J. reine angew. Math. 523 (2000), 1—54

Journal für die reine und angewandte Mathematik

On the Chow ring of the classifying stack of $PGL_{3,\mathbb{C}}$

By *Gabriele Vezzosi* at Bologna

Abstract. We compute generators for the Chow ring of the classifying space of $PGL_{3,\mathbb{C}}$ as defined by Totaro. We also find enough relations after inverting 3. We show that this ring is not generated by Chern classes (this is the first example of this kind among classical groups) and prove that Totaro's refined cycle class map to a quotient of complex cobordism of $BPGL_{3,\mathbb{C}}$ is surjective.

1. Introduction

Equivariant intersection theory is similar to Borel's equivariant cohomology. The common basic idea is simple. Let X be an algebraic scheme over a field k and let G be an algebraic group acting on X. Since invariant cycles are often too few to get a full-fledged intersection theory (e.g. to have a ring structure in smooth cases) we decide to enlarge this class to include invariant cycles not only on X but on $X \times V$ where V is any linear representation of G. If $k = \mathbb{C}$, equivariant cohomology can be defined along these lines and this definition agrees with the usual one given using the classifying space of G.

In particular, we get a non trivial equivariant intersection theory $A_G^* = A_G^*(\mathrm{pt})$ on $\mathrm{pt} = \operatorname{Spec} k$ which can be interpreted naturally as an intersection theory on the classifying stack of the group in the same way as equivariant cohomology of a point is naturally viewed as cohomology of the classifying space of the group.

Equivariant intersection theory (in the sense sketched above) was first defined by Totaro in [25] for $X = \operatorname{Spec} k$ and then extended to general X by Edidin and Graham in [4]. Totaro himself ([25]) and Pandharipande ([18], [19]) computed A_G^* in many interesting cases, for example $G = \mathrm{GL}_n$, SL_n (these two cases are trivial), $\mathrm{O}(n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(4)$.

Moreover, Totaro ([24], [25]) was able to define a remarkable refining of the classical cycle map from the Chow ring to the cohomology ring. In particular, he proved that for any complex algebraic group G, the equivariant version of the cycle class map

$$\mathrm{cl}_G : A_G^* \to H^*(\mathrm{B}G, \mathbb{Z}),$$

factors as

$$A_G^* \xrightarrow{\widetilde{\mathrm{cl}}_G} MU^*(BG) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^*(BG, \mathbb{Z})$$

where $MU^*(BG)$ is the complex cobordism of the classifying space of G,

$$MU^*(\mathrm{pt}) \equiv MU^* = \mathbb{Z}[x_1, x_2, x_3, \dots]$$

where $\deg x_i = -2i$ and $\mathbb{Z}$ is viewed as an MU^*-module via the map sending each x_i to zero. He conjectured that, if $MU^*(BG) \otimes_{MU^*} \mathbb{Z}$ is concentrated in even degrees, then $\widetilde{\mathrm{cl}}_G$ is an isomorphism.

The case $G = \mathrm{PGL}_n$ is of particular interest. One reason is its connection with Brauer-Severi varieties, whose Chow groups are quite mysterious (see [11] and [12] for some results on codimension 2 cycles). Also, many parameter spaces of interest are quotients of free actions of PGL_n, so the calculation of $A^*_{\mathrm{PGL}_n}$ would be a necessary first step to determine the Chow ring of some of these spaces.

Unfortunately, the ring $A^*_{\mathrm{PGL}_n}$ for general n seems extremely difficult to compute. It is a general principle that among all families of classical groups the series PGL_n is often the hardest to study. Thus, for example, while the cohomology and the complex cobordism ring of most classical groups have been determined, very little is known about the torsion part in the cohomology of the classifying space of PGL_n for $n \geqq 4$. Of course, given how much harder than cohomology the Chow ring usually is, this is not encouraging. On the other hand, the cohomology with $\mathbb{Z}/3$ coefficients of the classifying space of PGL_3, as well as its Brown-Peterson cohomology (relative to the prime 3) have been computed by Kono, Mimura and Shimada ([13]) and by Kono and Yagita ([14]).

The ring $A^*_{\mathrm{PGL}_2}$ was first computed by Pandharipande ([18]) through the isomorphism $\mathrm{PGL}_2 \simeq \mathrm{SO}(3)$. Pandharipande's method does not seem to extend to PGL_3.

In this paper we study $A^*_{\mathrm{PGL}_3}$. Our approach is completely different. The idea is that the adjoint representation sl_n of PGL_n can be stratified, using Jordan canonical form, in such a way that the equivariant Chow ring of each stratum is amenable to study. This determines completely $A^*_{\mathrm{PGL}_2}$ ([27]) and works fairly well for $n = 3$ yielding generators of $A^*_{\mathrm{PGL}_3}$. In principle this method could give generators for $A^*_{\mathrm{PGL}_n}$ for any n, but the calculations become extremely involved as n grows. Moreover, as usual, the stratification method is not very good for finding the relations. In the case $n = 3$, using also a recent general result by Totaro (Th. 2.1), we find some of the relations in section 5, but unfortunately we are not able to prove that our relations are sufficient.

We also prove some properties of the cycle map and of Totaro's refined cycle map. In particular, we are able to prove that $A^*_{\mathrm{PGL}_3}$, unlike $A^*_{\mathrm{PGL}_2}$, is not generated by Chern classes of representations, a result conjectured by Totaro in [25]. We have two proofs of this fact, one (Th. 4.2), relying on results of [25], [13], [14], carries more informations on the cycle and refined cycle maps while the other (Appendix) is self-contained not depending on cohomological arguments.

Most of the results in this paper constitute the core of [27].

Now we state the main results of this work in greater detail.

It is already clear "rationally", that Chern classes of the adjoint representation alone do not generate $A^*_{\mathrm{PGL}_3}$. So, if E is the standard representation of GL_3, we also consider $\mathrm{Sym}^3 E$, the PGL_3-representation defined by

$$[g] \cdot (v_1 \cdot v_2 \cdot v_3) \doteq \det g^{-1}(gv_1 \cdot gv_2 \cdot gv_3).$$

We prove the following (Theorem 4.6):

Theorem 1.1. *There exist elements* ρ *and* χ *with* $\deg \rho = 4$ *and* $\deg \chi = 6$, *such that* $A^*_{\mathrm{PGL}_3}$ *is generated by*

$$\{\lambda \doteq 2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), c_3(\mathrm{Sym}^3 E), \rho, \chi, c_6(\mathrm{sl}_3), c_8(\mathrm{sl}_3)\}.$$

The question of determining all the relations between this generators is hard. In this direction, we can prove the following (Th. 5.1):

Proposition 1.2. *The generators above satisfy*

$$3\rho = 3\chi = 3c_8(\mathrm{sl}_3) = 0,$$

$$\rho^2 = c_8(\mathrm{sl}_3),$$

$$3(27c_6(\mathrm{sl}_3) - c_3(\mathrm{Sym}^3 E)^2 - 4\lambda^3) = 0.$$

Moreover, if R^* denotes the graded ring

$$\frac{\mathbb{Z}[\lambda, c_3(\mathrm{Sym}^3 E), \rho, \chi, c_6(\mathrm{sl}_3), c_8(\mathrm{sl}_3)]}{\mathfrak{R}}$$

where $\mathfrak{R}$ is the ideal generated by the relations in Prop. 1.2 and $\deg \rho = 4$, $\deg \chi = 6$, we have (Theorem 5.3)

Theorem 1.3. *The composition*

$$R^* \longrightarrow A^*_{\mathrm{PGL}_3} \xrightarrow{\widetilde{\mathrm{cl}}} MU^*(B\mathrm{PGL}_3) \otimes_{MU^*} \mathbb{Z}$$

is surjective and its kernel is 3-torsion.

Note that this also proves that $R^*\left[\frac{1}{3}\right] \simeq A^*_{\mathrm{PGL}_3}\left[\frac{1}{3}\right]$. We also prove that while ρ is nonzero in cohomology, χ is zero in cohomology. Thus, by Remark 5.2, we also have $\widetilde{\mathrm{cl}}(\chi) = 0$. Note that if one was able to prove that $\chi \neq 0$ then Totaro's conjecture would be false. However, despite many efforts, we still do not know whether χ is zero or not.

By a result of Kono and Yagita ([13]), Totaro's conjecture predicts that $\widetilde{\mathrm{cl}}$ is actually an isomorphism. We are able to show that the generator ρ of Theorem 1.1 is not in the

Chern subring[1)] of $A^*_{PGL_3}$, thus proving the following consequence of Totaro's conjecture (Theorem 4.2):

Theorem 1.4. *$A^*_{PGL_3}$ is not generated by Chern classes.*

This same result is proved in the Appendix without using cohomology computations.

Conventions and notations. The word "scheme" will most of the time mean "algebraic scheme over a field k". In section 1, where we try to give some of the results in greater generality, we will allow a different base scheme S and the finiteness conditions needed will be properly specified.

We freely use the functorial point of view for schemes and group schemes (e.g. [2]) to be able to express maps, actions etc. as sending "elements to elements".

If s is a section of a vector bundle, we denote by $Z(s)$ its zero scheme.

Algebraic groups over a field k will always be linear. If G is an algebraic group over a field k, T_G (or simply T if no confusion is possible) denotes a maximal torus of G and $\widehat{T_G}$ its character group.

If $\varphi : G \to H$ is a morphism of algebraic groups over a field k and V is a representation of H, we denote by $V_{(\varphi)}$ or $V_{(G)}$ the obvious associated G-representation.

If E denotes the standard GL_3-representation, $\mathrm{Sym}^3 E$ becomes a PGL_3-representation via

$$[g] \cdot (v_1 \cdot v_2 \cdot v_3) \doteq \det g^{-1}(gv_1 \cdot gv_2 \cdot gv_3).$$

Acknowledgments. I wish to thank my thesis advisor Angelo Vistoli for his friendship, patience and constant attention to this work.

I also wish to thank Burt Totaro for generous advice and for many illuminating discussions we had in Cambridge. Among other things, he also explained to me the argument in Remark 5.2 and allowed me to include his still unpublished result Theorem 2.1. I hope this paper shows all its debts to his deep and original work.

2. Basic notations and results

In this section we mainly fix notations and collect some miscellaneous results on equivariant Chow groups we will need in the sequel; most of them (with one possible exception) are elementary or well known but we simply could not find proper references in the literature. For intersection theory the standard reference is [6] while for equivariant intersection theory we refer to [4] and [25].

1) I.e. in the subring generated by Chern classes of representations.

2.1. Equivariant intersection theory and Totaro's refined cycle map. Let G be an algebraic group over a field k and X a smooth[2)] G-scheme. Edidin and Graham ([4]), following an idea of Totaro ([25]), defined a G-equivariant version, $A^*_G(X)$, of the Chow ring $A^*(X)$. We will simply write A^*_G for $A^*_G(\mathrm{Spec}\, k)$. As a rule, if we do not mention explicitly the base field k, we are assuming $k = \mathbb{C}$.

We say that a pair (U, V), consisting of a k-representation V of G and an open subset U of V on which G acts freely, is a good pair (or simply a pair) relative to G if the codimension of $V \backslash U$ has sufficiently high codimension (see [4], 2.2, Definition-Proposition 1).

All the basic properties and constructions (Chern classes, localization sequence, proper pushforwards, Gysin maps, vector and projective bundle theorems, projection formula, self intersection formula, cycle class map, operational Chow groups etc.) of ordinary intersection theory ([6]) have their equivariant counterparts. Moreover, there are additional constructions one can do in the equivariant setting which simply do not exist in the ordinary case, for example those related to morphisms of algebraic groups. If

$$\varphi : G \to G'$$

is a morphism of algebraic groups and X a G'-scheme (which we suppose smooth just in order to state each result for Chow rings), then X is a G-scheme via φ and if (U, V) (respectively, (U', V')) is a good pair relative to G (resp., relative to G'), we let G act on $V \times V'$ as

$$g \cdot (v, v') = (g \cdot v, \pi(g) \cdot v'), \quad g \in G, v \in V, v' \in V'$$

and the projection

$$X \times U \times U' \to X \times U'$$

induces a flat map

$$(X \times U \times U')/G \to (X \times U')/G'.$$

Its pullback induces a restriction ring morphism

$$A^*_{G'}(X) \to A^*_G(X)$$

denoted by φ^*_X (or by $\mathrm{res}^G_{G',X}$ if φ is injective). Note that the same construction made in the topological case, defines the functoriality in G of the equivariant cohomology ring $H^*_G(X; \mathbb{Z})$.

Another construction which appears only in the equivariant setting is the following transfer construction for Chow groups; we will frequently use it. Let

$$1 \to H \xrightarrow{\phi} G \to F \to 1$$

2) We restrict our attention to smooth schemes for simplicity.

be an exact sequence of algebraic groups over a field k, with F finite. If X is an algebraic smooth G-scheme then $p_1 : X \times F \to X$ is proper G-equivariant and there is an equivariant push-forward

$$p_{1*} : A_*^G(X \times F) \to A_*^G(X).$$

If (U, V) is a good pair for G, we have:

$$\text{(1)} \qquad \frac{(X \times F) \times U}{G} \simeq \left(\frac{(X \times F) \times U}{H}\right) \Big/ F \simeq \left(\frac{X \times U}{H} \times F\right) \Big/ F \simeq \frac{X \times U}{H};$$

hence $A_G^*(X \times F) \simeq A_H^*(X)$ and p_{1*} induces a *transfer* morphism of graded groups

$$\text{(2)} \qquad \mathrm{tsf}_{H,X}^G : A_H^*(X) \to A_G^*(X),$$

which is natural in X with respect to pullbacks.

Observe that the pullback $A_G^*(X) \to A_H^*(X)$ has actually values in the F-invariant subring of $A_H^*(X)$ ([25])

$$\mathrm{res}_{G,X}^H : A_G^*(X) \to \left(A_H^*(X)\right)^F.$$

In exactly the same way as for group cohomology (e.g. [1], Prop. 9.5), we have

$$\mathrm{tsf}_{H,X}^G \circ \mathrm{res}_{G,X}^H = (\# F)$$

(by projection formula) and, since H is normal in G,

$$\text{(3)} \qquad (\mathrm{res}_{G,X}^H \circ \mathrm{tsf}_{H,X}^G)_{|(A_H^*(X))^F} = (\# F).$$

If we do not restrict to $\left(A_H^*(X)\right)^F$, we get

$$\text{(4)} \qquad \mathrm{res}_{G,X}^H \circ \mathrm{tsf}_{H,X}^G(\xi) = \sum_{f \in F} f_* \xi$$

for any ξ in $A_H^*(X)$.

Remark 2.1. For a general action of G on X, the quotient $[X/G]$ exists only as an Artin stack[3] ([15]). Edidin and Graham ([4], 5.3, Prop. 16, 17) showed that if $\mathscr{F}$ is a quotient Artin stack $\mathscr{F} \simeq [X/G]$, then the corresponding equivariant Chow groups do not depend on the presentation chosen for the quotient, enabling one to *define* $A_G^*(X)$ to be the (integral) Chow group of the stack $\mathscr{F}$. If moreover $\mathscr{F}$ is smooth, there is a ring structure on this Chow group and this applies to the classifying stack $\mathscr{B}G$ of any algebraic group G

[3] Not necessarily separated.

([15]), viewed as the quotient $[\mathrm{pt}/G]$,

$$A_G^* = A^*(\mathscr{B}G).$$

Theorem 2.1 (Gottlieb; Totaro). *Let G be an algebraic group over $\mathbb{C}$, T a maximal torus of G and $\mathrm{N}_G(T)$ its normalizer in G. The restriction maps*

$$\text{(5)} \qquad A_G^* \to A_{\mathrm{N}_G(T)}^*,$$

$$\text{(6)} \qquad H^*(\mathrm{B}G, \mathbb{Z}) \to H^*(\mathrm{BN}_G(T), \mathbb{Z})$$

are injective.

Proof. (6) is proved in [7]. We sketch the proof of (5) from [26]. If $f : Y \to B$ is a smooth proper morphism of relative dimension r between smooth, separated schemes of finite type over k, let us consider the following "modified" pushforward

$$f_\#(\alpha) \doteq f_* \circ (c_r(\mathscr{T}_f) \cdot \alpha) \in A^j(S)$$

for any $\alpha \in A^j(B)$, where $\mathscr{T}_f$ denotes the relative tangent bundle; by projection formula, we have

$$\text{(7)} \qquad f_\# \circ f^* = \chi(F)$$

where $\chi(F)$ denotes the Euler characteristic of "the fiber" of f (equal to the degree of the top Chern class of its tangent bundle). Now, let $g : X \to B$ be a smooth morphism between smooth schemes over a field k which admits a smooth relative compactification

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ & \searrow & \downarrow \\ & & B \end{array}$$

having divisors with normal crossing $\{D_i\}_{i=1,\ldots,n}$ as complement (smooth over B). If

$$D_{ij} \doteq D_i \cap D_j,$$

$D_{ijk} \doteq D_i \cap D_j \cap D_k$, etc., the previous construction yields modified pushforwards

$$f_\# : A^*(\overline{X}) \to A^*(B), \quad f_\#^{(1)} : \bigoplus_i A^*(D_i) \to A^*(B), \quad f_\#^{(2)} : \bigoplus_{i<j} A^*(D_{ij}) \to A^*(B), \ldots$$

satisfying (7). If $x \in A^*(X)$, lift it to some $\bar{x} \in A^*(\overline{X})$ and set

$$g_\#(x) \doteq f_\#(\bar{x}) - \sum_i f_\#^{(1)}(\bar{x}_{|D_i}) + \sum_{i<j} f_\#^{(2)}(\bar{x}_{|D_{ij}}) - \cdots$$

(alternating sum) which is an element in $A^*(B)$. This can be shown to be independent on the choice of the lifting and (7) holds for g by well-known properties of the Euler characteristic.

To prove (5), apply this construction to any approximation of the $G/\mathrm{N}_G(T)$-torsor

$$\mathrm{BN}_G(T) \to BG$$

recalling that $\chi(G/\mathrm{N}_G(T)) = 1$. Note that this proof works over any algebraically closed field k. □

In [25], Totaro proved the remarkable fact that, if G is a complex algebraic group, the cycle map

$$\mathrm{cl}_{BG} : A_G^* \to H^*(BG, \mathbb{Z})$$

factors as

$$(8) \qquad A_{BG}^* \xrightarrow{\widetilde{\mathrm{cl}}_{BG}} MU^*(BG) \otimes_{MU^*} \mathbb{Z} \xrightarrow{\underline{\mathrm{cl}}_{BG}} H^*(BG, \mathbb{Z}),$$

where $MU^*(BG)$ is the complex cobordism ring of BG ([23]) and $\underline{\mathrm{cl}}_{BG}$ is the natural morphism (since

$$MU^* \equiv MU^*(\mathrm{pt}) = \mathbb{Z}[x_1, x_2, \dots, x_n, \dots]$$

with $\deg x_i = -2i$, here $\mathbb{Z}$ is viewed as an MU^*-module via the map sending each generator x_i to zero). We call $\widetilde{\mathrm{cl}}_{BG}$ *Totaro's refined cycle map* for G. The kernel and cokernel of $\underline{\mathrm{cl}}_{BG}$, $\widetilde{\mathrm{cl}}_{BG}$ and cl_{BG} are torsion.

In [25], Totaro conjectures that if G is a complex algebraic group such that $MU^*(BG)$, localized at some prime p, is concentrated in even degrees, then the p-localization of $\widetilde{\mathrm{cl}}_{BG}$ should be an isomorphism. As a consequence of this conjecture, $A_{\mathrm{PGL}_3}^*$ should not be generated by Chern classes since, by [14], $MU^*(\mathrm{BPGL}_3)$ is concentrated in even degrees but not generated by Chern classes. This consequence of Totaro's conjecture will be proved in section 4 (see also the Appendix for a different proof).

Remark 2.2. Voevodsky ([28], [29]) defined an *algebraic cobordism* for an algebraic scheme over an *arbitrary* field k, so it would be interesting to investigate if there exists a generalization of Totaro's refined cycle map with values in (a quotient of) algebraic cobordism, for any algebraic group G over k. As M. Levine suggested to me, one may also ask more generally if Totaro's refined cycle map extends to a map from the entire motivic cohomology to algebraic cobordism.

2.2. Miscellaneous results.

Proposition 2.2. *Let k be algebraically closed. The pullback*

$$A_{\mathrm{PGL}_{n,k}}^* \otimes \mathbb{Q} \to A_{\mathrm{SL}_{n,k}}^* \otimes \mathbb{Q}$$

is an isomorphism.

Proof. By [5], Th. 1 (c),

$$A_G^* \otimes \mathbb{Q} \simeq \mathrm{Sym}_{\mathbb{Z}}(\hat{T})^W \otimes \mathbb{Q} = (A_T^*)^W \otimes \mathbb{Q}$$

for any connected reductive algebraic group G with maximal torus T and Weyl group W and $\mathrm{Sym}_{\mathbb{Z}}(\hat{T})^W \otimes \mathbb{Q}$ is the same for a group G and a quotient of G by a finite central subgroup. □

Remark 2.3. Let S be a locally noetherian base scheme. Since $\mathrm{Aut}(\mathbb{P}_S^n) \simeq \mathrm{PGL}_{n+1,S}$ as group-functors, for any S ([2] or [17], p. 20–21), the category of Brauer-Severi schemes ([16], p. 134) of relative dimension n over X for the étale (or fppf) topology is equivalent to that of PGL_{n+1}-torsors over X for the same topology and this equivalence actually extends to a 1-isomorphism of $\mathcal{BS}_{n,S}$ with the classifying stack $\mathcal{B}(\mathrm{PGL}_{n+1,S})$, where $\mathcal{BS}_{n,S}$ denotes the stack over S whose fibre category over X/S is the category of Brauer-Severi schemes of relative dimension n over X. Under this 1-isomorphism trivial[4)] Brauer-Severi schemes correspond to PGL_{n+1}-torsors induced by GL_{n+1}-torsors.

Proposition 2.3. *Let k be algebraically closed. Then $\ker(A_{\mathrm{PGL}_{n,k}}^* \to A_{\mathrm{SL}_{n,k}}^*)$ is n-torsion.*

Proof. By Prop. 2.2, our kernel is torsion and so it is enough to prove that

$$\ker(p^* : A_{\mathrm{PGL}_{n,k}}^* \to A_{\mathrm{GL}_{n,k}}^*)$$

is annihilated by n, $A_{\mathrm{GL}_{n,k}}^*$ being torsion free.

By [25], Th. 1.3 or [5], Th. 1, for any reductive algebraic group G, A_G^* can be identified with the ring $\mathscr{C}_G^*$ of characteristic classes for (étale) G-torsors over smooth, separated schemes of finite type over k. Via this identification p^* translates to

$$p^* : \mathscr{C}_{\mathrm{PGL}_{n,k}}^* \to \mathscr{C}_{\mathrm{GL}_{n,k}}^*,$$

$$F \mapsto p^*(F) : \begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix} \mapsto F\begin{pmatrix} P_E \\ \downarrow \\ X \end{pmatrix}$$

where $P_E \to X$ is the $\mathrm{PGL}_{n,k}$-torsor associated to $\mathbb{P}(\check{E}) \to X$, $\check{E} \to X$ being the vector bundle associated to the $\mathrm{GL}_{n,k}$-torsor $E \to X$ and slightly abusing notation in the argument of F

$$p^*F = 0 \Leftrightarrow F(\mathbb{P}(\check{E}) \to X) = 0, \quad \forall E \to X \text{ vector bundle of rk } n.$$

Now we use the 1-isomorphism of stacks $\mathcal{B}(\mathrm{PGL}_{n,k}) \simeq \mathcal{BS}_{n-1,k}$ (Remark 2.3). If

$$f : P \to X$$

is a $\mathrm{PGL}_{n,k}$-torsor and $\bar{f} : \bar{P} \to X$ the associated Brauer-Severi scheme, the base change of f via $\bar{f}$ is a $\mathrm{PGL}_{n,k}$-torsor induced by a $\mathrm{GL}_{n,k}$-torsor. Since $\chi(P_k^{n-1}) = n$, formula (7) in the proof of Theorem 2.1 yields

4) I.e. of the form $P(E) \to X$ for some vector bundle E over X.

$$nF\begin{pmatrix}P\\ \downarrow\\ X\end{pmatrix}=\bar{f}_{*}\bar{f}^{*}F\begin{pmatrix}P\\ \downarrow\\ X\end{pmatrix}=\bar{f}_{*}F\left(\bar{f}^{*}\begin{pmatrix}P\\ \downarrow\\ X\end{pmatrix}\right)=0$$

(by projection formula) if $p^*F=0$. □

Corollary 2.4. *$A^*_{\mathrm{PGL}_{n,k}}$ has only n-torsion.*

We conclude this section collecting some elementary results on equivariant Chow groups we will use in the sequel.

Proposition 2.5. $A^*_{\mu_{n,k}}\simeq\mathbb{Z}[t]/(nt)$.

Proof. From Kummer exact sequence

$$1\longrightarrow\mu_{n,k}\longrightarrow\mathbb{G}_{m,k}\xrightarrow{(\ )^n}\mathbb{G}_{m,k}\longrightarrow 1,$$

for any $N>0$ we get a $\mathbb{G}_{m,k}$-torsor

$$\frac{\mathbb{A}_k^{N+1}\backslash\{0\}}{\mu_{n,k}}\to\frac{\mathbb{A}_k^{N+1}\backslash\{0\}}{\mathbb{G}_{m,k}}=\mathbb{P}_k^N$$

whose associated line bundle is just $\mathcal{O}_{\mathbb{P}_k^N}(-n)$. By [8], Remark p. 4–35, we get

$$A^*\left(\frac{\mathbb{A}_k^{N+1}\backslash\{0\}}{\mu_{n,k}}\right)\simeq\frac{A^*(\mathbb{P}_k^N)}{(c_1(\mathcal{O}_{\mathbb{P}_k^N}(-n)))}$$

which implies the assert for $N\gg 0$. □

Proposition 2.6. *If G is a unipotent algebraic group over a field k of characteristic zero, then $A^*_G\simeq\mathbb{Z}$.*

Proof. Since G is unipotent it has a central composition series

$$G=G_0\rhd G_1\rhd G_2\rhd\cdots\rhd G_n=1$$

such that $G_i/G_{i+1}\simeq\mathbb{G}_{a,k}$ ([2], IV, §2, 2.5 (vii)). We proceed by induction on the length n of the composition series.

If $n=1$, $G\simeq\mathbb{G}_{a,k}$; if U is a G-free open subset of a G-representation such that $\pi:U\to U/G$ is a (fppf or étale) G-torsor then π is a Zariski G-torsor ($\mathbb{G}_{a,k}$ being special, [21]) and in particular a Zariski affine bundle with fiber $\mathbb{A}_k^1$ so that π^* is an isomorphism ([8], p. 4–35).

Suppose the assert true for any unipotent group whose central composition series has length $\leqq n$. If G is unipotent with a central decomposition series

$$G=G_0\rhd G_1\rhd G_2\rhd\cdots\rhd G_{n+1}=1$$

then G_1 is unipotent ([2], IV, §2, 2.3) and we have a short exact sequence

$$1\to G_1\to G\to G/G_1\simeq\mathbb{G}_{a,k}\to 1.$$

Therefore, if U is a G-free open subset of a G-representation which has a G-torsor quotient $U\to U/G$,

$$U/G_1\to U/G$$

is a $G/G_1\simeq\mathbb{G}_{a,k}$-torsor. As in case $n=1$, the pullback is an isomorphism $A^*_G\simeq A^*_{G_1}$ and we conclude since G_1 has a central decomposition series of length n. □

Proposition 2.7. *Let*

$$(9)\qquad 1\to H\to G\underset{\sigma}{\overset{\rho}{\rightleftarrows}}\mathbb{G}_m\to 1$$

be a split exact sequence of algebraic groups over a field k of characteristic zero, with H unipotent. Then the pullback

$$\rho^*:A^*_{\mathbb{G}_m}\to A^*_G$$

is an isomorphism.

Proof. Let U be a G-free open subset of a G-representation with complement of sufficiently high codimension and with a G-torsor quotient $U\to U/G$. Then

$$U/H\to U/G$$

is a $\mathbb{G}_m$-torsor which corresponds to some line bundle L over U/G and by [8], Remark p. 4–35,

$$A^*(U/H)\simeq\frac{A^*(U/G)}{c_1(L)}.$$

Since $A^*_H\simeq\mathbb{Z}$, by Proposition (2.6), A^*_G is then generated by $c_1(L)$. But the pullback $\mathbb{Z}[u]\simeq A^*_{\mathbb{G}_{m,k}}\to A^*_G$ sends u to $c_1(L)$, therefore ρ^* is surjective. Injectivity follows from the hypothesis that (9) is split. □

Proposition 2.8. *If G is an algebraic group over k, then $A^*_{G\times\mathbb{G}_{m,k}}\simeq A^*_G\otimes A^*_{\mathbb{G}_{m,k}}$.*

Proof. Straightforward using $(\mathbb{A}_k^{N+1}\backslash\{0\},\mathbb{A}_k^{N+1})$ as a good pair for $\mathbb{G}_{m,k}$, $N\gg 0$, and [6], Example 8.3.7. □

Proposition 2.9. *Let G be an algebraic group over k. If H is a closed algebraic subgroup of G, then there is a canonical isomorphism $A^*_G(G/H)\simeq A^*_H$.*

Proof. Straightforward. □

Proposition 2.10. *Let G be an algebraic group over a field k and X a smooth G-scheme. If $U \subset \mathbb{A}_k^n$ is an open subscheme with the trivial G-action, the pull-back*

$$\mathrm{pr}_2^* : A_G^*(X) \simeq A_G^*(U \times X)$$

is an isomorphism.

Proof. Since G acts trivially on U, we can reduce to the case of trivial G. By [6], Prop. 1.9, the pull back via $\mathbb{A}_k^n \times X \to X$ is surjective and so is pr_2^* by the localization exact sequence ([6], Prop. 1.8).

If k is infinite then pr_2 has always a section so that pr_2^* is injective. If k is finite, let $p \in U$ be a closed point with $r \doteq [k(p) : k]$. From the commutative diagram

$$\begin{array}{ccc} & & U \times X \\ & \nearrow & \downarrow \mathrm{pr}_2 \\ p \times X & \xrightarrow[\phi]{} & X \end{array}$$

and projection formula we get that $\ker(\mathrm{pr}_2^*)$ is r-torsion. Now observe that we can always find two closed points p and p' in U with residue fields of relatively prime degrees r and r' over k, so that $\ker(\mathrm{pr}_2^*)$ is indeed zero. □

Proposition 2.11. *Let G, H be algebraic groups having commuting actions on a smooth scheme X and suppose G acts freely. Then there is a canonical isomorphism*

$$A_H^*(X/G) \simeq A_{G\times H}^*(X).$$

Proof. If (U, V) is a good pair for H, with $\mathrm{codim}(V\backslash U) > i$, we have

$$A_H^i(X/G) \simeq A^i\left(\left(U \times \frac{X}{G}\right)\Big/ H\right)$$
$$\simeq A^i((U \times X)/G \times H) \simeq A_{G\times H}^i(X \times U),$$

by [4], Prop. 8. By the localization sequence,

$$A_{G\times H}^i(X \times U) \simeq A_{G\times H}^i(X \times V)$$

for $i < \mathrm{codim}(V\backslash U)$ and we conlude since for any $G \times H$-representation E, we have a pullback ring isomorphism $A_{G\times H}^*(X) \simeq A_{G\times H}^*(X \times E)$. □

3. Generators for $A_{\mathrm{PGL}_3}^*$

From now on, our base field will be $\mathbb{C}$.

By Prop. 2.2, we have

$$A_{\mathrm{PGL}_3}^* \otimes \mathbb{Q} \simeq A_{\mathrm{SL}_3}^* \otimes \mathbb{Q} = \mathbb{Q}[c_2(E), c_3(E)]$$

($E =$ standard representation of SL_3) and an easy computation shows that $c_3(E)$ is not in the image of the subring of $A_{\mathrm{PGL}_3}^* \otimes \mathbb{Q}$ generated by the Chern classes of sl_3. Therefore the Chern classes of the adjoint representation will certainly not suffice to generate $A_{\mathrm{PGL}_3}^*$.

In this section we find generators of $A_{\mathrm{PGL}_3}^*$ (Prop. 3.12) by stratifying the adjoint representation sl_3 using Jordan canonical forms.

Let G be a complex algebraic group. For our purposes a finite G-stratification of a G-scheme X will be a collection $\{X_i\}_{i=1,\dots,n}$ of disjoint smooth G-invariant subschemes, whose union is X and such that for each i the natural immersion

$$j_i : X_i \hookrightarrow X \backslash \bigcup_{k>i} X_k \doteq X^i$$

is closed. In particular, X_n is a closed subscheme of X, each X_i is topologically a locally closed subspace of X and all the maps

$$X_1 = X^1 \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots \hookrightarrow X^{n-1} \hookrightarrow X^n \hookrightarrow X$$

are open immersions. Any stratification $\{X_i\}_{i=1,\dots,n}$ gives then rise to the following exact sequences (of graded abelian groups, $\deg(j_i)_* = \mathrm{codim}_{X^i}(X_i)$):

$$\begin{array}{c} A_G^*(X_2) \xrightarrow{(j_2)_*} A_G^*(X^2) \xrightarrow{i_2^*} A_G^*(X_1) \longrightarrow 0, \\ A_G^*(X_3) \xrightarrow{(j_3)_*} A_G^*(X^3) \xrightarrow{i_3^*} A_G^*(X^2) \longrightarrow 0, \\ \vdots \\ A_G^*(X_n) \xrightarrow{(j_n)_*} A_G^*(X = X^n) \xrightarrow{i_n^*} A_G^*(X^{n-1} = X\backslash X_n) \longrightarrow 0. \end{array} \tag{10}$$

Note that if X is smooth, each graded group above is indeed a graded ring. This will be our case.

Let

$$U \doteq \{A \in \mathrm{sl}_3\backslash\{0\} \mid A \text{ has distinct eigenvalues}\} \underset{\text{open}}{\subset} \mathrm{sl}_3\backslash\{0\},$$

$$Z_{1,1} \doteq \left\{ A \in \mathrm{sl}_3\backslash\{0\} \mid A \text{ has Jordan form } \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \right\},$$

$$Z_{1,0} \doteq \left\{ A \in \mathrm{sl}_3\backslash\{0\} \mid A \text{ has Jordan form } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \right\},$$

$$Z_1 \doteq Z_{1,1} \cup Z_{1,0},$$

$$Z_{0,1} \doteq \left\{ \mathrm{PGL}_3\text{-orbit of } \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$Z_{0,0} \doteq \left\{ \mathrm{PGL}_3\text{-orbit of } \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$Z_0 \doteq Z_{0,1} \cup Z_{0,0}$$

(note that $Z_1 \cup Z_0 = \mathrm{sl}_3 \backslash (U \cup \{0\})$). Then

$$\{U, Z_{1,1}, Z_{1,0}, Z_{0,1}, Z_{0,0}, \{0\}\} \tag{11}$$

is a finite PGL_3-stratification of sl_3. In this case the first associated exact sequence of (10) is

$$A^*_{\mathrm{PGL}_3}(Z_{1,1}) \xrightarrow{(j_{1,1})_*} A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3 \backslash (Z_{1,0} \cup Z_0 \cup \{0\})) \xrightarrow{i^*_{1,1}} A^*_{\mathrm{PGL}_3}(U) \longrightarrow 0 \tag{12}$$

where $i_{1,1} : U \hookrightarrow \mathrm{sl}_3 \backslash (Z_{1,0} \cup Z_0 \cup \{0\})$ and $j_{1,1} : Z_{1,1} \hookrightarrow \mathrm{sl}_3 \backslash (Z_{1,0} \cup Z_0 \cup \{0\})$ are the natural immersions (open and closed, respectively).

To begin with, let us study $A^*_{\mathrm{PGL}_3}(U)$.

3.1. Generators coming from the open subset $U \subset \mathrm{sl}_3$. Let T be the maximal torus of PGL_3 and $\Gamma_3 \doteq N_{\mathrm{PGL}_3}(T) = S_3 \ltimes T$ its normalizer in PGL_3. Let $S_3 \hookrightarrow \mathrm{PGL}_3 : \sigma \mapsto \underline{\sigma}$ be the obvious inclusion (which identifies permutations with permutation matrices). Γ_3 acts on the subscheme $\mathrm{Diag}^*_{\mathrm{sl}_3} \subset \mathrm{sl}_3 \backslash \{0\}$ of diagonal matrices with distinct eigenvalues, through $S_3 \hookrightarrow \mathrm{PGL}_3$

$$(\sigma, [\underline{t}]) \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \doteq \underline{\sigma} \cdot \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \cdot \underline{\sigma}^{-1}$$

and we have[5]:

Proposition 3.1. *The composition of natural maps*

$$A^*_{\mathrm{PGL}_3}(U) \to A^*_{\Gamma_3}(U) \to A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

is a ring isomorphism.

Proof. Let T act by multiplication on the right of PGL_3 and $\dfrac{\mathrm{PGL}_3}{T}$ be the corresponding quotient. S_3 acts on the left of $\dfrac{\mathrm{PGL}_3}{T}$ via $\sigma \cdot [g] \doteq [g\sigma^{-1}]$, $g \in \mathrm{PGL}_3$, and on $\mathrm{Diag}^*_{\mathrm{sl}_3}$ as above. If we let PGL_3 act on $\mathrm{Diag}^*_{\mathrm{sl}_3} \times \dfrac{\mathrm{PGL}_3}{T}$ by left multiplication on $\dfrac{\mathrm{PGL}_3}{T}$ only, there is a PGL_3-equivariant isomorphism

$$U \simeq \left(\mathrm{Diag}^*_{\mathrm{sl}_3} \times \frac{\mathrm{PGL}_3}{T}\right) / S_3,$$

$$A \mapsto [\Delta, [g]_T]_{S_3}$$

where $g^{-1}Ag = \Delta$ (diagonal).

Since S_3 acts freely on $\mathrm{Diag}^*_{\mathrm{sl}_3} \times \dfrac{\mathrm{PGL}_3}{T}$, from Proposition 2.11, we get

$$A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\mathrm{PGL}_3 \times S_3}\left(\mathrm{Diag}^*_{\mathrm{sl}_3} \times \frac{\mathrm{PGL}_3}{T}\right).$$

Now, if W is a free open subset of a $\mathrm{PGL}_3 \times S_3$-representation with complement of sufficiently high codimension, we let Γ_3 act on W via the inclusion

$$(i, \pi) : \Gamma_3 \hookrightarrow \mathrm{PGL}_3 \times S_3 : (\sigma, [\underline{t}]) \mapsto ([\underline{t}]\underline{\sigma}, \sigma)$$

i being the natural inclusion $\Gamma_3 \hookrightarrow \mathrm{PGL}_3$. Then the morphisms

$$\frac{W \times \mathrm{Diag}^*_{\mathrm{sl}_3} \times \frac{\mathrm{PGL}_3}{T}}{\mathrm{PGL}_3 \times S_3} \underset{\varphi}{\overset{\psi}{\leftrightarrows}} \frac{W \times \mathrm{Diag}^*_{\mathrm{sl}_3}}{\Gamma_3},$$

$$\varphi : [w, \Delta, [g]_T]_{\mathrm{PGL}_3 \times S_3} \mapsto [w \cdot (g, 1), \Delta]_{\Gamma_3},$$

$$\psi : [w, \Delta]_{\Gamma_3} \mapsto [w, \Delta, [1]_T]_{\mathrm{PGL}_3 \times S_3}$$

are mutually inverse and we conclude. □

Lemma 3.2. *If T denotes the maximal torus of PGL_3 and A^*_T is viewed as a subring of $A^*_{T_{\mathrm{GL}_3}} = \mathbb{Z}[x_1, x_2, x_3]$, then the Weyl group-invariant subring $(A^*_T)^{S_3}$ is generated by*

$$\gamma_2 = s_1^2 - 3s_2,$$

$$\gamma_3 = 2s_1^3 - 9s_1s_2 + 27s_3,$$

$$\gamma_6 = \Delta \equiv (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$$

where s_i denotes the i-th elementary symmetric function on the x_j's and Δ is the discriminant.

Proof. We have $T = T_{\mathrm{PGL}_3} \simeq T_{\mathrm{GL}_3}/\mathbb{G}_m$, where $T_{\mathrm{GL}_3} = (\mathbb{G}_m)^3$ and $\mathbb{G}_m \hookrightarrow T_{\mathrm{GL}_3}$ diagonally. Therefore

$$A^*_T = \mathrm{Sym}_{\mathbb{Z}}(\hat{T}) \subset A^*_{T_{\mathrm{GL}_3}} = \mathrm{Sym}_{\mathbb{Z}}(\widehat{T_{\mathrm{GL}_3}}) = \mathbb{Z}[x_1, x_2, x_3]$$

[5] This proposition holds (with the same proof given below) for any PGL_n.

is the subring of polynomials $f(x_1, x_2, x_3)$ such that

$$f(x_1+t, x_2+t, x_3+t) = f(x_1, x_2, x_3).$$

Then

$$(A_T^*)^{S_3} = \{f \in \mathbb{Z}[x_1, x_2, x_3]^{S_3} | f(x_1+t, x_2+t, x_3+t) = f(x_1, x_2, x_3)\}$$
$$\equiv (\mathbb{Z}[s_1, s_2, s_3])^{\mathrm{inv}}$$

where S_3 permutes the x_i's. Now, if for any polynomial $f \in \mathbb{Z}[x_1, x_2, x_3]$ we let

$$f^t = f(x_1+t, x_2+t, x_3+t),$$

we get

$$s_1^t = s_1 + 3t, \tag{13}$$
$$s_2^t = s_2 + 2s_1t + 3t^2, \tag{14}$$
$$s_3^t = s_3 + s_2t + s_1t^2 + t^3, \tag{15}$$

and it is then easy to verify that γ_2, γ_3 and γ_6 are indeed in $(A_T^*)^{S_3}$.

Now, let $\varphi \in (A_T^*)^{S_3}$. We first claim that there exists $n_\varphi \geqq 0$ such that

$$3^{n_\varphi}\varphi \in \mathbb{Z}[\gamma_2, \gamma_3, \gamma_6].$$

By definition of γ_2 and γ_3, we have

$$(A_T^*)^{S_3}\left[\frac{1}{3}\right] \equiv \left(\mathbb{Z}\left[\frac{1}{3}\right][s_1, s_2, s_3]\right)^{\mathrm{inv}} = \left(\mathbb{Z}\left[\frac{1}{3}\right][s_1, \gamma_2, \gamma_3]\right)^{\mathrm{inv}}.$$

If

$$P(s_1, \gamma_2, \gamma_3) = P_0(\gamma_2, \gamma_3) + P_1(\gamma_2, \gamma_3)s_1 + \cdots + P_m(\gamma_2, \gamma_3)s_1^m$$

is in $\left(\mathbb{Z}\left[\frac{1}{3}\right][s_1, \gamma_2, \gamma_3]\right)^{\mathrm{inv}}$, using (13) and $\gamma_2^t = \gamma_2$, $\gamma_3^t = \gamma_3$, we easily get, by induction on m, $P_i = 0$, $\forall i \geqq 1$, i.e.

$$(A_T^*)^{S_3}\left[\frac{1}{3}\right] = \mathbb{Z}\left[\frac{1}{3}\right][\gamma_2, \gamma_3]$$

as claimed.

To prove that indeed $\varphi \in \mathbb{Z}[\gamma_2, \gamma_3, \gamma_6]$, we use induction on n_φ.

Suppose[6] $3\varphi = p(\gamma_2, \gamma_3, \gamma_6)$, for some polynomial p. Expanding p in powers of γ_6, we get

$$3\varphi = p_0(\gamma_2, \gamma_3) + p_1(\gamma_2, \gamma_3)\gamma_6 + \cdots$$

and reducing mod 3

$$0 \equiv p_0(s_1^2, -s_1^3) + p_1(s_1^2, -s_1^3)\gamma_6 + \cdots \pmod 3.$$

But s_1 and $\gamma_6 = \Delta$ are algebraically independent (over $\mathbb{Z}/3$), so $p_i(s_1^2, -s_1^3) \equiv 0 \pmod 3$, $\forall i$, i.e.

$$p_i(s_1^2, -s_1^3) \equiv ((s_1^2)^3 - (s_1^3)^2) \cdot q_i(s_1^2, s_1^3) \pmod 3$$

then

$$p_i(\gamma_2, \gamma_3) = (\gamma_2^3 - \gamma_3^2)q_i(\gamma_2, -\gamma_3) + 3r_i(\gamma_2, \gamma_3)$$

for each i. Thus

$$3\varphi = 3r(\gamma_2, \gamma_3, \gamma_6) + (\gamma_2^3 - \gamma_3^2)q(\gamma_2, \gamma_3, \gamma_6)$$

with an obvious notation. Straightforward computations yield

$$(\gamma_2^3 - \gamma_3^2) = -3(\gamma_2^3 - 9\gamma_6),$$

and the case $n_\varphi = 1$ is settled. The inductive step follows easily from the fact that we included a possible dependence of p on γ_6 in the above argument. □

Remark 3.1. Note that there is a (non canonical) isomorphism

$$T \to (\mathbb{G}_m)^2,$$
$$[t_1, t_2, t_3] \mapsto (t_1/t_3, t_2/t_3)$$

so that $A_T^* \simeq \mathbb{Z}[x, y]$, with action of the Weyl group given by

$$(12)x = y, \quad (12)y = x, \tag{16}$$
$$(123)x = -y, \quad (123)y = x - y.$$

Under this isomorphism, with the same notations as in Lemma 3.2, we have

$$\gamma_2 = (x+y)^2 - 3xy, \tag{17}$$
$$\gamma_3 = -9(x+y)xy + 2(x+y)^3$$
$$\gamma_6 = (x+y)^2x^2y^2 - 4x^3y^3.$$

[6] Note that we allow an explicit dependence of p on γ_6!

Moreover, there is an isomorphism of T with T_{SL_3}, the maximal torus of SL_3

$$T \to T_{\mathrm{SL}_3} : [t_1, t_2, t_3] \mapsto (t_2/t_3, t_3/t_1, t_1/t_2) \tag{18}$$

and an induced isomorphism $A_T^* \simeq A_{T_{\mathrm{SL}_3}}^* = \mathbb{Z}[u_1, u_2, u_3]/(u_1 + u_2 + u_3)$. The Weyl groups are isomorphic to S_3 in both cases but the isomorphism above on Chow rings is not S_3-equivariant, only A_3-equivariant. Rather, the action of S_3 on $A_{T_{\mathrm{SL}_3}}^*$ inherited from the Weyl group action on A_T^* via this isomorphism, is given by

$$(12)u_1 = -u_2, \quad (12)u_2 = -u_1, \quad (12)u_3 = -u_3,$$
$$(123)u_1 = u_3, \quad (123)u_2 = u_1, \quad (123)u_3 = u_1.$$

Corollary 3.3. *The canonical morphism $h : A_{\Gamma_3}^* \to (A_T^*)^{S_3}$ is surjective.*

Proof. Let $\phi : A_{\mathrm{PGL}_3}^* \to A_{\Gamma_3}^*$ be the restriction morphism, E the standard representation of GL_3 and $\mathrm{Sym}^3 E$ be the PGL_3-representation:

$$[g] \cdot (v_1 \cdot v_2 \cdot v_3) \doteq (\det g^{-1})(gv_1 \cdot gv_2 \cdot gv_3).$$

It is not difficult to verify that

$$\begin{aligned} h \circ \phi(c_2(\mathrm{sl}_3)) &= -2\gamma_2, \\ h \circ \phi(c_2(\mathrm{Sym}^3 E)) &= -5\gamma_2, \\ h \circ \phi(c_3(\mathrm{Sym}^3 E)) &= \gamma_3, \\ h \circ \phi(c_6(\mathrm{sl}_3)) &= \gamma_6 \end{aligned}$$

and the corollary follows from Lemma 3.2. □

Now consider the subgroup $A_3 \ltimes T \hookrightarrow \Gamma_3 = S_3 \ltimes T$; there is a transfer morphism (see (2), Section 2)

$$\mathrm{tsf} = \mathrm{tsf}_{A_3 \ltimes T}^{\Gamma_3}(\mathrm{Diag}_{\mathrm{sl}_3}^*) : A_{A_3 \ltimes T}^*(\mathrm{Diag}_{\mathrm{sl}_3}^*) \to A_{\Gamma_3}^*(\mathrm{Diag}_{\mathrm{sl}_3}^*)$$

and a restriction morphism:

$$\mathrm{res} = \mathrm{res}_{A_3 \ltimes T}^{\Gamma_3}(\mathrm{Diag}_{\mathrm{sl}_3}^*) : A_{\Gamma_3}^*(\mathrm{Diag}_{\mathrm{sl}_3}^*) \to (A_{A_3 \ltimes T}^*(\mathrm{Diag}_{\mathrm{sl}_3}^*))^{C_2}.$$

Lemma 3.4 (transfer-trick). res *induces an isomorphism*

$$_3A_{\Gamma_3}^*(\mathrm{Diag}_{\mathrm{sl}_3}^*) \to {}_3(A_{A_3 \ltimes T}^*(\mathrm{Diag}_{\mathrm{sl}_3}^*))^{C_2}$$

with inverse $(-\mathrm{tsf})$.

Proof. By projection formula, $\mathrm{tsf} \circ \mathrm{res} = 2$; so if ξ is 3-torsion, we have

$$\mathrm{tsf} \circ \mathrm{res}(\xi) = -\xi.$$

On the other hand, if $C_2 = \{1, \varepsilon\}$, we have $\mathrm{res} \circ \mathrm{tsf}(\eta) = \eta + \eta^{\varepsilon}$; so, if η is C_2-invariant and 3-torsion, we have $\mathrm{res} \circ \mathrm{tsf}(\eta) = -\eta$ and conclude. □

The isomorphism (18) of Remark 3.1 induces an isomorphism

$$A_3 \ltimes T \simeq A_3 \ltimes T_{\mathrm{SL}_3}$$

and hence an isomorphism

$$A_{A_3 \ltimes T}^* \simeq A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*. \tag{19}$$

We will consider the C_2-action on $A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*$ induced by the canonical action on $A_{A_3 \ltimes T}^*$ via this isomorphism. As already in Remark 3.1, we warn the reader that this is not the canonical action induced by the inclusion $A_3 \ltimes T_{\mathrm{SL}_3} \hookrightarrow \mathrm{N}_{\mathrm{SL}_3}(T_{\mathrm{SL}_3})$.

If $A_{A_3}^* = \mathbb{Z}[\alpha]/(3\alpha)$[7], we still denote by α the image of α in $A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*$ via the pullback induced by the projection $A_3 \ltimes T_{\mathrm{SL}_3} \to A_3$. We also recall the isomorphism

$$A_{T_{\mathrm{SL}_3}}^* \simeq \mathbb{Z}[u_1, u_2, u_3]/(u_1 + u_2 + u_3).$$

Then, if $W \simeq \mathbb{C}^3$ denotes the $A_3 \ltimes T_{\mathrm{SL}_3}$-representation

$$(\sigma, (\underline{s})) \cdot (\underline{x}) \doteq (s_1 x_{\sigma^{-1}(1)}, s_2 x_{\sigma^{-1}(2)}, s_3 x_{\sigma^{-1}(3)}), \tag{20}$$

we have the following basic result

Proposition 3.5. *The ring $A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*$ is generated by*

$$\{\alpha, c_2(W), c_3(W), \theta \doteq \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2^2 u_3)\}.$$

Proof. Throughout the proof we identify $A_3 \ltimes T$ with $A_3 \ltimes T_{\mathrm{SL}_3}$ (Remark 3.1). $A_3 \ltimes T_{\mathrm{SL}_3}$ acts on $\mathbb{P}(W)$ with a dense orbit $U \doteq D_+(x_1 x_2 x_3)$ with stabilizer isomorphic to $A_3 \times \mu_3$. If $j_2 : Y_2 \hookrightarrow \mathbb{P}(W)$ denotes the (closed) orbit of $[1, 0, 0] \in \mathbb{P}(W)$ and

$$Y_1 \doteq \mathbb{P}(W) \backslash U \cup Y_2 \underset{\text{closed}}{\overset{j_1}{\hookrightarrow}} \mathbb{P}(W) \backslash Y_2,$$

the orbit of $[1, 1, 0] \in \mathbb{P}(W)$, then $\{U, Y_1, Y_2\}$ is a finite $A_3 \ltimes T_{\mathrm{SL}_3}$-stratification of $\mathbb{P}(W)$ and the exact sequences (10) are

$$\begin{aligned} A_{\mathbb{G}_m}^* \simeq A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*(Y_1) &\xrightarrow{(j_1)_*} A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*(\mathbb{P}(W) \backslash Y_2) \\ &\xrightarrow{i^*} A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*(U) \simeq A_{A_3 \times \mu_3}^* \longrightarrow 0, \end{aligned} \tag{21}$$

$$A_{T_{\mathrm{SL}_3}}^* \simeq A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*(Y_2) \xrightarrow{(j_2)_*} A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*(\mathbb{P}(W)) \xrightarrow{i_2^*} A_{A_3 \ltimes T_{\mathrm{SL}_3}}^*(\mathbb{P}(W) \backslash Y_2) \longrightarrow 0, \tag{22}$$

---

[7] We use that $A_3 \simeq \mu_3$, which is true over any algebraically closed field of characteristic $\neq 3$. Note that in characteristic 3, it is no longer true that $A_{A_3}^* \simeq \mathbb{Z}[\alpha]/(3\alpha)$.

where we used Prop. 2.9 together with the fact that Y_1 (resp. Y_2, U) has stabilizer isomorphic to $\mathbb{G}_m$ (resp. T_{SL_3}, $A_3 \times \mu_3$). By [6], Th. 3.3 (b), we have ($c_1(W) = 0$):

$$A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W)) \simeq A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}[\ell]/(\ell^3 + c_2(W)\ell + c_3(W))$$

where $\ell = c_1(\mathcal{O}_{\mathbb{P}(W)}(1))$. Moreover, the Künneth morphism

$$A^*_{A_3} \otimes A^*_{\mu_3} \simeq \mathbb{Z}[\alpha]/(3\alpha) \otimes \mathbb{Z}[\beta]/(3\beta) \to A^*_{A_3 \times \mu_3}$$

is an isomorphism (e.g. [25], §6). It is not difficult to show that

$$i^*(\ell) = -\beta, \quad i^*(\alpha) = \alpha, \quad j_1^*(\ell) = -u$$

(where $A^*_{\mathbb{G}_m} = \mathbb{Z}[u]$ and with the usual abuse of notation, we write ℓ for $i_2^*(\ell)$ and α for its pullback to $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W) \backslash Y_2)$). So we can conclude the analysis of (21), by computing $(j_1)_*(1) = [Y_1]$. Y_1 is the zero scheme of the $A_3 \ltimes T_{\mathrm{SL}_3}$-invariant regular section

$$x_1 x_2 x_3 \in \Gamma(\mathcal{O}(3), \mathbb{P}(W) \backslash Y_2),$$

hence ([6], p. 61), $[Y_1] = 3\ell$ so that $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W) \backslash Y_2)$ is generated by $\{\alpha, \ell\}$.

Now let us turn our attention to (22). It is easy to verify that, with the usual abuse of notation, $i_2^*(\alpha) = \alpha$ and $i_2^*(\ell) = \ell$, so we are left to find generators of $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(Y_2) \simeq A^*_{T_{\mathrm{SL}_3}}$ as an $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$-module.

First of all, we have $j_2^*(\ell) = u_1$[8]. Therefore, by projection formula and the relation $u_1 + u_2 + u_3 = 0$, we see that $A^*_{T_{\mathrm{SL}_3}} \simeq A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(Y_2)$ is generated by

$$\{1, u_2^n \mid n > 0\} \qquad (23)$$

as an $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$-module. But

$$j_2^*(c_2(W)) = u_1 u_2 + u_2 u_3 + u_3 u_1 = -(u_1^2 + u_2^2 + u_1 u_2)$$

so that, by induction on n, $(j_2)_*(u_2^n)$, $n > 1$, belongs to the submodule generated by $(j_2)_*(1)$ and $(j_2)_*(u_2)$ (e.g.

[8] Of course this relation depends on the choice of the isomorphism

$$\mathbb{Z}[u_1, u_2, u_3]/(u_1 + u_2 + u_3) = A^*_{T_{\mathrm{SL}_3}} \simeq A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(Y_2)$$

which in its turn depends essentially on the choice of a point

$$p \in Y_2 = \{[1,0,0], [0,1,0], [0,0,1]\}.$$

The choice we are making here is $p = [1,0,0]$.

$$\begin{aligned}(j_2)_*(u_2^2) &= (j_2)_*(j_2^*(-c_2(W))) + (j_2)_*(j_2^*(-\ell^2)) - (j_2)_*(j_2^*(\ell) \cdot u_2) \\ &= -c_2(W) \cdot (j_2)_*(1) - \ell^2 \cdot (j_2)_*(1) - \ell \cdot (j_2)_*(u_2)\end{aligned}$$

and similarly for higher powers of u_2). Thus, the ideal

$$\mathrm{im}(j_2)_* \subset A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$$

is actually generated by $(j_2)_*(1)$ and $(j_2)_*(u_2)$.

Let us first compute $(j_2)_*(1)$ using a transfer argument (Section 2). Consider the $A_3 \ltimes T_{\mathrm{SL}_3}$-equivariant commutative diagram

$$\begin{array}{ccc} Y_2 & \xrightarrow{h_2} & \mathbb{P}(W) \times A_3 \\ & {\scriptstyle j_2} \searrow & \downarrow {\scriptstyle \mathrm{pr}_1} \\ & & \mathbb{P}(W) \end{array}$$

where

$$h_2([1,0,0]) = ([1,0,0], 1), \quad h_2([0,1,0]) = ([0,1,0], \sigma), \quad h_2([0,0,1]) = ([0,0,1], \sigma^2)$$

with $\sigma = (123)$. Using the canonical isomorphism

$$A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W) \times A_3) \simeq A^*_T(\mathbb{P}(W))$$

we see that

$$(j_2)_*(1) = (\mathrm{pr}_1)_* \circ (h_2)_*(1) = \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))([\{[1,0,0]\}]). \qquad (24)$$

But $[1,0,0] = Z(x_2) \cap Z(x_3)$, where the sections x_i, $i = 2,3$ are T_{SL_3}-semi-invariant ([3], Exposé $\mathrm{VI_B}$, p. 406) so that if we consider the T_{SL_3}-equivariant line bundles $L_i \to \operatorname{Spec} \mathbb{C}$ associated to the representations

$$(\underline{t})x = t_i x, \quad i = 2,3,$$

we have induced T_{SL_3}-invariant regular sections $\tilde{x}_i \in \Gamma(P(W), \mathcal{O}(1) \otimes p^*(L_i^\vee))$[9] with, obviously, $Z(\tilde{x}_i) = Z(x_i)$. Then

$$[\{[1,0,0]\}] = (\ell - u_2)(\ell - u_3) = \ell^2 + \ell u_1 + u_2 u_3 \qquad (25)$$

in $A^*_{T_{\mathrm{SL}_3}}(\mathbb{P}(W))$. Since $\ell = \mathrm{res}^{T_{\mathrm{SL}_3}}_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))(\ell)$ and the diagram

[9] Note that $p^*(L_i^\vee)$ is trivial but not T_{SL_3}-equivariantly trivial.

$$\begin{array}{ccc} A^*_{T_{\mathrm{SL}_3}} & \longrightarrow & A^*_{T_{\mathrm{SL}_3}}(\mathbb{P}(W)) \\ {\scriptstyle \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))} \\ A^*_{A_3 \ltimes T_{\mathrm{SL}_3}} & \longrightarrow & A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W)) \end{array}$$

is commutative, we have

$$\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))(\ell^2) = 3\ell^2, \tag{26}$$

$$\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))(\ell u_1) = \ell \cdot \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_1). \tag{27}$$

Now we claim $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_i) = 0$, $i = 1, 2, 3$. In fact, let $\pi : A^*_{A_3 \ltimes T_{\mathrm{SL}_3}} \to A_3$ be the projection and $\rho : A_3 \hookrightarrow A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}$ its right inverse. Since i_2^* in (22) is an isomorphism in degree 1 and $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W) \setminus Y_2)$ is generated by α and ℓ, $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_i) = n_i \pi^* \alpha$ for some integer n_i (in fact

$$\mathrm{res}_{A_3 \ltimes T_{\mathrm{SL}_3}}^{T_{\mathrm{SL}_3}} \circ \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_i) = u_1 + u_2 + u_3 = 0$$

thus $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_i)$ is 3-torsion). Since

$$\rho^* \circ \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}} \equiv \mathrm{res}_{A_3}^{A_3 \ltimes T_{\mathrm{SL}_3}} \circ \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}} = 0,$$

we get

$$n_i \rho^* \pi^*(\alpha) = n_i \alpha = 0$$

in $A^*_{A_3}$ and the claim follows.

Since $(j_2)_*(u_2)$ has degree 3, from (22) and the computations we have just done (in particular (24), (25), (26) and (27)), we know that the ring $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$ is generated up to degree 2 (included) by

$$\{\alpha, \ell, \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2 u_3)\}.$$

We will show that:

Claim. $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$ is generated up to degree 2 (included) by

$$\{\alpha, \ell, c_2(W)\}.$$

Proof of Claim. We write

$$\eta_{|T_{\mathrm{SL}_3}} \equiv \mathrm{res}_{A_3 \ltimes T_{\mathrm{SL}_3}}^{T_{\mathrm{SL}_3}}(\eta),$$

for any $\eta \in A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$.

Observe that

$$\mathrm{res}_{A_3 \ltimes T_{\mathrm{SL}_3}}^{T_{\mathrm{SL}_3}} \circ \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2 u_3) = u_2 u_3 + u_3 u_1 + u_1 u_2 = c_2(W)_{|T_{\mathrm{SL}_3}},$$

therefore $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2 u_3) - c_2(W) = \xi$, for some 3-torsion element[10)]

$$\xi \in A^2_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W)).$$

Since the group $A^2_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$ is generated by

$$\{\alpha^2, \ell^2, \alpha\ell, \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2 u_3) = c_2(W) + \xi\}$$

we have

$$c_2(W) = A(c_2(W) + \xi) + B\alpha^2 + C\ell^2 + D\alpha\ell. \tag{28}$$

Restricting to T_{SL_3}, we get

$$c_2(W)_{|T_{\mathrm{SL}_3}} = A c_2(W)_{|T_{\mathrm{SL}_3}} + C\ell^2;$$

but from

$$A^*_{T_{\mathrm{SL}_3}}(\mathbb{P}(W)) \simeq A^*_{T_{\mathrm{SL}_3}}[\ell] / (\ell^3 + \ell^2 c_1(W)_{|T_{\mathrm{SL}_3}} + \ell c_2(W)_{|T_{\mathrm{SL}_3}} + c_3(W)_{|T_{\mathrm{SL}_3}}),$$

we see that $c_2(W)_{|T_{\mathrm{SL}_3}}$ and ℓ^2 are algebraically independent, so we must have $A = 1$, $C = 0$. Thus (28) yields $\xi = B\alpha^2 + D\alpha\ell$ and this concludes the proof of Claim. □

So, the other possible generators of $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$ in degree >2 can only come from $(j_2)_*(u_2)$. Using the same arguments as in the computation of $(j_2)_*(1)$ above, we get

$$(j_2)_*(u_2) = \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))(u_2(\ell - u_2)(\ell - u_3)).$$

But, since we know that $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_i) = 0\ \forall i$, the only new generator is $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2^2 u_3)$.

To summarize, we have proved so far that $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W))$ is generated by

$$\{\alpha, \ell, c_2(W), \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2^2 u_3)\}.$$

Since

$$A^*_{A_3 \ltimes T_{\mathrm{SL}_3}} / (c_3(W)) \simeq A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(W \setminus \{0\}) \simeq A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}(\mathbb{P}(W)) / (\ell),$$

we conclude that $A^*_{A_3 \ltimes T_{\mathrm{SL}_3}}$ is generated by

$$\{\alpha, c_2(W), c_3(W), \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_2^2 u_3)\}. \quad \square$$

Recall (19) and the isomorphism

$${}_3A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \simeq {}_3(A^*_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}))^{C_2}$$

10) In fact $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}} \circ \mathrm{res}_{A_3 \ltimes T_{\mathrm{SL}_3}}^{T_{\mathrm{SL}_3}} = 3$.

from Lemma 3.4. If $C_2 = \{1, \varepsilon\}$, we denote by W^ε the $A_3 \ltimes T_{\mathrm{SL}_3}$-representation obtained from W twisting the action by ε. Let us also define the element

$$(29) \qquad \underline{\chi} \doteq \left(2\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_2^2u_3) + 3c_3(W)\right)^2 + 4c_2(W)^3 + 27c_3(W)^2 \in A^6_{A_3\ltimes T_{\mathrm{SL}_3}}$$

and denote $\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_2^2u_3)$ simply by θ.

Lemma 3.6. (i) *In* $A^*_{A_3\ltimes T}$ *we have*

$$3\underline{\chi} = 3\alpha = \alpha\theta = \alpha^3 + \alpha c_2(W) = 0.$$

(ii) *The kernel of the restriction map* $h' : A^*_{A_3\ltimes T} \to A^*_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})$ *is the ideal* (α^2).

(iii) *In* $A^*_{A_3\ltimes T}$, *we have*

$$c_2(W^\varepsilon) = c_2(W), \quad c_3(W^\varepsilon) = -c_3(W),$$
$$\theta^\varepsilon = \theta + 3c_3(W), \quad \underline{\chi}^\varepsilon = \underline{\chi}.$$

(iv) *Let*

$$q\left(c_2(W), c_3(W), \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_2^2u_3)\right) \in {}_3A^*_{A_3\ltimes T_{\mathrm{SL}_3}}$$

be a polynomial in the arguments indicated. Then there exists a polynomial

$$\bar{q} = \bar{q}(c_2(W), c_3(W), \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_2^2u_3))$$

such that $q = \underline{\chi}\bar{q}$.

Proof. (i) Since

$$\left(2\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_2^2u_3) + 3c_3(W)\right)^2_{|T_{\mathrm{SL}_3}} = \Delta(u_1, u_2, u_3),$$
$$c_2(W)_{|T_{\mathrm{SL}_3}} = s_2(u_1, u_2, u_3),$$
$$c_3(W)_{|T_{\mathrm{SL}_3}} = s_3(u_1, u_2, u_3)$$

in $A^*_{T_{\mathrm{SL}_3}} \simeq A^*_T$ (where Δ is the discriminant and s_i the i-th elementary symmetric function), it is well known that $\underline{\chi}_{|T} = 0$. Therefore $3\underline{\chi} = 0$. α is 3-torsion by definition and

$$\alpha \cdot \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_2^2u_3) = 0$$

by projection formula. Finally observe that

$$\left(c_2(W) - \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_1u_3)\right)_{|T_{\mathrm{SL}_3}} = 0$$

and therefore (Proposition 3.5) there exist $A, B \in \mathbb{Z}$ such that

$$c_2(W) - \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3\ltimes T_{\mathrm{SL}_3}}(u_1u_2) = A\alpha^2 + Bc_2(W)$$

is a 3-torsion element in $A^*_{A_3\ltimes T_{\mathrm{SL}_3}}$. Restricting to T_{SL_3} we get $B = 0$ while restricting to A_3 we get $A \equiv -1 \bmod 3$. Multiplying by α, we get

$$\alpha^3 + \alpha c_2(W) = 0$$

by projection formula.

(ii) A straightforward computation yields

$$c_2(\mathrm{Diag}_{\mathrm{sl}_3}) = -\alpha^2 \in A^*_{A_3\ltimes T}.$$

Consider then the two localization sequences:

$$(30) \quad A^*_{A_3\ltimes T} \xrightarrow{(-\alpha^2)\cdot} A^*_{A_3\ltimes T}(\mathrm{Diag}_{\mathrm{sl}_3}) \simeq A^*_{A_3\ltimes T} \longrightarrow A^*_{A_3\ltimes T}(\mathrm{Diag}_{\mathrm{sl}_3}\setminus\{0\}) \longrightarrow 0,$$
$$(31) \quad A^*_T \simeq A^*_{A_3\ltimes T}(Z) \xrightarrow{j_*} A^*_{A_3\ltimes T}(\mathrm{Diag}_{\mathrm{sl}_3}\setminus\{0\}) \longrightarrow A^*_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \longrightarrow 0$$

(where we used the obvious $A_3 \ltimes T$-equivariant isomorphism $Z \simeq A_3 \times \mathbb{C}^*$); (30) shows that $\alpha^2 \in \ker h'$ and the reverse inclusion will be established if we show that the push-forward j_* is zero.

Consider the projectivization $\mathbb{P}(\mathrm{Diag}_{\mathrm{sl}_3}) \simeq \mathbb{P}^1$ of $\mathrm{Diag}_{\mathrm{sl}_3}$. We have a cartesian diagram

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & \mathrm{Diag}_{\mathrm{sl}_3}\setminus\{0\} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z' & \underset{j'}{\hookrightarrow} & \mathbb{P}(\mathrm{Diag}_{\mathrm{sl}_3}) \end{array}$$

where

$$Z' = \{[1,1], [-2,1], [1,-2]\} \simeq A_3$$

$A_3 \ltimes T$-equivariantly. Since

$$j_* \circ p^* = \pi^* \circ j'_*$$

and p^* is obviously an isomorphism, it is enough to show that

$$(32) \qquad \mathrm{im}(j'_*) \subseteq \ker(\pi^*) = (\ell) \subset \frac{A^*_{A_3\ltimes T}[\ell]}{(\ell^2 - \alpha^2)}$$

by the projective bundle theorem.

To compute j'_* we translate it into a transfer map. Consider the $A_3 \ltimes T$-equivariant commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\ \rho\ } & A_3 \times \mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}) \\ & {\scriptstyle j'} \searrow & \downarrow {\scriptstyle \mathrm{pr}_2} \\ & & \mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}) \end{array}$$

where $(\sigma = (123) \in A_3)$

$$\rho([1,1]) = (1,[1,1]),$$
$$\rho([-2,1]) = (\sigma,[-2,1]),$$
$$\rho([1,-2]) = (\sigma^2,[1,-2]).$$

Since

$$A^*_{A_3 \ltimes T}(A_3 \times \mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3})) \simeq A^*_T(\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3})),$$

we have

$$j'_*(\xi) = \mathrm{pr}_{2*} \circ \rho_*(\xi) = \mathrm{tsf}_T^{A_3 \ltimes T}(\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}))(\xi \cdot [\{[1,1]\}])$$

for any $\xi \in A^*_T \simeq A^*_{A_3 \ltimes T}(Z')$, where $\{[1,1]\}$ is a T-invariant cycle on $\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3})$.

Now, $\{[1,1]\}$ is the zero scheme of the T-invariant regular section

$$(x_1 - x_2) \in \Gamma(\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}), \mathcal{O}(1)),$$

therefore

$$[\{[1,1]\}] = c_1(\mathcal{O}(1)) \equiv \ell' \in A^*_T(\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}))$$

and, obviously,

$$\mathrm{res}^T_{A_3 \ltimes T}(\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}))(\ell) = \ell'.$$

By projection formula, we then get

$$j'_*(\xi) = \mathrm{tsf}_T^{A_3 \ltimes T}(\mathbb{P}(\mathrm{Diag}_{\mathfrak{sl}_3}))(\xi \cdot \ell') = \ell \cdot \mathrm{tsf}_T^{A_3 \ltimes T}(\xi)$$

for any $\xi \in A^*_T \simeq A^*_{A_3 \ltimes T}(Z')$, which proves (32).

(iii) By Prop. 3.5, there are integers A, B such that

$$c_2(W^\varepsilon) = A\alpha^2 + Bc_2(W).$$

Restricting this to T, we get $B = 1$ and applying the involution $(\cdot)^\varepsilon$ we obtain $A \equiv 0 \bmod 3$.

Again by Prop. 3.5, there are integers A, B, C, D such that

$$c_3(W^\varepsilon) = A\alpha^3 + B\alpha c_2(W) + Cc_3(W) + D\theta$$

in $A^*_{A_3 \ltimes T}$. Restricting to T, we get

$$(C+1)u_1u_2u_3 + D(u_2^2u_3 + u_3^2u_1 + u_1^2u_2) = 0 \in A^*_{T_{\mathrm{SL}_3}} = \frac{\mathbb{Z}[u_1,u_2,u_3]}{(u_1+u_2+u_3)};$$

but $u_2^2u_3 + u_3^2u_1 + u_1^2u_2$ and $u_1u_2u_3$ are linearly independent, hence $C = -1$, $D = 0$. Now apply the involution $(\cdot)^\varepsilon$ to get

$$A\alpha^3 + B\alpha c_2(W) = 0.$$

Since (Remark 3.1)

$$\theta^\varepsilon = -\mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}}(u_1^2u_3),$$

an easy computation yields

$$(\theta - \theta^\varepsilon + 3c_3(W))_{|T_{\mathrm{SL}_3}} = 0.$$

Therefore (Proposition 3.5 and (i) of this lemma) there exist $A, B, C \in \mathbb{Z}$ such that

$$\theta - \theta^\varepsilon + 3c_3(W) = A\alpha^3 + Bc_3(W) + C\theta$$

is 3-torsion. Then, restricting to T_{SL_3} and observing that $c_3(W)_{|T}$ and $\theta_{|T}$ are linearly independent, we get $B = C = 0$; restricting now to A_3, we obtain $A \equiv 0 \bmod 3$ (since

$$\mathrm{res}^{A_3}_{A_3 \ltimes T_{\mathrm{SL}_3}} \circ \mathrm{tsf}_{T_{\mathrm{SL}_3}}^{A_3 \ltimes T_{\mathrm{SL}_3}} = 0).$$

The C_2-invariance of $\underline{\chi}$ is a consequence of the transformation rules of $c_2(W)$, $c_3(W)$ and θ.

(iv) Since q is 3-torsion, we may suppose 2 inverted. We have $q_{|T_{\mathrm{SL}_3}} = 0$ because $A^*_{T_{\mathrm{SL}_3}}$ is torsion-free. It is not difficult to verify that

$$(2\theta_{|T_{\mathrm{SL}_3}} + 3c_3(W)_{|T_{\mathrm{SL}_3}})^2 + 4c_2(W)^3_{|T_{\mathrm{SL}_3}} + 27c_3(W)^2_{|T_{\mathrm{SL}_3}} = 0.$$

Then it is enough to prove that the ideal $\mathcal{I}$ of relations between

$$\{c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}, \theta_{|T_{\mathrm{SL}_3}}\}$$

in $A^*_{T_{\mathrm{SL}_3}}\left[\frac{1}{2}\right]$ is generated by just this one.

Now, $\theta_{|T_{\mathrm{SL}_3}} = -\frac{3}{2}c_3(W)_{|T_{\mathrm{SL}_3}} + \frac{1}{2}\delta$, where $\delta = (u_1 - u_2)(u_2 - u_3)(u_1 - u_3)$, so we have to show that

$$\mathcal{I} = (\delta^2 + 4c_2(W)^3_{|T_{\mathrm{SL}_3}} + 27c_3(W)^2_{|T_{\mathrm{SL}_3}}).$$

Let $p \in Z\left[\frac{1}{2}\right][X,Y,Z]$ with

$$p\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}, \delta\big) = 0$$

in $A^*_{T_{\mathrm{SL}_3}}\left[\frac{1}{2}\right]$. We have

$$(33)\qquad p(X,Y,Z) = p_0(X,Y) + Zp_1(X,Y) \bmod (Z^2 + 4X^3 + 27Y^2).$$

If we let $C_2 = \{1, \varepsilon\}$ act on $A^*_{T_{\mathrm{SL}_3}}\left[\frac{1}{2}\right]$ permuting u_1 and u_2, we get

$$\big(c_2(W)_{|T_{\mathrm{SL}_3}}\big)^{\varepsilon} = c_2(W)_{|T_{\mathrm{SL}_3}},$$
$$\big(c_3(W)_{|T_{\mathrm{SL}_3}}\big)^{\varepsilon} = c_3(W)_{|T_{\mathrm{SL}_3}},$$
$$\delta^{\varepsilon} = -\delta$$

and then

$$p^{\varepsilon} = p\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}, -\delta\big) = 0$$

in $A^*_{T_{\mathrm{SL}_3}}\left[\frac{1}{2}\right]$ (note that $u_1 + u_2 + u_3$ is C_2-invariant). From (33) we get

$$\begin{cases} p_0\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}\big) + p_1\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}\big)\delta = 0, \\ p_0\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}\big) - p_1\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}\big)\delta = 0 \end{cases}$$

so $(\delta \neq 0)$

$$p_0\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}\big) = p_1\big(c_2(W)_{|T_{\mathrm{SL}_3}}, c_3(W)_{|T_{\mathrm{SL}_3}}\big) = 0.$$

But $c_2(W)_{|T_{\mathrm{SL}_3}}$ and $c_3(W)_{|T_{\mathrm{SL}_3}}$ are algebraically independent, thus

$$p_0(X,Y) = p_1(X,Y) = 0$$

as polynomials, as desired. □

Therefore both $h'\big(\alpha c_3(W)\big)$ and $h'(\underline{\chi})$ can be identified (via Lemma 3.4) with their transfers, which are elements of ${}_3A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3})$.

Proposition 3.7. *The natural morphism*

$$f : A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \to \big(A^*_T(\mathrm{Diag}^*_{\mathrm{sl}_3})\big)^{S_3} = (A^*_T)^{S_3}$$

is surjective with kernel $\big(h'(\alpha c_3(W)), h'(\underline{\chi})\big)$, *where*

$$h' : A^*_{A_3 \ltimes T} \to A^*_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

is the pullback.

Proof. Commutativity of

$$\begin{array}{ccc} & A^*_{\Gamma_3} & \\ {}^{h}\swarrow & & \searrow^{g} \\ A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) & \xrightarrow[f]{} & \big(A^*_T(\mathrm{Diag}^*_{\mathrm{sl}_3})\big)^{S_3} \end{array}$$

together with Lemma 3.3, prove that f is surjective. Moreover $h'\big(\alpha c_3(W)\big)$ and $h'(\underline{\chi})$ are 3-torsion so $\ker f \supseteq \big(h'(\alpha c_3(W)), h'(\underline{\chi})\big)$ since A^*_T is torsion-free. So we are left to prove the reverse inclusion.

Claim. $\ker f = {}_3A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \simeq {}_3\big(A^*_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})\big)^{C_2}$.

Proof of Claim. A^*_T is torsion-free, so $\ker f \supseteq {}_3A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3})$. The pullback

$$\pi : A^*_{\mathrm{PGL}_3} \to (A^*_T)^{S_3}$$

factors as

$$A^*_{\mathrm{PGL}_3} \overset{p}{\twoheadrightarrow} A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \overset{f}{\to} \big(A^*_T(\mathrm{Diag}^*_{\mathrm{sl}_3})\big)^{S_3} = (A^*_T)^{S_3}$$

and from Prop. 2.3, we get $\ker \pi = {}_3A^*_{\mathrm{PGL}_3}$; so $\ker(f \circ p) = {}_3A^*_{\mathrm{PGL}_3}$ and we conclude since p is surjective. □

Now, let $\xi \in_3 \big(A^*_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})\big)^{C_2}$. Omitting to write $h'(\cdot)$ everywhere and denoting $\mathrm{tsf}_T^{A_3\ltimes T}(u_2^2 u_3)$ by θ, we must have

$$\xi = \alpha \cdot p\big(c_2(W), c_3(W)\big) + \underline{\chi} \cdot q\big(c_2(W), c_3(W), \theta\big)$$

for some polynomials p and q, since ξ is 3-torsion (we used Prop. 3.5 and Lemma 3.6 (i), (ii), (iv)). But ξ is also C_2-invariant, so if $C_2 = \{1, \varepsilon\}$, we have:

$$\xi = -2\xi = -(\xi + \xi^{\varepsilon}) = \alpha \cdot (p^{\varepsilon} - p) + \underline{\chi} \cdot \big(-(q + q^{\varepsilon})\big)$$

(Lemma 3.6 (iii)). By Lemma 3.6 (iii), we have

$$\alpha \cdot p^{\varepsilon} = \alpha \cdot p\big(c_2(W), -c_3(W)\big)$$

and we can write $\alpha(p - p^{\varepsilon})$ as a polynomial of the form

$$\alpha c_3(W) \cdot p'\big(c_2(W), c_3(W)^2\big)$$

for some polynomial p'. By the Claim above, we conclude that $\ker f \subseteq \big(\alpha c_3(W), \underline{\chi}\big)$. □

Let us summarize the situation so far. We are studying the first step (12) of the stratification of sl_3. So we started studying $A^*_{\mathrm{PGL}_3}(U)$. We have an isomorphism

$$A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

(Prop. 3.1) and an exact sequence (Prop. 3.7):

$$0 \to (\alpha c_3(W), \underline{\chi}) \to A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \xrightarrow{f} (A^*_T(\mathrm{Diag}^*_{\mathrm{sl}_3}))^{S_3} = (A^*_T)^{S_3} \to 0.$$

To be precise, $\alpha c_3(W)$ and $\underline{\chi}$ belong to $A^*_{A_3 \ltimes T}$ but we denote by the same symbols the elements

$$(\mathrm{tsf}^{\Gamma_3}_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \circ h')(\alpha c_3(W)),$$
$$(\mathrm{tsf}^{\Gamma_3}_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \circ h')(\underline{\chi})$$

in $A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3})$, where

$$\mathrm{tsf}^{\Gamma_3}_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) : A^*_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \to A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

is the transfer morphism and

$$h' : A^*_{A_3\ltimes T} \to A^*_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

is the obvious pullback. Moreover, by the proof of Lemma 3.3 and with the same notations, the elements

$$\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), c_3(\mathrm{Sym}^3 E), c_6(\mathrm{sl}_3)\} \subset A^*_{\mathrm{PGL}_3}$$

project to the three generators (Lemma 3.2) of $(A^*_T)^{S_3}$ through the composition

$$A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \xrightarrow{f} (A^*_T(\mathrm{Diag}^*_{\mathrm{sl}_3}))^{S_3} = (A^*_T)^{S_3}.$$

If we lift the elements

$$(\mathrm{tsf}^{\Gamma_3}_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \circ h')(\alpha c_3(W)),$$
$$(\mathrm{tsf}^{\Gamma_3}_{A_3\ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \circ h')(\underline{\chi}) \in A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}),$$

respectively to elements $\rho, \chi \in A^*_{\mathrm{PGL}_3}$, via the surjective pullback

$$A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}),$$

we find the following 5 generators of $A^*_{\mathrm{PGL}_3}$ coming from the open subscheme $U \subset \mathrm{sl}_3$ (through the first step (12) of the stratification of sl_3)

(34) $$\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), c_3(\mathrm{Sym}^3 E), \rho, \chi, c_6(\mathrm{sl}_3)\},$$

with $\deg \rho = 4$ and $\deg \chi = 6$.

In the following subsection we will determine the other generators of $A^*_{\mathrm{PGL}_3}$ coming from the complement $\mathrm{sl}_3 \backslash U$, starting from $Z_{1,1}$.

3.2. Generators coming from the complement of $U \subset \mathrm{sl}_3$. Consider again the first step of the stratification (11):

(35) $$A^*_{\mathrm{PGL}_3}(Z_{1,1}) \xrightarrow{(j_{1,1})_*} A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3 \backslash (Z_{1,0} \cup Z_0 \cup \{0\})) \xrightarrow{i^*_{1,1}} A^*_{\mathrm{PGL}_3}(U) \longrightarrow 0$$

where $(j_{1,1})_*$ has degree 1, equal to the codimension of $Z_{1,1}$ in sl_3.

Lemma 3.8. *If $A \in Z_{1,1}$, let $g \in \mathrm{PGL}_3$ be such that*

$$g^{-1}Ag = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix};$$

then, the rule

$$A \mapsto (\lambda, [g])$$

defines a PGL_3-equivariant isomorphism $Z_{1,1} \to \mathbb{A}^1 \backslash \{0\} \times \dfrac{\mathrm{PGL}_3}{\mathrm{U}_2 \times \mathbb{G}_m}$, where U_2 is the full unipotent subgroup of GL_2 and PGL_3 acts trivially on $\mathbb{A}^1 \backslash \{0\}$.

Proof. Everything is a straightforward verification left to the interested reader. We only note that the stabilizer of

$$\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$$

(under the adjoint action of PGL_3) is

$$\left\{ [g] | g = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \alpha, \gamma \in \mathbb{G}_m \right\}$$

which is obviously isomorphic to $\mathrm{U}_2 \times \mathbb{G}_m$. □

By Corollary 2.7, Prop. 2.8, 2.9 and Lemma 2.10, we have

(36) $$A^*_{\mathrm{PGL}_3}(Z_{1,1}) \simeq A^*_{\mathbb{G}_m} = \mathbb{Z}[u].$$

It is not difficult to verify that

$$j^*_{1,1}(2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E)) = u^2,$$

where we abused notation writing $2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E)$ for its pullback to

$$A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3 \backslash (Z_{1,0} \cup Z_0 \cup \{0\})).$$

Moreover

$$(j_{1,1})_*(1) = [Z_{1,1}] = D^*([\{0\}]) = 0$$

where $D : sl_3\backslash(Z_{1,0} \cup Z_0 \cup \{0\}) \to \mathbb{A}^1$ is the discriminant; so, by projection formula, the ideal $\mathrm{im}(j_{1,1})_*$ is generated by $(j_{1,1})_*(u)$.

Let $\Theta_{1,1}^{(2)}$ be a lift of $(j_{1,1})_*(u) \in A^*_{PGL_3}(sl_3\backslash(Z_{1,0} \cup Z_0 \cup \{0\}))$ to $A^*_{PGL_3}$. The analysis we made of (35) has the following upshot (recall (34)): *$A^*_{PGL_3}(sl_3\backslash(Z_{1,0} \cup Z_0 \cup \{0\}))$ is generated by (the images via $A^*_{PGL_3} \twoheadrightarrow A^*_{PGL_3}(sl_3\backslash(Z_{1,0} \cup Z_0 \cup \{0\}))$ of)*

$$\{2c_2(sl_3) - c_2(\mathrm{Sym}^3 E), \Theta_{1,1}^{(2)}, c_3(\mathrm{Sym}^3 E), \rho, \chi, c_6(sl_3)\}. \tag{37}$$

Now let us proceed one step further in the analysis of stratification (11); the second exact sequence of (10) is:

$$A^*_{PGL_3}(Z_{1,0}) \xrightarrow{(j_{1,0})_*} A^*_{PGL_3}(sl_3\backslash(Z_0 \cup \{0\})) \xrightarrow{i^*_{1,0}} A^*_{PGL_3}(sl_3\backslash(Z_{1,0} \cup Z_0 \cup \{0\})) \longrightarrow 0 \tag{38}$$

where $(j_{1,0})_*$ has degree 3, equal to the codimension of $Z_{1,0}$ in sl_3. We omit the straightforward proof of the following:

Lemma 3.9. *If $A \in Z_{1,0}$, let $g \in PGL_3$ be such that*

$$g^{-1}Ag = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}.$$

Then, the rule

$$A \mapsto (\lambda, [g])$$

defines a PGL_3-equivariant isomorphism $Z_{1,0} \to \mathbb{A}^1\backslash\{0\} \times \frac{PGL_3}{GL_2}$, where GL_2 injects as

$$\begin{pmatrix} GL_2 & 0 \\ 0 & 1 \end{pmatrix}$$

and PGL_3 acts trivially on $\mathbb{A}^1\backslash\{0\}$.

Then, by Prop. 2.9 and Lemma 2.10, we have[11)]

$$A^*_{PGL_3}(Z_{1,0}) \simeq A^*_{GL_2} = \mathbb{Z}[\lambda_1, \lambda_2]. \tag{39}$$

Lemma 3.10. *$(j_{1,0})_*$ is 3-torsion.*

Proof. If $\xi \in A^*_{PGL_3}(Z_{1,0})$, let $\hat{\xi} \in A^*_{PGL_3}$ be a lift of $(j_{1,0})_*(\xi)$ via the surjective pullback

11) $\lambda_i \doteq c_i$ (standard representation).

$$\pi_{1,0} : A^*_{PGL_3} \to A^*_{PGL_3}(sl_3\backslash Z_0 \cup \{0\}).$$

It is enough to prove that $\hat{\xi}$ is 3-torsion i.e. that

$$\hat{\xi} \in \ker(A^*_{PGL_3} \to (A^*_T)^{S_3}),$$

since by [4], Prop. 6, the rational pullback

$$A^*_{PGL_3} \otimes \mathbb{Q} \to (A^*_T)^{S_3} \otimes \mathbb{Q}$$

is an isomorphism and $A^*_{PGL_3}$ has only 3-torsion by Cor. 2.4.

Now, observe that

$$(j_{1,0})_*(\xi) \in \ker(A^*_{PGL_3}(sl_3\backslash Z_0 \cup \{0\}) \to A^*_{PGL_3}(sl_3\backslash Z_{1,0} \cup Z_0 \cup \{0\}))$$

by the obvious localization sequence and therefore

$$\hat{\xi} \in \ker(A^*_{PGL_3} \to A^*_{PGL_3}(U)),$$

by (35). To conclude, we note that $A^*_{PGL_3} \to (A^*_T)^{S_3}$ factors as

$$A^*_{PGL_3} \to A^*_{PGL_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{sl_3}) \to (A^*_T)^{S_3}. \quad \square$$

Proposition 3.11. *The ideal $\mathrm{im}(j_{1,0})_*$ is generated by*

$$\{(j_{1,0})_*(1), (j_{1,0})_*(\lambda_1), (j_{1,0})_*(\lambda_2), (j_{1,0})_*(\lambda_2^2), (j_{1,0})_*(\lambda_1\lambda_2), (j_{1,0})_*(\lambda_1\lambda_2^2)\}.$$

Proof. Identifying $A^*_{PGL_3}(Z_{1,0})$ with $A^*_{GL_2} = \mathbb{Z}[\lambda_1, \lambda_2]$ via (39) and writing $(\cdot)_{|GL_2}$ for $j^*_{1,0}$, one can easily verify that

$$(\lambda \equiv 2c_2(sl_3) - c_2(\mathrm{Sym}^3 E))_{|GL_2} = \lambda_1^2 - 3\lambda_2 \doteq \tau_2, \tag{40}$$

$$c_6(sl_3)_{|GL_2} = -\lambda_1^2\lambda_2^2 + 4\lambda_2^3 \doteq \tau_6. \tag{41}$$

Therefore

$$\lambda_2^3 = \tau_6 + \tau_2\lambda_2^2, \tag{42}$$

and if $\mathfrak{I}$ denotes the ideal generated by

$$\{(j_{1,0})_*(1), (j_{1,0})_*(\lambda_1), (j_{1,0})_*(\lambda_2), (j_{1,0})_*(\lambda_2^2), (j_{1,0})_*(\lambda_1\lambda_2), (j_{1,0})_*(\lambda_1\lambda_2^2)\},$$

we have

$$(j_{1,0})_*(\lambda_2^m) \in \mathfrak{I}, \quad \forall m \geqq 0, \tag{43}$$

by an easy induction on m, using projection formula.

Now, consider the general monomial $\lambda_1^n\lambda_2^m$. If $n = 2r$, we have

$$\lambda_1^n\lambda_2^m = (\tau_2 + 3\lambda_2)^r\lambda_2^m \equiv \tau_2^r\lambda_2^m \pmod 3 \tag{44}$$

and then

$$(j_{1,0})_*(\lambda_1^{2r}\lambda_2^m) = \lambda^r \cdot (j_{1,0})_*(\lambda_2^m),$$

by Lemma 3.10 and projection formula; thus

$$(j_{1,0})_*(\lambda_1^{2r}\lambda_2^m) \in \mathfrak{I}, \quad \forall m, r \geqq 0,$$

by (43). If $n = 2r + 1$, (44), (42) and projection formula easily reduce the assert

$$(j_{1,0})_*(\lambda_1^{2r+1}\lambda_2^m) \in \mathfrak{I}, \quad \forall m, r \geqq 0$$

to the assert

$$(j_{1,0})_*(\lambda_1\lambda_2^m) \in \mathfrak{I}, \quad \forall m \geqq 0,$$

which is easily proved by induction on m.

Since the monomials $\lambda_1^n\lambda_2^m$ generate $A^*_{\mathrm{GL}_2}$ as a $\mathbb{Z}$-module, we conclude that

$$\mathfrak{I} = \mathrm{im}(j_{1,0})_*. \quad \square$$

Therefore, if we denote by $\Theta_{1,0}^{(3)}$ (respectively, $\Theta_{1,0}^{(4)}$, $\Theta_{1,0}^{(5)}$, $\Theta_{1,0}^{(6)}$, $\Theta_{1,0}^{(7)}$, $\Theta_{1,0}^{(8)}$) a lift of $(j_{1,0})_*(1)$ (respectively, of $(j_{1,0})_*(\lambda_1)$, $(j_{1,0})_*(\lambda_2)$, $(j_{1,0})_*(\lambda_1\lambda_2)$, $(j_{1,0})_*(\lambda_2^2)$, $(j_{1,0})_*(\lambda_1\lambda_2^2)$) to $A^*_{\mathrm{PGL}_3}$, from (37) and (38) we get that $A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_0 \cup \{0\}))$ *is generated by* (*the images via* $A^*_{\mathrm{PGL}_3} \twoheadrightarrow A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_0 \cup \{0\}))$ *of*)

$$\begin{aligned}&\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), \Theta_{1,1}^{(2)}, c_3(\mathrm{Sym}^3 E),\\ &\Theta_{1,0}^{(3)}, \rho, \Theta_{1,0}^{(4)}, \Theta_{1,0}^{(5)}, \chi, \Theta_{1,0}^{(6)}, c_6(\mathrm{sl}_3), \Theta_{1,0}^{(7)}, \Theta_{1,0}^{(8)}\}.\end{aligned} \tag{45}$$

Let us proceed one step further in the analysis of stratification (11); the third exact sequence of (10), in our case is:

$$A^*_{\mathrm{PGL}_3}(Z_{0,1}) \xrightarrow{(j_{0,1})_*} A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_{0,0} \cup \{0\})) \xrightarrow{i^*_{0,1}} A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_0 \cup \{0\})) \longrightarrow 0 \tag{46}$$

where $(j_{0,1})_*$ has degree 2, equal to the codimension of $Z_{0,1}$ in sl_3. $Z_{0,1}$ is a PGL_3-orbit with stabilizer

$$\left\{ [g] \middle| g = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \alpha & 1 \end{pmatrix}, \alpha, \beta \in \mathbb{C} \right\}$$

which is unipotent and then, by Prop. 2.6, we have $A^*_{\mathrm{PGL}_3}(Z_{0,1}) = \mathbb{Z}$. If we denote by $\Theta_{0,1}^{(2)}$ a lift of $(j_{0,1})_*(1) = [Z_{0,1}] \in A^2_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_{0,0} \cup \{0\}))$ via the surjective pullback

$$A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_{0,0} \cup \{0\})),$$

from (46) and (45) we get that $A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_{0,0} \cup \{0\}))$ *is generated by* (*the images via* $A^*_{\mathrm{PGL}_3} \twoheadrightarrow A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_{0,0} \cup \{0\}))$ *of*)

$$\begin{aligned}&\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), \Theta_{1,1}^{(2)}, \Theta_{0,1}^{(2)}, c_3(\mathrm{Sym}^3 E),\\ &\Theta_{1,0}^{(3)}, \rho, \Theta_{1,0}^{(4)}, \Theta_{1,0}^{(5)}, \chi, \Theta_{1,0}^{(6)}, c_6(\mathrm{sl}_3), \Theta_{1,0}^{(7)}, \Theta_{1,0}^{(8)}\}.\end{aligned} \tag{47}$$

We have come to the second-last step of stratification (11):

$$A^*_{\mathrm{PGL}_3}(Z_{0,0}) \xrightarrow{(j_{0,0})_*} A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash\{0\}) \xrightarrow{i^*_{0,0}} A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash(Z_{0,0} \cup \{0\})) \longrightarrow 0 \tag{48}$$

where $(j_{0,0})_*$ has degree 4, equal to the codimension of $Z_{0,0}$ in sl_3. $Z_{0,0}$ is a PGL_3-orbit with stabilizer

$$\left\{ [g] \middle| g = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \delta & 0 \\ \beta & \gamma & 1 \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{C}, \delta \in \mathbb{G}_m \right\}$$

which is a split extension of $\mathbb{G}_m$ by the full unipotent group $\mathrm{U}_3 \subset \mathrm{GL}_3$. By Cor. 2.7 we get an isomorphism $A^*_{\mathrm{PGL}_3}(Z_{0,0}) \simeq A^*_{\mathbb{G}_m} = \mathbb{Z}[u]$. Since

$$j^*_{0,0}(2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E)) = u^2,$$

$A^*_{\mathrm{PGL}_3}(Z_{0,0})$ is generated by $\{(j_{0,0})_*(1), (j_{0,0})_*(u)\}$ as an $A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash\{0\})$-module (by projection formula) and if we denote by $\Theta_{0,0}^{(4)}$ (respectively, $\Theta_{0,0}^{(5)}$) a lift of $(j_{0,0})_*(1)$ (respectively, of $(j_{0,0})_*(u)$) to $A^*_{\mathrm{PGL}_3}$, we get that $A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash\{0\})$ *is generated by* (*the images via* $A^*_{\mathrm{PGL}_3} \twoheadrightarrow A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash\{0\})$ *of*)

$$\begin{aligned}&\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), \Theta_{1,1}^{(2)}, \Theta_{0,1}^{(2)}, c_3(\mathrm{Sym}^3 E),\\ &\Theta_{1,0}^{(3)}, \rho, \Theta_{1,0}^{(4)}, \Theta_{0,0}^{(4)}, \Theta_{1,0}^{(5)}, \Theta_{0,0}^{(5)}, \chi, \Theta_{1,0}^{(6)}, c_6(\mathrm{sl}_3), \Theta_{1,0}^{(7)}, \Theta_{1,0}^{(8)}\}.\end{aligned} \tag{49}$$

The last step of (10) for stratification (11) is immediate because

$$A^*_{\mathrm{PGL}_3}(\mathrm{sl}_3\backslash\{0\}) \simeq A^*_{\mathrm{PGL}_3}/(c_8(\mathrm{sl}_3))$$

by self-intersection formula ([6], p. 103).

Therefore we conclude our analysis of the stratification (11) with the following result:

Proposition 3.12. *$A^*_{\mathrm{PGL}_3}$ is generated by*

$$\begin{aligned}&\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), \Theta_{1,1}^{(2)}, \Theta_{0,1}^{(2)}, c_3(\mathrm{Sym}^3 E),\\ &\Theta_{1,0}^{(3)}, \rho, \Theta_{1,0}^{(4)}, \Theta_{0,0}^{(4)}, \Theta_{1,0}^{(5)}, \Theta_{0,0}^{(5)}, \chi, \Theta_{1,0}^{(6)}, c_6(\mathrm{sl}_3), \Theta_{1,0}^{(7)}, \Theta_{1,0}^{(8)}, c_8(\mathrm{sl}_3)\},\end{aligned} \tag{50}$$

where $\deg \Theta_{0,1}^{(2)} = 2$, $\deg \Theta_{1,1}^{(2)} = 2$, $\deg \rho = 4$, $\deg \chi = 6$, $\deg \Theta_{1,0}^{(m)} = m$ *and* $\deg \Theta_{0,0}^{(r)} = r$.

We will make this result more precise in the following section by getting rid of all the Θ generators.

4. $A^*_{\mathrm{PGL}_3}$ is not generated by Chern classes. Elimination of some generators

In this section we first prove that $A^*_{\mathrm{PGL}_3}$ is not generated by Chern classes and then that all its Θ generators are zero.

Lemma 4.1. *Writing* $H^i_{\mathrm{PGL}_3}$ *for* $H^i(\mathrm{BPGL}_3, \mathbb{Z})$, *we have:*

(51)

$H^0_{\mathrm{PGL}_3} \simeq \mathbb{Z}$	$H^1_{\mathrm{PGL}_3} = 0$
$H^2_{\mathrm{PGL}_3} = 0$	$H^3_{\mathrm{PGL}_3} \simeq \mathbb{Z}/3$
$H^4_{\mathrm{PGL}_3} \simeq \mathbb{Z}$	$H^5_{\mathrm{PGL}_3} = 0$
$H^6_{\mathrm{PGL}_3} \simeq \mathbb{Z}$	$H^7_{\mathrm{PGL}_3} = 0$
$H^8_{\mathrm{PGL}_3} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$	$H^9_{\mathrm{PGL}_3} = 0$
$H^{10}_{\mathrm{PGL}_3} \simeq \mathbb{Z}$	$H^{11}_{\mathrm{PGL}_3} \simeq \mathbb{Z}/3$
$H^{12}_{\mathrm{PGL}_3} \simeq \mathbb{Z} \oplus \mathbb{Z}$	$H^{13}_{\mathrm{PGL}_3} = 0$
$H^{14}_{\mathrm{PGL}_3} \simeq \mathbb{Z}$	$H^{15}_{\mathrm{PGL}_3} \simeq \mathbb{Z}/3$
$H^{16}_{\mathrm{PGL}_3} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3$	...

Proof. It is a routine computation using the Universal Coefficients' Formula for cohomology (e.g. [22]), once one knows the following facts:

1. $H^*(\mathrm{BPGL}_3, \mathbb{Q}) \simeq H^*(\mathrm{BSL}_3, \mathbb{Q}) = \mathbb{Q}[c_2(E), c_3(E)]$, E being the standard representation of SL_3;

2. $H^*(\mathrm{BPGL}_3, \mathbb{Z})$ has only 3-torsion;

3. there is a ring isomorphism

$$H^*(\mathrm{BPGL}_3, \mathbb{Z}/3) \simeq \mathbb{Z}/3[y_2, y_8, y_{12}] \otimes \Lambda(y_3, y_7)/(y_2y_3, y_2y_7, y_2y_8 + y_3y_7)$$

where $\deg y_i = i$.

1. follows immediately from the Leray spectral sequence

$$H^p(\mathrm{BPGL}_3, H^q(\mathrm{B}\mu_3, \mathbb{Z})) \Rightarrow H^{p+q}(\mathrm{BSL}_3, \mathbb{Z});$$

2. is proved in [14], p. 790 and 3. was computed in [13]. □

Theorem 4.2. $A^*_{\mathrm{PGL}_3}$ *is not generated by Chern classes; more precisely,* ρ *is not a polynomial in Chern classes.*

Proof. We proceed in 4 steps:

(I) First we show that $\mathrm{cl}(\rho)$ is nonzero in $H^8(\mathrm{BPGL}_3, \mathbb{Z})_{\mathrm{tors}}$, where

$$\mathrm{cl} : A^*_{\mathrm{PGL}_3} \to H^*(\mathrm{BPGL}_3, \mathbb{Z})$$

is the cycle class map and ρ is one of the generators of $A^*_{\mathrm{PGL}_3}$ (see Prop. 3.12);

(II) then we use a spectral sequence argument to show that

$$\mathrm{im}\big(H^8(\mathrm{BPGL}_3, \mathbb{Z}) \to H^8(\mathrm{BSL}_3, \mathbb{Z})\big)$$

has index at least 9 in $H^8(\mathrm{BSL}_3, \mathbb{Z}) \simeq \mathbb{Z}$;

(III) next, we use the fact that $c_2(\mathrm{sl}_3)^2 \mapsto 36\alpha_2^2$ via

$$H^8(\mathrm{BPGL}_3, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3 \to H^8(\mathrm{BSL}_3, \mathbb{Z}) \simeq \mathbb{Z} \cdot \alpha_2^2$$

to conclude that

$$H^8(\mathrm{BPGL}_3, \mathbb{Z}_{(3)}) \simeq \mathbb{Z}_{(3)} \cdot c_2(\mathrm{sl}_3)^2 \oplus \mathbb{Z}/3 \cdot \mathrm{cl}(\rho)$$

(where we have written α in place of $j_\bullet(\alpha)$, with $j_\bullet : H^*(\mathrm{BPGL}_3, \mathbb{Z}) \to H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(3)})$ induced by the localization $j : \mathbb{Z} \to \mathbb{Z}_{(3)}$ and $\alpha_i \doteq c_i$ (standard repr. of SL_3));

(IV) finally we show that $\mathrm{cl}(\rho) \in H^8(\mathrm{BPGL}_3, \mathbb{Z})$ is not in the Chern subring of $H^*(\mathrm{BPGL}_3, \mathbb{Z})$ (implying that ρ itself is not in the Chern subring of $A^*_{\mathrm{PGL}_3}$).

(I) We freely use Remark 3.1. Recall that ρ is a lift to $A^*_{\mathrm{PGL}_3}$ of

$$\big(\alpha c_3(W)\big)_{|\mathrm{Diag}^*_{\mathrm{sl}_3}} \in {}_3\big(A^*_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})\big)^{C_2} \simeq {}_3A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \simeq {}_3A^*_{\mathrm{PGL}_3}(U),$$

with $\alpha c_3(W) \in {}_3A^*_{A_3 \ltimes T}$. To prove (I) it is then enough to show that

$$\mathrm{cl}\big(\alpha c_3(W)\big)_{|\mathrm{Diag}^*_{\mathrm{sl}_3}} \neq 0 \quad \text{in } H^8_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}, \mathbb{Z}).$$

If $A_3 \times \mu_3 \hookrightarrow A_3 \ltimes T$, we will get this by showing

$$\mathrm{cl}\big(\alpha c_3(W)\big)_{|\mathrm{Diag}^*_{\mathrm{sl}_3}} \neq 0 \quad \text{in } H^8_{A_3 \times \mu_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}, \mathbb{Z}) \tag{52}$$

(writing again $\mathrm{cl}\big(\alpha c_3(W)\big)$ for its restriction to $H^8_{A_3 \times \mu_3}$). Let us consider the localization exact sequences for cohomology, corresponding to $\mathrm{Diag}_{\mathrm{sl}_3} \supset \mathrm{Diag}_{\mathrm{sl}_3} \setminus \{0\} \supset \mathrm{Diag}^*_{\mathrm{sl}_3}$

$$H^4_{A_3 \times \mu_3} \xrightarrow{\cdot(-\alpha^2)} H^8_{A_3 \times \mu_3}(\mathrm{Diag}_{\mathrm{sl}_3}, \mathbb{Z}) \xrightarrow{p} H^8_{A_3 \times \mu_3}(\mathrm{Diag}_{\mathrm{sl}_3} \setminus \{0\}, \mathbb{Z}), \tag{53}$$

$$H^6_{\mu_3} \simeq H^6_{A_3 \times \mu_3}(Z, \mathbb{Z}) \xrightarrow{i_*} H^8_{A_3 \times \mu_3}(\mathrm{Diag}_{\mathrm{sl}_3} \setminus \{0\}, \mathbb{Z}) \xrightarrow{q} H^8_{A_3 \times \mu_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}, \mathbb{Z}) \tag{54}$$

where $i : Z \doteq (\mathrm{Diag}_{\mathfrak{sl}_3}\backslash\{0\})\backslash\mathrm{Diag}^*_{\mathfrak{sl}_3} \hookrightarrow \mathrm{Diag}_{\mathfrak{sl}_3}\backslash\{0\}$ and we used that $Z \simeq A_3 \times \mathbb{C}^*$, $A_3 \ltimes T$-equivariantly. If $\mathbb{C}_{\chi,\mu_3}$ (respectively, $\mathbb{C}^3_{\mathrm{perm},A_3}$) denotes the μ_3-representation given by multiplication by the character $\chi = \exp(i2\pi/3)$ (respectively, the A_3-permutation representation), we have $W \simeq \mathbb{C}_{\chi,\mu_3} \boxtimes \mathbb{C}^3_{\mathrm{perm},A_3}$ as $A_3 \times \mu_3$-representations. Then, if we let

$$H^*_{\mu_3} = \mathbb{Z}[\beta]/(3\beta), \quad H^*_{A_3} = \mathbb{Z}[\alpha]/(3\alpha),$$

the Chern roots of W are $\{\beta + \alpha, \beta - \alpha, \beta\}$ and

$$\mathrm{cl}(\alpha c_3(W)) = (\beta^2 - \alpha^2)\alpha\beta \in H^8_{A_3 \times \mu_3}.$$

Now we claim $i_* = 0$ in (54). In fact, consider the pullback E of $\mathbb{C}_{\chi,\mu_3}$ to $\mathrm{Diag}_{\mathfrak{sl}_3}\backslash\{0\}$ and view E as an $A_3 \times \mu_3$-equivariant vector bundle on $\mathrm{Diag}_{\mathfrak{sl}_3}\backslash\{0\}$, with A_3 acting trivially on E. Obviously, $i^*(c_1(E)) = \beta$. But we also have $i_*(1) = 0$ since

$$Z = D^{-1}(\{0\}),$$

where

$$D : \mathrm{Diag}_{\mathfrak{sl}_3}\backslash\{0\} \to \mathbb{A}^1,$$

(55) $$(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$$

is the square root of the discriminant (which is $A_3 \times \mu_3$-equivariant!). By projection formula, $i_* = 0$ and q is injective.

So, we are left to show that $p((\beta^2 - \alpha^2)\alpha\beta) = p(\alpha\beta^3) \neq 0$ in (53). Now observe that

$$H^{2n}_{A_3 \times \mu_3} \simeq (H^*_{A_3} \otimes H^*_{\mu_3})^{2n}$$

by Künneth formula, since

$$\bigoplus_{p+q=2n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^p_{A_3}, H^q_{\mu_3}) = 0$$

(either p or q being odd in every summand). So

$$H^8_{A_3 \times \mu_3} = \mathbb{Z}/3\langle \alpha^4, \alpha^3\beta, \alpha^2\beta^2, \alpha\beta^3, \beta^4 \rangle,$$
$$H^4_{A_3 \times \mu_3} = \mathbb{Z}/3\langle \alpha^2, \alpha\beta, \beta^2 \rangle$$

and $\alpha\beta^3 \notin \mathrm{im}(\cdot(-\alpha^2))$ i.e. $p(\alpha\beta^3) \neq 0$.

(II) Consider the Leray spectral sequence

$$E_2^{pq} = H^p(\mathrm{BPGL}_3, H^q(\mathrm{B}\mu_3, \mathbb{Z})) \Rightarrow H^{p+q}(\mathrm{BSL}_3, \mathbb{Z}).$$

By Lemma 4.1, its (first quadrant) E_2-term[12] is:

$\cdots$											$\vdots$
α^4	0	$\alpha^3\xi_2$	$\alpha^4 y_3$	$\alpha^4 y_4$	0	$\alpha^4 y_6$	$\alpha^3\xi_7$	$\alpha^4 x_8, \alpha^4 y_8$			
0	0	0	0	0	0	0	0	0	0	0	
α^3	0	$\alpha^2\xi_2$	$\alpha^3 y_3$	$\alpha^3 y_4$	0	$\alpha^3 y_6$	$\alpha^2\xi_7$	$\alpha^3 x_8, \alpha^3 y_8$			
0	0	0	0	0	0	0	0	0	0	0	
α^2	0	$\alpha\xi_2$	$\alpha^2 y_3$	$\alpha^2 y_4$	0	$\alpha^2 y_6$	$\alpha\xi_7$	$\alpha^2 x_8, \alpha^2 y_8$			
0	0	0	0	0	0	0	0	0	0	0	
α	0	ξ_2	αy_3	αy_4	0	αy_6	ξ_7	$\alpha x_8, \alpha y_8$			
0	0	0	0	0	0	0	0	0	0	0	
$\mathbb{Z}$	0	0	$y_3\mathbb{Z}/3$	$y_4\mathbb{Z}/3$	0	$y_6\mathbb{Z}$	0	$x_8\mathbb{Z} \oplus y_8\mathbb{Z}/3$	0	$\mathbb{Z}$	$\cdots$

where from the second row up, the coefficients are in $\mathbb{Z}/3$.

One of the edge maps is

$$H^8(\mathrm{BPGL}_3, \mathbb{Z}) = E_2^{8,0} \twoheadrightarrow E_\infty^{8,0} = F^8H^8(\mathrm{BSL}_3, \mathbb{Z}) \hookrightarrow H^8(\mathrm{BSL}_3, \mathbb{Z})$$

so we have to show that $F^8H^8(\mathrm{BSL}_3, \mathbb{Z})$ has index at least 9 in

$$H^8(\mathrm{BSL}_3, \mathbb{Z}) \simeq \mathbb{Z} \cdot \alpha_2^2.$$

First of all, note that $d_{(3)}(\alpha) = \pm y_3$ since

$$E_\infty^{3,0} = F^3H^3(\mathrm{BSL}_3, \mathbb{Z}) \hookrightarrow H^3(\mathrm{BSL}_3, \mathbb{Z}) = 0$$

and both α and y_3 are 3-torsion; we choose y_3 to have the plus sign. Therefore

$$d_{(3)}(\alpha^2 y_3) = 2\alpha y_3^2 + \alpha^2 d_{(3)}(y_3) = 0$$

since y_3^2 is 3-torsion in $H^6(\mathrm{BPGL}_3, \mathbb{Z}) \simeq \mathbb{Z}$, hence is zero.

Then

$$E_2^{62} = E_3^{62} = E_4^{62} = E_\infty^{62} \simeq \mathbb{Z}/3$$

and we have the first 3 factors of the desired index. Finally we have

$$d_{(3)}(\alpha^2 y_4) = 2\alpha y_3 y_4 + \alpha^2 d_{(3)}(y_4) = 0$$

[12] We write only the parts we'll need.

since $y_3y_4 \in H^7(\mathrm{BPGL}_3, \mathbb{Z}) = 0$; then

$$E_2^{44} = E_3^{44} = E_4^{44} = E_\infty^{44} \simeq \mathbb{Z}/3$$

yielding the other 3 factors in the index of $F^8H^8(\mathrm{BSL}_3, \mathbb{Z})$ in $H^8(\mathrm{BSL}_3, \mathbb{Z}) \simeq \mathbb{Z} \cdot \alpha_2^2$.

(III) As already observed, we have $c_2(\mathrm{sl}_3)^2 \mapsto 36\alpha_2^2$ via the pull back (use (I))

$$\phi : \mathbb{Z} \oplus \mathrm{cl}(\rho) \cdot (\mathbb{Z}/3) \simeq H^8(\mathrm{BPGL}_3, \mathbb{Z}) \to H^8(\mathrm{BSL}_3, \mathbb{Z}) \simeq \mathbb{Z} \cdot \alpha_2^2$$

whose kernel is 3-torsion; combining this with (II), we get that the image of ϕ has exactly index 9. Therefore

$$H^8(\mathrm{BPGL}_3, \mathbb{Z}_{(3)}) \simeq \mathbb{Z}_{(3)} \cdot j_\bullet(c_2(\mathrm{sl}_3)^2) \oplus (\mathbb{Z}/3) \cdot j_\bullet(\mathrm{cl}(\rho)),$$

where $j_\bullet : H^*(\mathrm{BPGL}_3, \mathbb{Z}) \to H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(3)})$ is the morphism induced by the localization $j : \mathbb{Z} \to \mathbb{Z}_{(3)}$.

(IV) By [14], Cor. 4.7, we know that

$$H^8(\mathrm{BPGL}_3, \mathbb{Z}/3) \simeq (\mathbb{Z}/3) \cdot y_2^4 \oplus (\mathbb{Z}/3) \cdot y_8,$$

and that the second generator y_8 is not in the Chern subring of $H^*(\mathrm{BPGL}_3, \mathbb{Z}/3)$. By the Bockstein exact sequence, the natural map

$$(j_{(3)})_\bullet : H^8(\mathrm{BPGL}_3, \mathbb{Z}_{(3)}) \to H^8(\mathrm{BPGL}_3, \mathbb{Z}/3)$$

is surjective since $H^9(\mathrm{BPGL}_3, \mathbb{Z}_{(3)}) = 0$. Therefore there exists an element

$$\xi = \alpha j_\bullet(c_2(\mathrm{sl}_3)^2) + \beta j_\bullet(\mathrm{cl}(\rho)) \in H^8(\mathrm{BPGL}_3, \mathbb{Z}_{(3)})$$

such that $(j_{(3)})_\bullet(\xi) = y_8$. In particular, $\mathrm{cl}(\rho)$ cannot be in the Chern subring of

$$H^*(\mathrm{BPGL}_3, \mathbb{Z}). \quad \square$$

Remark 4.1. For a different proof of Theorem 4.2, which does not depend on Kono-Yagita's results on $H^*(\mathrm{BPGL}_3, \mathbb{Z}/3)$ (and in fact does not depend on cohomology at all), see the Appendix.

Lemma 4.3. *$\Theta_{1,1}^{(2)}, \Theta_{0,1}^{(2)}, \Theta_{1,0}^{(3)}, \Theta_{1,0}^{(4)}, \Theta_{0,0}^{(4)}, \Theta_{1,0}^{(5)}, \Theta_{0,0}^{(5)}, \Theta_{1,0}^{(6)}, \Theta_{1,0}^{(7)}$ and $\Theta_{1,0}^{(8)}$ are 3-torsion.*

Proof. All the Θ's are supported on the complement of U and so they all go to zero via $A_{\mathrm{PGL}_3}^* \to A_T^*$, since this map factors through $A_{\mathrm{PGL}_3}^* \to A_{\mathrm{PGL}_3}^*(U)$. But, by [4], Prop. 6, the rational pullback

$$A_{\mathrm{PGL}_3}^* \otimes \mathbb{Q} \to (A_T^*)^{S_3} \otimes \mathbb{Q}$$

is an isomorphism, so the Θ's are torsion and hence 3-torsion by Cor. 2.4. $\square$

Remark 4.2. Note that $\mathrm{cl}(\chi) = 0$ since χ is torsion while $H^{12}(\mathrm{BPGL}_3, \mathbb{Z})$ is torsion free by Lemma 4.1.

Lemma 4.4. *$\Theta_{1,0}^{(4)}$ and $\Theta_{0,0}^{(4)}$ are in the kernel of the cycle map*

$$\mathrm{cl} : A_{\mathrm{PGL}_3}^* \to H^*(\mathrm{BPGL}_3, \mathbb{Z}).$$

Proof. By part (I) of the proof of Th. 4.2, $\mathrm{cl}(\rho)$ generates the 3-torsion of

$$H^8(\mathrm{BPGL}_3, \mathbb{Z})$$

and moreover $\mathrm{cl}(\rho)_{|U} \neq 0$ in $H_{\mathrm{PGL}_3}^8(U, \mathbb{Z})$, where $U \subset \mathrm{sl}_3$ is the open subscheme of matrices with distinct eigenvalues. Since $\Theta_{1,0}^{(4)}$ and $\Theta_{0,0}^{(4)}$ are both 3-torsion in $A_{\mathrm{PGL}_3}^4$, we must have

$$\mathrm{cl}(\Theta_{1,0}^{(4)}) = A \cdot \mathrm{cl}(\rho),$$
$$\mathrm{cl}(\Theta_{0,0}^{(4)}) = B \cdot \mathrm{cl}(\rho).$$

But $\Theta_{1,0}^{(4)}$ and $\Theta_{0,0}^{(4)}$ have supports in the complement of U, so $A = B = 0$. $\square$

Remark 4.3. Note that also the generator $\Theta_{1,0}^{(8)}$ can be chosen in such a way that

$$\mathrm{cl}(\Theta_{1,0}^{(8)}) = 0.$$

In fact $c_8(\mathrm{sl}_3) \neq 0$ in $H^{16}(\mathrm{BPGL}_3, \mathbb{Z})$ by [14], Lemma 3.18 and

$$H^{16}(\mathrm{BPGL}_3, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3$$

(Lemma. 4.1), therefore

$$\mathrm{cl}(\Theta_{1,0}^{(8)}) = Ac_8(\mathrm{sl}_3).$$

Now observe that

$$c_8(\mathrm{sl}_3)_{|\mathrm{sl}_3 \setminus Z_0 \cup \{0\}} = 0$$

while $\Theta_{1,0}^{(8)}$ is a lift of $(j_{1,0})_*(\lambda_1\lambda_2^2)$ where

$$j_{1,0} : Z_{1,0} \hookrightarrow \mathrm{sl}_3 \setminus Z_0 \cup \{0\};$$

thus we can choose a lift $\Theta_{1,0}^{(8)}$ such that $A = 0$.

Proposition 4.5. *The elements*

$$\{\Theta_{1,1}^{(2)}, \Theta_{0,1}^{(2)}, \Theta_{1,0}^{(3)}, \Theta_{1,0}^{(4)}, \Theta_{0,0}^{(4)}, \Theta_{1,0}^{(5)}, \Theta_{0,0}^{(5)}, \Theta_{1,0}^{(6)}, \Theta_{1,0}^{(7)}, \Theta_{1,0}^{(8)}\}$$

are all zero in $A_{\mathrm{PGL}_3}^*$.

Proof. We first prove that

$$\Theta^{(2)}_{1,1}=\Theta^{(2)}_{0,1}=\Theta^{(3)}_{1,0}=\Theta^{(4)}_{1,0}=\Theta^{(4)}_{0,0}=\Theta^{(5)}_{1,0}=\Theta^{(5)}_{0,0}=\Theta^{(7)}_{1,0}=0. \tag{56}$$

Consider the commutative diagram

$$\begin{array}{ccc} A^*_{\mathrm{PGL}_3} & \xrightarrow{\ \mathrm{cl}\ } & H^*(\mathrm{BPGL}_3,\mathbb{Z}) \\ \downarrow & & \downarrow \\ A^*_{\Gamma_3} & \xrightarrow[\ \mathrm{cl}\ ]{} & H^*(\mathrm{B}\Gamma_3,\mathbb{Z}) \end{array}$$

where the vertical arrows are injective by Theorem 2.1. We know that

$$\Theta^{(2)}_{1,1},\ \Theta^{(2)}_{0,1},\ \Theta^{(3)}_{1,0},\ \Theta^{(4)}_{1,0},\ \Theta^{(4)}_{0,0},\ \Theta^{(5)}_{1,0},\ \Theta^{(5)}_{0,0},\ \Theta^{(7)}_{1,0}$$

are 3-torsion and zero in cohomology (Lemmas 4.3, 4.1 and 4.4), so (56) will be proved if we show that

$${}_3\mathrm{cl}: {}_3A^*_{\Gamma_3}\to {}_3H^*(\mathrm{B}\Gamma_3,\mathbb{Z})$$

is injective up to degree 5 and in degree 7. But, by the usual transfer-trick, the restriction induces isomorphisms

$${}_3A^*_{\Gamma_3}\simeq ({}_3A^*_{A_3\ltimes T})^{C_2},\quad {}_3H^*(\mathrm{B}\Gamma_3,\mathbb{Z})\simeq \big({}_3H^*(\mathrm{B}(A_3\ltimes T),\mathbb{Z})\big)^{C_2}$$

and it will be (more than) enough to show that

$$\mathrm{cl}: A^*_{A_3\ltimes T}\to H^*(\mathrm{B}(A_3\ltimes T),\mathbb{Z})$$

is injective up to degree 5 and in degree 7.

Recall (Prop. 3.5) that $A^*_{A_3\ltimes T}$ is generated by

$$\{\alpha, c_2(W), c_3(W), \theta \doteq \mathrm{tsf}_T^{A_3\ltimes T}(u_2^2u_3)\} \tag{57}$$

where W is the representation defined in (20) and we identify A^*_T with

$$A^*_{T_{\mathrm{SL}_3}}\simeq \mathbb{Z}[u_1,u_2,u_3]/(u_1+u_2+u_3);$$

moreover (see Lemma 3.6), we have

$$3\alpha=0, \alpha\theta=0,\quad \alpha^3+\alpha c_2(W)=0, \tag{58}$$
$$3\big[(2\theta+3c_3(W))^2+4c_2(W)^3+27c_3(W)^2\big]=0.$$

For the duration of this proof, we will denote $c_2(W)$ and $c_3(W)$ simply by c_2 and c_3; moreover, if $\xi\in A^*_{A_3\ltimes T}$, we will write $\bar\xi$ for $\mathrm{cl}(\xi)$.

As shown in the proof of Th. 4.2, we have

$$H^{2n}(\mathrm{B}(A_3\times\mu_3),\mathbb{Z})\simeq \big(H^*(\mathrm{B}A_3,\mathbb{Z})\otimes(\mathrm{B}\mu_3,\mathbb{Z})\big)^{2n}$$

and

$$\overline{c_2(W)_{|A_3\times\mu_3}}=-\bar\alpha^2,\quad \overline{c_3(W)_{|A_3\times\mu_3}}=\bar\beta(\bar\beta^2-\bar\alpha^2) \tag{59}$$

where

$$H^*(\mathrm{B}A_3,\mathbb{Z})=\frac{\mathbb{Z}[\bar\alpha]}{(\bar\alpha)},\quad H^*(\mathrm{B}\mu_3,\mathbb{Z})=\frac{\mathbb{Z}[\bar\beta]}{(3\bar\beta)}.$$

In the following computations we will freely use that the cycle class map respects Chern classes, restrictions and transfers and that

$$\mathrm{cl}: A^*_T\to H^*(\mathrm{B}T,\mathbb{Z})$$

is an isomorphism.

If $\xi\in\ker\mathrm{cl}\cap A^1_{A_3\ltimes T}$, we have

$$\xi=A\alpha\quad\text{and}\quad A\bar\alpha=0$$

for some $A\in\mathbb{Z}$; restricting this to A_3 (in cohomology) we then get $A\equiv 0 \bmod 3$, hence $\xi=0$.

If $\xi\in\ker\mathrm{cl}\cap A^2_{A_3\ltimes T}$, we have

$$\xi=A\alpha^2+Bc_2,\quad A\bar\alpha^2+B\bar c_2=0$$

for some $A,B\in\mathbb{Z}$; restricting to T, we get $B=0$ then, restricting to A_3, we get $A\equiv 0\bmod 3$. Therefore, $\xi=0$.

If $\xi\in\ker\mathrm{cl}\cap A^3_{A_3\ltimes T}$, we have

$$\xi=A\alpha^3+Bc_3+C\theta,$$
$$A\bar\alpha^3+B\bar c_3+C\bar\theta=0$$

for some $A,B,C\in\mathbb{Z}$; restricting to T, we get $B=C=0$ since $\bar c_{3|T}$ and $\bar\theta_{|T}$ are linearly independent in $H^*(\mathrm{B}T,\mathbb{Z})$. Restricting then to A_3, we get $A\equiv 0\bmod 3$, hence $\xi=0$.

If $\xi\in\ker\mathrm{cl}\cap A^4_{A_3\ltimes T}$, we have

$$\xi=A\alpha^4+B\alpha c_3+Cc_2^2,$$
$$A\bar\alpha^4+B\overline{\alpha c_3}+C\bar c_2^{\,2}=0$$

for some $A,B,C\in\mathbb{Z}$; restricting to T, we get $C=0$. Restricting then to $A_3\times\mu_3$, from (59) we get $B\equiv A\equiv 0\bmod 3$, hence $\xi=0$.

If $\xi \in \ker \mathrm{cl} \cap A^5_{A_3 \ltimes T}$, we have

$$\xi = A\alpha^5 + B\alpha^2 c_3 + Cc_2c_3,$$
$$A\bar{\alpha}^5 + B\bar{\alpha}^2\bar{c_3} + C\overline{c_2c_3} = 0$$

for some $A, B, C \in \mathbb{Z}$; restricting to T, we get $C = 0$. Restricting then to $A_3 \times \mu_3$, from (59) we get $B \equiv A \equiv 0 \bmod 3$, hence $\xi = 0$.

Finally, if $\xi \in \ker \mathrm{cl} \cap A^7_{A_3 \ltimes T}$, we have

$$\xi = A\alpha^7 + B\alpha^4 c_3 + Cc_2^2 c_3 + Dc_2^2\theta + E\alpha c_3^2,$$
$$A\bar{\alpha}^7 + B\bar{\alpha}^4\bar{c_3} + C\bar{c_2}^2\bar{c_3} + D\bar{c_2}^2\bar{\theta} + E\bar{\alpha}\bar{c_3}^2 = 0$$

for some $A, B, C, D, E \in \mathbb{Z}$; restricting to T, we get $C = D = 0$ since $\bar{c}_{2|T} \neq 0$ and $(\bar{c}_{3|T}, \bar{\theta}_{|T})$ are linearly independent in the domain $H^*(BT, \mathbb{Z})$. Restricting then to $A_3 \times \mu_3$, from (59) we get $A \equiv B \equiv E \equiv 0 \bmod 3$, hence $\xi = 0$. This concludes the proof of (56).

Now we prove the remaining relations

$$\Theta^{(6)}_{1,0} = \Theta^{(8)}_{1,0} = 0. \tag{60}$$

First observe that $\Theta^{(6)}_{1,0}$ and $\Theta^{(8)}_{1,0}$ are 3-torsion and zero in cohomology (with $\Theta^{(8)}_{1,0}$ chosen as in Remark 4.3). Since they are lifts of elements having supports in the complement of $U \subset \mathrm{sl}_3$, their restrictions to $A^*_{\Gamma_3}$ are in the kernel of

$$A^*_{\Gamma_3} \to A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \simeq A^*_{\mathrm{PGL}_3}(U)$$

and in particular:

$$\{\Theta^{(6)}_{1,0|A_3 \ltimes T}, \Theta^{(8)}_{1,0|A_3 \ltimes T}\} \subset \ker(g : A^*_{A_3 \ltimes T} \to A^*_{A_3 \ltimes T}(\widetilde{\mathrm{Diag}}^*_{\mathrm{sl}_3})).$$

By Lemma 3.6 (ii) and (57), (58), we must have

$$\Theta^{(6)}_{1,0|A_3 \ltimes T} = \alpha^2(A\alpha^4 + B\alpha c_3 + Cc_2^3),$$
$$\Theta^{(8)}_{1,0|A_3 \ltimes T} = \alpha^2(D\alpha^6 + Ec_2^3 + Fc_3^2 + G\alpha^3 c_3)$$

for some $A, \dots, E \in \mathbb{Z}$. Using again (58), we get

$$\Theta^{(6)}_{1,0|A_3 \ltimes T} = A'\alpha^6 + B\alpha^3 c_3,$$
$$\Theta^{(8)}_{1,0|A_3 \ltimes T} = D'\alpha^6 + Fc_3^2 + G\alpha^3 c_3$$

for some $A', B, D', F, G \in \mathbb{Z}$. Again denoting $\mathrm{cl}(\xi)$ by $\bar{\xi}$ for $\xi \in A^*_{A_3 \ltimes T}$, we have

$$0 = \overline{\Theta^{(6)}_{1,0|A_3 \ltimes T}} = A'\bar{\alpha}^6 + B\bar{\alpha}^3\bar{c_3},$$
$$0 = \overline{\Theta^{(8)}_{1,0|A_3 \ltimes T}} = D'\bar{\alpha}^6 + F\bar{c_3}^2 + G\bar{\alpha}^3\bar{c_3}$$

in $H^*(B(A_3 \ltimes T), \mathbb{Z})$. Restricting these relations to $A_3 \times \mu_3$, by (59) we obtain:

$$A' \equiv B \equiv 0 \bmod 3,$$
$$D' \equiv F \equiv G \equiv 0 \bmod 3$$

i.e.

$$\Theta^{(6)}_{1,0|A_3 \ltimes T} = \Theta^{(8)}_{1,0|A_3 \ltimes T} = 0 \tag{61}$$

in $A^*_{A_3 \ltimes T}$. But the restriction map induces an isomorphism

$${}_3A^*_{\Gamma_3} \simeq ({}_3A^*_{A_3 \ltimes T})^{C_2}$$

and then, we also get

$$\Theta^{(6)}_{1,0|\Gamma_3} = \Theta^{(8)}_{1,0|\Gamma_3} = 0$$

in $A^*_{\Gamma_3}$. By Theorem 2.1 we finally get (60). □

Thus we can summarize the main result obtained so far in the following:

Theorem 4.6. *With the notation of* (50), $A^*_{\mathrm{PGL}_3}$ *is generated by*

$$\{2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), c_3(\mathrm{Sym}^3 E), \rho, \chi, c_6(\mathrm{sl}_3), c_8(\mathrm{sl}_3)\} \tag{62}$$

where $\deg \rho = 4$, $\deg \chi = 6$.

Remark 4.4. We point out that

$$2c_2(\mathrm{sl}_3) - c_2(\mathrm{Sym}^3 E), \quad c_3(\mathrm{Sym}^3 E), \quad c_6(\mathrm{sl}_3), \quad c_8(\mathrm{sl}_3)$$

are nonzero (by checking their images in $A^*_{\mathrm{SL}_3}$ or in cohomology) and we will show in the next section that $\rho \neq 0$. Unfortunately, we do not know whether χ is zero or not.

Note also that the generators ρ and χ, defined originally as lifts from the open subset U (therefore not unique *a priori*) are indeed uniquely defined since they have degrees <8 and $c_8(\mathrm{sl}_3)$ is the only generator coming from the complement of U.

5. Other relations and results on the cycle maps

With the notations established in the preceding sections we have:

Proposition 5.1. *The following relations hold among the generators of* $A^*_{\mathrm{PGL}_3}$:

$$3\rho = 3\chi = 3c_8(\mathrm{sl}_3) = 0,$$
$$3(27c_6(\mathrm{sl}_3) - c_3(\mathrm{Sym}^3 E)^2 - 4\lambda^3) = 0,$$
$$\rho^2 = c_8(\mathrm{sl}_3).$$

Proof. The pullback $\varphi : A^*_{\mathrm{PGL}_3} \to A^*_T$ factors through the composition

$$\pi : A^*_{\mathrm{PGL}_3} \twoheadrightarrow A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \twoheadrightarrow A^*_T(\mathrm{Diag}^*_{\mathrm{sl}_3})^{S_3} = (A^*_T)^{S_3},$$

and, by definition of χ and ρ, $\pi(\chi) = \pi(\rho) = 0$. Since ([4], Prop. 6) the rational pullback $\varphi_{\mathbb{Q}}$ is an isomorphism, χ and ρ are torsion and then 3-torsion by Cor. 2.4.

Since $\mathrm{sl}_3 = E \otimes E^\vee - \mathbf{1}$, as SL_3-representations (E being the standard representation), $c_8(\mathrm{sl}_3)$ is in the kernel of $A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{SL}_3}$, so it is 3-torsion (Prop. 2.3).

A long but straightforward computation[13] shows that

$$27c_6(\mathrm{sl}_3) - c_3(\mathrm{Sym}^3 E)^2 - 4\lambda^3 \in \ker(A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{SL}_3})$$

so that this element is 3-torsion (again by Prop. 2.3).

By definition of ρ and Lemma 3.6, we have

$$\rho_{|A_3 \ltimes T} = \alpha c_3(W) + A\alpha^4 \tag{63}$$

for some $A \in \mathbb{Z}/3$. Since

$$_3A^*_{\Gamma_3} \simeq ({}_3A^*_{A_3 \ltimes T})^{C_2},$$

by Lemma 3.6 (ii), ρ^2 belongs to the kernel of

$$A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{PGL}_3}(U) \simeq A^*_{\Gamma_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}).$$

Therefore, since by Proposition 4.5 all the generators of $A^*_{\mathrm{PGL}_3}$ coming from the complement of U are zero except for $c_8(\mathrm{sl}_3)$, we have

$$\rho^2 = Bc_8(\mathrm{sl}_3) \tag{64}$$

for some $B \in \mathbb{Z}/3$.

Let us determine A and B. Since $c_8(\mathrm{sl}_3)_{|A_3} = 0$, from (63) and (64), we get $A = 0$ i.e.

$$\rho_{|A_3 \ltimes T} = \alpha c_3(W). \tag{65}$$

Straightforward computations show that

$$c_8(\mathrm{sl}_3)_{|A_3 \times \mu_3} = \alpha^2\beta^2(\beta^2 - \alpha^2)^2,$$
$$c_3(W)_{|A_3 \times \mu_3} = \beta(\beta^2 - \alpha^2)$$

[13] The basic fact here is that $c_6(\mathrm{sl}_3)$ restricts to minus the discriminant, $4\alpha_2^3 + 27\alpha_3^2$, in $A^*_{\mathrm{SL}_3} = \mathbb{Z}[\alpha_2, \alpha_3]$, where $\alpha_i = c_i(E)$.

in $A^*_{A_3 \times \mu_3} = A^*_{A_3} \otimes A^*_{\mu_3} = \mathbb{Z}[\alpha]/(3\alpha) \otimes \mathbb{Z}[\beta]/(3\beta)$ and then (64) and (65) prove that $B = 1$. □

We define the graded ring

$$R^* \doteq \frac{\mathbb{Z}[\lambda, c_3(\mathrm{Sym}^3 E), \rho, \chi, c_6(\mathrm{sl}_3), c_8(\mathrm{sl}_3)]}{\mathfrak{R}}$$

where

$$\mathfrak{R} \doteq (3\rho, 3\chi, 3c_8(\mathrm{sl}_3), 3(27c_6(\mathrm{sl}_3) - c_3(\mathrm{Sym}^3 E)^2 - 4\lambda^3), \rho^2 - c_8(\mathrm{sl}_3))$$

and $\deg \rho = 4$, $\deg \chi = 6$.

This is our candidate for $A^*_{\mathrm{PGL}_3}$. What we do know is that the canonical morphism

$$\pi : R^* \to A^*_{\mathrm{PGL}_3}$$

is surjective (Th. 4.6).

Remark 5.1. Note that it is immediately clear that $\pi_{\mathbb{Q}} : R^* \otimes \mathbb{Q} \to A^*_{\mathrm{PGL}_3} \otimes \mathbb{Q}$ is an isomorphism. In fact

$$R^* \otimes \mathbb{Q} = \frac{\mathbb{Q}[\lambda, c_3(\mathrm{Sym}^3 E), c_6(\mathrm{sl}_3)]}{(27c_6(\mathrm{sl}_3) - c_3(\mathrm{Sym}^3 E)^2 - 4\lambda^3)} = \mathbb{Q}[\lambda, c_3(\mathrm{Sym}^3 E)].$$

Moreover, $\lambda \mapsto 3\alpha_2$ and $c_3(\mathrm{Sym}^3 E) \mapsto 27\alpha_3$ via

$$A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{SL}_3} = \mathbb{Z}[\alpha_2, \alpha_3]$$

which is rationally an isomorphism (Prop. 2.2). We will prove in Proposition 5.2 (ii) that more is true: R^* and $A^*_{\mathrm{PGL}_3}$ are isomorphic after inverting 3.

We will now establish some properties of the cycle map

$$\mathrm{cl} : A^*_{\mathrm{PGL}_3} \to H^*(\mathrm{BPGL}_3, \mathbb{Z})$$

and of Totaro's refined cycle map

$$\widetilde{\mathrm{cl}} : A^*_{\mathrm{PGL}_3} \to MU^*(\mathrm{BPGL}_3) \otimes_{MU^*} \mathbb{Z}.$$

Remark 5.2. In [14] Kono and Yagita proved that in the Atiyah-Hirzebruch spectral sequence for Brown-Peterson cohomology at the prime 3 ([30])

$$E_2^{pq} = H^p(\mathrm{BPGL}_3, \mathrm{BP}^q) \Longrightarrow \mathrm{BP}^{p+q}(\mathrm{BPGL}_3)$$

the E_∞-term is generated as a BP*-module by the top row i.e. by

$$\operatorname{im}\big(\mathrm{BP}^*(\mathrm{BPGL}_3) \to H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(3)})\big).$$

As a consequence, the natural map

$$\underline{\mathrm{cl}} : MU^*(\mathrm{BPGL}_3) \otimes_{MU^*} \mathbb{Z} \to H^*(\mathrm{BPGL}_3, \mathbb{Z})$$

is injective.

We have the following result[14]:

Proposition 5.2. (i) cl *and* $\widetilde{\mathrm{cl}}$ *are injective after inverting* 3.

(ii) π *is an isomorphism after inverting* 3.

Proof. (i) $A^*_{\mathrm{PGL}_3}$ has only 3-torsion and ker cl is torsion (Section 2). Therefore $\mathrm{cl} = \underline{\mathrm{cl}} \circ \widetilde{\mathrm{cl}}$ is injective after inverting 3 and the same is true for $\widetilde{\mathrm{cl}}$.

(ii) It is enough to prove that for any prime $p \neq 3$, the composition[15]

$$(R^*)_{(p)} \xrightarrow{\pi_{(p)}} (A^*_{\mathrm{PGL}_3})_{(p)} \xrightarrow{\mathrm{cl}_{(p)}} H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(p)})$$

is injective. Leray spectral sequence with $\mathbb{Z}_{(p)}$-coefficients:

$$E_2^{pq} = H^p(\mathrm{BPGL}_3, H^q(\mathrm{B}\mu_3, \mathbb{Z}_{(p)})) \Longrightarrow H^{p+q}(\mathrm{BSL}_3, \mathbb{Z}_{(p)})$$

collapses at the E_2-term since $H^*(\mathrm{B}\mu_3, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$, concentrated in degree zero, thus yielding an "edge" isomorphism (coinciding with the pullback):

$$\varphi_{(p)} : H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(p)}) \simeq H^*(\mathrm{BSL}_3, \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[\alpha_2, \alpha_3].$$

Now, consider the commutative diagram

$$\begin{array}{ccccc} (R^*)_{(p)} & \xrightarrow{\pi_{(p)}} & (A^*_{\mathrm{PGL}_3})_{(p)} & \xrightarrow{\mathrm{cl}_{(p)}} & H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(p)}) \\ & & \downarrow{\scriptstyle \phi_{(p)}} & & \downarrow{\scriptstyle \varphi_{(p)}} \\ & & (A^*_{\mathrm{SL}_3})_{(p)} & \xrightarrow[\mathrm{cl}_{\mathrm{SL}_3,(p)}]{\sim} & H^*(\mathrm{BSL}_3, \mathbb{Z}_{(p)}) \end{array} \tag{66}$$

and observe that for $p \neq 3$,

[14] A stronger version of (i) will be proved in Theorem 5.3.
[15] $(\cdot)_{(p)}$ denotes localization at the prime p.

$$(R^*)_{(p)} = \frac{\mathbb{Z}_{(p)}[\lambda, c_3(\mathrm{Sym}^3 E), c_6(\mathrm{sl}_3)]}{(27c_6(\mathrm{sl}_3) - c_3(\mathrm{Sym}^3 E)^2 - 4\lambda^3)} = \mathbb{Z}_{(p)}[\lambda, c_3(\mathrm{Sym}^3 E)].$$

Since, as we already computed, $\phi \circ \pi(\lambda) = 3\alpha_2$, $\phi \circ \pi(c_3(\mathrm{Sym}^3 E)) = 27\alpha_3$, commutativity of (66) concludes the proof. □

The stronger result we can prove about $\widetilde{\mathrm{cl}}$ is the following

Theorem 5.3. *Totaro's refined cycle class map*

$$\widetilde{\mathrm{cl}} : A^*_{\mathrm{PGL}_3} \to MU^*(\mathrm{BPGL}_3) \otimes_{MU^*} \mathbb{Z}$$

is surjective (*and has* 3-*torsion kernel*).

Proof. $\ker \widetilde{\mathrm{cl}}$ is 3-torsion since it is torsion and $A^*_{\mathrm{PGL}_3}$ has only 3-torsion. So we are left to prove surjectivity of $\widetilde{\mathrm{cl}}$. To do this, we first prove that $\widetilde{\mathrm{cl}}$ is surjective (thus an isomorphism by Prop. 5.2 (i)) after inverting 3 and then that $\widetilde{\mathrm{cl}}$ is surjective when localized at the prime 3.

$\underline{\mathrm{cl}}_{\mathrm{PGL}_3}$ is an isomorphism after inverting 3 since $H^*\left(\mathrm{BPGL}_3, \mathbb{Z}\left[\frac{1}{3}\right]\right)$ is torsion free ([24])[16]. So it is enough to prove that $\mathrm{cl}_{\mathrm{PGL}_3}$ is surjective when 3 is inverted. Now, in the commutative diagram

$$\begin{array}{ccc} A^*_{\mathrm{PGL}_3}\left[\frac{1}{3}\right] & \xrightarrow{\mathrm{cl}_{\mathrm{PGL}_3}\left[\frac{1}{3}\right]} & H^*_{\mathrm{PGL}_3}\left[\frac{1}{3}\right] \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ A^*_{\mathrm{SL}_3}\left[\frac{1}{3}\right] & \xrightarrow[\mathrm{cl}_{\mathrm{SL}_3}\left[\frac{1}{3}\right]]{} & H^*_{\mathrm{SL}_3}\left[\frac{1}{3}\right] \end{array}$$

φ' is an isomorphism since the corresponding Leray spectral sequence

$$E_2^{pq} = H^p(\mathrm{BPGL}_3, H^q(\mathrm{B}\mu_3, \mathbb{Z})) \Rightarrow H^{p+q}(\mathrm{BSL}_3, \mathbb{Z})$$

collapses after inverting 3, and $\mathrm{cl}_{\mathrm{SL}_3}$ is an isomorphism even without inverting 3. On the other hand, φ is injective because $\Phi : A^*_{\mathrm{PGL}_3} \to A^*_{\mathrm{SL}_3}$ has 3-torsion kernel and is surjective since

[16] We briefly sketch the argument. Since the differentials in the Atiyah-Hirzebruch spectral sequence

$$F_2^{pq} = H^p(\mathrm{BPGL}_3, MU^q) \Rightarrow MU^{p+q}(\mathrm{BPGL}_3)$$

are always torsion, they must be 0 if 3 is inverted since there is only 3-torsion (recall that MU^* is torsion-free). Therefore F_2^{pq} collapses when 3 is inverted.

$$c_2(\mathrm{sl}_3) \overset{\Phi}{\mapsto} 6\alpha_2,$$
$$c_2(\mathrm{Sym}^3 E) \overset{\Phi}{\mapsto} 15\alpha_2,$$
$$c_3(\mathrm{Sym}^3 E) \overset{\Phi}{\mapsto} 27\alpha_3.$$

Therefore $\mathrm{cl}_{\mathrm{PGL}_3}\left[\frac{1}{3}\right]$ is an isomorphism too.

So it remains to prove that the localization at the prime 3

$$(\widetilde{\mathrm{cl}}_{\mathrm{PGL}_3})_{(3)} : (A^*_{\mathrm{PGL}_3})_{(3)} \to MU^*(\mathrm{BPGL}_3) \otimes_{MU^*} \mathbb{Z}_{(3)}$$

is surjective. By [20],

$$MU^*(\mathrm{BPGL}_3) \otimes_{MU^*} \mathbb{Z}_{(3)} \simeq \mathrm{BP}^*(\mathrm{BPGL}_3) \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(3)}$$

where $\mathrm{BP}^*(X)$ denotes the Brown-Peterson cohomology of X localized at the prime 3 and

$$\mathrm{BP}^* = \mathrm{BP}^*(\mathrm{pt}) = \mathbb{Z}_{(3)}[v_1, \dots, v_n, \dots] \to \mathbb{Z}_{(3)}$$

($\deg v_i = -2(3^i - 1)$) sends each v_i to zero (see also [30]). Kono and Yagita computed $\mathrm{BP}^*(\mathrm{BPGL}_3)$ in [14], Th. 4.9, as a BP^*-module; it is a quotient of the following BP^*-module

$$(\mathrm{BP}^*\mathbb{Z}_{(3)}[[\widetilde{y}_2]]\widetilde{y}_2^2 \oplus \mathrm{BP}^* \oplus \mathrm{BP}^*\mathbb{Z}_{(3)}[[\widetilde{y}_8]]\widetilde{y}_8) \otimes \mathbb{Z}_{(3)}[[\widetilde{y}_{12}]]$$

and, if

$$r : \mathrm{BP}^*(\mathrm{BPGL}_3) \xrightarrow{s} H^*(\mathrm{BPGL}_3, \mathbb{Z}_{(3)}) \xrightarrow{j_\bullet} H^*(\mathrm{BPGL}_3, \mathbb{Z}/3)$$

(where s is the natural map of generalized cohomology theories and $j_\bullet$ is induced by $j : \mathbb{Z}_{(3)} \to \mathbb{Z}/3$), r has kernel $\mathrm{BP}^{<0} \cdot \mathrm{BP}^*(\mathrm{BPGL}_3)$ and

$$r(\widetilde{y}_2^2) = y_2^2 \equiv c_2(\mathrm{sl}_3),$$
$$r(\widetilde{y}_8) = y_8,$$
$$r(\widetilde{y}_{12}) = y_{12} \equiv c_6(\mathrm{sl}_3),$$

$y_8 \in H^8(\mathrm{BPGL}_3, \mathbb{Z}/3)$ being the same as in part (IV) of the proof of Th. 4.2. So we only need to show that $\widetilde{y}_8$ is in the image of $(\widetilde{\mathrm{cl}}_{\mathrm{PGL}_3})_{(3)}$. By part (IV) of the proof of Th. 4.2, y_8 is in the image of

$$j_\bullet \circ (\mathrm{cl}_{\mathrm{PGL}_3})_{(3)} : (A^4_{\mathrm{PGL}_3})_{(3)} \to H^8(\mathrm{BPGL}_3, \mathbb{Z}/3),$$

and this concludes the proof since r has kernel $\mathrm{BP}^{<0} \cdot \mathrm{BP}^*(\mathrm{BPGL}_3)$. □

Remark 5.3. We wish to point out that we do not know whether $\ker \widetilde{\mathrm{cl}}$ is zero or not. Moreover, since $\mathrm{cl}(\chi) = 0$ and $\underline{\mathrm{cl}}$ is injective (Remark 5.2), we also have $\widetilde{\mathrm{cl}}(\chi) = 0$. Therefore, if Totaro's conjecture was true (i.e $\widetilde{\mathrm{cl}}$ was an isomorphism) we should have $\chi = 0$; but, again, we are not able to prove whether $\chi = 0$ or not.

6. Appendix. A cohomology-independent proof that $A^*_{\mathrm{PGL}_3}$ is not generated by Chern classes

Here we give an alternative proof of Theorem 4.2 which is independent of Kono-Mimura-Shimada's results on the $\mathbb{Z}/3$-cohomology of BPGL_3 and deals only with Chow rings with no reference to cohomology. However, for the same reason, the following proof does not yield any direct information on the cycle or refined cycle map.

The notations are those of the previous sections.

Proposition 6.1. *The representation ring of* PGL_3 *is generated by*

$$\{\mathrm{sl}_3, \mathrm{Sym}^3 E, \mathrm{Sym}^3 E^\vee\}.$$

Proof. The exact sequence

$$1 \to \mu_3 \to \mathrm{SL}_3 \to \mathrm{PGL}_3 \to 1$$

induces an exact sequence of character groups

$$0 \to \widehat{T_{\mathrm{PGL}_3}} \equiv \hat{T} \to \widehat{T_{\mathrm{SL}_3}} \xrightarrow{\pi} \mathbb{Z}/3 \to 0$$

where $\widehat{T_{\mathrm{SL}_3}} = \mathbb{Z}^3/\mathbb{Z}$ ($\mathbb{Z} \hookrightarrow \mathbb{Z}^3$ diagonally) and $\pi : [n_1, n_2, n_3] \mapsto [n_1 + n_2 + n_3]$. Then

$$\mathbb{Z}[\hat{T}] \hookrightarrow \mathbb{Z}[\widehat{T_{\mathrm{SL}_3}}] = \mathbb{Z}[x_1, x_2, x_3]/(x_1x_2x_3 - 1)$$

is the subring generated by monomials $x_1^{n_1}x_2^{n_2}x_3^{n_3}$ with $n_1 + n_2 + n_3 \equiv 0 \bmod 3$. Therefore

$$R(\mathrm{PGL}_3) = (R(T))^{S_3} = (\mathbb{Z}[\hat{T}])^{S_3} \hookrightarrow R(\mathrm{SL}_3) = (\mathbb{Z}[\widehat{T_{\mathrm{SL}_3}}])^{S_3} = \mathbb{Z}[s_1, s_2]$$

(where s_i is the i-th elementary symmetric function on the x_i's) is the subring generated by $\{s_1^3, s_1s_2, s_2^3\}$. Then to prove the proposition it is enough to compute $\mathrm{sl}_3, \mathrm{Sym}^3 E$ and $\mathrm{Sym}^3 E^\vee$ in terms of s_1 and s_2 *in* $R(\mathrm{SL}_3)$.

If E is the standard representation and $\mathbf{1}$ the trivial one dimensional representation of SL_3, we have

$$E = x_1 + x_2 + x_3 = s_1,$$
$$E^\vee = x_1^{-1} + x_2^{-1} + x_3^{-1} = x_1x_2 + x_1x_3 + x_2x_3 = s_2,$$
$$\mathrm{sl}_3 = E \otimes E^\vee - \mathbf{1} = s_1s_2 - 1;$$

so

$$\mathrm{Sym}^3 E = s_1^3 - 2s_1 s_2 + 1, \quad \mathrm{Sym}^3 E^\vee = s_2^3 - 2s_1 s_2 + 1$$

and we conclude. □

Corollary 6.2. *The Chern subring* $A^*_{\mathrm{Ch,PGL}_3}$ *of* $A^*_{\mathrm{PGL}_3}$, *generated by Chern classes of representations, is generated by* $\{c_i(\mathrm{sl}_3), c_j(\mathrm{Sym}^3 E)\}_{i,j \geq 0}$.

Theorem 6.3. ρ *is not in the Chern subring of* $A^*_{\mathrm{PGL}_3}$.

Proof. By Prop. 6.1,

$$R(\mathrm{PGL}_3) = \mathbb{Z}[\mathrm{sl}_3, \mathrm{Sym}^3 E, \mathrm{Sym}^3 E^\vee]$$

and since sl_3 (respectively, $\mathrm{Sym}^3 E$) is isomorphic to the regular $A_3 \times \mu_3$-representation minus the trivial one (respectively, plus the trivial one), we have

$$c_i(\mathrm{sl}_3)_{|A_3 \times \mu_3} = c_j(\mathrm{Sym}^3 E)_{|A_3 \times \mu_3} = 0, \quad i, j = 1, 2, 3, 4. \tag{67}$$

Now recall (Section 3) that ρ is a lift to $A^*_{\mathrm{PGL}_3}$ of

$$\psi(\alpha c_3(W)) \in A^*_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

where

$$\psi : A^*_{A_3 \ltimes T} \to A^*_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3})$$

is the (surjective) pullback. So, the image of ρ under the restriction

$$A^*_{\mathrm{PGL}_3} \to A^*_{A_3 \ltimes T}$$

is of the form $\alpha c_3(W) + \xi$, for some $\xi \in \ker(\psi)$.

Now, let us suppose ρ is in the Chern subring $A^*_{\mathrm{Ch,PGL}_3}$. By (67), we have

$$\alpha c_3(W) + \xi \in \ker(\varphi : A^*_{A_3 \ltimes T} \to A^*_{A_3 \times \mu_3}).$$

From the commutative diagram

$$\begin{array}{ccc} A^*_{A_3 \ltimes T} & \xrightarrow{\varphi} & A^*_{A_3 \times \mu_3} \\ \psi \downarrow & & \downarrow \psi' \\ A^*_{A_3 \ltimes T}(\mathrm{Diag}^*_{\mathrm{sl}_3}) & \xrightarrow[\phi]{} & A^*_{A_3 \times \mu_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \end{array}$$

we get

$$\psi'(\alpha c_3(W)) = 0.$$

Therefore, if we show that $\alpha c_3(W)$ is not in the kernel of ψ', we will have proved that ρ cannot be in the Chern subring of $A^*_{\mathrm{PGL}_3}$. To do this, let us consider the two localization

sequences[17]:

$$A^*_{A_3 \times \mu_3} \xrightarrow{\cdot(-\alpha^2)\otimes 1} A^*_{A_3 \times \mu_3}(\mathrm{Diag}_{\mathrm{sl}_3}) \xrightarrow{p} A^*_{A_3 \times \mu_3}(\mathrm{Diag}_{\mathrm{sl}_3} \backslash \{0\}) \longrightarrow 0, \tag{68}$$

$$A^*_{\mu_3} \simeq A^*_{A_3 \times \mu_3}(Z) \xrightarrow{j_*} A^*_{A_3 \times \mu_3}(\mathrm{Diag}_{\mathrm{sl}_3} \backslash \{0\}) \xrightarrow{q} A^*_{A_3 \times \mu_3}(\mathrm{Diag}^*_{\mathrm{sl}_3}) \longrightarrow 0 \tag{69}$$

where we used that

$$Z \simeq A_3 \times \mathbb{C}^*,$$

$A_3 \ltimes T$-equivariantly. Since (Section 3),

$$W \simeq \mathbb{C}_{\chi,\mu_3} \boxtimes \mathbb{C}^3_{\mathrm{perm},A_3}$$

as $A_3 \times \mu_3$-representations (where $\mathbb{C}_{\chi,\mu_3}$ is the μ_3-representation of character $\chi = \exp(i2\pi/3)$ and $\mathbb{C}^3_{\mathrm{perm},A_3}$ is the A_3-permutation representation), its Chern roots are

$$\{\beta + \alpha, \beta - \alpha, \beta\}$$

and then

$$\alpha c_3(W)_{|A_3 \times \mu_3} = (\beta^2 - \alpha^2)\alpha\beta. \tag{70}$$

By (70) and (68), it is enough to prove that $j_* = 0$.

Let us consider the pullback E of $\mathbb{C}_{\chi,\mu_3}$ to $X = \mathrm{Diag}_{\mathrm{sl}_3} \backslash \{0\}$ as an $A_3 \times \mu_3$-equivariant vector bundle, with A_3 acting trivially on $\mathbb{C}_{\chi,\mu_3}$ and $A_3 \times \mu_3$ acting as usual on X (i.e. μ_3 acting trivially and A_3 by permutations). We have

$$j^*(c_1(E)) \equiv c_1(E)_{|\mu_3} = \beta.$$

But we also have $j_*(1) = 0$, since

$$Z = D^{-1}(\{0\})$$

where

$$D : X \to \mathbb{A}^1,$$

$$(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \tag{71}$$

---

17) Here $\mathrm{Diag}_{\mathrm{sl}_3}$ are the diagonal matrices in sl_3 and we identify $A^*_{A_3 \times \mu_3} \simeq A^*_{A_3} \otimes A^*_{\mu_3}$ with

$$\frac{\mathbb{Z}[\alpha]}{(3\alpha)} \otimes \frac{\mathbb{Z}[\beta]}{(3\beta)}.$$

is the square root of the discriminant (which is $A_3 \times \mu_3$-equivariant). So $j_* = 0$ and we conclude. □

References

[1] *K. Brown*, Cohomology of groups, Springer-Verlag, Berlin 1982.
[2] *M. Demazure, P. Gabriel*, Groupes Algébriques, Tome I, North-Holland, 1970.
[3] *M. Demazure, A. Grothendieck*, Schémas en Groupes, Tome I, Springer Lect. Notes Math. **151** (1970).
[4] *D. Edidin, W. Graham*, Equivariant intersection theory, Inv. Math. **131** (1998), 595–634.
[5] *D. Edidin, W. Graham*, Characteristic classes in the Chow ring, J. Alg. Geom. **6** (1997), 431–443.
[6] *W. Fulton*, Intersection theory, Second edition, Springer Verlag, 1998.
[7] *D. H. Gottlieb*, Fiber bundles and the Euler characteristic, J. Diff. Geom. **10** (1975), 39–48.
[8] *A. Grothendieck*, Sur quelques propriétés fondamentales en théorie des intersections, Exp. **4**, Séminaire C. Chevalley 2e année 1958 "Anneaux de Chow et Applications".
[9] *A. Grothendieck*, Torsion homologique et sections rationelles, Exp. **5**, Séminaire C. Chevalley 2e année 1958 "Anneaux de Chow et Applications".
[10] *A. Grothendieck*, Le groupe de Brauer I, II, III, in: Dix exposés sur la cohomologie des schémas, North-Holland, 1968.
[11] *N. A. Karpenko*, Torsion in CH^2 of Severi-Brauer varieties and indecomposability of generic algebras, Manuscr. Math. **88** (1995), 109–117.
[12] *N. A. Karpenko*, Codimension 2 cycles on Severi-Brauer varieties, preprint 1995.
[13] *A. Kono, M. Mimura, N. Shimada*, Cohomology of classifying spaces of certain associative *H*-spaces, J. Math. Kyoto Univ. **15** (1975), 607–617.
[14] *A. Kono, N. Yagita*, Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups, Trans. A.M.S. **339** (1993), 781–798.
[15] *G. Laumon, L. Moret-Bailly*, Champs Algébriques, Prépubl. Math. Université de Paris-Sud, 1995.
[16] *J. S. Milne*, Étale cohomology, Princeton Math. Ser. **33**, Princeton 1980.
[17] *D. Mumford, J. Fogarty, F. Kirwan*, Geometric invariant theory, Third enlarged edition, Springer Verlag, 1992.
[18] *R. Pandharipande*, The Chow ring of the Hilbert scheme of rational normal curves, preprint.
[19] *R. Pandharipande*, The equivariant Chow rings of $\mathbf{O}(k)$, $\mathbf{SO}(2k+1)$ and $\mathbf{SO}(4)$, J. reine. angew. Math. **496** (1998), 131–148.
[20] *D. Quillen*, On the formal group laws of oriented and unoriented cobordism theory, Bull. A.M.S. **75** (1969), 1293–1298.
[21] *J. P. Serre*, Espaces fibrés algebriques, Exp. **1**, Séminaire C. Chevalley 2e année 1958 "Anneaux de Chow et Applications".
[22] *E. H. Spanier*, Algebraic topology, Mc Graw-Hill, 1966.
[23] *R. Stong*, Notes on cobordism theory, Princeton Univ. Press, Princeton 1968.
[24] *B. Totaro*, Torsion algebraic cycles and complex cobordism, J. AMS **10** (1997), 467–493.
[25] *B. Totaro*, The Chow ring of classifying spaces, Proc. Symp. Pure Math., *K*-Theory (Seattle, 1997), to appear.
[26] *B. Totaro*, preprint 1999.
[27] *G. Vezzosi*, PhD Thesis, Scuola Normale Superiore, Pisa 1999.
[28] *V. Voevodsky*, The Milnor conjecture, preprint MPI, 1996.
[29] *V. Voevodsky*, A^1-homotopy theory, Doc. Math. Extra Volume ICM **I** (1998), 417–442.
[30] *W. S. Wilson*, Brown-Peterson Homology: An Introduction and Sampler, CBMS Regional Conf. Ser. **48**, A.M.S., Providence 1982.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy
e-mail: vezzosi@dm.unibo.it

Eingegangen 3. Mai 1999