Symmetric algebra of a cochain complex: cheat-sheet

Gabriele Vezzosi

July 2019

Abstract

These are notes for students. In the cheat-sheet tradition, the focus will be on giving correct and general enough definitions and statements, rather than on proofs, which are most of the times straightforward or in any case elementary; they are thus mostly left to the student. This should hopefully insert a grain of pedagogical virtue in the weird act of a teacher handling cheat-sheets to his own students.

Contents

1 Overview **1**

2 Adjunctions **2**

3 Bigradings and formulae **3**

4 Graded derivations **5**

5 Symmetric algebra of a dg-module **5**

6 Examples **7**

7 Comparison with Σ-coinvariants **10**

8 Another approach **11**

9 Graded modules over a graded-commutative algebra **12**

10 dg-modules over a cdga **13**

1 Overview

We work over an arbitrary commutative (ungraded) ring R. In order to compute the symmetric algebra of an R-dg-module $(E^\bullet, d)$, we proceed in two steps: first we forget about the differential d and compute the symmetric algebra of the graded R-module $E^\bullet$, then we use a further property of this symmetric algebra in order to deduce from d a graded differential ∂_d on the symmetric algebra of $E^\bullet$. The pair consisting of the symmetric algebra of the graded R-module $E^\bullet$ and ∂_d will satisfy the universal property of the symmetric algebra of the dg-module $(E^\bullet, d)$.
We also state the straightforward generalizations to the case of dg-modules over a commutative differential graded R-algebra (= R-cdga).

2 Adjunctions

Let $(\mathcal{C}, \otimes, \varphi)$ be a symmetric monoidal R-linear category (φ denoting the symmetry morphism) having arbitrary coproducts, and with $\otimes$ distributing along coproducts.

Definition 2.1 *We define the symmetric monoidal category $(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ as follows:*

- $$gr\mathcal{C} := \prod_{\mathbb{Z}} \mathcal{C} = \mathsf{Fun}(\mathbb{Z}_{discr}, \mathcal{C})$$

- $$(X^i)_{i\in\mathbb{Z}} \otimes^{gr} (Y^j)_{j\in\mathbb{Z}} := (\oplus_{p+q=n}(X^p \otimes Y^q))_{n\in\mathbb{Z}}$$

- $$gr\varphi_{X,Y} : (X^i)_{i\in\mathbb{Z}} \otimes^{gr} (Y^j)_{j\in\mathbb{Z}} \longrightarrow (Y^j)_{j\in\mathbb{Z}} \otimes^{gr} (X^i)_{i\in\mathbb{Z}}$$

  is the map in $gr\mathcal{C}$ whose n-th component $(gr\varphi_{X,Y})_n$ is determined by the maps $(gr\varphi_{X,Y})_{n;(p,q)}$, for $(p+q) = n$, defined by the compositions

  $$(gr\varphi_{X,Y})_{n;(p,q)} : \ X^p \otimes Y^q \xrightarrow{\varphi_{X^p,Y^q}} Y^q \otimes X^p \xrightarrow{can} \oplus_{p'+q'=n}(Y^{q'} \otimes X^{p'}) \ .$$

- *The coproduct functor $\oplus_{i\in\mathbb{Z}} : gr\mathcal{C} \longrightarrow \mathcal{C}$ will sometimes be referred to as the* underlying $\mathcal{C}$-object *functor.*

- *For any $r \in \mathbb{Z}$, there is a r-shift functor $(-)(r) : gr\mathcal{C} \longrightarrow gr\mathcal{C}$ sending $(X^i)_{i\in\mathbb{Z}}$ to $(X^{i+r})_{i\in\mathbb{Z}}$ and $f : (X^i)_{i\in\mathbb{Z}} \to (Y^i)_{i\in\mathbb{Z}}$ to $(-1)^r f$ (in the unique obvious sense).*

- *For $r \in \mathbb{Z}$, $\underline{X}, \underline{Y} \in gr\mathcal{C}$, a* map of weight r in $gr\mathcal{C}$ *from $\underline{X}$ to $\underline{Y}$ is a morphism $\underline{X} \to \underline{Y}(r)$ in $gr\mathcal{C}$.*

Remark 2.2 Our notion of graded object in $\mathcal{C}$, as an object in $gr\mathcal{C}$, compares as follows to the usual one, defined as an object $X \in \mathcal{C}$ plus a decomposition $X = \oplus_{i\in\mathbb{Z}} X^i$. To $(X^i)_{i\in\mathbb{Z}} \in gr\mathcal{C}$, we associate its underlying $\mathcal{C}$-object together with its tautological decomposition.

Proposition 2.3 *Let $(\mathcal{C}, \otimes, \varphi)$ be as above.*

1. *The functor*
   $$(-)^1 : \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi) \longrightarrow \mathcal{C}$$
   defined as the composition
   $$\mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi) \xrightarrow{U} gr\mathcal{C} \xrightarrow{(-)^1} \mathcal{C}$$
   has a left adjoint that will be denoted by $\mathsf{Sym}_{gr,\mathcal{C}}$.

2. *The* coproduct functor $\oplus_{i\in\mathbb{Z}} : (gr\mathcal{C}, \otimes^{gr}, gr\varphi) \longrightarrow (\mathcal{C}, \otimes, \varphi)$ *is symmetric monoidal.*

Remark 2.4 Note that $\mathsf{Sym}_{gr,\mathcal{C}}$ is actually $\mathbb{N}$-graded, i.e. it factors as

$$\mathsf{grmod}_R \longrightarrow \mathsf{CAlg}(gr_{\mathbb{N}}\mathcal{C}, \otimes^{gr_{\mathbb{N}}}, gr_{\mathbb{N}}\varphi) \longrightarrow \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi).$$

One could have also chosen to work with $(gr_{\mathbb{N}}\mathcal{C}, \otimes^{gr_{\mathbb{N}}}, gr_{\mathbb{N}}\varphi)$ rather than with $\mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ from the beginning.

Note that, in particular, $\mathsf{Sym}_{gr,\mathcal{C}}$ preserves coproducts, and the coproduct functor induces a functor

$$(-)^{\mathrm{int}} : \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi) \longrightarrow \mathsf{CAlg}(\mathcal{C}, \otimes, \varphi).$$

Here the super-script "int" stands for *internal*, in the sense that we are tracing out (i.e. summing up on) all the "external weights": e.g. $\mathcal{C}$ might be itself "graded", by some "internal weights", as it will be the case in our main $\mathcal{C}$ of interest ($\mathcal{C} = \mathsf{grmod}_R$, see below).

Corollary 2.5 *The functor* $(\mathsf{Sym}_{gr,\mathcal{C}})^{\mathrm{int}} : \mathcal{C} \longrightarrow \mathsf{CAlg}(\mathcal{C}, \otimes, \varphi)$ *is left adjoint to the forgetful functor.*

Remark 2.6 Both Proposition 2.3 and Corollary 2.5 are quite classical when $\mathcal{C} = \mathsf{mod}_R$ (endowed with its classical monoidal structure and its classical obviously-no-signs-switch symmetry). In this case, $\mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ is the category of strictly commutative (not graded-commutative !) graded R-algebras, since $gr\varphi$ is again the no-signs-switch symmetry, and, for M an R-module, $\mathsf{Sym}_{gr}(M)$ is the commutative algebra $S^\bullet(M)$, equipped with its canonical $\mathbb{N}$-grading, while $(\mathsf{Sym}_{gr}(M))^{\mathrm{int}}$ is $S(M)$, i.e. $S^\bullet(M)$ without its grading. One can check that $M \mapsto S^\bullet(M)$ is left adjoint to the functor sending a strictly commutative graded R-algebra to its weight 1 part, while $M \mapsto S(M)$ is left adjoint to the forgetful functor sending a(n ungraded) commutative R-algebra A to its underlying R-module.

We will be mainly interested in the special case where $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi = \text{Koszul symmetry})$, i.e.

- $$(E^i)_{i\in\mathbb{Z}} \otimes_{gr} (F^j)_{j\in\mathbb{Z}} := (\oplus_{p+q=n}(E^p \otimes_R F^q))_{n\in\mathbb{Z}}$$
- $\varphi(x^i \otimes y^j) := (-1)^{ij} y^j \otimes x^i$, for $\mathrm{wt}(x^i) = i$, $\mathrm{wt}(y^j) = j$.

In this situation, Proposition 2.3 gives us the left adjoint functor

$$\mathsf{Sym}_{gr} := \mathsf{Sym}_{gr,\,\mathsf{grmod}_R} : \mathsf{grmod}_R \longrightarrow \mathsf{CAlg}(gr(\mathsf{grmod}_R), \otimes^{gr}, gr\varphi)$$

to the $A \mapsto (A)^1$ functor, and Corollary 2.5 provides us the left adjoint functor

$$(\mathsf{Sym}_{gr})^{\mathrm{int}} = \oplus_{n\geq 0}\mathsf{Sym}^n_{gr} : \mathsf{grmod}_R \longrightarrow \mathsf{CAlg}(\mathsf{grmod}_R, \otimes_{gr}, \varphi = \text{Koszul symmetry})$$

to the forgetful functor.
Note that $\mathsf{CAlg}(\mathsf{grmod}_R, \otimes_{gr}, \varphi = \text{Koszul symmetry})$ is the usual category of $\mathbb{Z}$-graded-commutative R-algebras, hence, if $E^\bullet$ is a $\mathbb{Z}$-graded R-module, then $(\mathsf{Sym}_{gr}(E^\bullet))^{\mathrm{int}}$ is a $\mathbb{Z}$-graded-commutative R-algebra. Moreover, Sym_{gr} is fully faithful, and, equivalently, the unit map

$$E^\bullet \longrightarrow \mathsf{Sym}^1_{gr}(E^\bullet) := (\mathsf{Sym}_{gr}(E^\bullet))^1$$

is an isomorphism in grmod_R.

3 Bigradings and formulae

In our case of interest, i.e. $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi = \text{Koszul symmetry})$, we have equivalences of categories

$$gr\mathcal{C} \simeq \prod_{\mathbb{Z}} \prod_{\mathbb{Z}} \mathsf{mod}_R \simeq \prod_{\mathbb{Z}\times\mathbb{Z}} \mathsf{mod}_R.$$

Elements in the first $\mathbb{Z}$ factor in $\mathbb{Z}\times\mathbb{Z}$ are called *external* weights, while elements in the second $\mathbb{Z}$ are called *internal* weights: they form bi-weights $(i,j)\in\mathbb{Z}\times\mathbb{Z}$. Direct summation over external weights gives the underlying graded module functor.
In particular, any $A\in\mathsf{CAlg}(gr\mathcal{C},\otimes^{gr},gr\varphi)$ has bi-weighted components $(A^{i,j})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$, where i is the external weight, and j the internal weight. The algebra product in A is bi-weighted, i.e. satisfies

$$A^{i,j}\cdot A^{h,k}\subseteq A^{i+h,j+k}$$

and the graded-commutativity property reads as

$$a^{i,j}\cdot a'^{h,k}=(-1)^{jk}a'^{h,k}\cdot a^{i,j}\qquad\text{for } a^{i,j}\in A^{i,j}\,,\ a'^{h,k}\in A^{h,k}.$$

Note that the external grading is *immaterial* in the graded commutativity constraint.
Moreover, we have $A^{\mathrm{int}}=(\oplus_{i\in\mathbb{Z}}A^{i,j})_{j\in\mathbb{Z}}$, and this formula together with the above commutativity constraint give another proof of the fact that A^{int} is indeed a graded-commutative R-algebra. For the same reason, $A^{0,\bullet}$ is a graded-commutative R-algebra, as well.

If $E^\bullet\in\mathcal{C}=\mathsf{grmod}_R$, we have

$$\mathsf{Sym}_{gr}(E^\bullet)=\left(\mathsf{Sym}^n_{gr}(E^\bullet):=(\mathsf{Sym}_{gr}(E^\bullet))^{n,\bullet}\right)_{n\in\mathbb{N}}$$

where each $\mathsf{Sym}^n_{gr}(E^\bullet)\in\mathsf{grmod}_R$ (its weight-grading is given by the *internal* weight) is given by

$$\mathsf{Sym}^n_{gr}(E^\bullet)=\left(\mathsf{Sym}^{n,m}_{gr}(E^\bullet):=(\mathsf{Sym}_{gr}(E^\bullet))^{n,m}\right)_{m\in\mathbb{Z}}$$

where $\mathsf{Sym}^{n,m}_{gr}(E^\bullet)\in\mathsf{mod}_R$ is defined as

$$\bigoplus_{\left\{(\underline{p},\underline{q})\in\mathbb{N}^{\mathbb{Z}_{\mathrm{odd}}}\times\mathbb{N}^{\mathbb{Z}_{\mathrm{even}}}\,\mid\,\sum_{i\in\mathbb{Z}_{\mathrm{odd}}}ip_i+\sum_{j\in\mathbb{Z}_{\mathrm{even}}}jq_j=m\,,\,\sum_{i\in\mathbb{Z}_{\mathrm{odd}}}p_i+\sum_{j\in\mathbb{Z}_{\mathrm{even}}}q_j=n\right\}}\left(\bigotimes_{i\in\mathbb{Z}_{\mathrm{odd}}}\wedge^{p_i}(E^i)\bigotimes_{j\in\mathbb{Z}_{\mathrm{even}}}S^{q_j}(E^j)\right)$$

Remark 3.1 According to the convention that a graded object in $\mathcal{C}$ is a direct sum of its components (in our language this graded object is the underlying $\mathcal{C}$ object), a more common way of writing the previous formula is the following

$$\mathsf{Sym}^n_{gr}(E^\bullet)=\bigoplus_{p+q=n}\ \bigoplus_{\sum_{i\in\mathbb{Z}_{\mathrm{odd}}}p_i=p,\,\sum_{j\in\mathbb{Z}_{\mathrm{even}}}q_j=q}\ \bigotimes_{i\in\mathbb{Z}_{\mathrm{odd}}}\wedge^{p_i}(E^i)\bigotimes_{j\in\mathbb{Z}_{\mathrm{even}}}S^{q_j}(E^j)\qquad\in\mathsf{grmod}_R.$$

However, according to our definition of $gr\mathcal{C}$ as $\prod_{\mathbb{Z}}\mathcal{C}$, the formula before this remark is definitely more correct (and it gives explicitly the internal grading).

The multiplication in $\mathsf{Sym}_{gr}(E^\bullet)$ is given by a family $(\mu_{n,m,n',m'})_{(n,m,n',m')\in\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}}$ of maps of R-modules

$$\mu_{n,m,n',m'}:\mathsf{Sym}^{n,m}_{gr}(E^\bullet)\otimes_R\mathsf{Sym}^{n',m'}_{gr}(E^\bullet)\longrightarrow\mathsf{Sym}^{n+n',m+m'}_{gr}(E^\bullet)$$

while its commutativity constraint is expressed by the following family, indexed by $(n,m,n',m')\in\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}$, of commutative diagrams of R-modules

$$\begin{tikzcd}
\mathsf{Sym}^{n,m}_{gr}(E^\bullet)\otimes_R\mathsf{Sym}^{n',m'}_{gr}(E^\bullet) \arrow[r, "\sigma"] \arrow[d, "\mu_{n,m,n',m'}"'] & \mathsf{Sym}^{n',m'}_{gr}(E^\bullet)\otimes_R\mathsf{Sym}^{n,m}_{gr}(E^\bullet) \arrow[d, "\mu_{n',m',n,m}"] \\
\mathsf{Sym}^{n+n',m+m'}_{gr}(E^\bullet) \arrow[r, "(-1)^{mm'}"'] & \mathsf{Sym}^{n+n',m+m'}_{gr}(E^\bullet)
\end{tikzcd}$$

where σ is the usual no-sign-switch symmetry in the symmetric monoidal category mod_R.

4 Graded derivations

Definition 4.1 *Let* $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi =$ *Koszul symmetry*$)$*,* $A \in \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$*, and* $M \in \mathsf{Mod}_A(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$*. A* graded derivation of weight $r \in \mathbb{Z}$ of A into M *is a map* $\partial : A \to M$ *of* R*-modules of bi-weight* $(0, r)$*, i.e. a family of maps of* R*-modules*

$$\left(\partial^{i,j} : A^{i,j} \longrightarrow M^{i,j+r}\right)_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$$

satisfying the following internal-weighted Leibniz rule

$$\partial(a^{i,j} \cdot a'^{h,k}) = \partial(a^{i,j})a'^{h,k} + (-1)^{rj} a^{i,j}\partial(a'^{h,k})$$

where on the r.h.s. we used mere juxtaposition for the A*-module structure on* M*.*

Remark 4.2 If 2 is invertible in R, then any graded derivation $\partial : A \to M$ vanishes on R (i.e. along the canonical bi-weight $(0,0)$ map $R \to A$). If 2 is not invertible in R, the reader should add this condition to Definition 4.1.

Note that the external degree is immaterial both in the bi-weight of a graded derivation and in its weighted Leibniz rule. As a consequence, we have the following result

Proposition 4.3 *Let* $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi =$ *Koszul symmetry*$)$*. If* $A \in \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$*,* $M \in \mathsf{Mod}_A(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$*, and* $\partial : A \to M$ *is graded derivation of weight* $r \in \mathbb{Z}$ *as in Definition 4.1, then* $\partial^{\mathrm{int}} : A^{\mathrm{int}} \longrightarrow M^{\mathrm{int}}$ *is a graded derivation of weight* r *of the* $\mathbb{Z}$*-graded-commutative* R*-algebra* A^{int} *into the graded* A^{int}*-module* M^{int}*, in the usual sense of the literature.*

Remark 4.4 Note that if $\partial : A \to A$ is a graded self-derivation such that $\partial \circ \partial = 0$, then $A^{n,\bullet}$ is a R-dg-module, for any $n \in \mathbb{Z}$, and moreover A^{int} is a graded-commutative differential R-algebra (R-cdga).

The following result will allow us to induce a differential on the symmetric algebra of a dg-module.

Proposition 4.5 *Let* $E^\bullet$ *be a graded* R*-module,* $A \in \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$*, and suppose that* A *is a module over* $\mathsf{Sym}_{gr}(E^\bullet)$ *(i.e.* $A \in \mathsf{Mod}_{\mathsf{Sym}_{gr}(E^\bullet)}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ *). Then, for any map* $u : E^\bullet \longrightarrow (A)^1$ *of weight* $r \in \mathbb{Z}$ *between graded* R*-modules, there is a unique graded derivation* $\partial_u : \mathsf{Sym}_{gr}(E^\bullet) \longrightarrow A$ *of weight* r*, as in Definition 4.1, such that* ∂_u *restricts to* u *on* $(-)^1$*.* [1]

Proof. Since $\mathsf{Sym}_{gr}(E^\bullet)$ is generated by its external weight 1 part ($\simeq E^\bullet$), we just use the given module structure of A over $\mathsf{Sym}_{gr}(E^\bullet)$ and the formulae defining graded derivations of weight r (Definition 4.1), to construct the extension ∂_d. Uniqueness is left to the reader (use that, in the appropriate sense, $\mathsf{Sym}_{gr}(E^\bullet)$ is *freely* generated by $E^\bullet$ in external weight 1). □

5 Symmetric algebra of a dg-module

Let $(E^\bullet, d) \in \mathsf{dgmod}_R$. By viewing the differential d as an endomorphism of weight 1 of $E^\bullet \in \mathsf{grmod}_R$, Proposition 4.5, with $A = \mathsf{Sym}_{gr}(E^\bullet)$, gives us a graded self-derivation $\partial_d : \mathsf{Sym}_{gr}(E^\bullet) \longrightarrow \mathsf{Sym}_{gr}(E^\bullet)$ of weight 1. Since $d^2 = 0$, again Proposition 4.5 tells us that $\partial_d \circ \partial_d = 0$: we say that ∂_d is a *graded differential* on $\mathsf{Sym}_{gr}(E^\bullet) \in \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ (here $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi =$ Koszul symmetry$)$). Let's look at the

[1] Recall that the unit of adjunction gives a functorial isomorphism $\mathsf{Sym}^1_{gr}(E^\bullet) \simeq E^\bullet$ in grmod_R, for any graded R-module $E^\bullet$, and that a graded derivation has external weight 0, so it maps the external weight 1 part to the external weight 1 part.

pair $(\mathsf{Sym}_{gr}(E^\bullet), \partial_d)$. By Remark 4.4, we have that:

(i) for any $n \in \mathbb{N}$, the pair $(\mathsf{Sym}^n_{gr}(E^\bullet), \partial^n_d)$ is a R-dg-module ;

(ii) the pair $(\mathsf{Sym}_{gr}(E^\bullet)^{\mathrm{int}}, \partial^{\mathrm{int}}_d) = (\oplus_{n\in\mathbb{N}}\mathsf{Sym}^n_{gr}(E^\bullet), \oplus_{n\in\mathbb{N}}\partial^n_d)$ is a R-cdga, that will be denoted as $\mathsf{Sym}_{dg}(E^\bullet, d)^{\mathrm{int}}$.

Let $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi = \text{Koszul symmetry})$, and $(\mathcal{C}_{dg}, \otimes_{dg}, \varphi_{dg}) = (\mathsf{dgmod}_R, \otimes_{dg}, \text{Koszul symmetry})$ (these are the usual symmetric monoidal structures, nothing fancier). Note that, in particular, $\mathsf{CAlg}(\mathcal{C}_{dg}, \otimes_{dg}, \varphi_{dg})$ is the category cdga_R of graded-commutative differential R-algebras (R-cdga's).
Recall that by Proposition 2.3 (2), the coproduct functor $\oplus_{i\in\mathbb{Z}} : (gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \longrightarrow (\mathcal{C}_{dg}, \otimes_{dg}, \varphi_{dg})$ is symmetric monoidal, so that it induces a functor

$$(-)^{\mathrm{int}} : \mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \longrightarrow \mathsf{CAlg}(\mathcal{C}_{dg}, \otimes_{dg}, \varphi_{dg}) = \mathsf{cdga}_R$$

This is notationally consistent with **(ii)** above, and yields **(ii)** without using explicit formulae.
It is also easy to verify that $\mathsf{Sym}_{dg}(E^\bullet, d) := (\mathsf{Sym}_{gr}(E^\bullet), \partial_d) \in \mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg})$.

Proposition 5.1 *1. The functor*

$$(-)^1 : \mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \longrightarrow \mathcal{C}_{dg}$$

defined as the composition

$$\mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \xrightarrow{U_{dg}} gr\mathcal{C}_{dg} \xrightarrow{(-)^1} \mathcal{C}_{dg}$$

has $\mathsf{Sym}_{dg}(-)$ as its left adjoint .

2. The functor $\mathsf{Sym}_{dg}(-)^{\mathrm{int}} : \mathsf{dgmod}_R = \mathcal{C}_{dg} \longrightarrow \mathsf{CAlg}(\mathcal{C}_{dg}, \otimes_{dg}, \varphi_{dg}) = \mathsf{cdga}_R$ is left adjoint to the forgetful functor.

Proof. (2) follows from (1) and Corollary 2.5. As usual, let $(\mathcal{C}, \otimes, \varphi) = (\mathsf{grmod}_R, \otimes_{gr}, \varphi = \text{Koszul symmetry})$. Proposition 2.3 yields the existence of the left adjoint in (1). In order to identify this left adjoint with $(E^\bullet, d) \mapsto (\mathsf{Sym}_{gr}(E^\bullet), \partial_d)$, we first observe that the forgetful functor $U : (gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \longrightarrow (gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ (forgetting the differential) is symmetric monoidal, and that the induced forgetful functor

$$U_{alg} : \mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \longrightarrow \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$$

makes the following diagram commute

$$\begin{tikzcd} \mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg}) \arrow[r, "U_{alg}"] \arrow[d, "(-)^1"'] & \mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi) \arrow[d, "(-)^1"] \\ \mathcal{C}_{dg} \arrow[r] & \mathcal{C} \end{tikzcd}$$

where the horizontal arrows are the forgetful ones. Since we already know that Sym_{gr} is left adjoint to $(-)^1 : \mathsf{CAlg}(gr(\mathcal{C}), \otimes^{gr}, gr\varphi) \longrightarrow \mathcal{C}$, we are reduced to prove the following statement: for any $(E^\bullet, d) \in \mathcal{C}_{dg}$, $(A, \partial_A) \in \mathsf{CAlg}(gr\mathcal{C}_{dg}, \otimes^{gr}_{dg}, gr\varphi_{dg})$, and any map $v : (E^\bullet, d) \to A^1$ in $\mathcal{C}_{dg}$, the unique induced map $f_v : \mathsf{Sym}_{gr}(E^\bullet) \to U_{alg}(A)$ in $\mathsf{CAlg}(gr\mathcal{C}, \otimes^{gr}, gr\varphi)$ (given by the left adjointness of Sym_{gr}) commutes with the graded differentials ∂_d and ∂_A. To see this, first notice that f_v induces a module structure of $U_{alg}(A)$ over $\mathsf{Sym}_{gr}(E^\bullet)$. Now, both $f_v \circ \partial_d$ and $\partial_A \circ f_v$ are graded derivations $\mathsf{Sym}_{gr}(E^\bullet) \to U_{alg}(A)$ of weight 1, and $f_v \circ \partial_d$ restricts to $v \circ d$ on $(-)^1$, while $\partial_A \circ f$ restricts to $\partial^1_A \circ v$ on $(-)^1$. But v is a map of dg-modules, so $v \circ d = \partial^1_A \circ v$, thus, by the uniqueness statement in Proposition 4.5, we indeed have $f_v \circ \partial_d = \partial_A \circ f_v$.

$\square$

6 Examples

Example 6.1 Single even or odd weight $(E^\bullet, d) := (E(k), d = 0)$, for $E \in \mathsf{Mod}_R$, i.e. $E^\bullet$ consists of just E sitting in weight $(-k)$. Then

$$\mathsf{Sym}^n_{gr}(E(k)) = S^n_R(E)[nk]\,,\ \text{for}\ k\ \text{even (i.e.}\ S^n_R(E)\ \text{sitting in degree/internal weight}\ (-kn)\,)$$

and

$$\mathsf{Sym}^n_{gr}(E(k)) = \wedge^n_R(E)[nk]\,,\ \text{for}\ k\ \text{odd (i.e.}\ \wedge^n_R(E)\ \text{sitting in degree/internal weight}\ (-kn)\,).$$

The differentials are all, obviously, zero.

Example 6.2 $(E^\bullet, d) := (\ E^{-1} \xrightarrow{d} E^0\ )$.

- $\boxed{\mathsf{Sym}_{gr}(E^\bullet)}$ external weights and internal weights (denoted by deg):

$$\mathsf{Sym}^0_{gr}(E^\bullet) = \overset{\deg 0}{R}$$

$$\mathsf{Sym}^1_{gr}(E^\bullet) = \left(\overset{\deg -1}{E^{-1}}\,,\ \overset{\deg 0}{E^0} \right)$$

$$\mathsf{Sym}^2_{gr}(E^\bullet) = \left(\overset{\deg -2}{\wedge^2_R E^{-1}}\,,\ \overset{\deg -1}{E^{-1} \otimes_R E^0}\,,\ \overset{\deg 0}{S^2_R(E^0)} \right)$$

$$\mathsf{Sym}^3_{gr}(E^\bullet) = \left(\overset{\deg -3}{\wedge^3_R E^{-1}}\,,\ \overset{\deg -2}{\wedge^2_R E^{-1} \otimes_R E^0}\,,\ \overset{\deg -1}{E^{-1} \otimes_R S^2_R(E^0)}\,,\ \overset{\deg 0}{S^3_R(E^0)} \right)$$

etc.

(Non-zero) differentials:

$$\mathsf{Sym}^0_{gr}(E^\bullet) = \overset{\deg 0}{R}$$

$$\mathsf{Sym}^1_{gr}(E^\bullet) = \left(\overset{\deg -1}{E^{-1}} \xrightarrow{\partial_d^{1,-1} := d} \overset{\deg 0}{E^0} \right)$$

$$\mathsf{Sym}^2_{gr}(E^\bullet) = \left(\overset{\deg -2}{\wedge^2_R E^{-1}} \xrightarrow{\partial_d^{2,-2}} \overset{\deg -1}{E^{-1} \otimes_R E^0} \xrightarrow{\partial_d^{2,-1}} \overset{\deg 0}{S^2_R(E^0)} \right)$$

where

$$\partial_d^{2,-2}(x \wedge x') = d(x) \otimes x' + (-1)^{-1} x \otimes d(x') \overset{(*)}{=} x' \otimes d(x) - x \otimes d(x')$$

($(*)$ comes from the Koszul symmetry for $E^0 \otimes_R E^{-1} \simeq E^{-1} \otimes_R E^0$) and

$$\partial_d^{2,-1}(x \otimes y) = d(x) \odot y.$$

$$\mathsf{Sym}^3_{gr}(E^\bullet) = \left(\overset{\deg -3}{\wedge^3_R E^{-1}} \xrightarrow{\partial_d^{3,-3}} \overset{\deg -2}{\wedge^2_R E^{-1} \otimes_R E^0} \xrightarrow{\partial_d^{3,-2}} \overset{\deg -1}{E^{-1} \otimes_R S^2_R(E^0)} \xrightarrow{\partial_d^{3,-1}} \overset{\deg 0}{S^3_R(E^0)} \right)$$

where

$$\partial_d^{3,-3}(x\wedge x'\wedge x'')=d(x)\otimes(x'\wedge x'')+(-1)^{-1}x\wedge\partial_d^{2,-2}(x'\wedge x'')\overset{(*)}{=}(x'\wedge x'')\otimes d(x)-x\wedge\partial_d^{2,-2}(x'\wedge x'')$$

($(*)$ comes from the Koszul symmetry for $\wedge_R^2 E^{-1}\otimes_R E^0\simeq E^0\otimes_R\wedge_R^2 E^{-1}$),

$$\partial_d^{3,-2}((x\wedge x')\otimes y)=x'\otimes(d(x)\odot y)+(-1)^{-1}x\otimes(d(x')\odot y),$$

and

$$\partial_d^{3,-1}(x\otimes(y\odot y'))=d(x)\odot y\odot y'.$$

etc.

- $\boxed{\mathsf{Sym}^{\mathrm{int}}_{gr}(E^\bullet)}$ internal weights denoted by deg:

$$\begin{aligned}\deg 0\ &: S(E^0)\\ \deg -1\ &: E^{-1}\otimes_R S(E^0)\\ \deg -2\ &: \wedge_R^2(E^{-1})\otimes_R S(E^0)\\ \deg -3\ &: \wedge_R^3(E^{-1})\otimes_R S(E^0)\end{aligned}$$

etc.

Remark 6.3 Suppose that $(E^\bullet,d):=(\ E^{-1}=E\xrightarrow{d=\varphi}E^0=R\ )$ with E free of rank r over R. Then it is easy to check that $\mathsf{Sym}^m_{gr}(E^\bullet,d)=\mathsf{Sym}^n_{dg}(E^\bullet,d)$ for any $m\geq n$. Moreover, for any $m\geq n$, $\mathsf{Sym}^m_{dg}(E^\bullet,d)=\mathsf{Kos}(\varphi:E\to R)$, the Koszul complex associated to $\varphi:E\to R$. Since $\mathsf{Sym}^m_{gr}(E^\bullet)$ stabilizes for $m\geq n$, the algebra product rule

$$\mathsf{Sym}^{m,i}_{dg}(E^\bullet,d)\otimes_R\mathsf{Sym}^{m,j}_{dg}(E^\bullet,d)\longrightarrow\mathsf{Sym}^{2m,i+j}_{dg}(E^\bullet,d)=\mathsf{Sym}^{m,i+j}_{dg}(E^\bullet,d)$$

defines a cdga structure on $\mathsf{Sym}^m_{dg}(E^\bullet,d)$, for any $m\geq n$. This coincides with the usual cdga structure on $\mathsf{Kos}(\varphi:E\to R)$.

Example 6.4 $(E^\bullet,d):=(\ E^1\xrightarrow{d}E^2\ )$.

- $\boxed{\mathsf{Sym}_{gr}(E^\bullet)}$ external weights and internal weights (denoted by deg):

$$\mathsf{Sym}^0_{gr}(E^\bullet)=\overset{\deg 0}{R}$$

$$\mathsf{Sym}^1_{gr}(E^\bullet)=\left(\overset{\deg 1}{E^1}\,,\ \overset{\deg 2}{E^2}\right)$$

$$\mathsf{Sym}^2_{gr}(E^\bullet)=\left(\overset{\deg 2}{\wedge_R^2E^1}\,,\ \overset{\deg 3}{E^1\otimes_R E^2}\,,\ \overset{\deg 4}{S_R^2(E^2)}\right)$$

$$\mathsf{Sym}^3_{gr}(E^\bullet)=\left(\overset{\deg 3}{\wedge_R^3E^1}\,,\ \overset{\deg 4}{\wedge_R^2E^1\otimes_R E^2}\,,\ \overset{\deg 5}{E^1\otimes_R S_R^2(E^2)}\,,\ \overset{\deg 6}{S_R^3(E^2)}\right)$$

etc.

(Non-zero) differentials:

$$\mathsf{Sym}^0_{gr}(E^\bullet) = \overset{\deg 0}{R}$$

$$\mathsf{Sym}^1_{gr}(E^\bullet) = \left(\overset{\text{deg1}}{E^1} \xrightarrow{\partial_d^{1,1} := d} \overset{\deg 2}{E^2} \right)$$

$$\mathsf{Sym}^2_{gr}(E^\bullet) = \left(\overset{\text{deg2}}{\wedge^2_R E^1} \xrightarrow{\partial_d^{2,2}} \overset{\text{deg3}}{E^1 \otimes_R E^2} \xrightarrow{\partial_d^{2,3}} \overset{\deg 4}{S^2_R(E^2)} \right)$$

where

$$\partial_d^{2,2}(x \wedge x') = d(x) \otimes x' + (-1)^{-1} x \otimes d(x') \overset{(*)}{=} x' \otimes d(x) - x \otimes d(x')$$

($(*)$ comes from the Koszul symmetry for $E^1 \otimes_R E^2 \simeq E^2 \otimes_R E^1$) and

$$\partial_d^{2,3}(x \otimes y) = d(x) \odot y.$$

$$\mathsf{Sym}^3_{gr}(E^\bullet) = \left(\overset{\text{deg3}}{\wedge^3_R E^1} \xrightarrow{\partial_d^{3,3}} \overset{\text{deg4}}{\wedge^2_R E^1 \otimes_R E^2} \xrightarrow{\partial_d^{3,4}} \overset{\text{deg5}}{E^1 \otimes_R S^2_R(E^2)} \xrightarrow{\partial_d^{3,5}} \overset{\deg 6}{S^3_R(E^2)} \right)$$

where

$$\partial_d^{3,3}(x \wedge x' \wedge x'') = d(x) \otimes (x' \wedge x'') + (-1)^1 x \wedge \partial_d^{2,2}(x' \wedge x'') \overset{(*)}{=} (x' \wedge x'') \otimes d(x) - x \wedge \partial_d^{2,2}(x' \wedge x'')$$

($(*)$ comes from the Koszul symmetry for $\wedge^2_R E^1 \otimes_R E^2 \simeq E^2 \otimes_R \wedge^2_R E^1$),

$$\partial_d^{3,2}((x \wedge x') \otimes y) = x' \otimes (d(x) \odot y) + (-1)^1 x \otimes (d(x') \odot y),$$

and

$$\partial_d^{3,1}(x \otimes (y \odot y')) = d(x) \odot y \odot y'.$$

etc.

- $\boxed{\mathsf{Sym}^{\text{int}}_{gr}(E^\bullet)}$ internal weights denoted by deg:

$$\deg 0 \ : R$$

$$\deg 1 \ : E^1$$

$$\deg 2 \ : \wedge^2_R(E^1) \oplus E^2$$

$$\deg 3 \ : \wedge^3_R(E^1) \oplus (E^1 \otimes_R E^2)$$

$$\deg 4 \ : \wedge^4_R(E^1) \oplus (\wedge^2_R E^1 \otimes_R E^2) \oplus S^2_R(E^2)$$

etc.

Remark 6.5 *Koszul complex via* 2*-periodization.* Consider a map $\varphi : E \to R$ of R-modules. This fits into the case above with $E^\bullet := (E^1 = E, E^2 = R)$, and $d := \varphi$. We are going to briefly describe an alternative way of recovering the Koszul complex $\mathsf{Kos}(\varphi : E \to R)$ via a "2-periodization" of $\mathsf{Sym}_{dg}(E^\bullet)$, with no hypothesis on the R-module E (compare with Remark 6.3, where we needed E to be free of finite rank).
First of all, notice that

$$\mathsf{Sym}_{dg}(\overset{\mathrm{wt2}}{R}) \simeq \mathsf{Sym}_{gr}(\overset{\mathrm{wt2}}{R}) \equiv \mathsf{Sym}_{gr}(R(-2)) \simeq R[u]$$

where u has bi-weight $(1,2)$, and that $\mathsf{Sym}_{dg}(E^\bullet)$ is a module over $R[u]$. Now consider, the localization $R[u,u^{-1}]$: this is now a $(\mathbb{Z}\times\mathbb{Z})$-graded commutative algebra (while $R[u]$ is only $(\mathbb{N}\times\mathbb{Z})$-graded), where u^{-1} has bi-weight $(-1,-2)$; as usual, we endow both $R[u]$, and $R[u,u^{-1}]$ with trivial differential.
The reader is invited to verify that we have an isomorphism of dg-R-modules[2]

$$(\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}])^{\mathrm{wt}\, n} \simeq \mathsf{Kos}(\varphi : E \to R)[-2n] \quad \forall n \in \mathbb{Z}$$

where we view $\mathsf{Kos}(\varphi : E \to R)$ as a dg-R-module in degrees $(-\infty, 0]$. In particular, we get

$$H^i((\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}])^{\mathrm{int}}) = \begin{cases} H^{\mathrm{ev}}(\mathsf{Kos}(\varphi : E \to R)), & \text{i even} \\ H^{\mathrm{odd}}(\mathsf{Kos}(\varphi : E \to R)), & \text{i odd} \end{cases} \tag{1}$$

where $H^{\mathrm{ev}} := \oplus_{n\,\mathrm{even}} H^n$, and $H^{\mathrm{odd}} := \oplus_{n\,\mathrm{odd}} H^n$.
Note that, as it is always the case, the weight 0 part $(\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}])^{\mathrm{wt}\,0}$ is in fact a cdga, and it is easy to check that the above dg-R-modules isomorphism

$$(\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}])^{\mathrm{wt}\,0} \simeq \mathsf{Kos}(\varphi : E \to R)$$

is in fact an isomorphism of (negatively graded) R-cdga's.

Exercise 6.6 Verify that cohomology commutes with filtered colimits by computing formula (1). More precisely, prove that one may apply the statement "cohomology commutes with filtered colimits" to compute the cohomology of $(\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}])^{\mathrm{int}}$ in terms of the cohomology of $\mathsf{Sym}_{dg}(E^\bullet)$, then compute the cohomology of $(\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}])^{\mathrm{int}}$ using this statement, and finally compare this computation to formula (1).

Exercise 6.7 Compute the R-dg-modules $\mathsf{Sym}^n_{dg}(E^\bullet, d)$, for $n \leq 4$, in the case $(E^\bullet, d) := (\ E^{-1} \xrightarrow{d_{-1}} E^0 \xrightarrow{d_0} E^1\ )$.

7 Comparison with Σ-coinvariants

Let $(\mathcal{C}, \otimes, \varphi)$ be a symmetric monoidal R-linear category (φ denoting the symmetry morphism) having arbitrary coproducts, with $\otimes$ distributing along coproducts. For an arbitrary $X \in \mathcal{C}$, consider the following action of Σ_n on $X^{\otimes n}$. Any $\sigma \in \Sigma_n$ can be decomposed as a product of adjacent transpositions (recall: such a decomposition exists but is not unique in general); for $\tau(i,i+1)$ the adjacent transposition exchanging i and $(i+1)$, we define $\tau(i,i+1)_X \in Aut_{\mathcal{C}}(X^{\otimes n})$ as the following composition

$$X^{\otimes n} = X \otimes \cdots \otimes \overset{i}{X} \otimes \overset{i+1}{X} \otimes \cdots \otimes X \xrightarrow{\mathrm{id}\otimes\varphi_{\mathrm{X,X}}\otimes\mathrm{id}} X \otimes \cdots \otimes \overset{i}{X} \otimes \overset{i+1}{X} \otimes \cdots \otimes X = X^{\otimes n}\,.$$

Showing that the induced action of Σ_n on $X^{\otimes n}$ is well defined (i.e. that it is independent of the decomposition of an arbitrary $\sigma \in \Sigma_n$ into a product of adjacent transposition) is long and boring but possible[3].

[2]Since $R[u,u^{-1}]$ is $\mathbb{Z}\times\mathbb{Z}$-graded, we have that $\mathsf{Sym}_{dg}(E^\bullet)\otimes_{R[u]} R[u,u^{-1}]$ is also $\mathbb{Z}\times\mathbb{Z}$-graded rather than $\mathbb{N}\times\mathbb{Z}$-graded, so the external weight is $\mathbb{Z}$. Also recall that for M, N graded modules over a graded-commutative algebra A, the weight n-part of $M\otimes_A N$ is given by the quotient of $\oplus_{p+q=n} M^p \otimes_{A^0} N^q$ by the submodule generated by elements of the form $(ma \otimes_{A^0} n - m \otimes_{A^0} an)$, where $m \in M^\alpha$, $n \in N^\beta$, and $a \in A^{n-\alpha-\beta}$ are arbitrary (in particular, we do not require $\alpha + \beta = n$).

[3]Using the description of symmetric monoidal categories as special Γ-categories, this Σ_n-action appears more naturally. However, if the focus is on being able to compute this action, this alternative approach is not that helpful.

Definition 7.1 *If $\mathcal{C}$ has finite colimits, se define*

$$(X^{\otimes n})_{\Sigma_n} := \mathrm{colim}(u(X,n) : \mathrm{B}\Sigma_n \to \mathcal{C})$$

the Σ_n-coinvariants object *of* $X^{\otimes n}$*. Here the functor* $u(X,n)$ *sends the unique object of* $\mathrm{B}\Sigma_n$ *to* $X^{\otimes n}$*, and* $\sigma \in \Sigma_n$ *to the automorphism* σ_X *of* $X^{\otimes n}$ *defined above.*

If $\mathcal{C} = \mathsf{grmod}_R$, and $X = E^\bullet \in \mathsf{grmod}_R$, then

$$((E^\bullet)^{\otimes_{gr} n})_{\Sigma_n} \simeq (E^\bullet)^{\otimes_{gr} n}/N$$

where N is the graded submodule of $(E^\bullet)^{\otimes_{gr} n}$ generated by $\{\underline{x} - \sigma\underline{x} \,|\, \sigma \in \Sigma_n,\ \underline{x} \in (E^\bullet)^{\otimes_{gr} n}\}$.
If $\mathcal{C} = \mathsf{dgmod}_R$, and $X = (E^\bullet, d) \in \mathsf{dgmod}_R$, then

$$((E^\bullet, d)^{\otimes_{dg} n})_{\Sigma_n} \simeq (E^\bullet, d)^{\otimes_{dg} n}/N'$$

where N' is the dg-submodule of $(E^\bullet, d)^{\otimes_{dg} n}$ generated by $\{\underline{x} - \sigma\underline{x} \,|\, \sigma \in \Sigma_n,\ \underline{x} \in (E^\bullet)^{\otimes_{dg} n}\}$.

Proposition 7.2

- *For* $E^\bullet \in \mathsf{grmod}_R$*, we have a canonical isomorphism*

  $$\mathsf{Sym}_{gr}(E^\bullet) \simeq (((E^\bullet)^{\otimes^{gr} n})_{\Sigma_n})_{n \in \mathbb{N}}$$

  in $\mathsf{CAlg}(gr(\mathsf{grmod}_R), \otimes^{gr}_{gr}, gr\varphi = gr(Koszul\ \ symmetry))$*. This isomorphism is functorial in* $E^\bullet$*.*

- *For* $(E^\bullet, d) \in \mathsf{dgmod}_R$*, we have a canonical isomorphism*

  $$\mathsf{Sym}_{dg}(E^\bullet, d) \simeq (((E^\bullet, d)^{\otimes^{dg} n})_{\Sigma_n})_{n \in \mathbb{N}}$$

  $\mathsf{CAlg}(gr(\mathsf{dgmod}_R), \otimes^{gr}_{dg}, gr\varphi_{dg}))$*. This isomorphism is functorial in* $(E^\bullet, d)$*.*

8 Another approach

Let $R[\varepsilon] = R \oplus R\varepsilon$ the graded-commutative R-algebra where ε sits in weight 1, and $\varepsilon^2 = 0$. Then, there is an equivalence of categories

$$\mathsf{grmod}_{R[\varepsilon]} \simeq \mathsf{dgmod}_R$$

where multiplication by ε gives the differential. This equivalence becomes a symmetric monoidal equivalence

$$\alpha : (\mathsf{grmod}_{R[\varepsilon]}, \otimes^{\varepsilon}_R, \varphi_\varepsilon) \simeq (\mathsf{dgmod}_R, \otimes, \text{Koszul symmetry})$$

where

- for $E, F \in \mathsf{grmod}_{R[\varepsilon]}$, their tensor product $E \otimes^{\varepsilon}_R F$ is given by their graded tensor product $E \otimes^{gr}_R F$ over R (which is canonically a $R[\varepsilon] \otimes^{gr}_R R[\varepsilon]$-module), with the $R[\varepsilon]$-graded module structure induced by the canonical comultiplication map $\Delta : R[\varepsilon] \longrightarrow R[\varepsilon] \otimes^{gr}_R R[\varepsilon]$ sending ε to $\varepsilon \otimes 1 + 1 \otimes \varepsilon$.
- the symmetry φ_ε is the usual Koszul symmetry on $E \otimes^{gr}_R F$.

The symmetric monoidal equivalence

$$\alpha : (\mathsf{grmod}_{R[\varepsilon]}, \otimes^{\varepsilon}_R, \varphi_\varepsilon) \simeq (\mathsf{dgmod}_R, \otimes, \text{Koszul symmetry})$$

induces an equivalence

$$\mathsf{CAlg}(\alpha) : \mathsf{CAlg}(\mathsf{grmod}_{R[\varepsilon]}, \otimes^{\varepsilon}_R, \varphi_\varepsilon) \simeq \mathsf{cdga}_R.$$

Proposition 8.1 *The forgetful functor*

$$\mathsf{CAlg}(\mathsf{grmod}_{R[\varepsilon]}, \otimes_R^{\varepsilon}, \varphi_{\varepsilon}) \longrightarrow \mathsf{grmod}_{R[\varepsilon]}$$

admits a left adjoint denoted by $\mathsf{Sym}_R^{\varepsilon}$*, and the following diagram commutes*

$$\begin{tikzcd}
\mathsf{grmod}_{R[\varepsilon]} \arrow[r, "\alpha"] \arrow[d, "\mathsf{Sym}_R^{\varepsilon}"'] & \mathsf{dgmod}_R \arrow[d, "((\mathsf{Sym}_{gr})^{\mathrm{int}}, \partial_?)"] \\
\mathsf{CAlg}(\mathsf{grmod}_{R[\varepsilon]}, \otimes_R^{\varepsilon}, \varphi_{\varepsilon}) \arrow[r, "\mathsf{CAlg}(\alpha)"'] & \mathsf{cdga}_R
\end{tikzcd}$$

Proof. The existence of the left adjoint follows from the adjoint functor theorem. For the commutativity of the square, first observe that the square

$$\begin{tikzcd}
\mathsf{grmod}_{R[\varepsilon]} & \mathsf{dgmod}_R \arrow[l, "\alpha^{-1}"'] \\
\mathsf{CAlg}(\mathsf{grmod}_{R[\varepsilon]}, \otimes_R^{\varepsilon}, \varphi_{\varepsilon}) \arrow[u, "U"] & \mathsf{cdga}_R \arrow[l, "\mathsf{CAlg}(\alpha^{-1})"] \arrow[u, "U'"']
\end{tikzcd}$$

(where U and U' denote the obvious forgetful functors) commutes by definition of $\mathsf{CAlg}(\alpha)$. In other words, the square

$$\begin{tikzcd}
\mathsf{grmod}_{R[\varepsilon]} \arrow[r, "\alpha"] & \mathsf{dgmod}_R \\
\mathsf{CAlg}(\mathsf{grmod}_{R[\varepsilon]}, \otimes_R^{\varepsilon}, \varphi_{\varepsilon}) \arrow[u, "U"] \arrow[r, "\mathsf{CAlg}(\alpha)"'] & \mathsf{cdga}_R \arrow[u, "U'"']
\end{tikzcd}$$

is horizontally right-adjointable. But this is equivalent to it being vertically left-adjointable, which is exactly the commutativity of the square in the statement. $\square$

Remark 8.2 The weakness of this approach lies in the fact that, without further analysis, the only concrete way of computing $\mathsf{Sym}_R^{\varepsilon}(E^{\bullet})$ seems to be via the commutativity of the square in Prop. 8.1, and the computations shown in the previous section for $((\mathsf{Sym}_{gr}(E^{\bullet}))^{\mathrm{int}}, \partial_{d_E})$. I would be curious if some student could refute this weakness by making this approach more constructive and useful.

9 Graded modules over a graded-commutative algebra

We quickly treat here the case of graded modules over a graded-commutative R-algebras: this is not so different from the case of graded R-modules that we have already described in details.

Let $A \in \mathsf{CAlg}(\mathsf{grmod}_R, \otimes_{gr}, \varphi =$ Koszul symmetry), i.e. a graded-commutative R-algebra. The category $\mathsf{Mod}_A(\mathsf{grmod}_R, \otimes_{gr}, \varphi)$ of graded-modules over A has a symmetric monoidal structure that we will denote as $(\mathcal{C}_A, \otimes_{A,gr}, \varphi_A)$. By applying Proposition 2.3 and Corollary 2.5 to $(\mathcal{C}_A, \otimes_{A,gr}, \varphi_A)$, we get

Proposition 9.1 *1. The functor*

$$(-)^1 : \mathsf{CAlg}(gr\mathcal{C}_A, \otimes_{A,gr}^{gr}, gr\varphi_A) \longrightarrow \mathcal{C}_A$$

defined as the composition

$$\mathsf{CAlg}(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A) \xrightarrow{forget} gr\mathcal{C}_A \xrightarrow{(-)^1} \mathcal{C}_A$$

has a left adjoint denoted by $\mathsf{Sym}_{A,gr}(-)$.

2. *The* coproduct functor $\oplus_{i\in\mathbb{Z}} : (gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_{\mathbf{A}}) \longrightarrow (\mathcal{C}_A, \otimes_{A,gr}, \varphi_A)$ *is symmetric monoidal. The induced functor on commutative algebras objects is denoted by*

$$(-)^{\mathrm{int}} : \mathsf{CAlg}(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A) \longrightarrow \mathsf{CAlg}(\mathcal{C}_A, \otimes_{A,gr}, \varphi_A).$$

3. *The functor*

$$\mathsf{Sym}_{A,gr}(-)^{\mathrm{int}} : \mathsf{grmod}_A = \mathcal{C}_A \longrightarrow \mathsf{CAlg}(\mathcal{C}_A, \otimes_{A,gr}, \varphi_A) = \mathsf{grcalg}_A$$

is left adjoint to the forgetful functor.

Note that $B \in \mathsf{CAlg}(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A)$ has bi-weights $B^{i,j}$, where i is called the external weight, and j the internal one. We have

- The A-module structure of B is given by maps $A^n \otimes_R B^{i,j} \to B^{i,n+j}$ (with properties).
- The behaviour of the algebra multiplication $\cdot$ on B with respect to its bi-weights is given by $B^{i,j} \cdot B^{h,k} \subseteq B^{i+h,j+k}$ and is linear with respect to the previous A-module structure.
- For $M \in \mathsf{Mod}_B(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A)$, a *graded derivation of weight* $r \in \mathbb{Z}$ from B to M, is a map $\partial : B \to M$ of bi-weight $(0, r)$ in $gr\mathcal{C}_A$, such that $\partial(b^{i,j} \cdot b'^{h,k}) = \partial(b^{i,j})b'^{h,k} + (-1)^{jr} b^{i,j}\partial(b'^{h,k})$ where on the r.h.s. we used mere juxtaposition for the B-module structure on M.
- The unit of adjunction gives an isomorphism $E^\bullet \simeq \mathsf{Sym}^1_{A,gr}(E^\bullet) := (\mathsf{Sym}_{A,gr}(E^\bullet))^1$ in $gr\mathcal{C}_A$.
- Let $E^\bullet \in gr\mathcal{C}_A$ be a graded A-module, $B \in \mathsf{CAlg}(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A)$, and suppose that B is a module over $\mathsf{Sym}_{A,gr}(E^\bullet)$ (i.e. $B \in \mathsf{Mod}_{\mathsf{Sym}_{A,gr}(E^\bullet)}(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A)$). Then, for any map $u : E^\bullet \longrightarrow (B)^1$ of weight $r \in \mathbb{Z}$ in $gr\mathcal{C}_A$, there is a unique graded derivation $\partial_u : \mathsf{Sym}_{A,gr}(E^\bullet) \longrightarrow B$ of weight r, as above, such that ∂_u restricts to u on $(-)^1$.

10 dg-modules over a cdga

We quickly treat here the case of dg-modules over a cdga over R: this is not so different from the case of dg-modules over R that we have already described in details (Section 5).

Let $\mathbf{A} := (A, d_A) \in \mathsf{cdga}_R = \mathsf{CAlg}(\mathsf{dgmod}_R, \otimes_{dg}, \varphi_{dg} = \text{Koszul symmetry})$. The category

$$\mathsf{dgmod}_{\mathbf{A}} := \mathsf{Mod}_{\mathbf{A}}(\mathsf{dgmod}_R, \otimes_{dg}, \varphi_{dg})$$

of $\mathbf{A}$-dg-modules has a symmetric monoidal structure that we will denote by $(\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A},dg}, \varphi_{\mathbf{A}})$. We will put

$$\mathsf{cdga}_{\mathbf{A}} := \mathsf{CAlg}(\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A},dg}, \varphi_{\mathbf{A}}).$$

By applying Proposition 2.3 and Corollary 2.5 to $(\mathcal{C}, \otimes, \varphi) = (\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A},gr}, \varphi_{\mathbf{A}})$ we get

Proposition 10.1 *1. The functor*

$$(-)^1 : \mathsf{CAlg}(gr\mathcal{M}_{\mathbf{A}}, \otimes^{gr}_{\mathbf{A},dg}, gr\varphi_{\mathbf{A}}) \longrightarrow \mathcal{M}_{\mathbf{A}}$$

defined as the composition

$$\mathsf{CAlg}(gr\mathcal{M}_{\mathbf{A}}, \otimes^{gr}_{\mathbf{A},dg}, gr\varphi_{\mathbf{A}}) \xrightarrow{forget} gr\mathcal{M}_{\mathbf{A}} \xrightarrow{(-)^1} \mathcal{M}_{\mathbf{A}}$$

has a left adjoint denoted by $\mathsf{Sym}_{\mathbf{A},dg}(-)$.

2. *The* coproduct functor $\oplus_{i\in\mathbb{Z}} : (gr\mathcal{M}_{\mathbf{A}}, \otimes^{gr}_{\mathbf{A},dg}, gr\varphi_{\mathbf{A}}) \longrightarrow (\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A},gr}, \varphi_{\mathbf{A}})$ *is symmetric monoidal. The induced functor on commutative algebras objects is denoted by*

$$(-)^{\mathrm{int}} : \mathsf{CAlg}(gr\mathcal{M}_{\mathbf{A}}, \otimes^{gr}_{\mathbf{A},dg}, gr\varphi_{\mathbf{A}}) \longrightarrow \mathsf{CAlg}(\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A}}, \varphi_{\mathbf{A}}).$$

3. *The functor*

$$\mathsf{Sym}_{\mathbf{A},dg}(-)^{\mathrm{int}} : \mathsf{dgmod}_{\mathbf{A}} = \mathcal{M}_{\mathbf{A}} \longrightarrow \mathsf{CAlg}(\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A},gr}, \varphi_{\mathbf{A}}) = \mathsf{cdga}_{\mathbf{A}}$$

is left adjoint to the forgetful functor.

Let $(E^\bullet, d) \in \mathsf{dgmod}_{\mathbf{A}}$. By viewing the differential d as an endomorphism of weight 1 of $E^\bullet \in \mathsf{grmod}_A$, the extension property listed in the previous Section, applied to $B = \mathsf{Sym}_{A,gr}(E^\bullet)$, gives us a graded self-derivation $\partial_d : \mathsf{Sym}_{A,gr}(E^\bullet) \longrightarrow \mathsf{Sym}_{A,gr}(E^\bullet)$ of weight 1. Since $d^2 = 0$, the same property tells us that $\partial_d \circ \partial_d = 0$: we say that ∂_d is a *graded differential* on $\mathsf{Sym}_{A,gr}(E^\bullet) \in \mathsf{CAlg}(gr\mathcal{C}_A, \otimes^{gr}_{A,gr}, gr\varphi_A)$. Let's look at the pair $(\mathsf{Sym}_{A,gr}(E^\bullet), \partial_d)$:

(i) for any $n \in \mathbb{N}$, the pair $(\mathsf{Sym}^n_{A,gr}(E^\bullet) := (\mathsf{Sym}_{A,gr}(E^\bullet))_n, \partial^n_d)$ is a $\mathbf{A}$-dg-module

(ii) the pair $(\mathsf{Sym}_{gr}(E^\bullet)^{\mathrm{int}}, \partial^{\mathrm{int}}_d) = (\oplus_{n\in\mathbb{N}}\mathsf{Sym}^n_{gr}(E^\bullet), \oplus_{n\in\mathbb{N}}\partial^n_d)$ is a $\mathbf{A}$-cdga, that will be denoted as $\mathsf{Sym}_{A,dg}(E^\bullet, d)^{\mathrm{int}}$.

(iii) it is easy to verify that the pair $(\mathsf{Sym}_{A,gr}(E^\bullet), \partial_d)$ is an object in $\mathsf{CAlg}(gr\mathcal{M}_{\mathbf{A}}, \otimes^{gr}_{\mathbf{A},dg}, gr\varphi_{\mathbf{A}})$.

Proposition 10.2 *1. For any* $(E^\bullet, d) \in \mathcal{M}_{\mathbf{A}}$*, there is a canonical isomorphism*

$$\mathsf{Sym}_{\mathbf{A},dg}(E^\bullet, d) \simeq (\mathsf{Sym}_{A,gr}(E^\bullet), \partial_d)$$

in $\mathsf{CAlg}(gr\mathcal{M}_{\mathbf{A}}, \otimes^{gr}_{\mathbf{A},dg}, gr\varphi_{\mathbf{A}})$*, functorial in* $(E^\bullet, d)$.

2. *For any* $(E^\bullet, d) \in \mathcal{M}_{\mathbf{A}}$*, there is a canonical isomorphism*

$$\mathsf{Sym}^{\mathrm{int}}_{\mathbf{A},dg}(E^\bullet, d) \simeq (\mathsf{Sym}^{\mathrm{int}}_{A,gr}(E^\bullet), (\partial_d)^{\mathrm{int}})$$

in $\mathsf{cdga}_{\mathbf{A}} = \mathsf{CAlg}(\mathcal{M}_{\mathbf{A}}, \otimes_{\mathbf{A},dg}, \varphi_{\mathbf{A}})$*, functorial in* $(E^\bullet, d)$.

Gabriele Vezzosi, DIMAI, Università di Firenze, gabriele.vezzosi@unifi.it