Quadratic forms and Clifford algebras on derived stacks

Gabriele Vezzosi

Institut de Mathématiques de Jussieu

Paris - France

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Abstract

In this paper we present an approach to quadratic structures in derived algebraic geometry. We define derived n-shifted quadratic complexes, over derived affine stacks and over general derived Artin stacks, and give examples of those. We define the associated notion of derived Clifford algebra, in all these contexts, and compare it with its classical version, when they both apply. Finally, we prove two existence results for derived shifted quadratic forms over derived Artin stacks, and define a derived version of the Grothendieck-Witt group.

Contents

Introduction

Prompted by the recent introduction of *symplectic forms* in derived algebraic geometry ([PTVV]), we present an approach to *quadratic forms* on derived stacks. In the case where the derived stack is a the spectrum of a ring k (where 2 is invertible) these quadratic forms are defined on complexes C of k-modules, and are maps, in the derived category of k, from $\mathrm{Sym}_k^2 C$ to $k[n]$, where $\mathrm{Sym}_k^2(-)$ denotes the derived functor of the second symmetric power over k. There is an obvious notion of non-degeneracy for such a quadratic form. The derived features are therefore two: first of all the map is a morphism in the *derived category* of k, and secondly, and most importantly, we allow for a *shift* in the target. These features accommodate for various symmetric shifted duality situations in topology, the motivating one being classical Poincaré duality, and the globalization to quadratic forms on Modules over a derived stack is rather straightforward. When the Module in question is the tangent complex, we obtain what we call *shifted quadratic stacks*. We remark that the main definitions of derived quadratic forms and derived quadratic stacks are easy modifications of the notion of derived symplectic structure from [PTVV] (without the complication coming from closedness data). In particular, we are able to reproduce in the quadratic case, two of the main existence theorems in [PTVV]: the existence of a shifted quadratic form on the stack of maps from a $\mathcal{O}$-compact , $\mathcal{O}$-oriented derived stack to a shifted quadratic stack, and the existence of a quadratic form on the homotopy fiber product of two null-mappings to a shifted quadratic stack. We also observe that shifted symplectic structures give rise to shifted quadratic Modules.
A new feature we investigate here is the definition and study of a derived version of the *Clifford algebra* associated to a shifted quadratic Module over a derived stack (so, in particular, to any shifted quadratic stack). We prove various basic properties of this derived Clifford algebra and give a theorem comparing it to the classical Clifford algebra, when they are both defined: the classical Clifford algebra happens to be the truncation at H^0 of the derived one. Some of the leftovers and future directions of this work are discussed below.

Alternative approaches. An alternative approach with respect to the one used in this paper is the following, originally due in the underived and unshifted case to A. Preygel ([Pre]). It applies, to our present understanding, only in the even-shifted case, which is however the interesting one for the existence of derived Clifford algebras. Let (E, q) be a $2n$-shifted quadratic complex on X (see Definition 1.1 and 3.1). Here X can be any derived Artin stack locally of finite presentation over the base ring k, but the reader could stick to the derived affine case of Def. 1.1 without loosing any essential feature. The derived quadratic form q induces on the linear derived stack

$$\mathbb{V}(E[-n]) := \mathbb{R}\mathbf{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(E^\vee[n]))$$

a global function $f_q : \mathbb{V}(E[-n]) \to \mathbb{A}_k^1$. Then we may view the pair $(\mathbb{V}(E[n]), f_q)$ as a *Landau-Ginzburg pair*. One can then extract from the corresponding matrix factorization category $\mathbf{MF}((\mathbb{V}(E[n]), f_q))$, a derived Clifford algebra for (E, q), via a (derived, shifted) variant of [Pre, Thm. 9.3.4]. Its $\mathbb{Z}/2$-grading appears here in a natural way. Note that for derived quadratic stacks, i.e. when $E = \mathbb{T}_X$, then $\mathbb{V}(E[-n])$ is the $(-n)$-shifted tangent stack of X.

Leftovers. Many interesting topics are not present in this first paper on the derived theory of quadratic forms. Namely, a treatment of involutions, analogues of classical classification theorems (like Witt cancellation), more details and applications of the derived Grothendieck-Witt group (defined in Section 6), and a more thorough investigation of the relations to derived Azumaya algebras ([To–deraz]), especially in the $\mathbb{Z}/2$-graded case where we may expect a map from the derived Grothendieck-Witt group to the derived Brauer-Wall group (classifying derived $\mathbb{Z}/2$-graded Azumaya

up to $\mathbb{Z}/2$-graded Morita equivalences). We plan to come back at least to some of these leftovers in a future paper.

Generalizations. One natural, mild, extension of the theory developed in this paper can be obtained by working over a base ring k *with involution*. This would allow to treat, for example, derived (shifted) hermitian forms, objects that arise naturally (together with closed shifted differential forms) in derived Kähler geometry, a topic yet to be investigated.

A wider generalization of the present theory involves a categorification process: define a derived quadratic form on a dg-category. In order to do this one needs a homotopy invariant notion of the second symmetric power of a dg-category, i.e. a slight reformulation of the notion introduced by Ganter and Kapranov in [G-K]. As further explained by M. Schlichting, the role of the shifted $k[n]$ will be taken here by $\mathcal{C}_k^{[n]}$ (see [Schl, §1.9]). We plan to come back to this categorification in a future paper. One could expect e.g. that the quasi-coherent dg-category of a derived quadratic stack carries a derived quadratic form, but it might be the case that the converse is not true. This would of course make the categorification more interesting.

Notations.

- k will denote the base commutative ring such that $2 \neq 0$ in k. When k will be needed to be a field, we will also use the alternative notation $k = \mathbb{F}$.

- when we say that (V, Q) is a (classical) *quadratic module* over k, we actually mean that V is a k-module, and that Q is a symmetric bilinear form on V over k (these might bear the name of symmetric bilinear modules, but this name is almost never used in the literature). Note that classical quadratic modules, as defined e.g. in [Mi-Re], coincide with our if 2 is invertible in k.

- $\mathsf{C}(k)$ will denote the model category of unbounded cochain complexes of k-modules with surjections as fibrations, and quasi-isomorphisms s equivalences. It is a symmetric monoidal model category with the usual tensor product $\otimes_k$ of complexes over k. The corresponding ∞-category will be denoted by $\mathbf{C}(k)$.

- $\mathsf{C}^{\leq 0}(k)$ will denote the model category of cochain complexes of k-modules in non-positive degrees, with surjections in strictly negative degrees as fibrations, and quasi-isomorphisms s equivalences. It is a symmetric monoidal model category with the usual tensor product $\otimes_k$ of complexes over k. The corresponding ∞-category will be denoted by $\mathbf{C}^{\leq 0}(k)$.

- $\mathsf{cdga}_k^{\leq 0}$ denotes the category of differential *non-positively* graded algebras over k, with differential increasing the degree by 1. We will always consider $\mathsf{cdga}_k^{\leq 0}$ endowed with the usual model

structure for which fibrations are surjections in negative degrees, and equivalences are quasi-isomorphisms (see [HAGII, §2.2.1]).

- dga_k denotes the category of unbounded differential graded cochain algebras over k. We will always consider dga endowed with the usual model structure for which fibrations are surjections, and equivalences are quasi-isomorphisms (see [Sch-Shi-1]).

- $\mathsf{dga}_k^{\leq 0}$ denotes the category of differential non-positively graded cochain algebras over k. We will always consider $\mathsf{dga}_k^{\leq 0}$ endowed with the usual model structure for which fibrations are surjections in strictly negative degrees, and equivalences are quasi-isomorphisms (see [HAGII]).

- salg_k denotes the model category of simplicial commutative k-algebras where weak equivalences and fibrations are detected on the underlying morphisms of simplicial sets (see [HAGII, §2.2]).

- $\bigwedge \equiv \bigwedge_k, \otimes \equiv \otimes_k, \mathrm{Sym}_k^2 \equiv \mathrm{Sym}_k^2, \bigwedge_A, \otimes_A, \mathrm{Sym}_A^2$ (for $A \in \mathsf{salg}_k$, identified with its normalization cdga $\mathrm{N}(A)$) will always be *derived* functors.

- By ∞-category, we mean a Segal category, and we use the results and notations from [To-Ve –Traces]. Equivalently, one might work in the framework of quasi-categories (see [Lu–HTT]).

- We will denote by $\mathcal{S}$ the ∞-category of spaces or simplicial sets, i.e. $\mathcal{S}$ is the Dwyer-Kan localization of the category SSets with respect to weak equivalences.

1 Derived quadratic complexes

If $A \in \mathsf{salg}_k$, we denote by $\mathrm{N}(A)$ its normalization (see...). In particular $\mathrm{N}(A) \in \mathsf{cdga}_k^{\leq 0}$ (see [Sch-Shi-2]) and we will write

- $A - \mathsf{dgmod}$ for the model category of unbounded dg-modules over $\mathrm{N}(A)$. This is a symmetric monoidal model category satisfying the monoid axiom ([Sch-Shi-1]).

- $\mathrm{D}(A) := \mathrm{Ho}(A - \mathsf{dgmod})$ is the unbounded derived category of $\mathrm{N}(A)$, $\otimes_A^{\mathbb{L}} \equiv \otimes_A$ its induced monoidal structure, and $\mathbb{R}\underline{\mathrm{Hom}}_A$ the adjoint internal Hom-functor.

- $A - \mathsf{dga}$ the model category of differential graded algebras over A, i.e. the category of monoids in the symmetric monoidal model category $A - \mathsf{dgmod}$ (see [Sch-Shi-2]).

- For any $n \in \mathbb{Z}$, $A[n] := \mathrm{N}(A)[n]$ as an object in $A - \mathsf{dgmod}$.

Definition 1.1 *Let* $C \in A - \mathsf{dgmod}$, *and* $n \in \mathbb{Z}$.

- *The* space of n-shifted derived quadratic forms on C *is the mapping space*

$$\mathbf{QF}_A(C; n) := \mathrm{Map}_{A-\mathsf{dgmod}}(\mathrm{Sym}_A^2(C), A[n]) \; ;$$

- *The* set of n-shifted derived quadratic forms on C *is*

$$\mathrm{QF}_A(C; n) := \pi_0 \mathbf{QF}_A(C; n)$$

of connected components of $\mathbf{QF}_A(C; n)$, *and an* n-shifted quadratic form on C *is by definition an element* $q \in \mathrm{QF}_A(C; n)$.

- *The* space $\mathbf{QF}_A^{nd}(C;n)$ of n-shifted derived non-degenerate quadratic forms on C *is defined by the following homotopy pullback diagram of simplicial sets*

$$\begin{array}{ccc} \mathbf{QF}_A^{nd}(C;n) & \longrightarrow & \mathbf{QF}_A(C;n) \\ \downarrow & & \downarrow \\ [\operatorname{Sym}_A^2(C), A[n]]^{nd} & \longrightarrow & [\operatorname{Sym}_A^2(C), A[n]] \end{array}$$

  where $[-,-]$ *denotes the hom-sets in the homotopy category of* $A-\mathsf{dgmod}$*, and* $[\operatorname{Sym}_A^2(C), A[n]]^{nd}$ *is the subset of* $[\operatorname{Sym}_A^2(C), A[n]]$ *consisting of maps* $v : \operatorname{Sym}_A^2(C) \to A[n]$ *such that the associated, adjoint map* $v^\flat : C \to C^\vee[n]$ *is an isomorphism in* $\operatorname{Ho}(A-\mathsf{dgmod})$ *(where* $C^\vee := \mathbb{R}\underline{\operatorname{Hom}}_A(C, A)$ *is the derived dual of* C *over* A*).*

- *The* set $\operatorname{QF}_A^{nd}(C;n)$ of n-shifted derived non-degenerate quadratic forms on C *is the set*

$$\operatorname{QF}_A^{nd}(C;n) := \pi_0 \mathbf{QF}_A(C;n)^{nd}$$

  of connected components of $\mathbf{QF}(C;n)_A^{nd}$*, and an* n*-shifted non-degenerate quadratic form on* C *is by definition an element* $q \in \operatorname{QF}_A(C;n)^{nd}$*.*

- *an* n*-shifted derived (resp. non-degenerate) quadratic complex* is a pair (C, q) *where* $C \in A-\mathsf{dgmod}$ *and* $q \in \operatorname{QF}_A(C;n)$ *(resp.* $q \in \operatorname{QF}_A^{nd}(C;n)$*)*

Note that, by definition, an n-shifted quadratic form q on C is a map $q : \operatorname{Sym}_A^2(C) \to A[n]$ in the homotopy category $\operatorname{Ho}(A-\mathsf{dgmod})$.

When working over a fixed base A, we will often write $\mathbf{QF}(C;n)$ for $\mathbf{QF}_A(C;n)$, $\mathbf{QF}(C;n)^{\mathrm{nd}}$ for $\mathbf{QF}_A(C;n)^{\mathrm{nd}}$, and so on.

Remark 1.2 Note that Definition 1.1 makes perfect sense if the cdga $\operatorname{N}(A)$ is replaced by an arbitrary unbounded commutative differential graded algebra (over k). Note also that, if $A = k$, the constant simplicial algebra with value k, then $\operatorname{N}(A) = k$, and $A-\mathsf{dgmod} = \mathsf{C}(k)$. However, the only nicely behaved situation is when C is a connective dg module (i.e. with vanishing cohomologies in strictly positive degrees) over a non-positively graded cdga.

Examples 1.3

1. For any $n \in \mathbb{Z}$, any $C \in A-\mathsf{dgmod}$ is an n-shifted derived quadratic complex when endowed with the zero derived quadratic form, and for the loop space $\Omega_0(\mathbf{QF}(C;n))$ at 0 of $\mathbf{QF}(C;n))$, we have an isomorphism in $\operatorname{Ho}(\mathsf{SSets})$

$$\Omega_0(\mathbf{QF}(C;n)) \simeq \mathbf{QF}(C;n-1),$$

   since $\Omega_0\operatorname{Map}_{A-\mathsf{dgmod}}(E, F) \simeq \operatorname{Map}_{A-\mathsf{dgmod}}(E, F[-1])$, for any $E, F \in A-\mathsf{dgmod}$

2. Suppose that C is a connective m-connected cochain complex over k, for $m > 0$ (i.e. $\operatorname{H}^i(C) = 0$ for $i > 0$ and $-m \le i \le 0$). Then, if $n > m + 2$, there are no non-zero derived n-shifted quadratic forms on C over k. In other words, $\mathbf{QF}_k(C;n)$ is connected, for any $n > m + 2$. This follows immediately from the fact that, under the connectivity hypotheses on C, $\operatorname{Sym}_k^2(C)$ is $(m+2)$-connected.

3. If $(C, q, n = 0)$ is a derived non-degenerate quadratic complex in $\mathsf{C}^{\le 0}(k)$ over $A = k$, then C is discrete (i.e. cohomologically concentrated in degree 0).

4. If $n, m \in \mathbb{Z}$, and $C \in A - \mathsf{dgmod}$ is a perfect complex, the associated (n, m)-*hyperbolic space of* C is the derived $(n + m)$-shifted non-degenerate quadratic complex $\mathsf{hyp}(C; n, m) := (C[n] \oplus C^\vee[m], q_{(n,m)}^{\mathrm{hyp}})$, where $q_{(n,m)}^{\mathrm{hyp}}$ is induced by the identity map

$$C[n] \oplus C^\vee[m] \longrightarrow (C[n] \oplus C^\vee[m])^\vee[n + m] = (C^\vee[-n] \oplus C[-m])[n + m] = C[n] \oplus C^\vee[m].$$

Note that, if $A = k$, here we may also take $C = V[0]$, with V a projective finitely generated k-module.

5. Let $A = k$ and (V, ω) be a symplectic projective and finitely generated k-module. Since $\mathrm{Sym}^2(V[\pm 1]) \simeq \wedge^2 V[\pm 2]$, we get induced derived quadratic non-degenerate structures on $V[1]$ (with shift -2), and on $V[-1]$ (with shift 2).

6. Let $M = M^n$ be a compact oriented manifold of dimension n, and let $C = C^\bullet(M; \mathbb{Z})$ its singular cochain complex. Consider the composite

$$q : C \otimes C \longrightarrow C \xrightarrow{-\cap[M]} \mathbb{Z}[n] \ .$$

Then, q is symmetric (in the derived sense) and non-degenerate (in the derived sense) by Poincaré duality. Hence q is an n-shifted non-degenerate derived quadratic form on $C = C^\bullet(M; \mathbb{Z})$, i.e. an element in $\mathrm{QF}_{\mathbb{Z}}^{\mathrm{nd}}(C; n)$.

7. Other symmetric dualities in topology give rise to derived quadratic forms (see, e.g. [Ma, Thm 2.5.2, Thm. 3.1.1]).

Let $n \in \mathbb{Z}$. If $\varphi : A \to B'$ a morphism in salg_k, and (C, q) is a derived n-shifted quadratic complex over A, the derived base change complex $\varphi^* C = C \otimes_A B \in \mathrm{Ho}(B - \mathsf{dgmod})$ comes naturally endowed with a derived n-shifted quadratic form

$$\varphi^* q : \mathrm{Sym}_B^2(C \otimes_A B) \xrightarrow{\ \sim\ } \mathrm{Sym}_A^2(C) \otimes_A B \xrightarrow{q \otimes \mathrm{id}} A[n] \otimes_A B \simeq B[n] \ .$$

Definition 1.4 *Let* $n \in \mathbb{Z}$, (C, q) *a derived* n-*shifted quadratic complex over* A, *and* $\varphi : A \to B$ *a morphism in* salg_k. *The derived* n-*shifted quadratic complex* $(\varphi^* C, \varphi^* q)$ *over* B *is called the base-change of* (C, q) *along* φ.

More generally, a morphism $A \to B$ in salg_k defines base change maps in $\mathrm{Ho}(\mathsf{SSets})$

$$\varphi^* : \mathbf{QF}_A(C; n) \longrightarrow \mathbf{QF}_B(\varphi^* C; n),$$

and

$$\varphi^* : \mathbf{QF}_A^{\mathrm{nd}}(C; n) \longrightarrow \mathbf{QF}_B^{\mathrm{nd}}(\varphi^* C; n).$$

Let (C_1, q_1) and (C_2, q_2) be two n-shifted derived quadratic complexes over A. Since

$$\mathrm{Sym}_A^2(C_1 \oplus C_2) \simeq \mathrm{Sym}_A^2(C_1) \oplus \mathrm{Sym}_A^2(C_2) \oplus (C_1 \otimes_A C_2),$$

we have a canonical projection

$$\pi_\oplus : \mathrm{Sym}_A^2(C_1 \oplus C_2) \longrightarrow \mathrm{Sym}_A^2(C_1) \oplus \mathrm{Sym}_A^2(C_2)$$

in $\mathrm{Ho}(A - \mathsf{dgmod})$, that can be used to give the following

6

Definition 1.5 *Let (C_1, q_1) and (C_2, q_2) be two n-shifted derived quadratic complexes over A. The orthogonal sum $(C_1 \oplus C_2, q_1 \perp q_2)$ is the derived n-shifted quadratic complex over A where $q_1 \perp q_2$ is defined by the composition*

$$\mathrm{Sym}^2_A(C_1 \oplus C_2) \xrightarrow{\pi_\oplus} \mathrm{Sym}^2_A(C_1) \oplus \mathrm{Sym}^2_A(C_2) \xrightarrow{q_1 \oplus q_2} A[n] \oplus A[n] \xrightarrow{sum} A[n] \ .$$

Definition 1.6 *If $C_1 \in A - \mathsf{dgmod}$, (C_2, q_2) is a derived n-shifted quadratic complexes over A, and $f : C_1 \to C_2$ is a map in $\mathrm{Ho}(A - \mathsf{dgmod})$, then the composite*

$$\mathrm{Sym}^2_A C_1 \xrightarrow{\mathrm{Sym}^2 f} \mathrm{Sym}^2_A C_2 \xrightarrow{q_2} A[n]$$

defines an n-shifted quadratic form on C_1, that we denote by $f^* q_2$. $f^* q_2$ is called the pull-back *or* restriction *of q_2 along f.*

In the present derived setting, the concept of f being an isometry, is not a property of but rather a datum on f. More precisely, we give the following

Definition 1.7 • *The* space of derived isometric structures *on a map $f : (C_1, q_1) \to (C_2, q_2)$, between two n-shifted derived quadratic complexes over A, is by definition the space*

$$\mathsf{Isom}(f; (C_1, q_1), (C_2, q_2)) := \mathrm{Path}_{q_1, f^* q_2}(\mathbf{QF}_A(C_1; n)).$$

- *The space $\mathsf{Isom}(f; (C_1, 0), (C_2, q_2))$ is called the* space of derived null-structures *on f.*

- *A* derived isometric structure *on a map $f : (C_1, q_1) \to (C_2, q_2)$, between two n-shifted derived quadratic complexes, is an element in $\pi_0(\mathsf{Isom}(f; (C_1, q_1), (C_2, q_2)))$.*

- *A* derived null structure *on a map $f : (C_1, 0) \to (C_2, q_2)$, between two n-shifted derived quadratic complexes, is an element in $\pi_0(\mathsf{Isom}(f; (C_1, 0), (C_2, q_2)))$.*

Remark 1.8 *Derived symplectomorphism structures.* A similar idea as in the definition above yields a natural notion of *derived symplectomorphism structure* in the theory developed in [PTVV]. If ω and ω' are derived n-shifted symplectic forms on a derived stack X, the space of *derived symplectic equivalences* between ω and ω' is the space

$$\mathsf{SymplEq}(X; \omega, \omega'; n) := \mathrm{Path}_{\omega, \omega'}(\mathbf{Sympl}(X; n)).$$

A *derived symplectic equivalence* between ω and ω' is then an element $\gamma_{\omega, \omega'} \in \pi_0 \mathsf{SymplEq}(X; \omega, \omega'; n)$. Let now $f : X_1 \to X_2$ be a map, and $\omega_i \in \mathbf{Sympl}(X_i; n)$, $i = 1, 2$. Then, in general, $f^* \omega_2 \in \mathcal{A}^{2, \mathrm{cl}}(X_1; n)$ ([PTVV, §2.2]); if moreover $f^* \omega_2 \in \mathbf{Sympl}(X_1; n)$, then the space of *derived symplectomorphism structures* on f is the space

$$\mathsf{SymplMor}(f; (X, \omega_1), (X_2, \omega_2)) := \mathsf{SymplEq}(X; \omega_1, f^* \omega_2; n),$$

and a *derived symplectomorphism structure* on f is a derived symplectic equivalence

$$\gamma_{\omega_1, f^* \omega_2} \in \pi_0 \mathsf{SymplEq}(X_1; \omega_1, f^* \omega_2; n).$$

2 Derived Clifford algebra of a derived quadratic complex

For *evenly* shifted derived quadratic complexes, it is possible to define a derived version of the Clifford algebra.

2.1 Derived Clifford algebra of a derived quadratic complex

Let $n \in \mathbb{Z}$, and (C, q) be derived $2n$-shifted quadratic complex over $A \in \mathsf{salg}_k$. We denote by $2\widetilde{q}$: $C \otimes_A C \to A[2n]$ the composite of $2q$ with the canonical map $C \otimes_A C \to \mathrm{Sym}_A^2(C)$.

If now $B \in A - \mathsf{dga}$, $\widetilde{q}$ induces a map $\widetilde{q}_B : * \to \mathrm{Map}_{A-\mathsf{dgmod}}(C \otimes_A C, B[2n])$, and the rule

$$(\varphi : C \to B[n]) \longmapsto (C \otimes_A C \xrightarrow{(\varphi, \varphi \circ \sigma)} (B[n] \otimes_A B[n]) \oplus (B[n] \otimes_A A[n]) \xrightarrow{\mu \oplus \mu} B[2n] \oplus B[2n] \xrightarrow{+} B[2n])$$

determines a map

$$s_B : \mathrm{Map}_{A-\mathsf{dgmod}}(C, B[n]) \to \mathrm{Map}_{A-\mathsf{dgmod}}(C \otimes_A C, B[2n]).$$

Here,

$$\sigma : C \otimes_A C \to C \otimes_A C : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

is the Koszul sign involution, μ denotes the multiplication map on B, and $+$ the sum in B, so, essentially, the image of φ sends $x \otimes y$ to $\varphi(x)\varphi(y) + (-1)^{|x||y|}\varphi(y)\varphi(x)$.

By using these maps we may define the derived Clifford algebra functor associated to the derived $2n$-shifted quadratic space (C, q), as

$$\underline{\mathbf{Cliff}}(C, q, 2n) : A - \mathsf{dga} \longrightarrow \mathsf{SSets} \ : \ B \longmapsto \underline{\mathbf{Cliff}}(C, q, 2n)(B)$$

where $\underline{\mathbf{Cliff}}(C, q, 2n)(B)$ is defined by the following homotopy pull back in SSets

$$
\begin{array}{ccc}
\underline{\mathbf{Cliff}}(C, q, 2n)(B) & \longrightarrow & \mathrm{Map}_{A-\mathsf{dgmod}}(C, B[n]) \\
\downarrow & & \downarrow s_B \\
* & \xrightarrow{\ \widetilde{q}_B\ } & \mathrm{Map}_{A-\mathsf{dgmod}}(C \otimes_A C, B[2n])
\end{array}
$$

Proposition 2.1 *The functor* $\underline{\mathbf{Cliff}}(C, q, 2n)$ *is homotopy co-representable, i.e. there exists a well defined* $\mathbf{Cliff}_A(C, q, 2n) \in \mathrm{Ho}(A - \mathsf{dga})$ *and a canonical isomorphism in* $\mathrm{Ho}(\mathsf{SSets})$

$$\underline{\mathbf{Cliff}}(C, q, 2n)(B) \simeq \mathrm{Map}_{A-\mathsf{dga}}(\mathbf{Cliff}_A(C, q, 2n), B).$$

Proof. Since the notion of ideal is not well-behaved in derived geometry, we need to reformulate the existence in a homotopical meaningful way. This leads us to the following construction. Let $\mathsf{Free}_A : A - \mathsf{dgmod} \to A - \mathsf{dga}$ be the left derived functor of the free dga-functor (Quillen left adjoint to the forgetful functor). Then $\mathbf{Cliff}(C, q; 2n)$ is defined by the following homotopy push-out square in $A - \mathsf{dga}$

$$
\begin{array}{ccc}
\mathsf{Free}_A(C \otimes_A C[-2n]) & \xrightarrow{\ u\ } & \mathsf{Free}_A(C[-n]) \\
t_q \downarrow & & \downarrow \\
A & \longrightarrow & \mathbf{Cliff}(C, q, 2n)
\end{array}
$$

where

- t_q is induced, by adjunction, by the map $2\widetilde{q}[-2n] : C[-n] \otimes_A C[-n] \to A[2n - 2n] = A$

- u is induced, by adjunction, by the map

$$C[-n] \otimes_A C[-n] \xrightarrow{(id, \sigma)} (C[-n] \otimes_A C[-n]) \oplus (C[-n] \otimes_A C[-n]) \xrightarrow{+}$$

$$\xrightarrow{+} C[-n] \otimes_A C[-n] \longrightarrow \mathsf{Free}_A(C[-n]).$$

$\square$

Definition 2.2 *The dga* $\mathbf{Cliff}_A(C, q; 2n)$, *defined up to isomorphism in* $\mathrm{Ho}(A - \mathsf{dga})$, *is called the derived Clifford algebra of the derived* $2n$-*shifted quadratic space* (C, q).

When the base simplicial algebra A is clear from the context, we will simply write $\mathbf{Cliff}(C, q, 2n)$ for $\mathbf{Cliff}_A(C, q, 2n)$.

Remark 2.3 To understand classically the proof of Proposition 5.1, observe that the classical Clifford algebra of a quadratic k-module (V, Q) is defined as the quotient

$$\frac{\mathrm{T}_k(V)}{\mathrm{I} := < x \otimes y + y \otimes x - 2Q(x, y) >} ,$$

$\mathrm{T}_k(-)$ denoting the tensor k-algebra functor. Then it is easy to verify that the following square is a (strict) push-out in the category of k-algebras

$$
\begin{array}{ccc}
\mathrm{T}_k(V \otimes_k V) & \overset{u}{\longrightarrow} & \mathrm{T}_k(V) \\
{\scriptstyle t}\downarrow & & \downarrow \\
k & \longrightarrow & \mathrm{T}_k(V)/\mathrm{I}
\end{array}
$$

where u is defined by $u(x) := x \otimes y + y \otimes x$, and t by $t(x \otimes y) := 2Q(x, y)$. For a more detailed study of the relation between the classical and the derived Clifford algebra when they both apply, see Section 2.2.

Remark 2.4 Note that it follows immediately from the description of $\mathbf{Cliff}(C, q, 2n)$ in the proof of Proposition 5.1, that $\mathbf{Cliff}(C, q, 2n)$ is a dg algebra *homotopically of finite type* over A whenever C is a perfect A-dg module.

Proposition 2.5 *Let* $n \in \mathbb{Z}$, $f : C_1 \to C_2$ *be a map in* $A - \mathsf{dgmod}$, *and* q_2 *a derived* $2n$-*shifted quadratic form on* C_2 *over* A. *Then there is a canonical map in* $\mathrm{Ho}(\mathsf{dga})$

$$\mathbf{Cliff}_A(C_1, f^*q_2, 2n) \longmapsto \mathbf{Cliff}_A(C_2, q_2, 2n)$$

where f^*q_2 *is the pull-back quadratic form of Definition 1.6.*

Proof. We use the proof of Proposition 5.1. By definition of f^*q_2, there is a map (in the homotopy category of diagrams of that shape in dga) from the diagram

$$
\begin{array}{ccc}
\mathsf{Free}_A(C_1 \otimes_A C_1[-2n]) & \overset{u}{\longrightarrow} & \mathsf{Free}_A(C_1[-n]) \\
{\scriptstyle t_{f^*q_2}}\downarrow & & \\
A & &
\end{array}
$$

to the diagram

$$
\begin{array}{ccc}
\mathsf{Free}_A(C_2 \otimes_A C_2[-2n]) & \overset{u}{\longrightarrow} & \mathsf{Free}_A(C_2[-n]) \ . \\
{\scriptstyle t_{q_2}}\downarrow & & \\
A & &
\end{array}
$$

9

By definition of homotopy push-out, we get the induced map. □

Note that, by composition with the natural adjunction map of complexes $C[-n] \to \mathsf{Free}_A(C[-n])$, the derived Clifford algebra $\mathbf{Cliff}_A(C, q, 2n)$ of the derived $2n$-shifted quadratic complex (C, q), comes equipped with a natural map of A-dg modules

$$C[-n] \longrightarrow \mathbf{Cliff}(C, q, 2n).$$

Using this, and the universal property of the derived Clifford algebra, we get the following

Proposition 2.6 *Let $n \in \mathbb{Z}$. If (C_1, q_1) and (C_2, q_2) are $2n$-shifted derived quadratic complexes over A, $f : C_1 \to C_2$ is a map in $\mathrm{Ho}(A - \mathsf{dgmod})$ and $\gamma \in \pi_0(\mathsf{Isom}(f; (C_1, q_1), (C_2, q_2)))$ is a derived isometric structure on f, then there is a canonical induced map in $\mathrm{Ho}(A - \mathsf{dga})$*

$$f_\gamma : \mathbf{Cliff}(C_1, q_1, 2n) \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n).$$

If moreover f is a quasi-isomorphism, f_γ is an isomorphism.

Proof. Let us consider the following two maps

$$h : \mathsf{Free}_A(C_1 \otimes_A C_1[-2n]) \xrightarrow{u_{C_1}} \mathsf{Free}_A(C_1[-n]) \xrightarrow{\mathsf{Free}(f[-n])} \mathsf{Free}_A(C_2[-n]) \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n),$$

and

$$g_1 : \mathsf{Free}_A(C_1 \otimes_A C_1[-2n]) \xrightarrow{\tilde{q}_1} A \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n),$$

By definition of $\mathbf{Cliff}(C_1, q_1, 2n)$ as a homotopy push-out (proof of Proposition 5.1), it is enough to show that our data give a path between h and g_1. To start with, the homotopy push-out defining $\mathbf{Cliff}(C_2, q_2, 2n)$, provides us with a path between the composite maps

$$\mathsf{Free}_A(C_2 \otimes_A C_2[-2n]) \xrightarrow{u_{C_2}} \mathsf{Free}_A(C_1[-n]) \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n),$$

and

$$\mathsf{Free}_A(C_2 \otimes_A C_2[-2n]) \xrightarrow{\tilde{q}_2} A \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n),$$

and, by precomposing with $\mathsf{Free}_A(f[-n])$, we get a path δ between the maps

$$h' : \mathsf{Free}_A(C_1 \otimes_A C_1[-2n]) \xrightarrow{\mathsf{Free}(f[-n])} \mathsf{Free}_A(C_2 \otimes_A C_2[-2n]) \xrightarrow{u_{C_2}} \mathsf{Free}_A(C_1[-n]) \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n),$$

and

$$g_2' : \mathsf{Free}_A(C_1 \otimes_A C_1[-2n]) \xrightarrow{\mathsf{Free}(f[-n])} \mathsf{Free}_A(C_2 \otimes_A C_2[-2n]) \xrightarrow{\tilde{q}_2} A \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n).$$

Now observe that $h' = h$, while g_2' is equal to the composite

$$g_2 : \mathsf{Free}_A(C_1 \otimes_A C_1[-2n]) \xrightarrow{\widetilde{f^* q_2}} A \longrightarrow \mathbf{Cliff}(C_2, q_2, 2n),$$

by definition of $\widetilde{f^* q_2}$. Hence, δ gives us a path between h and g_2. We conclude by using the further path between $\tilde{q}_1$ and $\widetilde{f^* q_2}$ (hence between g_2 and g_1), induced by the derived isometric structure on f, i.e. by the path γ between q_1 and $f^* q_2$.

□

In the following Proposition, recall the definition of base change for a derived quadratic complex along a base ring morphism (Definition 1.4).

Proposition 2.7 *Let $n \in \mathbb{Z}$, (C, q) be a $2n$-shifted derived quadratic complex over A, and $\varphi : A \to B$ a morphism in salg_k. Then there is a canonical isomorphism in $\mathrm{Ho}(B - \mathsf{dga})$*

$$\mathbf{Cliff}_A(C, q, 2n) \otimes_A B \simeq \mathbf{Cliff}_B(\varphi^* C, \varphi^* q, 2n).$$

Proof. This follows immediately from the proof of Proposition 5.1, and the observation that there exists a natural isomorphism $\mathsf{Free}_A(E) \otimes_A B \simeq \mathsf{Free}_B(E \otimes_A B)$ in $\mathrm{Ho}(B - \mathsf{dga})$, for any $E \in A - \mathsf{dgmod}$. $\square$

As shown in Appendix 7, if $n \in \mathbb{Z}$, and (C, q) be a $2n$-shifted derived quadratic complex over A, then $\mathbf{Cliff}_A(C, q, 2n)$ admits a natural $\mathbb{Z}/2$-weight grading, i.e it is naturally an object $\mathbf{Cliff}_A^w(C, q, 2n) \in \mathrm{Ho}(A - \mathsf{dga}^w)$ (see the Appendix for the notations). Then, a slight modification of classical proof of the corresponding classical statement (see e.g. [Mi-Re, Prop. 2.2.1]), yields the following

Proposition 2.8 *Let $n \in \mathbb{Z}$, (C_i, q_i) be $2n$-shifted derived quadratic complexes over A, and $(C_1 \oplus C_2, q_1 \perp q_2)$ be the corresponding orthogonal sum (Def. 1.5). Then there is a canonical isomorphism in $\mathrm{Ho}(A - \mathsf{dga}^w)$*

$$\mathbf{Cliff}_A^w(C_1 \oplus C_2, q_1 \perp q_2, 2n) \simeq \mathbf{Cliff}_A^w(C_1, q_1, 2n) \otimes_A^w \mathbf{Cliff}_A^w(C_2, q_2, 2n),$$

where $\otimes_A^w$ denotes the derived tensor product of $\mathbb{Z}/2$-dg algebras over A (see §7).

Remark 2.9 From derived Clifford algebras to derived Azumaya algebras ? Classically, Clifford algebras are Azumaya algebras: this is literally true when the rank of the quadratic module is even, true for its even graded piece in the odd rank case, and always true if we replace Azumaya algebras by $\mathbb{Z}/2$-graded Azumaya algebras (i.e. those classified by the Brauer-Wall group). Recently, Toën introduced the notion of derived Azumaya algebras over a (derived) stack, and proved e.g. that they are classified, up to Morita equivalence, by $H^1_{\mathrm{\acute{e}t}}(X, \mathbb{Z}) \times H^2_{\mathrm{\acute{e}t}}(X, \mathbb{G}_m)$ on a quasi-separated, quasi-compact scheme X ([To–deraz, Cor. 3.8]).

As suggested by Toën it might be interesting to investigate the exact relation between derived Clifford algebras and derived Azumaya algebras. The first problem we meet in such a comparison is the presence of a non-zero shift in the derived quadratic structure. There is no evident place for a shift in the current definition of derived Azumaya algebras, and it is not clear to us how one could modify the definition of a derived Azumaya algebra in order to accommodate such a shift. If we limit ourselves to the 0-shifted case, one can give at least one (admittedly not very relevant). Let C be a perfect complex of k-modules, and consider the associated 0-hyperbolic space $\mathsf{hyp}(C; n = 0, m = 0) := (C \oplus C^\vee, q_{0,0}^{\mathrm{hyp}}; 0)$ of Example 1.3 (3). Then there is a canonical isomorphism

$$\mathbf{Cliff}(\mathsf{hyp}(C; 0, 0)) \simeq \mathbb{R}\underline{\mathrm{Hom}}_k(\bigwedge C, \bigwedge C)$$

in $\mathrm{Ho}(\mathsf{dga})$, where $\mathbb{R}\underline{\mathrm{Hom}}_k$ denotes the derived internal Hom's in $\mathsf{C}(k)$. Thus $\mathbf{Cliff}(\mathsf{hyp}(C; 0, 0))$ is indeed a derived Azumaya algebra, but its class is obviously trivial in the derived Brauer group introduced by Toën ([To–deraz, Def. 2.14]). Note this case is morally an even rank case. A slightly more interesting statement would be the following (still sticking to the 0-shifted case). By systematically replacing $\mathbb{Z}/2$-graded dga's (see Appendix 7) in Toën's definition of derived Azumaya algebras, we get a notion of $\mathbb{Z}/2$-*graded derived Azumaya algebras*. Note that the extra $\mathbb{Z}/2$-grading affects both the notion of opposite algebra and of tensor product. As shown in Appendix 7 below, the derived Clifford algebra of derived $2n$-shifted quadratic complex over k is naturally an object in the homotopy category of $\mathbb{Z}/2$-graded dga's over k. So we may formulate the following question that has an affirmative answer

in the underlying, classical case : is the $\mathbb{Z}/2$-graded derived Clifford algebra of a 0-shifted quadratic complex over k a $\mathbb{Z}/2$-graded derived Azumaya algebra?

Of course, one could formulate a similar question by replacing k with a base scheme or an arbitrary derived Artin stack. This question is probably not too hard to settle but we do not have neither a proof nor a disproof at the moment.

A related interesting question is to extend Toën cohomological identification of the derived Brauer group to the corresponding *derived Brauer-Wall group* i.e. the Morita equivalence classes of derived $\mathbb{Z}/2$-graded Azumaya algebras.

2.2 Comparison with the classical Clifford algebra

In this §, we will work over $A = k$. Given a a classical quadratic (projective and finitely generated) k-module (V, Q), natural question is how can we get back the usual Clifford algebra $\mathbf{Cliff}^{\mathrm{class}}(V, Q)$ from its derived (0-shifted) Clifford algebra $\mathbf{Cliff}_k(V[0], Q[0], 0)$.

Proposition 2.10 *Let (C, q) be a derived 0-shifted quadratic complex in $\mathsf{C}^{\leq 0}(k)$. Then $(H^0(C), H^0(q))$ is a classical quadratic k-module, and there is a natural isomorphism of associative unital algebras*

$$\mathbf{Cliff}^{\mathrm{class}}(H^0(C), H^0(q)) \simeq H^0(\mathbf{Cliff}_k(C, q, 0)).$$

Proof. Let $\mathsf{Free} : \mathsf{C}(k) \to \mathsf{dga}_k$ be the free dga-functor (left adjoint to the forgetful functor $\mathsf{For} : \mathsf{dga}_k \to \mathsf{C}(k)$), $\mathsf{Free}^{\leq 0} : \mathsf{C}^{\leq 0}(k) \to \mathsf{dga}_k^{\leq 0}$ the free dga-functor (left adjoint to the forgetful functor $\mathsf{For}^{\leq 0} : \mathsf{dga}_k^{\leq 0} \to \mathsf{C}^{\leq 0}(k))$, $j_- : \mathsf{C}^{\leq 0}(k) \to \mathsf{C}(k)$ the inclusion functor, $[-]_{\leq 0} : \mathsf{C}(k) \to \mathsf{C}^{\leq 0}(k)$ its right adjoint (given by the intelligent truncation in degrees ≤ 0), $i_- : \mathsf{dga}_k^{\leq 0} \to \mathsf{dga}_k$ the inclusion functor, and $(-)_{\leq 0} : \mathsf{dga}_k \to \mathsf{dga}_k^{\leq 0}$ its right adjoint (given by the intelligent truncation in degrees ≤ 0). Note that the pairs $(j_-, [-]_{\leq 0})$ and $(i_-, (-)_{\leq 0})$, are Quillen pairs of type (left, right).

Lemma 2.11   *1. Let*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

be a homotopy push-out diagram in $\mathsf{dga}_k^{\leq 0}$, and

$$
\begin{array}{ccc}
i_- A & \longrightarrow & i_- B \\
\downarrow & & \downarrow \\
i_- C & \longrightarrow & D'
\end{array}
$$

the homotopy push-out in dga_k. Then the canonical map $i_- D' \to D$ is a weak equivalence.

2. If $E \in \mathsf{C}^{\leq 0}(k)$, then the canonical map $i_- \mathsf{Free}^{\leq 0}(E) \to \mathsf{Free}(j_- E)$ is a weak equivalence.

Proof of Lemma. Statement 1 follows immediately from the fact that i_- is left Quillen, hence preserves homotopy push-outs.

To prove 2, we start from the adjunction map $j_- E \to \mathsf{For}\,\mathsf{Free}(j_- E)$ in $\mathsf{C}(k)$. Since j_- is left adjoint, we get a map $E \to [\mathsf{For}\,\mathsf{Free}(j_- E)]_{\leq 0}$ in $\mathsf{C}^{\leq 0}(k)$. Note that the canonical transformation

$[-]_{\leq 0} \circ \mathsf{For} \to \mathsf{For}^{\leq 0} \circ (-)_{\leq 0}$ is an isomorphism, hence we get a map $E \to \mathsf{For}^{\leq 0}((\mathsf{Free}(j_- E))_{\leq 0})$ in $\mathsf{C}^{\leq 0}(k)$. Since $\mathsf{For}^{\leq 0}$ is right-adjoint, we get an induced map $\mathsf{Free}^{\leq 0}(E) \to (\mathsf{Free}(j_- E))_{\leq}$ in $\mathsf{dga}_k^{\leq 0}$, and finally the desired canonical map $i_- \mathsf{Free}^{\leq 0}(E) \to \mathsf{Free}(j_- E)$. This map is a weak equivalence since j_- preserves tensor products (?? CHECK). □

Let us consider now the following homotopy push-out square in $\mathsf{dga}_k^{\leq 0}$ (defining $\mathbf{Cliff}^{\leq 0}(C, q, 0)$)

$$
\begin{array}{ccc}
\mathsf{Free}^{\leq 0}(C \otimes_k C) & \xrightarrow{u^{\leq 0}} & \mathsf{Free}^{\leq 0}(C) \\
{\scriptstyle t^{\leq 0}}\downarrow & & \downarrow \\
\mathsf{Free}^{\leq 0}(0) = k & \longrightarrow & \mathbf{Cliff}^{\leq 0}(C, q, 0).
\end{array}
$$

By applying the left Quillen functor i_-, by point 1 in Lemma, we then get a homotopy push-out in dga_k

$$
\begin{array}{ccc}
i_- \mathsf{Free}^{\leq 0}(C \otimes_k C) & \xrightarrow{i_- u^{\leq 0}} & i_- \mathsf{Free}^{\leq 0}(C) \\
{\scriptstyle i_- t^{\leq 0}}\downarrow & & \downarrow \\
i_- \mathsf{Free}^{\leq 0}(0) = k & \longrightarrow & i_- \mathbf{Cliff}^{\leq 0}(C, q, 0).
\end{array}
$$

By point 2 in Lemma above, we also have a homotopy push-out diagram in dga_k

$$
\begin{array}{ccc}
\mathsf{Free}(j_- C \otimes_k j_- C) & \xrightarrow{u} & i_- \mathsf{Free}(j_- C) \\
{\scriptstyle t}\downarrow & & \downarrow \\
k & \longrightarrow & i_- \mathbf{Cliff}^{\leq 0}(C, q, 0),
\end{array}
$$

hence, by definition of $\mathbf{Cliff}(C, q, 0)$, we get an equivalence $i_- \mathbf{Cliff}^{\leq 0}(C, q, 0) \simeq \mathbf{Cliff}(j_- C, q, 0)$ in dga_k (therefore $\mathbf{Cliff}(j_- C, q, 0)$ is cohomologically concentrated in non-positive degrees). Since i_- preserves weak equivalences, we also get an induced map

$$\mathbf{Cliff}(j_- C, q, 0) \to H^0(\mathbf{Cliff}(j_- C, q, 0)) = H^0(\mathbf{Cliff}^{\leq 0}(C, q, 0))$$

that is an isomorphism on H^0. Since $H^0 : \mathsf{dga}_k^{\leq 0} \to \mathsf{alg}$ is left Quillen and $H^0 \mathsf{Free}^{\leq 0}(E) \simeq \mathsf{Free}^0(H^0 E)$ (where E is any object in $\mathsf{C}^{\leq 0}(k)$ and Free^0 is the free k-algebra functor defined on k-modules), by definition of the classical Clifford algebra (see Remark 2.3) we get an isomorphism

$$H^0(\mathbf{Cliff}^{\leq 0}(C, q, 0)) \simeq \mathbf{Cliff}^{\mathrm{class}}(H^0(C), H^0(q)).$$

□

Corollary 2.12 *If (V, Q) is a quadratic k-module, then the canonical map of dga's*

$$\mathbf{Cliff}_k(V[0], Q[0], 0) \longrightarrow \mathbf{Cliff}^{class}(V, Q)$$

induce an isomorphism on H^0.

One obvious natural question is now the following: suppose that (V, Q) is a quadratic k-module, is the dga $\mathbf{Cliff}_k(V[0], Q[0], 0)$ (which we know being cohomologically concentrated in non-positive degrees) 0-truncated (i.e. discrete) ? In other words, we are asking whether the derived Clifford algebra of a classical quadratic k-module contains or not strictly more information than its classical Clifford algebra.

The following example shows that the answer to this question is that, in general, the derived Clifford algebra of a classical quadratic k-module is *not* discrete, so it contains a priori more information than its classical Clifford algebra. The further question, i.e. whether or not this extra information might be relevant to the classical theory of quadratic forms - by giving new invariants or just a reinterpretation of known ones - is interesting but will not be addressed in the present paper.

Example 2.13 Let $V = k \oplus k$ be endowed with the quadratic form Q given by the symmetric matrix $(Q_{ij})_{i,j=1,2}$. We will show that $\mathrm{H}^{-1}(\mathbf{Cliff}_k(V[0], Q[0], 0)) \neq 0$ by simply using the symmetry of (Q_{ij}). Let

$$A := k < x_{ij} \,|\, i, j = 1, 2 >,$$

$$B_1 := k < x_{ij}, y_{ij} \,|\, i, j = 1, 2 > \qquad B := k < t_l \,|\, l = 1, 2 >$$

where $k < ... >$ denotes the free associative graded algebra on the specified generators, and $\deg(x_{ij}) = \deg(t_l) = 0$, for any $i, j, l = 1, 2$, while $\deg(y_{ij}) = -1$, for any $i, j = 1, 2$, and B_1 is endowed with the unique differential d such that $d(y_{ij}) = x_{ij}$ and making B_1 into a differential graded algebra. The map

$$t' : k < x_{ij} \,|\, i, j = 1, 2 > \longrightarrow B_1 \; : \; x_{ij} \longmapsto x_{ij} + 2Q_{ij}$$

is a cofibrant replacement in $k - \mathsf{dga}$ of the map

$$t : k < x_{ij} \,|\, i, j = 1, 2 > \longrightarrow k \; : \; x_{ij} \longmapsto 2Q_{ij}$$

as the factorization of t as

$$k < x_{ij} \,|\, i, j = 1, 2 > \xrightarrow{\;t'\;} B_1 \xrightarrow[\sim]{\;x_{ij} \mapsto 0\;} k$$

shows. Then $\mathbf{Cliff}_k(V[0], Q[0], 0)$ can be identified with the strict pushout of the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;u\;} & B \\
{\scriptstyle t'} \downarrow & & \\
B_1 & &
\end{array}
$$

in $k - \mathsf{dga}$ (where $u(x_{ij}) := x_i x_j + x_j x_i$). The pushout of dga's is a rather complicated object to describe concretely, so we use the following trick.

Let C be the dga over k whose underlying graded algebra is the free graded algebra on generators $\{y_{ij}\}_{i,j=1,2}$ in degree -1, with the unique differential making it into a dga over k such that $d(y_{ij}) := -2Q_{ij}$). Consider the following maps in $k - \mathsf{dga}$

$$\psi : B = k < t_l \,|\, l = 1, 2 > \longrightarrow C \; : \; x_l \longmapsto 0$$

$$\varphi : B_1 = k < x_{ij}, y_{ij} \,|\, i, j = 1, 2 > \longrightarrow C \; : \; x_{ij} \mapsto -2Q_{ij} \, , \; y_{ij} \mapsto y_{ij} \, ;$$

it is easy to verify that the two composites

$$A \xrightarrow{\;t'\;} B_1 \xrightarrow{\;\varphi\;} C$$

14

$$A \xrightarrow{\ u\ } B \xrightarrow{\ \psi\ } C,$$

coincide. Hence we obtain a canonical map of dga's

$$f : \mathbf{Cliff}_k(V[0], Q[0], 0) \longrightarrow C.$$

Suppose now that $Q_{12} = Q_{21} = 0$, and that neither $2Q_{11}$ nor $2Q_{22}$ are invertible in k. Then, by a long but straightforward computation, one checks that the element $(y_{ij} - y_{ji})$ is a (-1)-cycle and gives a non-zero class in $\overline{\alpha} \in \mathrm{H}^{-1}(C)$, for $i \neq j$ [1]. Note that $\varphi(\alpha := y_{ij} - y_{ji}) = y_{ij} - y_{ji}$. Hence, if

$$p : B_1 \longrightarrow \mathbf{Cliff}_k(V[0], Q[0], 0) \qquad q : B \longrightarrow \mathbf{Cliff}_k(V[0], Q[0], 0)$$

are the natural maps, in order to show that $\mathrm{H}^{-1}(\mathbf{Cliff}_k(V[0], Q[0], 0)) \neq 0$, it will be enough to show that the element $p(\alpha)$ is a (-1)-cycle in $\mathbf{Cliff}_k(V[0], Q[0], 0)$. But this is easy to check, by just using that $Q_{ij} = Q_{ji}$; more precisely:

$$d(p(\alpha)) = p(x_{ij} - x_{ji}) = p((x_{ij} + 2Q_{ij}) - (x_{ji} + 2Q_{ji})) = p(t'(x_{ij} - x_{ji})) =$$

$$= q(u(x_{ij} - x_{ji})) = q(x_i x_j + x_j x_i - (x_j x_i + x_i x_j)) = 0.$$

So, the (-1)-cycle $p(\alpha)$ in $\mathbf{Cliff}_k(V[0], Q[0], 0)$, has image a non zero cycle via

$$f : \mathbf{Cliff}_k(V[0], Q[0], 0) \longrightarrow C,$$

hence its class is non-zero in $\mathrm{H}^{-1}(\mathbf{Cliff}_k(V[0], Q[0], 0))$. Note that this apply also to the case where k is a field with characteristic different from 2, and $Q = 0$.

Remark 2.14 Recently, Bertrand Toën (private communication), simplified the previous example as follows. Start with $V = k$ a field of characteristic zero, with the zero quadratic form. Then it is easy to verify that $\mathbf{Cliff}_k(V[0], 0, 0)$ is the free graded k-algebra generated by x in degree 0 and y in degree -1, with differential defined by $d(y) = x^2$. One then verifies, as in the above example, that this dga has non vanishing H^{-1}, while its truncation is the corresponding exterior algebra i.e. the k-algebra of dual numbers $k[x]/x^2$.

3 Derived quadratic complexes and derived quadratic stacks

3.1 Derived quadratic complexes on a derived stack

The globalization of Definition 1.1 over a derived stack is straightforward. If X is a derived stack over k, let $\mathsf{L}_{\mathsf{Perf}}(X)$ (resp. $\mathsf{L}_{\mathsf{QCoh}}(X)$) be the symmetric monoidal Segal category of perfect (respectively, of quasi-coherent) complexes on X ([To–Seattle]). For any Segal category T, and any pair of objects (x, y) in T, we denote by $\mathrm{Map}_T(x, y)$ the corresponding mapping space (well defined in $\mathrm{Ho}(\mathsf{SSets})$).

Definition 3.1 Let X be a derived stack over k, $E \in \mathsf{L}_{\mathsf{QCoh}}(X)$, and $n \in \mathbb{Z}$.

- The space of n-shifted derived quadratic forms on E is the mapping space

$$\mathbf{QF}(E; n) := \mathrm{Map}_{\mathsf{L}_{\mathsf{QCoh}}(X)}(\mathrm{Sym}^2_{\mathcal{O}_X}(E), \mathcal{O}_X[n]) \ ;$$

---

[1]There are probably other, non-diagonal, cases for which the statement holds: the general answer boils down to showing that two matrices of sizes 4×16 and 4×17 have different ranks over k. I hope this explains my choice of a particular solution.

- *The set of n-shifted derived quadratic forms on E is*

$$\mathrm{QF}(E;n) := \pi_0 \mathbf{QF}(E;n)$$

of connected components of $\mathbf{QF}(E;n)$, and an n-shifted quadratic form on E is by definition an element $q \in \mathrm{QF}(E;n)$.

- *The space $\mathbf{QF}^{nd}(E;n)$ of n-shifted derived non-degenerate quadratic forms on E is defined by the following homotopy pullback diagram of simplicial sets*

$$
\begin{array}{ccc}
\mathbf{QF}^{nd}(E;n) & \longrightarrow & \mathbf{QF}(E;n) \\
\downarrow & & \downarrow \\
[\,\mathrm{Sym}^2_{\mathcal{O}_X}(E), \mathcal{O}_X[n]\,]^{nd} & \longrightarrow & [\,\mathrm{Sym}^2_{\mathcal{O}_X}(E), \mathcal{O}_X[n]\,]
\end{array}
$$

where $[-,-]$ denotes the hom-sets in the homotopy category of $\mathsf{L}_{\mathsf{QCoh}}(X)$, and $[\,\mathrm{Sym}^2_{\mathcal{O}_X}(E), \mathcal{O}_X[n]\,]^{nd}$ is the subset of $[\,\mathrm{Sym}^2_{\mathcal{O}_X}(E), \mathcal{O}_X[n]\,]$ consisting of maps $v : \mathrm{Sym}^2_{\mathcal{O}_X}(E) \to \mathcal{O}_X[n]$ such that the adjoint map $v^\flat : E \to E^\vee[n]$ is an isomorphism in $\mathrm{Ho}(\mathsf{L}_{\mathsf{QCoh}}(X))$.

- *The set $\mathrm{QF}^{nd}(E;n)$ of n-shifted derived non-degenerate quadratic forms on E is the set*

$$\mathrm{QF}^{nd}(E;n) := \pi_0 \mathbf{QF}(E;n)^{nd}$$

of connected components of $\mathbf{QF}(E;n)^{nd}$, and an n-shifted non-degenerate quadratic form on E is by definition an element $q \in \mathrm{QF}(E;n)^{nd}$.

- *an n-shifted derived (resp. non-degenerate) quadratic complex on X is a pair (E,q) where $E \in \mathsf{L}_{\mathsf{QCoh}}(X)$ and $q \in \mathrm{QF}(E;n)$ (resp. $q \in \mathrm{QF}^{nd}(E;n)$).*

Remark 3.2 Though Definition 3.1 makes sense for any $E \in \mathsf{L}_{\mathsf{QCoh}}(X)$ (actually, even quasi-coherence is not necessary), we will be mostly interested in the case where $E \in \mathsf{L}_{\mathsf{Perf}}(X)$.

Note that, by definition, a derived n-shifted quadratic form q on E is a map $q : \mathrm{Sym}^2_{\mathcal{O}_X}(E) \to \mathcal{O}_X[n]$ in the homotopy category $\mathrm{Ho}(\mathsf{L}_{\mathsf{QCoh}}) = \mathrm{D}_{\mathrm{QCoh}}(X)$.

Example 3.3 A *symmetric obstruction theory*, according to [Be-Fa, Def. 1.10], is an example of a derived 1-shifted quadratic complex.

Definition 3.4 If $E_1 \in \mathsf{L}_{\mathsf{QCoh}}(X)$, (E_2, q_2) is a derived n-shifted quadratic complex over X, and $f : E_1 \to E_2$ is a map in $\mathsf{L}_{\mathsf{QCoh}}(X)$, then the composite

$$\mathrm{Sym}^2_{\mathcal{O}_X} E_1 \xrightarrow{\ \mathrm{Sym}^2 f\ } \mathrm{Sym}^2_{\mathcal{O}_X} E_2 \xrightarrow{\ q_2\ } \mathcal{O}_X[n]$$

defines an n-shifted quadratic form on E_1, that we denote by $f^* q_2$. $f^* q_2$ is called the pull-back *or* restriction *of q_2 along f.*

The next result establishes a first link between derived symplectic structures and (non-degenerate) derived quadratic forms on derived stacks. We use the notations of [PTVV].

Proposition 3.5 *Let be a derived stack over k, char $k = 0$. There are canonical maps in* $\mathrm{Ho}(\mathsf{SSets})$

$$\mathbf{Sympl}(X; n) \longrightarrow \mathbf{QF}^{nd}(\mathbb{T}_X[1]; n - 2),$$

$$\mathbf{Sympl}(X; n) \longrightarrow \mathbf{QF}^{nd}(\mathbb{T}_X[-1]; n + 2).$$

Proof. It's enough to recall that $\wedge^2_{\mathcal{O}_X} \mathbb{T}_X \simeq \mathrm{Sym}^2_{\mathcal{O}_X}(\mathbb{T}_X[\pm 1])[\mp 2]$; this yields maps

$$\mathcal{A}^{2,\mathrm{nd}}(X; n) \longrightarrow \mathbf{QF}^{\mathrm{nd}}(\mathbb{T}_X[\pm 1]; n \mp 2),$$

and we just precompose each of these with the canonical underlying-2-form map $\mathbf{Sympl}(X; n) \longrightarrow \mathcal{A}^{2,\mathrm{nd}}(X; n)$.

$\square$

As done in [PTVV] for shifted symplectic forms, we may give the following

Definition 3.6 *For a derived m-shifted quadratic (not necessarily non-degenerate) form q on $\mathbb{T}_X[\pm 1]$, we define the* space of keys *of q, as the homotopy fiber at q of the composite map*

$$\mathcal{A}^{p,cl}(X; m \pm 2) \longrightarrow \mathcal{A}^2(X; m \pm 2) \longrightarrow \mathbf{QF}(\mathbb{T}_X[\pm 1]; m).$$

Exactly in the same way as done in Section 2.1 for the case of complexes over simplicial commutative algebra (i.e the case $X = \mathbb{R}\mathsf{Spec}(A)$), for a map of derived stacks $\varphi : Y \to X$, and $E \in \mathsf{L}_{\mathsf{QCoh}}(X)$ we may define the *base-change* maps $\mathrm{Ho}(\mathsf{SSets})$

$$\varphi^* : \mathbf{QF}_X(E; n) \longrightarrow \mathbf{QF}_Y(\varphi^* E; n),$$

and

$$\varphi^* : \mathbf{QF}_X^{\mathrm{nd}}(E; n) \longrightarrow \mathbf{QF}_Y^{\mathrm{nd}}(\varphi^* E; n).$$

Similarly, we define the *orthogonal sum* of derived derived n-shifted quadratic complexes over X and the notion of *derived isometric structure* on a map $f : (E, q) \to (E', q')$ between derived quadratic complexes in $\mathsf{L}_{\mathsf{QCoh}}$, by globalizing Definition 1.5 and 1.7.

3.2 Derived quadratic stacks

Definition 3.7 *Let X be a derived Artin stack locally finitely presented ($\equiv$ lfp) over k, and $n \in \mathbb{Z}$.*

- *The* space of n-shifted derived quadratic forms over X *is the space*

$$\mathbf{QF}(X; n) := \mathbf{QF}(\mathbb{T}_X; n).$$

- *The* set of n-shifted derived quadratic forms over X *is*

$$\mathrm{QF}(X; n) := \pi_0 \mathbf{QF}(X; n)$$

of connected components of $\mathbf{QF}(X; n)$, and an n-shifted quadratic form over X is by definition an element $q \in \mathrm{QF}(X; n)$.

- *The space $\mathbf{QF}^{nd}(X; n) := \mathbf{QF}^{nd}(\mathbb{T}_X; n)$ is the space of n-shifted derived non-degenerate quadratic forms over X*

17

- *The* set $\mathrm{QF}^{nd}(X; n) := \pi_0 \mathbf{QF}(E; n)^{nd}$ *is the set* of n-shifted derived non-degenerate quadratic forms over X, *and an* n-shifted non-degenerate quadratic form over X *is by definition an element* $q \in \mathrm{QF}(X; n)^{nd}$.

- *an* n-shifted derived *(resp.* non-degenerate*)* quadratic stack *is a pair* (X, q), *where X is a derived stack locally of finite presentation over k, and* $q \in \mathrm{QF}(X; n)$ *(resp.* $q \in \mathrm{QF}^{nd}(X; n)$*).*

Definition 3.8 *If X_1 is a derived Artin stack lfp over k, (X_2, q_2) an n-shifted derived quadratic stack over k, and $f : X_1 \to X_2$ is a map in $\mathsf{Ho}(\mathsf{dSt}_k)$, then the composite*

$$\mathrm{Sym}^2_{\mathcal{O}_{X_1}} \mathbb{T}_{X_1} \longrightarrow \mathrm{Sym}^2_{\mathcal{O}_{X_1}} f^* \mathbb{T}_{X_2} \xrightarrow{\sim} f^* \mathrm{Sym}^2_{\mathcal{O}_{X_2}} \mathbb{T}_{X_2} \xrightarrow{f^* q_2} f^* \mathcal{O}_{X_2}[n] \simeq \mathcal{O}_{X_1}[n]$$

defines an n-shifted quadratic form on X_1, that we denote by $f^q q_2$. More generally, we still denote by f^q the induced map

$$f^q : \mathbf{QF}(X_2; n) \longrightarrow \mathbf{QF}(X_1; n)$$

in $\mathsf{Ho}(\mathsf{SSets})$.

Remark 3.9 Note that $f^q q_2$ has the following equivalent description. Let us denote by $\mathrm{T}f : \mathbb{T}_X \to f^* \mathbb{T}_Y$ the induced tangent map. Then

$$f^q q = (\mathrm{T}f)^*(f^* q)$$

were $f^* q$ denotes the base-change of q (a quadratic form on $f^* \mathbb{T}_Y$ over X), and $(\mathrm{T}f)^*(\varphi^* q)$ the restriction of $f^* q$ along f (as in Definition 3.4).

The derived versions of f being an isometry or a null-map, are not properties of but rather data on f. More precisely, we give the following

Definition 3.10 *Let (X_1, q_1) and (X_2, q_2) be n-shifted derived quadratic stack sover k*

- *The* space of derived isometric structures *on a map $f : X_1 \to X_2$ is by definition the space*

$$\mathsf{Isom}(f; (X_1, q_1), (X_2, q_2)) := \mathrm{Path}_{q_1, f^* q_2}(\mathbf{QF}(X_1; n)).$$

- *The space* $\mathsf{Null}(f; X_1, X_2, q_2) := \mathsf{Isom}(f; (X_1, 0), (X_2, q_2))$ *is called the* space of derived null-structures *on f.*

- *A* derived isometric structure *on a map $f : X_1 \to X_2$ is an element in $\pi_0(\mathsf{Isom}(f; (X_1, q_1), (X_2, q_2)))$.*

- *A* derived null structure *on a map $f : X_1 \to X_2$ is an element in $\pi_0(\mathsf{Isom}(f; (X_1, 0), (X_2, q_2)))$.*

Now we want to define a condition of *non-degeneracy* on a null-structure on a given map $f : X_1 \to (X_2, q_2)$ that will prove useful later. This is a quadratic analog of the notion of Lagrangian structure for derived symplectic forms ([PTVV, §2.2]). Let $\gamma \in \mathsf{Null}(f; X_1, X_2, q_2)$ be a fixed null-structure on f. By definition, γ is a path between 0 and the composite morphism

$$\mathrm{Sym}^2_{\mathcal{O}_{X_1}} \mathbb{T}_{X_1} \longrightarrow f^* \mathrm{Sym}^2_{\mathcal{O}_{X_2}} \mathbb{T}_{X_2}) \longrightarrow \mathcal{O}_{X_1}[n].$$

If $\mathbb{T}_f$ is the relative tangent complex of f, so that we have the transitivity exact triangle of perfect complexes on X_1

$$\mathbb{T}_f \longrightarrow \mathbb{T}_{X_1} \longrightarrow f^*(\mathbb{T}_{X_2}).$$

The null-structure γ induces also a path γ' between 0 and the composite morphism

$$\varphi : \mathbb{T}_f \otimes \mathbb{T}_{X_1} \longrightarrow \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_1} \longrightarrow \mathrm{Sym}^2_{\mathcal{O}_{X_1}} \mathbb{T}_{X_1} \xrightarrow{f^* q_2} \mathcal{O}_{X_1}[n].$$

But the morphism $\mathbb{T}_f \longrightarrow f^*(\mathbb{T}_F)$ comes itself with a canonical (independent of γ) path from itself to 0, so we get another induced path δ from φ to 0. By composing γ' and δ, we then obtain a loop pointed at 0 in the space

$$\mathrm{Map}_{\mathsf{L}_{\mathsf{QCoh}}(X_1)}(\mathbb{T}_f \otimes \mathbb{T}_{X_1}, \mathcal{O}_{X_1}[n]).$$

This loop defines an element in

$$\pi_1(\mathrm{Map}_{\mathsf{L}_{\mathsf{QCoh}}(X_1)}(\mathbb{T}_f \otimes \mathbb{T}_{X_1}, \mathcal{O}_{X_1}[n]), 0) \simeq \mathrm{Hom}_{\mathsf{D}_{\mathsf{QCoh}}}(\mathbb{T}_f \otimes \mathbb{T}_{X_1}, \mathcal{O}_{X_1}[n-1]].$$

By adjunction, we get a morphism of perfect complexes on X_1

$$\Theta_\gamma : \mathbb{T}_f \longrightarrow \mathbb{L}_{X_1}[n-1],$$

depending on the null-structure structure γ.

Definition 3.11 *Let $f : X_1 \longrightarrow X_2$ be a morphism of derived Artin stacks and q_2 a derived n-shifted quadratic form on X_2. An null-structure γ on $f : X_1 \to (X_2, q_2)$ is* non-degenerate *if the induced morphism*

$$\Theta_\gamma : \mathbb{T}_f \longrightarrow \mathbb{L}_{X_1}[n-1]$$

is an isomorphism in $\mathsf{D}(X_1)$.

4 Existence theorems

In this Section we prove two existence theorems for derived quadratic forms on derived stacks, directly inspired by the analogous results for derived symplectic structures ([PTVV, Thms. 2.5 and 2.9]). For notation and definitions of $\mathcal{O}$-compact derived stack and $\mathcal{O}$-orientation, we refer the reader to [PTVV, §2].

Theorem 4.1 *Let (X, q) be an n-shifted quadratic derived stack, Y be an $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation η of degree d, and suppose that the derived mapping stack $\mathsf{MAP}_{\mathsf{dSt}_k}(Y, X)$ is a derived Artin stack locally of finite presentation over k. Then, $\mathsf{MAP}_{\mathsf{dSt}_k}(Y, X)$ admits a canonical $(n-d)$-shifted derived quadratic form $q' \equiv q'_\eta$. If moreover q is non-degenerate, then so is q'_η.*

    Proof. We borrow the notations from [PTVV, §2.1], and define q' as follows. For $x : \mathbb{R}\mathrm{Spec}\, A \longrightarrow \mathsf{Map}(Y, X)$ an A-point corresponding to a morphism of derived stacks

$$f : Y_A := Y \times \mathbb{R}\mathrm{Spec}\, A \longrightarrow X,$$

the tangent complex of $\mathsf{Map}(Y, X)$ at the point x is given by

$$\mathbb{T}_x \mathsf{Map}(Y, X) \simeq \mathbb{R}\mathrm{Hom}(\mathcal{O}_{Y_A}, f^*(\mathbb{T}_X)).$$

The quadratic form

$$q : \mathrm{Sym}^2_{\mathcal{O}_X}(\mathbb{T}_X) \longrightarrow \mathcal{O}_X[n],$$

induces by pullback a map of A-dg-modules

$$\rho_A : \mathrm{Sym}^2_A(\mathbb{R}\mathrm{Hom}(\mathcal{O}_{Y_A}, f^*(\mathbb{T}_X))) \longrightarrow \mathbb{R}\mathrm{Hom}(\mathcal{O}_{Y_A}, \mathcal{O}_{Y_A}[n]).$$

Now we use the d-orientation η on Y ([PTVV, Def. 2.4]): it induces, by definition a map of A-dg modules

$$\eta_A : \mathbb{R}\mathrm{Hom}(\mathcal{O}_{Y_A}, \mathcal{O}_{Y_A}) \longrightarrow A[-d]$$

By composing $\eta_A[n]$ with ρ_A, we get

$$q'_A : \mathrm{Sym}^2_A(\mathbb{T}_x\mathsf{Map}(Y, X)) \simeq \mathrm{Sym}^2_A(\mathbb{R}\mathrm{Hom}(\mathcal{O}_{Y_A}, f^*(\mathbb{T}_X))) \longrightarrow A[n - d].$$

This defines the quadratic form q' on $\mathsf{Map}(Y, X)$. When q is non-degenerate, then q' is also non-degenerate, since η is an orientation. $\qquad\square$

Corollary 4.2 *If (X, q) is an n-shifted quadratic derived stack, its derived loop space $\mathsf{L}X := MAP_{\mathsf{dSt}_k}(S^1, X)$ has an induced derived $(n-1)$-shifted quadratic form.*

Proof. This follows from the previous Theorem, since for any compact, oriented d- dimensional manifold M, the constant derived stack with value its singular simplicial set $\mathrm{Sing}(M)$ is canonically a $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation η of degree d induced from the fundamental class.
Another proof can be given using Theorem 4.3 by noticing that

$$\mathsf{L}X \simeq X \times_{X \times X} X$$

and endowing $X \times X$ with the quadratic form $(q, -q)$. We leave the details of this alternative proof to the interested reader. $\qquad\square$

Theorem 4.3 *Let (X, q) be an n-shifted quadratic derived stack, and $(f_i : Y_i \to X, \gamma_i)$, $i = 1, 2$ two null-structures relative to (X, q). Then, the homotopy fiber product $Y_1 \times^h_X Y_2$ admits a canonical $(n-1)$-shifted derived quadratic form. If moreover, the null structures $(f_i : Y_i \to X, \gamma_i)$, $i = 1, 2$ are non-degenerate, and (X, q) is non-degenerate, so is $Y_1 \times^h_X Y_2$ with this induced $(n-1)$-shifted derived quadratic form.*

Proof. The proof is analogous to that of [PTVV, Thm. 2.9]. To ease notations we will write, for the duration of the proof, g^* instead of g^q (Def. 3.8), for an arbitrary map g in dSt_k.
Let $Z := Y_1 \times^h_X Y_2$. By definition of homotopy fiber product, the two morphisms

$$p_1 : Z \longrightarrow Y_1 \xrightarrow{\ f_1\ } X \qquad p_2 : Z \longrightarrow Y_2 \xrightarrow{\ f_2\ } X$$

come equipped with a natural path u between them. Now, u gives rise to a path between the induced morphisms on the spaces of derived n-shifted quadratic forms

$$u^* : p_1^* \rightsquigarrow p_2^* : \mathbf{QF}(X; n) \longrightarrow \mathbf{QF}(Z; n).$$

Moreover, γ_i defines a path in the space $\mathbf{QF}(Z; n)$ between 0 and $p_i^* q$, for $i = 1, 2$. By composing γ_1, $u^*(q)$ and γ_2^{-1}, we get a loop at 0 in the space $\mathbf{QF}(Z; n)$, thus a well defined element

$$Q = Q(q, \gamma_1, \gamma_2) \in \pi_1(\mathbf{QF}(Z; n); 0) \simeq \pi_0(\mathbf{QF}(Z; n-1)) = \mathrm{QF}(Z; n-1).$$

Let us now suppose that (X, q) is non-degenerate and so are the null-structures. We will prove that $Q = Q(q, \gamma_1, \gamma_2)$ is also non-degenerate. Let $\Theta_{\gamma_i} : \mathbb{T}_{f_i} \longrightarrow \mathbb{L}_{Y_i}[n-1]$, $i = 1, 2$ be the induced maps (see Def. 3.11), and $pr_i : Z \longrightarrow X_i$, $i = 1, 2$ the two projections. We have a morphism of exact triangles in $\mathsf{L}_{\mathsf{QCoh}}(Z)$

$$
\begin{array}{ccccc}
\mathbb{T}_Z & \longrightarrow & pr_1^*(\mathbb{T}_{Y_1}) \oplus pr_2^*(\mathbb{T}_{Y_2}) & \longrightarrow & p_1^*(\mathbb{T}_X) \\
\Big\downarrow {\scriptstyle Q^\flat} & & \Big\downarrow {\scriptstyle pr_1^*\Theta_{\gamma_1} \oplus pr_2^*\Theta_{\gamma_2}} & & \Big\downarrow {\scriptstyle p_1^*(q^\flat)} \\
\mathbb{L}_Z[n-1] & \longrightarrow & pr_1^*(\mathbb{L}_{f_1})[n-1] \oplus pr_2^*(\mathbb{L}_{f_2})[n-1] & \longrightarrow & p_1^*(\mathbb{L}_X[n]).
\end{array}
$$

Now, the morphism $p_1^*(q^\flat)$ is a quasi-isomorphism since q is non-degenerate, and the morphism $pr_1^*\Theta_{\gamma_1} \oplus pr_2^*\Theta_{\gamma_2}$ is a quasi-isomorphism because the two null-structures are non-degenerate. This implies that $Q^\flat$ is a quasi-isomorphism too, and thus that $Q = Q(q, \gamma_1, \gamma_2) \in \mathrm{QF}^{\mathrm{nd}}(Z; n-1)$.

$\square$

5 Derived Clifford algebra of derived quadratic complexes and stacks

This section is rather brief and sketchy, since we mainly observe that the main definitions and results of Section 2.1 go through over a base derived Artin stack X. More details will probably be added in the next version of this paper.

All derived stacks even when not explicitly stated will be Artin and locally of finite presentation over k.

Let X be a derived Artin stack locally finitely presented over k. We denote by Alg_X (respectively, $\mathsf{Alg}_X^{\mathrm{perf}}$) the ∞-category of associative algebra objects in the symmetric monoidal ∞- category $(\mathsf{L}_{\mathsf{QCoh}}(X), \otimes_{O_X})$ (respectively, $(\mathsf{L}_{\mathsf{Perf}}(X), \otimes_{O_X})$). Its objects will be simply called *Algebras* (respectively, *perfect Algebras*) over X. For any ∞-category T, and any pair (x, y) of objects in T, we denote by $\mathrm{Map}_{\mathsf{T}}(x, y)$ the corresponding mapping space. Note that, up to isomorphisms in $\mathrm{Ho}(\mathsf{SSets})$, this is compatible withe notations used before when T is the Dwyer-Kan localization of a model category with respect to weak equivalences.

Exactly like done in the case of X being the derived spectrum of a simplicial commutative k-algebra, for *evenly* shifted derived quadratic complexes over X, it is possible to define a derived Clifford Algebra. We will sketch briefly definitions and results, leaving to the reader the necessary changes with respect to the derived affine case.

Let $n \in \mathbb{Z}$, and (E, q) be derived $2n$-shifted quadratic complex over X. The derived Clifford algebra functor associated to (E, q) is defined by

$$\underline{\mathbf{Cliff}}_X(E, q, 2n) : \mathsf{Alg}_X \longrightarrow \mathsf{SSets} \; : \; B \longmapsto \underline{\mathbf{Cliff}}(E, q, 2n)(B)$$

where $\underline{\mathbf{Cliff}}(E, q, 2n)(B)$ is defined by the following homotopy pull back in SSets

$$
\begin{array}{ccc}
\underline{\mathbf{Cliff}}(E, q, 2n)(B) & \longrightarrow & \mathrm{Map}_{\mathsf{L}_{\mathsf{QCoh}}(X)}(E, B[n]) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle s_B} \\
* & \xrightarrow{\;\; \widetilde{q}_B \;\;} & \mathrm{Map}_{\mathsf{L}_{\mathsf{QCoh}}(X)}(E \otimes_A E, B[2n])
\end{array}
$$

where the maps s_B and q_B are defined analogously as in the case where $X = \mathbb{R}\mathsf{Spec}(A)$, for $A \in \mathsf{salg}_k$.

Proposition 5.1 *The functor* $\underline{\mathbf{Cliff}}(E, q, 2n)$ *is co-representable, i.e. there exists a well defined* $\mathbf{Cliff}_A(E, q, 2n) \in \mathrm{Ho}(\mathsf{Alg}_X)$ *and a canonical isomorphism in* $\mathrm{Ho}(\mathsf{SSets})$

$$\underline{\mathbf{Cliff}}(E, q, 2n)(B) \simeq \mathrm{Map}_{\mathsf{Alg}_X}(\mathbf{Cliff}_A(E, q, 2n), B).$$

Proof. The proof is analogous to the one of Proposition 5.1. $\qquad\square$

Definition 5.2 *The Algebra* $\mathbf{Cliff}_X(E, q; 2n)$*, defined up to isomorphism in* Alg_X*, is called the* derived Clifford Algebra *of the derived $2n$-shifted quadratic complex (E, q). When $E = \mathbb{T}_X$, we will write*

$$\mathbf{Cliff}(X, q; 2n) := \mathbf{Cliff}_X(\mathbb{T}_X, q; 2n).$$

When the base derived stack X is clear from the context, we will simply write $\mathbf{Cliff}(E, q, 2n)$ for $\mathbf{Cliff}_X(E, q, 2n)$.

Proposition 5.3 *Let X be a derived Artin stack, lfp over k, $n \in \mathbb{Z}$, $f : E_1 \to E_2$ be a map in* $\mathsf{L}_{\mathsf{QCoh}}(X)$*, and q_2 a derived $2n$-shifted quadratic form on E_2 over X. Then there is a canonical map in* $\mathrm{Ho}(\mathsf{dga})$

$$\mathbf{Cliff}_X(E_1, f^*q_2, 2n) \longmapsto \mathbf{Cliff}_X(E_2, q_2, 2n)$$

where f^*q_2 is the pull-back quadratic form of Definition 3.4.

Proof. Same proof as for Proposition 2.5. $\qquad\square$

Note that, by composition with the natural adjunction map in $\mathsf{L}_{\mathsf{QCoh}}(X)$

$$E[-n] \to \mathsf{Free}_{\mathcal{O}_X}(E[-n]),$$

the derived Clifford algebra $\mathbf{Cliff}_X(E, q, 2n)$ of the derived $2n$-shifted quadratic complex (E, q), comes equipped with a natural map in $\mathsf{L}_{\mathsf{QCoh}}(X)$

$$E[-n] \longrightarrow \mathbf{Cliff}_X(E, q, 2n).$$

The base-change of derived quadratic complexes over stacks (along maps of derived stacks), and derived isometric structures on a map of derived quadratic complexes (over a fixed derived stack), induce the following behavior on derived Clifford Algebras.

Proposition 5.4 *Let $n \in \mathbb{Z}$.*

1. *If (E_1, q_1) and (E_2, q_2) are $2n$-shifted derived quadratic complexes over X, $f : E_1 \to E_2$ a map in $\mathsf{L}_{\mathsf{QCoh}}(X)$ and $\gamma \in \pi_0(\mathsf{Isom}(f; (E_1, q_1), (E_2, q_2)))$ is a derived isometric structure on f, then there is an induced map in $\mathsf{L}_{\mathsf{QCoh}}(X)$*

$$f_\gamma : \mathbf{Cliff}_X(E_1, q_1, 2n) \longrightarrow \mathbf{Cliff}_X(E_2, q_2, 2n).$$

   If moreover f is a quasi-isomorphism, f_γ is an isomorphism.

2. *If (E, q) is a $2n$-shifted derived quadratic complex over Y, and $\varphi : X \to Y$ a morphism in dSt_k. Then there is a canonical isomorphism in $\mathsf{L}_{\mathsf{QCoh}}(X)$*

$$\varphi^*(\mathbf{Cliff}_Y(E, q, 2n)) \overset{\sim}{\longrightarrow} \mathbf{Cliff}_X(\varphi^*E, \varphi^*q, 2n) .$$

3. Let $\varphi : X \to Y$ be a map in dSt_k, and q a derived $2n$-shifted quadratic form on Y. By Definition 3.8, $(X, \varphi^q q)$ is a derived $2n$-shifted quadratic stack, and there is a canonical map in Alg_X

$$\mathbf{Cliff}(X, \varphi^q q; 2n) \longrightarrow \varphi^* \mathbf{Cliff}(Y, q; 2n)$$

Proof. (1) and (2) are analogous to the proofs of propositions 2.6 and 2.7, respectively. To establish the map in (3), we first observe that the map $\mathbb{T}_X \to \varphi^* \mathbb{T}_Y$ gives us, by Prop. 5.3, an induced canonical map

$$\mathbf{Cliff}(X, \varphi^q q; 2n) = \mathbf{Cliff}_X(\mathbb{T}_X, \varphi^q q; 2n) \longrightarrow \mathbf{Cliff}_X(\varphi^* \mathbb{T}_Y, \varphi^* q; 2n)$$

and we conclude by point (2) applied to $E = \mathbb{T}_Y$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.5 In the situation, and notations, of Theorem 4.1, with both n and d even, it can be proved that there exists a canonical map in $\mathsf{Alg}_{\mathsf{MAP}_{\mathsf{dSt}_k}(Y,X)}$

$$h_\eta : \mathbf{Cliff}(\mathsf{MAP}_{\mathsf{dSt}_k}(Y, X), q'_\eta; n - d) \longrightarrow \mathrm{pr}_* \mathrm{ev}^* \mathbf{Cliff}(X, q; n)$$

where $\mathrm{pr} : Y \times \mathsf{MAP}_{\mathsf{dSt}_k}(Y, X) \to \mathsf{MAP}_{\mathsf{dSt}_k}(Y, X)$ is the projection, and $\mathrm{ev} : Y \times \mathsf{MAP}_{\mathsf{dSt}_k}(Y, X) \to X$ the evaluation map. It is an open question whether this map is an isomorphism in $\mathsf{Alg}_{\mathsf{MAP}_{\mathsf{dSt}_k}(Y,X)}$.

6 Derived Grothendieck-Witt groups

In this short section we define a derived version of the Grothendieck-Witt group of a derived stack. Applications will be given elsewhere.

Let $A, B \in \mathsf{salg}_k$ and $n \in \mathbb{Z}$. Define the ∞-category $\underline{\mathsf{QC}}(A; n)$ via the following fiber product of ∞-categories

$$
\begin{array}{ccc}
\underline{\mathsf{QC}}(A; n) & \longrightarrow & A - \underline{\mathsf{dgmod}} \ . \\
\downarrow & & \downarrow {\scriptstyle \mathrm{Sym}^2_A} \\
A - \underline{\mathsf{dgmod}} \, / A[n] & \xrightarrow{\ \mathrm{source}\ } & A - \underline{\mathsf{dgmod}}
\end{array}
$$

The ∞-category $\underline{\mathsf{QC}}(A; n)$ is called the *category of quadratic complexes over A*. The full ∞-sub category of non degenerate complexes is denoted by $\underline{\mathsf{QC}}^{\mathrm{nd}}(A; n)$. For any morphism $\varphi : A \to B$ in salg_k, there is an induced base-change ∞-functor

$$\varphi^* : \underline{\mathsf{QC}}(A; n) \longrightarrow \underline{\mathsf{QC}}(B; n),$$

so that we get (as explained in [To-Ve –Traces, §1]) a cofibered ∞-category $\underline{\mathsf{QC}}(n)$ over salg_k, whose associated ∞-functor will be denoted by

$$\underline{\mathbb{QC}}(n) : \mathsf{salg}_k \longrightarrow \infty - \mathsf{Cat}.$$

Since the fiber product of (derived) stacks is a (derived) stack, we have that $\underline{\mathbb{QC}}(n)$ is a derived étale stack with values in ∞-categories, called the *stack of derived n-shifted quadratic complexes*. The *stack of derived n-shifted non-degenerate quadratic complexes* is the sub-stack $\underline{\mathbb{QC}}^{\mathrm{nd}}(n)$ obtained by working with $\underline{\mathsf{QC}}^{\mathrm{nd}}(A; n)$ instead of $\underline{\mathsf{QC}}(A; n)$. By composing with the underlying space (or maximal ∞-groupoid) functor $\mathcal{I}$ ([To-Ve –Traces, §1]), we obtain usual (i.e. with values in the ∞-category $\mathcal{S}$ of simplicial sets) derived stacks

$$\mathbb{QC}(n) := \mathcal{I}(\underline{\mathbb{QC}}(n)) \in \mathsf{dSt}_k,$$

$$\mathbb{QC}^{\mathrm{nd}}(n) := \mathcal{I}(\underline{\mathbb{QC}^{\mathrm{nd}}(n)}) \in \mathsf{dSt}_k.$$

Note that, if X is a derived Artin stack lfp over k, then there is a bijection between k-points of the stack $\mathsf{MAP}_{\mathsf{dSt}_k}(X, \mathbb{QC}(n))$ (respectively, $\mathsf{MAP}_{\mathsf{dSt}_k}(X, \mathbb{QC}^{\mathrm{nd}}(n))$) and (equivalence classes) of derived n-shifted (respectively, non degenerate) quadratic complexes on X.

Definition 6.1 *Let $n \in \mathbb{Z}$, and X be a derived Artin stack lfp over k. The* classifying space of derived n-shifted quadratic complexes *over X is the object in $\mathcal{S}$*

$$\mathsf{QC}(X; n) := \mathrm{Map}_{\mathsf{dSt}_k}(X, \mathbb{QC}(n)).$$

The classifying space of derived n-shifted non degenerate quadratic complexes *over X is the object in $\mathcal{S}$*

$$\mathsf{QC}^{\mathrm{nd}}(X; n) := \mathrm{Map}_{\mathsf{dSt}_k}(X, \mathbb{QC}^{\mathrm{nd}}(n)).$$

The orthogonal sum $\perp$ of derived quadratic complexes over a fixed simplicial algebra (Definition 1.5), induces both on $\pi_0(\mathsf{QC}(X; n))$ and on $\pi_0(\mathsf{QC}^{\mathrm{nd}}(X; n))$ a commutative monoid structure, still denoted by $\perp$.

Definition 6.2 *Let $n \in \mathbb{Z}$, and X be a derived Artin stack lfp over k. The* extended derived Grothendieck-Witt group *of X is the Grothendieck group of the commutative monoid $\pi_0(\mathsf{QC}(X; n))$*

$$\widehat{\mathsf{GW}^{\mathrm{ext}}}(X; n) := K_0(\pi_0(\mathsf{QC}(X; n)), \perp).$$

The derived Grothendieck-Witt group *of X is the Grothendieck group of the commutative monoid $\pi_0(\mathsf{QC}^{\mathrm{nd}}(X; n))$*

$$\widehat{\mathsf{GW}}(X; n) := K_0(\pi_0(\mathsf{QC}(X; n)^{\mathrm{nd}}), \perp).$$

Remark 6.3 Unlike the classical, unshifted case, the derived (extended) Grothendieck-Witt will only be a ring if we consider all even shifts at the same time, the tensor product of an n-shifted quadratic complex with an m-shifted quadratic complex being naturally an $(n+m)$-shifted quadratic complex.

Proposition 6.4 *We have a canonical isomorphism of abelian groups*

$$\widehat{\mathsf{GW}}(k; 0) \simeq \widehat{GW}(k),$$

where $\widehat{GW}(k)$ denotes the classical Grothendieck-Witt group of k (see, e.g. [Mi-Re, §1.8]).

7 Appendix: Superstuff

In this Appendix, we recall a few basic facts about $\mathbb{Z}$- and $\mathbb{Z}/2$-graded dg-modules and dg-algebras, mainly to fix our notations and establish the background for Proposition 2.8 in the main text, where we need to know that the derived Clifford algebra is naturally an object in the homotopy category of $\mathbb{Z}/2$-graded dg-algebras.

Let $A \in \mathsf{cdga}_k^{\leq 0}$, and $A - \mathsf{dgmod}^w \equiv A - \mathsf{dgmod}^{\mathbb{Z}/2-\mathrm{gr}}$ denote the category of $\mathbb{Z}/2$-*weighted A-dg modules*, whose objects are triples $C_*^\bullet = (C; C_0, C_1)$ where $C \in A - \mathsf{dgmod}$, and (C_0, C_1) are sub A-dg modules that provide a $\mathbb{Z}/2$-graded decomposition in $A - \mathsf{dgmod}$ $C = C_0 \oplus C_1$. Elements of C_0 (resp. of C_1) are said to have *weight* 0 (respectively, 1. The morphisms in $A - \mathsf{dgmod}^w$ $(C; C_0, C_1) \to (D; D_0, D_1)$

are morphisms $f : C \to D$ in $A - \mathsf{dgmod}$ preserving the weights: $f(C_i) \subset D_i$, $i = 0, 1$. There is a symmetric monoidal structure on $A - \mathsf{dgmod}^w$, defined by

$$(C; C_0, C_1) \otimes_A^w (C; C_0, C_1) := (C \otimes_A D, (C_0 \otimes_A D_0) \oplus (C_1 \otimes_A D_1), (C_0 \otimes_A D_1) \oplus (C_1 \otimes_A D_0)),$$

with commutativity constraint given, on $(\mathbb{Z} \times \mathbb{Z}/2)$-homogeneous elements, by

$$\sigma_{(C_*^\bullet, D_*^\bullet)} : C_*^\bullet \otimes^w D_*^\bullet \longrightarrow D_*^\bullet \otimes^w C_*^\bullet \; : \; x \otimes y \longmapsto (-1)^{w(x)w(y) + \deg(x)\deg(y)} y \otimes x$$

where $w(-)$ denotes the weight, and $\deg(-)$ the (cohomological) degree. Alternatively, we could have written

$$\sigma_{(C_*^\bullet, D_*^\bullet)}(x_w^i \otimes y_{w'}^j) = (-1)^{ww' + ij} y_{w'}^j \otimes x_w^i,$$

with the standard obvious meaning of the symbols.

There are functors

$$\mathsf{i} : A - \mathsf{dgmod} \longrightarrow A - \mathsf{dgmod}^w \; : \; C \longmapsto (C; C_0 = C, 0) \,,$$

$$\mathsf{forget} : A - \mathsf{dgmod}^w \longrightarrow A - \mathsf{dgmod} \; : \; (C, C_0, C_1) \longmapsto C \,,$$

but observe that the first one is symmetric monoidal, while the second one is not.

The category $A - \mathsf{dgmod}^w$ is endowed with a symmetric monoidal model structure where equivalences and fibrations are morphisms mapped to quasi-isomorphisms and fibrations via the forgetful functor to $A - \mathsf{dgmod}$: the classical proof in the unweighted case goes through. This model structure satisfies the monoidal axiom ([Sch-Shi-1, Def. 3.3]), hence the category $A - \mathsf{dga}^w$ of monoids in $(A - \mathsf{dgmod}^w, \otimes^w)$ have an induced model category structure where equivalences and fibrations are detected on the underlying $\mathbb{Z}/2$-weighted A-dg modules ([Sch-Shi-1, Thm. 4.1]). The objects of $A - \mathsf{dga}^w$ will be called $\mathbb{Z}/2$-*weighted A-dg algebras*. Note that both $A - \mathsf{dgmod}^w$ and $A - \mathsf{dga}^w$ are cofibrantly generated model categories. The forgetful functor

$$\mathsf{Forget} : A - \mathsf{dga}^w \longrightarrow A - \mathsf{dgmod}^w$$

is Quillen right adjoint to the free $\mathbb{Z}/2$-weighted A-dg algebra functor

$$\mathsf{T}_A : A - \mathsf{dgmod}^w \longrightarrow A - \mathsf{dga}^w$$

(given by the tensor algebra construction in $(A - \mathsf{dgmod}^w, \otimes^w)$). Note that T acquires also an additional $\mathbb{Z}$-grading but we will not use it.

Note that moreover, since A is (graded commutative), the category $\mathrm{Ho}(A - \mathsf{dga}^w)$ comes equipped with a (derived) tensor product, defined by

$$(B; B_0, B_1) \otimes_A^w (D; D_0, D_1) := Q(B; B_0, B_1) \otimes_A^w (D; D_0, D_1)$$

where $Q(-)$ denotes a cofibrant replacement functor in $A - \mathsf{dga}^w$, and the algebra product is defined by

$$(x_w^i \otimes y_{w'}^j) \cdot (z_p^h \otimes t_q^k) := (-1)^{w'p} (x_w^i z_p^h) \otimes (y_{w'}^j t_q^k).$$

Completely analogous (notations and) results holds if we start with the category $A - \mathsf{dgmod}^{\mathbb{Z}-\mathrm{gr}}$ of $\mathbb{Z}$-*weighted A-dg modules*, whose objects are pairs $C_*^\bullet = (C; C_w)_{w \in \mathbb{Z}}$ where $C \in A - \mathsf{dgmod}$, and

$(C_w)_{w\in\mathbb{Z}}$ are sub A-dg modules providing a $\mathbb{Z}$-graded decomposition in $A-\mathsf{dgmod}$ $C = \oplus_w C_w$. The symmetric monoidal structure is given by

$$(C; C_w)_{w\in\mathbb{Z}} \otimes_A^w (D; D_{w'})_{w'\in\mathbb{Z}} := (C\otimes_A D, ((C\otimes_A D)_p := \oplus_{w+w'=p} C_w \otimes_A D_{w'})_{p\in\mathbb{Z}})$$

while the commutativity is given by the same formula as in the $\mathbb{Z}/2$-graded case (except that the weights are now in $\mathbb{Z}$). Thus we dispose of a (cofibrantly generated) model category $A-\mathsf{dga}^{\mathbb{Z}-\mathrm{gr}}$ of monoids in the symmetric monoidal (cofibrantly generated) model category $(A-\mathsf{dgmod}^{\mathbb{Z}-\mathrm{gr}}, \otimes^w)$.

Proposition 7.1 *The functor* $\mathsf{Free}_A : A-\mathsf{dgmod} \to A-\mathsf{dga}$ *(used in the main text) naturally factors through the weight-forgetful functor* $A-\mathsf{dga}^{\mathbb{Z}-gr} \longrightarrow A-\mathsf{dga}$ *(that simply forgets the weight-grading). And, if we denote by the same symbol the induced functor*

$$\mathsf{Free}_A : A-\mathsf{dgmod} \to A-\mathsf{dga}^{\mathbb{Z}-gr},$$

this functor sends weak equivalences between cofibrant A-dg modules to weak equivalences.

Proof. Let $C \in A-\mathsf{dgmod}$. By giving to $\mathsf{Free}_A(C)$ the $\mathbb{Z}$-grading

$$(\mathsf{Free}_A(C))_w := \mathsf{T}_A^w(C) = C\otimes_A \cdots \otimes_A C \ \ (w \text{ times})$$

with the convention that $\mathsf{T}_A^w(C) = 0$ for $w < 0$, it is easy to verify that $(\mathsf{Free}_A(C), (\mathsf{Free}_A(C))_w)_{w\in\mathbb{Z}})$ defines an element in $A-\mathsf{dga}^{\mathbb{Z}-\mathrm{gr}}$. The induced functor $\mathsf{Free}_A : A-\mathsf{dgmod} \to A-\mathsf{dga}^{\mathbb{Z}-\mathrm{gr}}$ sends weak equivalences between cofibrant objects to weak equivalences since the left Quillen functor $\mathsf{Free}_A : A-\mathsf{dgmod} \to A-\mathsf{dga}$ does ([Ho, Lemma 1.1.12]). $\qquad\square$

By definition of $\otimes^w$ both in the $\mathbb{Z}/2$- and in the $\mathbb{Z}$-weighted case, the functor

$$A-\mathsf{dgmod}^{\mathbb{Z}-\mathrm{gr}} \longrightarrow A-\mathsf{dgmod}^w$$

given by

$$(C; C_w)_{w\in\mathbb{Z}} \longmapsto (C; C_{\mathrm{even}} := \oplus_w C_{2w}, C_{\mathrm{odd}} := \oplus_w C_{2w+1})$$

is symmetric monoidal, and preserves weak equivalences, hence it induces a weak-equivalences preserving functor between the corresponding monoid objects

$$(-)^w : A-\mathsf{dga}^{\mathbb{Z}-\mathrm{gr}} \longrightarrow A-\mathsf{dga}^w.$$

Thus the composite functor

$$\mathsf{Free}_A^w : A-\mathsf{dgmod} \xrightarrow{\ \mathsf{Free}_A\ } A-\mathsf{dga}^{\mathbb{Z}-\mathrm{gr}} \xrightarrow{\ (-)^w\ } A-\mathsf{dga}^w$$

sends weak equivalences between cofibrant objects to weak equivalences.

Now we come back to explain the statement of Proposition 2.8. Let (C, q) be a derived $2n$-shifted quadratic complex over A, where C is cofibrant in $A-\mathsf{dgmod}$. Note that, then, also any shift of C, and any shift of $C\otimes_A C$ is cofibrant, too. Consider the homotopy push-out square in $A-\mathsf{dga}$

$$
\begin{array}{ccc}
\mathsf{Free}_A(C\otimes_A C[-2n]) & \xrightarrow{\ u\ } & \mathsf{Free}_A(C[-n]) \\
{\scriptstyle t}\downarrow & & \downarrow \\
\mathsf{Free}_A(0) = A & \longrightarrow & \mathbf{Cliff}(C, q; 2n)
\end{array}
$$

26

used in the proof of Proposition 5.1 in order to define the derived Clifford algebra $\mathbf{Cliff}(C, q; 2n)$. By upgrading Free_A to Free_A^w, we observe that the maps u and t are no longer maps in $A - \mathsf{dga}^w$ (they have odd degree) but they both land into the even parts of $\mathsf{Free}_A^w(C[-n])$ and $\mathsf{Free}_A^w(0)$, respectively. This implies that $\mathbf{Cliff}(C, q; 2n)$ admits a $\mathbb{Z}/2$-weight grading

$$(\mathbf{Cliff}(C, q; 2n); \mathbf{Cliff}(C, q; 2n)_0, \mathbf{Cliff}(C, q; 2n))$$

such that the induced maps

$$\mathsf{Free}_A^w(C[-n]) \longrightarrow (\mathbf{Cliff}(C, q; 2n); \mathbf{Cliff}(C, q; 2n)_0, \mathbf{Cliff}(C, q; 2n)),$$

$$\mathsf{Free}_A^w(A) = (A; A, 0) \longrightarrow (\mathbf{Cliff}(C, q; 2n); \mathbf{Cliff}(C, q; 2n)_0, \mathbf{Cliff}(C, q; 2n)),$$

are maps in $\mathrm{Ho}(A - \mathsf{dga}^w)$. In other words the derived Clifford algebra functor

$$(C, q : 2n) \longmapsto \mathbf{Cliff}(C, q; 2n)$$

extend to a functor

$$\mathbf{Cliff}_A^w : \underline{\mathsf{QC}}(A; 2n) \longrightarrow L^{DK}(A - \mathsf{dga}^w),$$

where

- $A - \mathsf{Qdgmod}$ is the ∞-category defined in §6, and

- $L^{DK}(A - \mathsf{dga}^w)$ is the Dwyer-Kan localization of the model category $L^{DK}(A - \mathsf{dga}^w)$ along its weak equivalences.

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