k which is $\mathrm{GL}_{n,k}$-conjugate to ρ since $T_0 \to \operatorname{Spec}\Omega$ has a section and $\mathrm{GL}_{n,k}$ satisfies our rationality condition (RC) (see Rem. 2.2(i)). By definition of $\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$, there exists a unique $\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$ which is G-conjugated to σ' over k; that is,

$$g\sigma' g^{-1} = \sigma$$

(functorially) for some $g \in G(k)$. Since $\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$, there is an element $u \in \mathrm{GL}_{n,k}(k)$ such that $u\sigma u^{-1} = \rho$. Therefore, $u^{-1}Ag^{-1}$ restricted to T_0 is in $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)(T_0)$, $gx \in X^\sigma(T_0)$, and if $[u^{-1}Ag^{-1}, gx]^\sim$ denotes the element in $(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma)(\Omega)$ represented by the element $[u^{-1}Ag^{-1}, gx]$ in $\mathscr{N}_\sigma(T_0)$, we have $j_{\rho,\sigma}(\Omega)([u^{-1}Ag^{-1}, gx]^\sim) = \xi$ by definition of $j_{\rho,\sigma}$. Thus, $j_\rho(\Omega)$ is surjective.

Now, let $\eta \in (\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma)(\Omega)$ $\big($resp., $\eta' \in (\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma') \times^{\mathrm{N}_G(\sigma')} X^{\sigma'})(\Omega)\big)$ for σ and σ' in $\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$. Choosing a common refinement, we can assume that there exists an fppf cover $T_0 \to \operatorname{Spec}\Omega$ such that η (resp., η') is represented by an element $[B, y] \in \mathscr{N}_\sigma(T_0)$ (resp., $[B', y'] \in \mathscr{N}_{\sigma'}(T_0)$). If $j_\rho(\Omega)(\eta) =$

$j_\rho(\Omega)(\eta')$, there exists an fppf cover $T_1 \to T_0$ such that $[u_{\rho,\sigma}B, y] = [u_{\rho,\sigma'}B', y']$ in $\mathrm{GL}_{n,k}(T_1) \times X(T_1)/G(T_1)$; that is, there is an element $g \in G(T_1)$ such that

$$u_{\rho,\sigma}Bg^{-1} = u_{\rho,\sigma'}B' \quad \text{in } \mathrm{GL}_{n,k}(T_1),$$
$$gy = y' \quad \text{in } X(T_1).$$

Then it is easy to check that $\sigma = g^{-1}\sigma' g$ over T_1 and, as in the proof of surjectivity of $j_\rho(\Omega)$, since $T_1 \to \operatorname{Spec}\Omega$ has a section and G satisfies our rationality condition (RC), σ and σ' are G-conjugated over k as well, and therefore $\sigma = \sigma'$ as elements in $\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)$. In particular, $g \in \mathrm{N}_G(\sigma)(T_1)$ and $[B, y] = [B', y']$ in $\mathscr{N}_\sigma(T_1)$. Since $T_1 \to \operatorname{Spec}\Omega$ is still an fppf cover, we have $\eta = \eta'$ and $j_\rho(\Omega)$ is injective.

(c) *Each $j_{\rho,\sigma}$ is a closed and open immersion.* It is enough to show that each $j_{\rho,\sigma}$ is an open immersion because in this case it is also a closed immersion, Y^ρ being quasi-compact. Since $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$ acts on both $\coprod_{\sigma\in\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)} \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma$ and Y^ρ and since j_ρ is equivariant, it will be enough to prove that $j_{\rho,\rho}$ is an open immersion. We prove first that $j_{\rho,\rho}$ is injective and unramified and then conclude the proof by showing that it is also flat (in fact, an étale injective map is an open immersion).

(c$_1$) *The map $j_{\rho,\rho}$ is injective and unramified.* It is enough to show that the inverse image under $j_{\rho,\rho}$ of a geometric point is a (geometric) point. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^\rho \arrow[r, "l"] \arrow[d, "p"'] & \mathrm{GL}_{n,k} \times X \arrow[d, "\pi"] \\
\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times^{\mathrm{N}_G(\rho)} X^\rho \arrow[r, "i_\rho\circ j_{\rho,\rho}"'] & Y
\end{tikzcd}$$

where l and $i_\rho : Y^\rho \hookrightarrow Y$ are the natural inclusions and where p, π are the natural projections. Let y_0 be a geometric point of Y in the image of $i_\rho \circ j_{\rho,\rho}$; using the action of $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$ on $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times^{\mathrm{N}_G(\rho)} X^\rho$ and Y^ρ, we may suppose that y_0 is of the form $[1, x_0] \in Y^\rho(\Omega)$, with Ω an algebraically closed field and $x_0 \in X^\rho(\Omega)$. Obviously, $\overline{(1, x_0)} \in \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times^{\mathrm{N}_G(\rho)} X^\rho(\Omega)$ is contained in $j_{\rho,\rho}^{-1}(y_0)$, and, by faithful flatness of p, $j_{\rho,\rho}^{-1}(y_0) = \overline{(1, x_0)}$ if

$$p^{-1}\big(\overline{(1, x_0)}\big) = \pi^{-1}(y_0) \cap \big(\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times^{\mathrm{N}_G(\rho)} X^\rho\big). \tag{29}$$

But $G(\Omega) \simeq \pi^{-1}(y_0)$ via $g \mapsto (g^{-1}, gx_0)$ and $\mathrm{N}_G(\rho)(\Omega) \simeq p^{-1}(\overline{(1, x_0)})$ via $h \mapsto (h^{-1}, hx_0)$; therefore, (29) follows from $\mathrm{N}_G(\rho) = \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \cap G$.

(c$_2$) *The map $j_{\rho,\rho}$ is flat.* This fact is proved in Section 5.2, where we also single out a more general technical hypothesis for the action of G on X under which Proposition 5.6 still holds. □

The remaining part of this subsection is devoted to the conclusion of the proof of Theorem 5.4 using Proposition 5.6. First we show that Proposition 5.6(ii) allows one to define a canonical isomorphism

$$\prod_{\rho\in\mathscr{C}(\mathrm{GL}_{n,k})} \big(K_*(Y^\rho, \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\rho)_\Lambda\big)^{w_{\mathrm{GL}_{n,k}}(\rho)}$$
$$\simeq \prod_{\sigma\in\mathscr{C}(\mathrm{G})} \big(K_*(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_{\mathrm{GL}_{n,k}}(\sigma)};$$

next we show, using Lemma 2.10, that each factor in the right-hand side is isomorphic to

$$\big(K_*(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\big)^{w_{\mathrm{G}}(\sigma)}.$$

The conclusion (i.e., the isomorphism (28)) is then accomplished by establishing, for any regular Noetherian separated algebraic space Z on which G acts with finite stabilizers, a "geometric" Morita equivalence

$$K_*(\mathrm{GL}_{n,k} \times^G Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \simeq K_*(Z, G)_{\mathrm{geom}}.$$

First, note that the choice of a family $\{u_{\rho,\sigma} \mid \sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)\}$ of elements $u_{\rho,\sigma} \in \mathrm{GL}_{n,k}(k)$ such that $u_{\rho,\sigma}\sigma u_{\rho,\sigma}^{-1} = \rho$, which uniquely defines j_ρ in Proposition 5.6, also determines a unique family of isomorphisms

$$\big\{\mathrm{int}(u_{\rho,\sigma}) : \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho) \to \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \big| \sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)\big\}$$

(where $\mathrm{int}(u_{\rho,\sigma})$ denotes conjugation by $u_{\rho,\sigma}$), and this family gives us an action of $\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)$ on

$$\coprod_{\sigma\in\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)} \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma$$

(since $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)$, and then $\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)$, acts naturally on $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma$ by left multiplication on $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)$). With this action, j_ρ becomes a $\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)$-equivariant isomorphism, and since $\mathrm{int}(u_{\rho,\sigma})$ induces an isomorphism $\mathrm{R}(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)) \simeq \mathrm{R}(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))$ commuting with rank morphisms, j_σ induces an isomorphism

$$K_*\big(Y^\rho, \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)\big)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\rho)_\Lambda$$
$$\simeq \prod_{\sigma\in\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)} K_*\big(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\big)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda$$

which, by definition of the action of $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$ on each $\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_G(\sigma)} X^\sigma$, induces an isomorphism

$$\begin{aligned}\bigl(K_*(Y^\rho, \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\rho)_\Lambda\bigr)^{w_{\mathrm{GL}_{n,k}}(\rho)} \\ \simeq \prod_{\sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)} \bigl(K_*(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\bigr)^{w_{\mathrm{GL}_{n,k}}(\sigma)}.\end{aligned} \tag{30}$$

Now, if j'_ρ is induced, as in Proposition 5.6, by another choice of a family $\{v_{\rho,\sigma} \mid \sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)\}$ of elements $v_{\rho,\sigma} \in \mathrm{GL}_{n,k}(k)$ such that $v_{\rho,\sigma}\sigma v_{\rho,\sigma}^{-1} = \rho$, then $v_{\rho,\sigma}^{-1}u_{\rho,\sigma} \in \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)(k)$ and there is a commutative diagram

$$\begin{tikzcd} \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma \arrow[rr, "(v_{\rho,\sigma}^{-1}u_{\rho,\sigma})\cdot"] \arrow[dr, "j_\rho"'] & & \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma \arrow[dl, "j'_\rho"] \\ & Y^\rho & \end{tikzcd}$$

Therefore, isomorphism (30) on the invariants is actually *independent* of the choice of the family $\{u_{\rho,\sigma} \mid \sigma \in \mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho)\}$. Since $\mathscr{C}_{\mathrm{GL}_{n,k},G}(\rho) = \alpha^{-1}(\rho)$ and, as already observed, $Y^\rho = \emptyset$ unless $\rho \in \mathrm{im}(\alpha)$, this gives us a canonical isomorphism

$$\begin{aligned}\prod_{\rho \in \mathscr{C}(\mathrm{GL}_{n,k})} \bigl(K_*(Y^\rho, \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\rho)_\Lambda\bigr)^{w_{\mathrm{GL}_{n,k}}(\rho)} \\ \simeq \prod_{\sigma \in \mathscr{C}(\mathrm{G})} \bigl(K_*(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\bigr)^{w_{\mathrm{GL}_{n,k}}(\sigma)}.\end{aligned}$$

Now, let us fix $\sigma \in \mathscr{C}(\mathrm{G})$, and let us choose a set $\mathscr{A} \subset \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma)(k)$ such that the classes in $w_{\mathrm{GL}_{n,k}}(\sigma)$ of the elements in $\mathscr{A}$ constitute a set of representatives for the $w_{\mathrm{G}}(\sigma)$-orbits in $w_{\mathrm{GL}_{n,k}}(\sigma)$; $\mathscr{A}$ is a finite set. Since

$$\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma \hookrightarrow \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma$$

is an open and closed immersion, the morphism

$$\begin{aligned}\coprod_{\mathscr{A}} \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma &\longrightarrow \mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \\ [C, x]_{A_i \in \mathscr{A}} &\longmapsto [A_i \cdot C, x]\end{aligned}$$

(in the obvious functor-theoretic sense), which is easily checked to induce an isomorphism on geometric points, is an isomorphism. Therefore, there is an induced isomorphism

$$\begin{aligned}\prod_{\mathscr{A}} K_*\bigl(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\bigr)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda \\ \simeq K_*\bigl(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\bigr)_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda.\end{aligned}$$

Since $w_{\mathrm{GL}_{n,k}}(\sigma)$ acts transitively on $\mathscr{A}$ with stabilizer $w_{\mathrm{G}}(\sigma)$, by Lemma 2.10 we get a canonical isomorphism

$$\begin{aligned}&\bigl(K_*(\mathrm{N}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{N}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\bigr)^{w_{\mathrm{GL}_{n,k}}(\sigma)}\\ &\quad\simeq \bigl(K_*(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma))_{\mathrm{geom}} \otimes \widetilde{\mathrm{R}}(\sigma)_\Lambda\bigr)^{w_{\mathrm{G}}(\sigma)}.\end{aligned}$$

Since, by Morita equivalence (see [Th3, Prop. 6.2]),

$$K_*\bigl(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\bigr) \simeq K_*\bigl(X^\sigma, \mathrm{C}_{\mathrm{G}}(\sigma)\bigr), \tag{31}$$

to conclude the proof of Theorem 5.4 we need only show that the natural morphism

$$K_*\bigl(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\bigr)_{\mathrm{geom}} \simeq K_*\bigl(X^\sigma, \mathrm{C}_{\mathrm{G}}(\sigma)\bigr)_{\mathrm{geom}} \tag{32}$$

induced by (31) is still an isomorphism. Since the diagram

$$\begin{tikzcd} K_*\bigl(\mathrm{GL}_{n,k} \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{GL}_{n,k}\bigr)_{\mathrm{geom}} \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & K_*\bigl(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\bigr)_{\mathrm{geom}} \arrow[d, "\beta"] \\ & K_*\bigl(X^\sigma, \mathrm{C}_{\mathrm{G}}(\sigma)\bigr)_{\mathrm{geom}} \end{tikzcd}$$

is commutative and, by Morita equivalence,

$$\begin{aligned}&K_*\bigl(\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)\bigr)\\ &\quad\simeq K_*\bigl(\mathrm{GL}_{n,k} \times^{\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)} \mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma) \times^{\mathrm{C}_{\mathrm{G}}(\sigma)} X^\sigma, \mathrm{GL}_{n,k}\bigr)\\ &\quad\simeq K_*\bigl(\mathrm{GL}_{n,k} \times^{\mathrm{C}_{\mathrm{GL}_{n,k}}(\sigma)} X^\sigma, \mathrm{GL}_{n,k}\bigr),\end{aligned}$$

to show that β is an isomorphism it is enough to prove that for any regular separated algebraic space Z on which G acts with finite stabilizers, Morita equivalence induces an isomorphism

$$K_*(\mathrm{GL}_{n,k} \times^G Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \simeq K_*(Z, G)_{\mathrm{geom}} \tag{33}$$

since in this case both α and γ are isomorphisms.

Let $\pi : \mathrm{R}(\mathrm{GL}_{n,k}) \to \mathrm{R}(G)$ be the restriction morphism, let $\rho : \mathrm{R}(G) \to K_0(Z, G)$ be the pullback along $Z \to \operatorname{Spec} k$, let $\mathrm{rk}' : \mathrm{R}(\mathrm{GL}_{n,k}) \to \Lambda$ and $\mathrm{rk} : \mathrm{R}(G) \to \Lambda$ be the rank morphisms, and let $S' \doteq (\mathrm{rk}')^{-1}(1)$, $S \doteq (\mathrm{rk})^{-1}(1)$, and $T \doteq \pi(S') \subseteq S$; the following diagram commutes:

$$\begin{tikzcd} T^{-1}K_0(Z, G)_\Lambda \arrow[r, "\mathrm{rk}_{0,T}"] \arrow[d] & \Lambda \\ K_0(Z, G)_{\mathrm{geom}} \arrow[ur, "\mathrm{rk}_{\mathrm{geom}}"'] & \end{tikzcd}$$

where $\mathrm{rk}_{\mathrm{geom}}$ and $\mathrm{rk}_{0,T}$ denote the localizations of the rank morphism $\mathrm{rk}_0 : K_0(Z, G)_\Lambda \to \Lambda$. By Morita equivalence, the natural map (which commutes with the induced rank morphisms)

$$K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \longrightarrow T^{-1}K_0(Z, G)_\Lambda$$

is an isomorphism, and then, by Proposition 5.1, $\ker(\mathrm{rk}_{0,T} : T^{-1}K_0(Z, G)_\Lambda \to \Lambda)$ is nilpotent. Now, if $s \in S$, then $\mathrm{rk}_{0,T}(\rho(s)/1) = \mathrm{rk}(s) = 1$, and therefore

$$T^{-1}K_0(Z, G)_\Lambda \to K_0(Z, G)_{\mathrm{geom}}$$

and

$$K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \to K_0(Z, G)_{\mathrm{geom}}$$

are both isomorphisms. Since $K_*(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_\Lambda$ is naturally a $K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_\Lambda$-module and an $\mathrm{R}(\mathrm{GL}_{n,k})_\Lambda$-module via the pullback ring morphism

$$\rho' : \mathrm{R}(\mathrm{GL}_{n,k})_\Lambda \to K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_\Lambda,$$

we have

$$\begin{aligned}
&K'_*(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \\
&\quad \simeq K'_*(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_\Lambda \otimes_{K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_\Lambda} K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \\
&\quad \simeq K'_*(Z, G)_\Lambda \otimes_{K_0(Z,G)_\Lambda} K_0(\mathrm{GL}_{n,k} \times^{\mathrm{G}} Z, \mathrm{GL}_{n,k})_{\mathrm{geom}} \\
&\quad \simeq K'_*(Z, G)_\Lambda \otimes_{K_0(Z,G)_\Lambda} K_0(Z, G)_{\mathrm{geom}} \\
&\quad \simeq K'_*(Z, G)_{\mathrm{geom}},
\end{aligned}$$

which proves (32), and we conclude the proof of Theorem 5.4. □

5.2. Hypotheses on G

In this subsection we conclude the proof of Proposition 5.6, showing that (this is part (c_2) of the proof)

$$j_{\rho,\rho} : \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times^{\mathrm{N}_G(\rho)} X^\rho \longrightarrow Y^\rho$$

is flat. This is the only step in the proof of Proposition 5.6 where we make use of the hypothesis that the quotient algebraic space $G/\mathrm{C}_G(\rho)$ is smooth over k. Actually, our proof works under the following weaker hypothesis. Let S denote the spectrum of the dual numbers over k,

$$S = \mathrm{Spec}(k[\varepsilon]),$$

and for any k-group scheme H, let $\overline{\mathrm{H}}^1(S, H)$ denote the k-vector space of isomorphism classes of pairs $(P \to S, y)$, where $P \to S$ is an H-torsor and y is a k-rational point on the closed fiber of P. Then Proposition 5.6, and hence Theorem 5.4, still holds with hypothesis

(S) for any essential dual cyclic subgroup scheme $\sigma \subseteq G$, the quotient $G/\mathrm{C}_G(\sigma)$ is smooth over k

replaced by the following:

(S′) for any essential dual cyclic k-subgroup scheme $\sigma \subseteq G$, we have

$$\dim \overline{\mathrm{H}}^1\bigl(S, \mathrm{C}_G(\sigma)\bigr) = \dim \bigl(\overline{\mathrm{H}}^1(S, G)\bigr)^\sigma.$$

First we prove that $j_{\rho,\rho}$ is flat assuming (S′) holds. Then we show that (S) implies (S′). Since $p : \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^\rho \to \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times^{\mathrm{N}_G(\rho)} X^\rho$ is faithfully flat, it is enough to prove that $\underline{j}_\rho \doteq j_{\rho,\rho} \circ p$ is flat. Let $\pi : \mathrm{GL}_{n,k} \times X \to Y$ be the projection, and let

$$\begin{aligned} f : \mathrm{GL}_{n,k} \times X \times G &\longrightarrow \mathrm{GL}_{n,k} \times X, \\ (A, x, g) &\longmapsto (Ag^{-1}, gx). \end{aligned}$$

Consider the following Cartesian squares:

$$\begin{tikzcd} U \arrow[r, "u_\rho"] \arrow[d] & \pi^{-1}(Y^\rho) \arrow[r, hook] \arrow[d] & \mathrm{GL}_{n,k} \times X \arrow[d, "\pi"'] \\ \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^\rho \arrow[r, "\underline{j}_\rho"'] & Y^\rho \arrow[r, hook] & Y \end{tikzcd} \tag{34}$$

Since π is faithfully flat, it is enough to prove that u_ρ is flat. But the squares

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & \mathrm{GL}_{n,k} \times X \times G \arrow[r, "f"] \arrow[d, "\mathrm{pr}_{12}"'] & \mathrm{GL}_{n,k} \times X \arrow[d, "\pi"] \\ \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^\rho \arrow[r, hook] & \mathrm{GL}_{n,k} \times X \arrow[r, "\pi"'] & Y \end{tikzcd} \tag{35}$$

are Cartesian and (in the obvious functor-theoretic sense)

$$U = \bigl\{(A, x, g) \in \mathrm{GL}_{n,k} \times X \times G \bigm| A^{-1}\rho A = \rho, x \in X^\rho\bigr\} \simeq \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^\rho \times G.$$

Moreover, if $P \doteq \{A \in \mathrm{GL}_{n,k} \mid A^{-1}\rho A \subseteq G\}$, the map

$$\begin{aligned} \pi^{-1}(Y^\rho) = \bigl\{(A, x) \in \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X \bigm| A^{-1}\rho A \subseteq G, x \in X^{A^{-1}\rho A}\bigr\} &\longrightarrow P \times X^\rho, \\ (A, x) &\longmapsto (A, Ax) \end{aligned}$$

is an isomorphism. Therefore, we are reduced to proving that the map

$$\begin{aligned} v_\rho : \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^\rho \times G &\longrightarrow P \times X^\rho, \\ (A, x, g) &\longmapsto (Ag^{-1}, Ax) \end{aligned}$$

is flat. But since the diagram

$$\begin{tikzcd} \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times X^{\rho} \times G \arrow[r, "v_\rho"] \arrow[d, "\mathrm{pr}_{13}"'] & P \times X^{\rho} \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times G \arrow[r, "\Theta_\rho"'] & P \end{tikzcd}$$

(where $\Theta_\rho(A, g) \doteq (Ag^{-1})$) is easily checked to be Cartesian, it is enough to show that F_ρ is flat. To do this, let us observe that ρ acts by conjugation on $\mathrm{GL}_{n,k}/G$ (quotient by the G-action on $\mathrm{GL}_{n,k}$ by right multiplication), and we have a Cartesian diagram

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, "\tau"'] & \mathrm{GL}_{n,k} \arrow[d] \\ (\mathrm{GL}_{n,k}/G)^{\rho} \arrow[r, hook] & \mathrm{GL}_{n,k}/G \end{tikzcd}$$

Then τ is a G-torsor and Θ_ρ is G-equivariant. Thus, the following commutative diagram, in which the vertical arrows are G-torsors,

$$\begin{tikzcd} \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \times G \arrow[r, "\Theta_\rho"] \arrow[d, "\mathrm{pr}_1"'] & P \arrow[d, "\tau"] \\ \mathrm{N}_{\mathrm{GL}_{n,k}}(\rho) \arrow[r, "\chi_\rho"'] & (\mathrm{GL}_{n,k}/G)^{\rho} \end{tikzcd}$$

(where $\chi_\rho(A) \doteq [A] \in \mathrm{GL}_{n,k}/G$) is Cartesian, and we may reduce ourselves to prove that χ_ρ is flat. Now, observe that $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$ acts on the left of both $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$ and $(\mathrm{GL}_{n,k}/G)^{\rho}$ in such a way that χ_ρ is $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$-equivariant. Therefore, it is enough to prove that χ_ρ is flat when restricted to the connected component of the identity in $\mathrm{N}_{\mathrm{GL}_{n,k}}(\rho)$, that is, that the map

$$\chi'_\rho : \mathrm{C}_{\mathrm{GL}_{n,k}}(\rho) \longrightarrow (\mathrm{GL}_{n,k}/G)^{\rho}$$

is flat. Now, $\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho) = (\mathrm{GL}_{n,k})^{\rho}$, where ρ acts by conjugation; both $(\mathrm{GL}_{n,k})^{\rho}$ and $(\mathrm{GL}_{n,k}/G)^{\rho}$ are smooth by [Th5, Prop. 3.1] (since $\mathrm{GL}_{n,k}$ and $\mathrm{GL}_{n,k}/G$ are smooth); each fiber of χ'_ρ has dimension equal to $\dim(\mathrm{C}_G(\rho))$ because χ'_ρ is $\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)$-equivariant for the natural actions; and all the fibers are obtained from $(\chi'_\rho)^{-1}([1]) = \mathrm{C}_G(\rho)$ by the $\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)$-action. Therefore, χ'_ρ is flat if

$$\dim\big(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)\big) = \dim\big(\mathrm{C}_G(\rho)\big) + \dim\big((\mathrm{GL}_{n,k}/G)^{\rho}\big). \tag{36}$$

Note that, in any case,

$$\dim\big(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)\big) \leq \dim\big(\mathrm{C}_G(\rho)\big) + \dim\big((\mathrm{GL}_{n,k}/G)^{\rho}\big). \tag{37}$$

Since $\mathrm{GL}_{n,k}$ is smooth, $\dim((\mathrm{GL}_{n,k}/G)^\rho) = \dim_k(\mathrm{T}_1(\mathrm{GL}_{n,k}/G)^\rho)$, where T_1 denotes the tangent space at the class of $1 \in \mathrm{GL}_{n,k}$. Moreover, since $\overline{\mathrm{H}}^1(S, \mathrm{GL}_{n,k}) = 0$, there is an exact sequence of k-vector spaces

$$0 \to \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(\mathrm{GL}_{n,k}) \longrightarrow \mathrm{T}_1(\mathrm{GL}_{n,k}/G) \longrightarrow \overline{\mathrm{H}}^1(S, G) \to 0 \tag{38}$$

which, ρ being linearly reductive over k, yields an exact sequence of ρ-invariants

$$0 \to \mathrm{Lie}(G)^\rho \longrightarrow \mathrm{Lie}(\mathrm{GL}_{n,k})^\rho \longrightarrow \mathrm{T}_1(\mathrm{GL}_{n,k}/G)^\rho \longrightarrow \overline{\mathrm{H}}^1(S, G)^\rho \to 0. \tag{39}$$

But $\mathrm{GL}_{n,k}$ is smooth, so

$$\dim_k \left(\mathrm{Lie}(\mathrm{GL}_{n,k})^\rho\right) = \dim(\mathrm{GL}_{n,k})^\rho = \dim\left(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)\right),$$

and, since $\mathrm{Lie}(\mathrm{C}_G(\rho)) = \mathrm{Lie}(G) \cap \mathrm{Lie}(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho))$, we get

$$\dim_k \left(\mathrm{Lie}(G)^\rho\right) = \dim_k \left(\mathrm{Lie}(\mathrm{C}_G(\rho))\right).$$

By (39), we get

$$\begin{aligned}
\dim_k \left(\overline{\mathrm{H}}^1(S, G)^\rho\right) &= \dim_k \left(\mathrm{T}_1(\mathrm{GL}_{n,k}/G)^\rho\right) - \dim\left(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)\right) \\
&\quad + \dim_k \left(\mathrm{Lie}(\mathrm{C}_G(\rho))\right) \\
&= \dim\left((\mathrm{GL}_{n,k}/G)^\rho\right) - \dim\left(\mathrm{C}_{\mathrm{GL}_{n,k}}(\rho)\right) + \dim\left(\mathrm{C}_G(\rho)\right) \\
&\quad + \dim_k \left(\mathrm{Lie}(\mathrm{C}_G(\rho))\right) - \dim\left(\mathrm{C}_G(\rho)\right);
\end{aligned} \tag{40}$$

hence (36) is satisfied if

$$\dim_k \left(\overline{\mathrm{H}}^1(S, G)^\rho\right) = \dim_k \left(\mathrm{Lie}(\mathrm{C}_G(\rho))\right) - \dim\left(\mathrm{C}_G(\rho)\right). \tag{41}$$

But

$$\dim_k \left(\overline{\mathrm{H}}^1(S, \mathrm{C}_G(\rho))\right) = \dim_k \left(\mathrm{Lie}(\mathrm{C}_G(\rho))\right) - \dim\left(\mathrm{C}_G(\rho)\right)$$

by the exact sequence (analogous to (38) with G replaced by $\mathrm{C}_G(\rho)$)

$$\begin{aligned}
0 \to \mathrm{Lie}\big(\mathrm{C}_G(\rho)\big) \longrightarrow \mathrm{Lie}(\mathrm{GL}_{n,k}) \longrightarrow \mathrm{T}_1\big(\mathrm{GL}_{n,k}/\mathrm{C}_G(\rho)\big) \\
\longrightarrow \overline{\mathrm{H}}^1\big(S, \mathrm{C}_G(\rho)\big) \to 0;
\end{aligned}$$

hence (41) holds by hypothesis (S′). We complete the proof of Proposition 5.6 by showing that (S) implies (S′). Since $\mathrm{C}_G(\rho) \subseteq G$, we have a natural map

$$\epsilon : \overline{\mathrm{H}}^1\big(S, \mathrm{C}_G(\rho)\big) \longrightarrow \overline{\mathrm{H}}^1(S, G)^\rho,$$

and by (40) and (37), we get

$$\dim_k \left(\overline{\mathrm{H}}^1(S, G)^\rho\right) \geq \dim_k \left(\overline{\mathrm{H}}^1(S, \mathrm{C}_G(\rho))\right). \tag{42}$$

Now, if (S) holds, that is, if $G/\mathrm{C}_G(\rho)$ is smooth, and if $[P \to S, y]$ is a class in $\overline{\mathrm{H}}^1(S, G)^\rho$, $P/\mathrm{C}_G(\rho) \to S$ is smooth and y induces a point in the closed fiber of $P/\mathrm{C}_G(\rho) \to S$; we may reduce the structure group to $\mathrm{C}_G(\rho)$, thus showing that ϵ is surjective. By (42), we conclude that ϵ is an isomorphism, and this implies (S′).

5.3. *Final remarks*

PROPOSITION 5.7
Let X be a Noetherian regular separated algebraic space over k, and let G be a finite group acting on X. There is a canonical isomorphism of $R(G)$-algebras

$$K_*(X, G)_{\mathrm{geom}} \otimes \mathbb{Z}\big[1/|G|\big] \simeq K_*(X)^G \otimes \mathbb{Z}\big[1/|G|\big].$$

Proof
Since $\ker\big(\mathrm{rk} : K_0(X) \to \mathrm{H}^0(X, \mathbb{Z}[1/|G|])\big)$ is nilpotent by Corollary 5.2, the canonical homomorphism

$$\pi^* : K_*(X, G) \longrightarrow K_*(X)^G$$

induces a ring homomorphism (still denoted by π^*)

$$\pi^* : K_*(X, G)_{\mathrm{geom}} \otimes \mathbb{Z}\big[1/|G|\big] \longrightarrow K_*(X)^G \otimes \mathbb{Z}\big[1/|G|\big].$$

Moreover, the functor

$$\pi_* : \mathscr{F} \longmapsto \bigoplus_{g \in G} g^*\mathscr{F},$$

defined on coherent $\mathscr{O}_X$-modules, induces a homomorphism

$$\pi_* : K'_*(X)^G \otimes \mathbb{Z}[1/|G|] \longrightarrow K'_*(X, G)_{\mathrm{geom}} \otimes \mathbb{Z}[1/|G|],$$

and (recalling that $K_*(X, G) \simeq K'_*(X, G)$) we obviously get

$$\pi^*\pi_*(\mathscr{F}) = |G| \cdot \mathscr{F}.$$

On the other hand, we have

$$\pi_*\pi^*(\mathscr{F}) \simeq \mathscr{F} \otimes \pi_*\mathscr{O}_X.$$

But $\mathrm{rk}(\pi_*\mathscr{O}_X) = |G|$, and therefore $\pi_*\pi^*$ is an isomorphism, too, because of Corollary 5.2. □

As a corollary of this result and of Theorem 5.4, we recover [Vi1, Th. 1], which was proved there in a completely different way.

We conclude the paper with a conjecture expressing the fact that $K_*(X, G)_{\mathrm{geom}}$ should be the K-theory of the quotient X/G, if X/G is regular, after inverting the orders of all the essential dual cyclic subgroups of G.

CONJECTURE 5.8
Let X be a Noetherian regular separated algebraic space over a field k, and let G be a linear algebraic k-group acting on X with finite stabilizers in such a way that the quotient X/G exists as a regular algebraic space. Let N denote the least common multiple of the orders of all the essential dual cyclic subgroups of G, and let $\Lambda = \mathbb{Z}[1/N]$. If $p : X \to X/G$ is the quotient map, the composition

$$K_*(X/G)_\Lambda \xrightarrow{p^*} K_*(X, G)_\Lambda \longrightarrow K_*(X, G)_{\mathrm{geom}}$$

is an isomorphism.

Remark 5.9
Bertrand Toen pointed out to us that if X/G is smooth, it follows from the results of [EG] that the composition

$$K_0(X/G) \otimes \mathbb{Q} \longrightarrow K_0(X, G) \otimes \mathbb{Q} \longrightarrow K_0(X, G)_{\mathrm{geom}} \otimes \mathbb{Q}$$

is an isomorphism.

Appendix. Higher equivariant K-theory of Noetherian regular separated algebraic spaces

In this appendix we describe the K-theories we use in the paper and their relationships. We essentially follow the example of [ThTr, Sec. 3]. We also adopt the language of [ThTr].

Let us remark that it is strongly probable that there exist equivariant versions of most of the results in [ThTr, Sec. 3]. In particular, there should exist a higher K-theory of G-equivariant cohomologically bounded pseudocoherent complexes on Z (resp., of G-equivariant perfect complexes on Z) for any quasi-compact algebraic space Z having most of the alternative models described in [ThTr, Pars. 3.5 – 3.12]. The arguments below can also be considered as a first step toward an extension of [ThTr, Lems. 3.11, 3.12] to the equivariant case on algebraic spaces. However, to keep the paper to a reasonable size, we have decided to give only the results we need, and, moreover, we have made almost no attempt to optimize the hypotheses.

We would also like to mention the paper [J] (in particular, Section 1) in which, among many other results, the general techniques of [ThTr] are used as guidelines for the K-theory of arbitrary Artin stacks.

We work in a slightly more general situation than required in the rest of the paper.

Let S be a separated Noetherian scheme, and let G be a group scheme affine over S which is finitely presented, separated, and flat over S. We denote by $G\text{-AlgSp}_{\text{reg}}$ the category of regular Noetherian algebraic spaces separated over S with an action of G over S and equivariant maps.

Definition A.1

If $X \in G\text{-AlgSp}_{\text{reg}}$, we denote by $K_*(X,G)$ (resp., $K'_*(X,G)$, resp., $K_*^{\text{naive}}(X,G)$) the Waldhausen K-theory of the complicial bi-Waldhausen category (see [ThTr]) $\mathscr{W}_{1,X}$ of complexes of quasi-coherent G-equivariant $\mathscr{O}_X$-modules with bounded coherent cohomology (resp., the Quillen K-theory of the abelian category of G-equivariant coherent $\mathscr{O}_X$-modules, resp., the Quillen K-theory of the exact category of G-equivariant locally free coherent $\mathscr{O}_X$-modules).

PROPOSITION A.2

Let $Z \to S$ be a morphism of Noetherian algebraic spaces such that the diagonal $Z \to Z \times_S Z$ is affine, and let $H \to S$ be an affine group space acting on Z. Let $\mathscr{F}$ be an equivariant quasi-coherent sheaf on Z of finite flat dimension; then there exists a flat equivariant quasi-coherent sheaf $\mathscr{F}'$ on Z together with a surjective H-equivariant homomorphism $\mathscr{F}' \to \mathscr{F}$.

In particular, if Z is regular, this holds for all equivariant quasi-coherent sheaves $\mathscr{F}$ on Z.

The hypotheses of Proposition A.2 ensure that the usual morphism $Z \times_S H \to Z \times_S Z$ is affine. In fact, the projection $Z \times_S H \to Z$ is obviously affine, the projection $Z \times_S Z \to Z$ has affine diagonal, so this follows from the elementary fact that if $Z \to U \to V$ are morphisms of algebraic spaces, $Z \to V$ is affine, and $U \to V$ has affine diagonal, then $Z \to U$ is affine. Consider the quotient stack $\mathscr{Z} = [Z/H]$ (see [LMB]); the argument above implies that the diagonal $\mathscr{Z} \to \mathscr{Z} \times_S \mathscr{Z}$ is affine. Since an H-equivariant quasi-coherent $\mathscr{O}_Z$-module is the same as a quasi-coherent module over $\mathscr{Z}$, now Proposition 5.3 follows from the more general result below.

PROPOSITION A.3

Let S be a Noetherian algebraic space, and let $\mathscr{X}$ be a Noetherian algebraic stack over S with affine diagonal. Let $\mathscr{F}$ be a quasi-coherent sheaf of finite flat dimension on $\mathscr{X}$; then there exists a flat quasi-coherent sheaf $\mathscr{F}'$ on $\mathscr{X}$ together with a surjective homomorphism $\mathscr{F}' \to \mathscr{F}$.

Proof

Take an affine scheme U with a flat morphism $f : U \to \mathscr{X}$; then f is affine, and, in particular, the pushforward f_* on quasi-coherent sheaves is exact. Consider a quasi-coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ of finite flat dimension, with the adjunction map $\mathscr{F} \to f_* f^* \mathscr{F}$. This map is injective; call $\mathscr{Q}$ its cokernel. Clearly, the flat dimension of $f_* f^* \mathscr{F}$ is the same as the flat dimension of $\mathscr{F}$; we claim that the flat dimension of $\mathscr{Q}$ is at most equal to the flat dimension of $\mathscr{F}$. Now, if there were a section $\mathscr{X} \to U$ of f, then the sequence

$$0 \to \mathscr{F} \to f_* f^* \mathscr{F} \to \mathscr{Q} \to 0$$

would split and this would be clear. However, to compute the flat dimension of $\mathscr{Q}$, we can pull back to any flat surjective map to $\mathscr{X}$; in particular, after pulling back to U, we see that f acquires a section, and the statement is checked. Now U is an affine scheme, so we can take a flat quasi-coherent sheaf $\mathscr{P}$ on U with a surjective map $u : \mathscr{P} \to f^* \mathscr{F}$. Call $\mathscr{F}'$ the kernel of the composition $f_* \mathscr{P} \to f_* f^* \mathscr{F} \to \mathscr{Q}$; then $\mathscr{F}'$ surjects onto $\mathscr{F}$, and it fits into an exact sequence $0 \to \mathscr{F}' \to f_* \mathscr{P} \to \mathscr{Q} \to 0$. But $f_* \mathscr{P}$ is flat over $\mathscr{X}$, so the flat dimension of $\mathscr{F}'$ is less than the flat dimension of $\mathscr{Q}$, unless $\mathscr{Q}$ is flat. But since the flat dimension of $\mathscr{Q}$ is at most equal to the flat dimension of $\mathscr{F}$, we see that the flat dimension of $\mathscr{F}'$ is less than the flat dimension of $\mathscr{F}$, unless $\mathscr{F}$ is flat. The proof is completed with a straightforward induction on the flat dimension of $\mathscr{F}$. $\square$

THEOREM A.4

Let X be an object in G-$\mathrm{AlgSp}_{\mathrm{reg}}$. The obvious inclusions of the following complicial biWaldhausen categories induce homotopy equivalences on the Waldhausen K-theory spectra $K^{(i)}(X) \doteq K(\mathscr{W}_{i,X})$, $i = 1, 2, 3$. In particular, the corresponding Waldhausen K-theories $K_*^{(i)}(X, G)$ coincide.

(i) $\mathscr{W}_{1,X}$ = (*complexes of quasi-coherent G-equivariant $\mathscr{O}_X$-modules with bounded coherent cohomology*).

(ii) $\mathscr{W}_{2,X}$ = (*bounded complexes in* $G - \mathrm{Coh}_X$).

(iii) $\mathscr{W}_{3,X}$ = (*complexes of flat quasi-coherent G-equivariant $\mathscr{O}_X$-modules with bounded coherent cohomology*).

Moreover, the Waldhausen K-theory of any of the categories above coincides with Quillen K-theory $K'_*(X, G)$ of G-equivariant coherent $\mathscr{O}_X$-modules.

Proof

By [Th2, Par. 1.13], the inclusion of $\mathscr{W}_{2,X}$ in $\mathscr{W}_{1,X}$ induces an equivalence of K-theory spectra. Proposition 5.3, together with [ThTr, Lem. 1.9.5] (applied to $\mathscr{D}$ = (flat G-equivariant $\mathscr{O}_X$-modules) and $\mathscr{A}$ = (G-equivariant $\mathscr{O}_X$-modules)), implies that for any object E^* in $\mathscr{W}_{1,X}$ there exist an object F^* in $\mathscr{W}_{3,X}$ and a quasi-isomorphism

$F^* \xrightarrow{\sim} E^*$. Therefore, by [ThTr, Par. 1.9.7 and Th. 1.9.8], the inclusion of $\mathscr{W}_{3,X}$ in $\mathscr{W}_{1,X}$ induces an equivalence of K-theory spectra.

The last statement of the theorem follows immediately from [Th2, Par. 1.13, p. 518]. □

Since any complex in $\mathscr{W}_{3,X}$ is degreewise flat and X is regular (hence boundedness of cohomology is preserved under tensor product*), the tensor product of complexes makes the Waldhausen K-theory spectrum of $\mathscr{W}_{3,X}$ into a functor $K^{(3)}$ from G-AlgSp$_{\text{reg}}$ to ring spectra, with product

$$K^{(3)} \wedge K^{(3)} \longrightarrow K^{(3)},$$

exactly as described in [ThTr, Par. 3.15]. In particular, by Theorem 5.3, K_* is a functor from G-AlgSp$_{\text{reg}}$ to graded rings. In the same way, the tensor product with complexes in $\mathscr{W}_{3,X}$ gives a pairing

$$K^{(3)} \wedge K^{(1)} \longrightarrow K^{(1)}$$

between the corresponding functors from G-AlgSp$_{\text{reg}}$ to spectra, so that $K_*^{(1)}(X, G)$ becomes a module over the ring $K_*^{(3)}(X, G)$ functorially in $(X, G) \in G$-AlgSp$_{\text{reg}}$. We denote the corresponding cap-product by

$$\cap : K_*^{(3)}(X, G) \otimes K_*^{(1)}(X, G) \longrightarrow K_*^{(1)}(X, G),$$

which becomes the ring product in $K_*(X, G)$ with the identifications allowed by Theorem 5.3. Note that there is an obvious ring morphism $\eta : K_*^{\text{naive}}(X, G) \to K_*^{(3)}(X, G)$, and if

$$\cap^{\text{naive}} : K_*^{\text{naive}}(X, G) \otimes K'_*(X, G) \longrightarrow K'_*(X, G)$$

denotes the usual "naive" cap-product on Quillen K-theories, there is a commutative diagram

$$\begin{tikzcd} K_*^{\text{naive}}(X, G) \otimes K'_*(X, G) \arrow[r, "\cap^{\text{naive}}"] \arrow[d, "\eta\otimes u"'] & K'_*(X, G) \arrow[d, "u"] \\ K_*^{(3)}(X, G) \otimes K_*^{(1)}(X, G) \arrow[r, "\cap"'] & K_*^{(1)}(X, G) \end{tikzcd}$$

where u is the isomorphism of Theorem 5.3. Because of that, we simply write $\cap$ for both the naive and nonnaive cap-products. Note that, as shown in [Th2, Par. 1.13, p. 519], $K'_*(-, G)$ (and therefore $K_*(X, G)$ under our hypotheses) is a covariant functor

*In fact, this is a nonequivariant statement and a local property in the flat topology, so it reduces to the same statement for regular affine schemes, which is elementary (see also [SGA6]).

for proper maps in G-AlgSp$_{\mathrm{reg}}$; on the other hand, since any map in G-AlgSp$_{\mathrm{reg}}$ has finite Tor-dimension, $K_*(-, G)$ is a controvariant functor from G-AlgSp$_{\mathrm{reg}}$ to (graded) rings. In fact, if $f : X \longrightarrow Y$ is a morphism in G-AlgSp$_{\mathrm{reg}}$, the same argument in [ThTr, Par. 3.14.1] shows that there is an induced pullback exact functor $f^* : \mathscr{W}_{3,Y} \to \mathscr{W}_{3,X}$, and then we use Theorem A.4 to identify $K_*^{(3)}(-, G)$ with $K_*(-, G)$.

PROPOSITION A.5 (Projection formula)
*Let $j : Z \longrightarrow X$ be a closed immersion in G-*AlgSp$_{\mathrm{reg}}$*. Then, if α is in $K_*(X, G)$ and β in $K'_*(Z, G)$, we have*

$$j_*\big(j^*(\alpha) \cap \beta\big) = \alpha \cap j_*(\beta)$$

in $K'_*(X, G)$.

Proof
Since j is affine, j_* is exact on quasi-coherent modules and therefore induces an exact functor of complicial bi-Waldhausen categories $j_* : \mathscr{W}_{1,Z} \to \mathscr{W}_{1,X}$ (the condition of bounded coherent cohomology being preserved by regularity of Z and X). Therefore, the maps

$$\begin{aligned} (\alpha, \beta) &\longmapsto j_*\big(j^*(\alpha) \cap \beta\big), \\ (\alpha, \beta) &\longmapsto \alpha \cap j_*(\beta) \end{aligned}$$

from $K_*(X, G) \times K_*(Z, G)$ to $K'_*(X, G) \simeq K_*(X, G)$ are induced by the exact functors $\mathscr{W}_{3,X} \times \mathscr{W}_{1,Z} \longrightarrow \mathscr{W}_{1,X}$,

$$\begin{aligned} (F^*, E^*) &\longmapsto j_*\big(j^*(F^*) \otimes E^*\big), \\ (F^*, E^*) &\longmapsto F^* \otimes j_*(E^*). \end{aligned} \tag{43}$$

But for any equivariant quasi-coherent sheaf $\mathscr{F}$ on X and $\mathscr{G}$ on Z, there is a natural (hence, equivariant) isomorphism

$$j_*(j^*\mathscr{F} \otimes \mathscr{G}) \simeq \mathscr{F} \otimes j_*\mathscr{G}$$

which, again by naturality, induces an isomorphism between the two functors in (43); therefore, we conclude by [ThTr, Par. 1.5.4]. □

Remark A.6
Since we need the projection formula only for (regular) closed immersion in this paper, we have decided to state the result only in this case. However, since, by [Th2,

Par. 1.13 p. 519], $K_*(X, G)$ coincides also with Waldhausen K-theory of the category $\mathscr{W}_{4,X}$ of complexes of G-equivariant quasi-coherent injective modules on X with bounded coherent cohomology, therefore, by Theorem 5.3, it also coincides with Waldhausen K-theory of the category $\mathscr{W}_{5,X}$ complexes of G-equivariant quasi-coherent flasque modules on X with bounded coherent cohomology. For any proper map $f : X \to Y$ in G-$\mathrm{AlgSp}_{\mathrm{reg}}$, we have an exact functor $f_* : \mathscr{W}_{5,X} \to \mathscr{W}_{5,Y}$, which therefore gives a "model" for the pushforward $f_* : K_*(X, G) \to K_*(Y, G)$ (cf. [ThTr, Par. 3.16]). Now, the proof of [ThTr, Prop. 3.17] should also give a proof of Proposition 3 with j replaced by any proper map in G-$\mathrm{AlgSp}_{\mathrm{reg}}$ because it only uses [ThTr, Th. 2.5.5], which obviously holds for X and Y Noetherian algebraic spaces, and [SGA4, Exp. XVII, par. 4.2], which should give a canonical G-equivariant Godement flasque resolution of any complex of G-equivariant modules on any algebraic space in G-$\mathrm{AlgSp}_{\mathrm{reg}}$ since it is developed in a general topos.

It is very probable that Theorem 5.3 and therefore the functoriality with respect to morphisms of finite Tor-dimension still hold without the regularity assumption on the algebraic spaces. On the other hand, it should also be true that with G and X as above (therefore, X regular), the Waldhausen K-theory of the category of G-equivariant perfect complexes on X coincides with $K'_*(X, G)$. This last statement should follow (with a bit of work to identify $K'_*(X, G)$ with the Waldhausen K-theory of G-equivariant pseudocoherent complexes with bounded cohomology on X) from [J, Th. 1.6.2].

Acknowledgments. We wish to thank the referee for useful and precise remarks. We also thank Bertrand Toen, who pointed out the content of Remark 5.9 to us.

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