Reduced obstruction theory for the stack of stable maps to a $K3$-surface via derived algebraic geometry

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Abstract

Let $S$ be an algebraic $K3$-surface, $g \in \mathbb{N}$, and $\beta \neq 0$ in $H_2(S, \mathbb{Z})$ a curve class. We construct a derived stack $\mathbb{M}_g^{\text{red}}(S; \beta)$ whose truncation is the usual stack $\overline{M}_g(S; \beta)$ of stable maps from curves of genus $g$ to $S$ hitting the class $\beta$, and such that the inclusion $\overline{M}_g(S; \beta) \hookrightarrow \mathbb{M}_g^{\text{red}}(S; \beta)$ induces on $\overline{M}_g(S; \beta)$ a perfect obstruction theory whose tangent and obstruction spaces coincide with the corresponding reduced spaces of Okounkov-Maulik-Pandharipande-Thomas [O-P2, M-P, M-P-T]. The approach we present here yields not only a full rigorous proof of the existence of a reduced obstruction theory, not relying on any result on semiregularity maps, but also a new global geometric interpretation. An important ingredient of our construction is a perfect determinant map from the stack of perfect complexes to the stack of line bundles whose tangent morphism is Illusie’s trace map for perfect complexes. We expect that this determinant map might be useful in other contexts as well.

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Introduction

It is a distinguished feature of derived algebraic geometry that any derived extension of a given algebraic stack $F$ induces a canonical obstruction theory on $F$ (see §1). In other words, a suitable reformulation of the moduli problem in derived algebraic geometry, immediately gives us an obstruction theory, in a completely geometric way, with no need of clever choices. And, under suitable conditions, also the converse is expected to hold. The present paper is a sample application of this construction, not relying on the conjectural general equivalence between a class of derived stacks and a class of underived stacks endowed with a properly structured obstruction theory.

We concentrate on the first geometrically interesting occurrence of a given stack having two different obstruction theories: the stack $\overline{\mathcal{M}}_g(S; \beta)$ of stable maps of type $(g, \beta)$ to a smooth projective complex $K3$-surface $S$. The stack $\overline{\mathcal{M}}_g(S; \beta)$ has a standard obstruction theory, yielding trivial Gromov-Witten invariants in the $n$-pointed case, and a so-called reduced obstruction theory, considered by Okounkov-Maulik-Pandharipande-Thomas (often abbreviated to O-M-P-T in the text), giving interesting curve counting invariants in the $n$-pointed case ([P, M-P, M-P-T]). We use derived algebraic geometry to give a construction of a global reduced obstruction theory on $\overline{\mathcal{M}}_g(S; \beta)$, and compare its deformation and obstruction spaces with those of Okounkov-Maulik-Pandharipande-Thomas. More precisely, we use a perfect determinant map from the derived stack of perfect complexes to the derived stack of line bundles, and exploit the peculiarities of the derived stack of line bundles on a $K3$-surface, to produce a derived extension $\mathbb{R}\overline{\mathcal{M}}_g^{\text{red}}(S; \beta)$ of $\overline{\mathcal{M}}_g(S; \beta)$.

The derived stack $\mathbb{R}\overline{\mathcal{M}}_g^{\text{red}}(S; \beta)$ arises as the canonical homotopy fiber over the unique derived factor of the derived stack of line bundles on $S$, so it is, in a very essential way, a purely derived geometrical object. We prove quasi-smoothness of $\mathbb{R}\overline{\mathcal{M}}_g^{\text{red}}(S; \beta)$, and this immediately gives us a global reduced obstruction theory on $\overline{\mathcal{M}}_g(S; \beta)$. Our proof is self contained (inside derived algebraic geometry), and does not rely on any previous result on semiregularity maps.

An important thing to remark is that, in order to simplify the exposition, we have chosen not to treat explicitly the $n$-pointed case, leaving to the reader the obvious straightforward modifications to be made in the statements. No substantial difference in the proofs is required.
One of the main ingredients of our construction is the perfect determinant map $\det_{\text{perf}} : \text{Perf} \to \text{Pic}$, whose definition requires the use of a bit of Waldhausen $K$-theory, and whose tangent map can be identified with Illusie’s trace map of perfect complexes ([III Ch. 5]). We expect that this determinant map might be useful in other contexts as well (e.g. in establishing a full Gromov-Witten/Donaldson-Thomas correspondence).

The way we approach the problem of finding a reduced obstruction theory on the stack of stable maps to a $K3$-surface is somewhat interesting from a conceptual and geometrical point of view, in that it shows that there is a neatly defined derived stack inducing this otherwise somehow ad-hocly defined obstruction theory. Let us try to explain why our construction might also be considered useful from a more practical point of view. First of all, the existence of the reduced obstruction theory follows directly, with no need of results about semi-regularity maps (which might however be interpreted derived-geometrically in our context). Related to this, we could only find a partially explicit definition of O-M-P-T global reduced obstruction theory in the literature\footnote{In [M-P, §2.2] the argument gives a uniquely defined obstruction theory with target the $\tau_{2,-1}$-truncation of the cotangent complex of the stack of maps (from a fixed domain curve).}, while there is certainly a complete description of the corresponding tangent and obstruction spaces. Therefore our construction might also be seen as establishing rigorously such a reduced global obstruction theory. Finally, the obstruction theory coming from a derived extension possesses all the functoriality properties that sometimes are missing in the purely obstruction-theoretic approach. This might prove useful in getting similar results in families or even relative to the whole moduli stack of $K3$-surfaces.

As a side remark, let us observe that the emerging picture seems to suggest that most of the natural maps of complexes arising in moduli problems can be realized as tangent or cotangent maps associated to morphisms between appropriate derived moduli stacks. This suggestion is confirmed in the present paper for the standard obstruction theories associated to the stack of maps between a fixed algebraic scheme and a smooth projective target, to the stack of stable maps to a smooth projective scheme or to the Picard stack of a smooth projective scheme, for the trace map and the first Chern class map for perfect complexes ([III Ch. 5]), and for the map inducing O-M-P-T’s reduced obstruction theory.

Description of contents. The first three sections are written for an arbitrary smooth complex projective scheme $X$. We explain how a derived extension induces an obstruction theory on its truncation, how to define the standard derived extensions of the Picard stack of $X$, and of the stack of stable maps to $X$, and finally define the perfect determinat map. In the last section, we specialize to the case where $X = S$ is a smooth complex projective $K3$ surface. We first give a self-contained description of O-M-P-T’s pointwise reduced tangent and obstruction spaces. Then, by exploiting the features of the derived Picard stack of $S$, we define a derived extension $\mathbb{R}\mathcal{M}^\text{red}_g(S; \beta)$ of the usual stack $\mathcal{M}_g(S; \beta)$ of stable maps of type $(g, \beta \neq 0)$ to $S$, having the property that, for the canonical inclusion $j^\text{red}_* : \mathcal{M}_g(S; \beta) \hookrightarrow \mathbb{R}\mathcal{M}^\text{red}_g(S; \beta)$, the induced map

$$j^\text{red}_* \mathbb{L}\mathbb{R}\mathcal{M}^\text{red}_g(S; \beta) \longrightarrow \mathbb{L}\mathcal{M}_g(S; \beta)$$

is a perfect obstruction theory with the same tangent and obstruction spaces as the reduced theory introduced by Maulik-Okounkov-Pandharipande-Thomas (Thm. 4.8).
Acknowledgments. Our initial interest in the possible relationships between reduced obstruction theories and derived algebraic geometry was positively boosted by comments and questions by B. Fantechi, D. Huybrechts and R. Thomas. We are grateful to R. Pandharipande for pointing out a useful classical statement, and to H. Flenner for some important remarks. We especially thanks A. Vistoli for generously sharing his expertise on stable maps with us.

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Notations. For background and basic notations in derived algebraic geometry we refer the reader to [HAG-II] and to the overview [To-2]. We will most often omit the functor $i : \text{St}_C \rightarrow d\text{St}_C$ – from the homotopy category of stacks over $C$ to the homotopy category of derived stacks over $C$ – from our notations, since it is fully faithful. Recall however that $i$ does not commute with taking internal HOM (derived) stacks nor with taking homotopy limits. In particular, we will write $t_0(F) \hookrightarrow F$ for the adjunction morphism $i t_0(F) \hookrightarrow F$.

All fibered products will be implicitly derived.

When useful, we will freely switch back and forth between simplicial commutative $k$-algebras and commutative differential non-positively graded $k$-algebras, where $k$ is a field of characteristic 0 ([To-Ve, App. A]).

All complexes will be cochain complexes and, for such a complex $C^\bullet$, either $C_{\leq n}$ or $C^{\leq n}$ (depending on typographical convenience) will denote its good truncation in degrees $\leq n$.

Analogously for either $C_{\geq n}$ or $C^{\geq n}$.

To ease notation we will often write $\otimes$ for $\otimes^L$ whenever no confusion is likely to arise.

$X$ will denote a smooth complex projective scheme while $S$ a smooth complex projective $K3$-surface.

As a terminological remark, for a given obstruction theory, we will call its deformation space what is usually called its tangent space (while we keep the terminology obstruction space). We do this to avoid confusion with tangent spaces, tangent complexes or tangent cohomologies of related (derived) stacks.

We will often abbreviate the list of authors Okounkov-Maulik-Pandharipande-Thomas to O-M-P-T.

1 Derived extensions and obstruction theories

We briefly recall here the basic observation that a derived extension of a given stack $X$ induced an obstruction theory (in the sense of [B-F]) on $X$.

Definition 1.1 Given a stack $X \in \text{Ho}(\text{St}_C)$, a derived extension of $X$ is a derived stack $X^{\text{der}}$ together with an isomorphism

$$X \simeq t_0(X^{\text{der}}).$$
Proposition 1.2 Let $X^{\text{der}}$ be a derived geometric stack which is a derived extension of the (geometric) stack $X$. Then, the closed immersion

$$j : X \simeq t_0(X^{\text{der}}) \hookrightarrow X^{\text{der}}$$

induces a morphism

$$j^*(L_{X^{\text{der}}}) \to L_X$$

which is 2-connective, i.e. its cone has vanishing cohomology in degrees $\geq -1$.

Proof. The proof follows easily from the remark that if $A$ is a simplicial commutative $C$-algebra and $A \to \pi_0(A)$ is the canonical surjection, then the cotangent complex $L_{\pi_0(A)/A}$ is 2-connective, i.e. has vanishing cohomology in degrees $\geq -1$. $\square$

The previous Proposition shows that a derived extension always induces an obstruction theory (whenever such a notion is defined by [B-F, Def. 4.4], e.g. when $X$ is a Deligne-Mumford stack). In particular, recalling that a derived stack is quasi-smooth if its cotangent complex is perfect of amplitude in $[-1, 0]$, we have the following result

Corollary 1.3 Let $X^{\text{der}}$ be a quasi-smooth derived Deligne-Mumford stack which is a derived extension of a (Deligne-Mumford) stack $X$. Then

$$j^*(L_{X^{\text{der}}}) \to L_X$$

is a $[-1, 0]$-perfect obstruction theory as defined in [B-F, Def. 5.1].

Remark 1.4 We expect that also the converse is true, i.e. that given any stack $X$ locally of finite presentation over a field $k$, endowed with a map of co-dg-Lie algebroids $E \to L_X$, there should exist a derived extension inducing the given obstruction theory. We will come back to this in a future work and will not use it in the rest of this paper, although it should be clear that it was exactly such an expected result that first led us to think about the present work.

2 Derived stack of stable maps and derived Picard stack

In this section we prove a correspondence between derived open substacks of a derived stack and open substacks of its truncation, and use it to construct the derived Picard stack $\mathbb{R}\text{Pic}(X; \beta)$ of type $\beta \in H^2(X, \mathbb{Z})$, for any complex projective smooth variety $X$. After recalling the derived version of the stack of (pre-)stable maps to $X$, the same correspondence will lead us to defining the derived stack $\mathbb{R}\mathcal{M}_g(X; \beta)$ of stable maps of type $(g, \beta)$ to $X$.

Throughout the section $X$ will denote a smooth complex projective scheme, $g$ a nonnegative integer, $c_1$ a class in $H^2(X, \mathbb{Z})$ (which, for our purposes, may be supposed to belong to the image of $\text{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$, i.e. belonging to $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$), and $\beta \in H_2(X, \mathbb{Z})$ an effective curve class.

We will frequently use of the following
Proposition 2.1 Let $F$ be a derived stack and $t_0(F)$ its truncation. There is a bijective correspondence

$$\phi_F : \{\text{Zariski open substacks of } t_0(F)\} \longrightarrow \{\text{Zariski open derived substacks of } F\}.$$ 

For any Zariski open substack $U_0 \hookrightarrow t_0(F)$, we have a homotopy cartesian diagram in $\mathrm{dSt}_C$

$$\begin{array}{ccc}
U_0 & \longrightarrow & t_0(F) \\
\downarrow & & \downarrow \\
\phi_F(U_0) & \longrightarrow & F
\end{array}$$

where the vertical maps are the canonical closed immersions.

Proof. The statement is an immediate consequence of the fact that $F$ and $t_0(F)$ have the same topology (\cite[Cor. 2.2.2.9]{HAG-II}). More precisely, let us define $\phi_F$ as follows. If $U_0 \hookrightarrow t_0(F)$ is an open substack, $\phi_F(U_0)$ is the functor

$$\mathbf{SAlg}_C \longrightarrow \mathbf{SSets} : A \mapsto F(A) \times_{t_0(F)(\pi_0(A))} U_0(\pi_0(A))$$

where $F(A)$ maps to $t_0(F)(\pi_0(A))$ via the morphism (induced by the truncation functor $t_0$)

$$F(A) \simeq \mathbb{R}\text{HOM}_{\mathbf{St}_C}(\mathbb{R}\text{Spec}(A), F) \longrightarrow \mathbb{R}\text{HOM}_{\mathbf{St}_C}(t_0(\mathbb{R}\text{Spec}(A)), t_0(F)) \simeq t_0(F)(\pi_0(A)).$$

The inverse to $\phi_F$ is simply induced by the truncation functor $t_0$. \hfill \Box

2.1 The derived Picard stack

Definition 2.2 The Picard stack of $X/\mathbb{C}$ is the stack

$$\mathbf{Pic}(X) := \mathbb{R}\text{HOM}_{\mathbf{St}_C}(X, B\mathbb{G}_m).$$

The derived Picard stack of $X/\mathbb{C}$ is the derived stack

$$\mathbb{R}\mathbf{Pic}(X) := \mathbb{R}\text{HOM}_{\mathbf{dSt}_C}(X, B\mathbb{G}_m).$$

By definition we have a natural isomorphism $t_0(\mathbb{R}\mathbf{Pic}(X)) \simeq \mathbf{Pic}(X)$ in $\mathrm{Ho}(\mathbf{dSt}_C)$. Note that even though $\mathbf{Pic}(X)$ is smooth, it is not true that $\mathbb{R}\mathbf{Pic}(X) \simeq \mathbf{Pic}(X)$, if $\dim(X) > 1$; this can be seen on tangent spaces since

$$T_L\mathbb{R}\mathbf{Pic}(X) \simeq C^\bullet(X, \mathcal{O}_X)[1]$$

for any global point $x_L : \text{Spec}(\mathbb{C}) \to \mathbb{R}\mathbf{Pic}(X)$ corresponding to a line bundle $L$ over $X$.

Given $c_1 \in H^2(X, \mathbb{Z})$, we denote by $\mathbf{Pic}(X; c_1)$ the open substack of $\mathbf{Pic}(X)$ classifying line bundles with first Chern class $c_1$. More precisely, for any $R \in \mathbf{Alg}_C$, let us denote by $\text{Vect}_1(R; c_1)$ the groupoid of line bundles $L$ on $\text{Spec}(R) \times X$ such that, for any point $x : \text{Spec}(\mathbb{C}) \to \text{Spec}(R)$ the pullback line bundle $L_x$ on $X$ has first Chern class equal to $c_1$. Then, $\mathbf{Pic}(X; c_1)$ is the stack:

$$\mathbf{Alg}_C \longrightarrow \mathbf{SSets} : R \mapsto \text{Nerve}((\text{Vect}_1(R; c_1)))$$
where \( \text{Nerve}(C) \) is the nerve of the category \( C \).

Note that we have

\[
\text{Pic}(X) = \coprod_{c_1 \in H^2(X, \mathbb{Z})} \text{Pic}(X; c_1).
\]

**Definition 2.3** Let \( c_1 \in H^2(X, \mathbb{Z}) \). The derived Picard stack of type \( c_1 \) of \( X/\mathbb{C} \) is the derived stack

\[
\mathbb{R}\text{Pic}(X; c_1) := \phi_{\mathbb{R}\text{Pic}(X)}(\text{Pic}(X; c_1)).
\]

In particular, we have a natural isomorphism \( t_0(\mathbb{R}\text{Pic}(X; c_1)) \simeq \text{Pic}(X; c_1) \), and a homotopy cartesian diagram in \( \text{dSt}_\mathbb{C} \)

\[
\begin{array}{ccc}
\text{Pic}(X; c_1) & \longrightarrow & \text{Pic}(X) \\
\downarrow & & \downarrow \\
\mathbb{R}\text{Pic}(X; c_1) & \longrightarrow & \mathbb{R}\text{Pic}(X)
\end{array}
\]

### 2.2 The derived stack of stable maps

We recall from [To-2, 4.3 (4.d)] the construction of the derived stack \( \mathbb{R}M^\text{pre}_g(X) \) of prestable maps of genus \( g \) to \( X \), and of its open derived substack \( \mathbb{R}M^\text{pre}_g(X) \) of stable maps of genus \( g \) to \( X \). Then we move to define the derived version of the stack of stable maps of type \((g, \beta)\) to \( X \).

Let \( M^\text{pre}_g \) be the stack of pre-stable curves of genus \( g \), and \( C^\text{pre}_g \longrightarrow M^\text{pre}_g \) its universal family (see e.g. [Be]).

**Definition 2.4** The derived stack \( \mathbb{R}M^\text{pre}_g(X) \) of prestable maps of genus \( g \) to \( X \) is defined as

\[
\mathbb{R}M^\text{pre}_g(X) := \text{RHom}_{\text{dSt}_{\mathbb{C}}}((C^\text{pre}_g \times X) \times M^\text{pre}_g).
\]

\( \mathbb{R}M^\text{pre}_g(X) \) is then canonically a derived stack over \( M^\text{pre}_g \), and the corresponding derived universal family \( \mathbb{R}C^\text{pre}_{g; X} \) is defined by the following homotopy cartesian square

\[
\begin{array}{ccc}
\mathbb{R}C^\text{pre}_{g; X} & \longrightarrow & \mathbb{R}M^\text{pre}_g(X) \\
\downarrow & & \downarrow \\
C^\text{pre}_g & \longrightarrow & M^\text{pre}_g
\end{array}
\]

Note that, by definition, \( \mathbb{R}C^\text{pre}_{g; X} \) comes also equipped with a canonical map

\[
\mathbb{R}C^\text{pre}_{g; X} \longrightarrow \mathbb{R}M^\text{pre}_g(X) \times X.
\]

We also have \( t_0(\mathbb{R}M^\text{pre}_g(X)) \simeq M^\text{pre}_g(X) \) (the stack of prestable maps of genus \( g \) to \( X \)), and \( t_0(\mathbb{R}C^\text{pre}_{g; X}) \simeq C^\text{pre}_{g; X} \) (the universal family over the stack of pre-stable maps of genus \( g \) to \( X \)), since the truncation functor \( t_0 \) commutes with homotopy fibered products.

We can now use Proposition 2.1 to define the derived stable versions. Let \( M^\text{open}_g(X) \) be the open substack of \( M^\text{pre}_g(X) \), of stable maps of genus \( g \) to \( X \), and \( C^\text{open}_{g; X} \longrightarrow M^\text{pre}_g(X) \) the (induced) universal family ( [Be] ).
Definition 2.5 The derived stack $\mathbb{M}_g(X)$ of stable maps of genus $g$ to $X$ is defined as

$$\mathbb{M}_g(X) := \phi_{\mathbb{M}_g^\text{pre}(X)}(\mathcal{M}_g(X)).$$

The derived stable universal family

$$\mathbb{C}_{g;X} \rightarrow \mathbb{M}_g(X)$$

is the derived restriction of $\mathbb{C}_{g;X} \rightarrow \mathbb{M}_g^\text{pre}(X)$ to $\mathbb{M}_g(X)$.

Recall that

- $t_0(\mathbb{M}_g(X)) \simeq M_g(X);$ 
- $t_0(\mathbb{C}_{g;X}) \simeq C_{g;X};$
- $\mathbb{C}_{g;X}$ comes equipped with a canonical map
  $$\pi : \mathbb{C}_{g;X} \rightarrow \mathbb{M}_g(X) \times X;$$
- we have a homotopy cartesian diagram in $\text{dSt}_\mathbb{C}$

$$\xymatrix{ M_g(X) \ar[r] & \mathcal{M}_g^\text{pre}(X) \ar[d] \\
\mathbb{M}_g(X) \ar[r] & \mathbb{M}_g^\text{pre}(X) \ar[u] }$$

Let $g$ a non-negative integer, $\beta \in H_2(X, \mathbb{Z})$, and $\mathcal{M}_g(X;\beta)$ be the stack of stable maps of type $(g,\beta)$ to $X$ (see e.g. [Be]); its derived version is given by the following

Definition 2.6 The derived stack of stable maps of type $(g,\beta)$ to $X$ is defined as the open substack of $\mathbb{M}_g(X)$

$$\mathbb{M}_g(X;\beta) := \phi_{\mathbb{M}_g(X)}(\mathcal{M}_g(X;\beta)).$$

The derived stable universal family of type $(g,\beta)$,

$$\mathbb{C}_{g,\beta;X} \rightarrow \mathbb{M}_g(X;\beta),$$

is the (derived) restriction of $\mathbb{C}_{g;X} \rightarrow \mathbb{M}_g(X)$ to $\mathbb{M}_g(X;\beta)$.

Note that, by definition, $t_0(\mathbb{M}_g(X;\beta)) \simeq \mathcal{M}_g(X;\beta)$, therefore $\mathbb{M}_g(X;\beta)$ is a proper derived Deligne-Mumford stack ([HAG-II 2.2.4]). Moreover, the derived stable universal family $\mathbb{C}_{g,\beta;X}$ comes, by restriction, equipped with a natural map

$$\pi : \mathbb{C}_{g,\beta;X} \rightarrow \mathbb{M}_g(X;\beta) \times X.$$

We have a homotopy cartesian diagram in $\text{dSt}_\mathbb{C}$

$$\xymatrix{ M_g(X;\beta) \ar[r] & \mathcal{M}_g(X) \\
\mathbb{M}_g(X;\beta) \ar[r] & \mathbb{M}_g(X) \ar[u] }.$$
The tangent complex of \( R^g(X; \beta) \) at a stable map \( (f : C \to X) \) of type \((g, \beta)\) (corresponding to a classical point \( x_f : \text{Spec}(\mathbb{C}) \to R^g(X; \beta) \)) is given by
\[
T_{(f:C \to X)} \simeq \mathbb{R}\Gamma(C, \text{Cone}(T_C \to f^*T_X)),
\]
where \( T_C \) is the tangent complex of \( C \) and \( T_X \) is the tangent sheaf of \( X \).

The canonical map \( \mathbb{R}M^g(X; \beta) \to \mathbb{M}^g_{\text{pre}} \) is quasi-smooth. In fact, the fibre at a geometric point, corresponding to prestable curve \( C \), is the derived stack \( \mathbb{R}\text{HOM}_{\beta}(C, X) \) whose tangent complex at a point \( f : C \to X \) is \( \mathbb{R}\Gamma(C, f^*T_S) \) which, obviously, has cohomology only in degrees \([0, 1]\). But \( \mathbb{M}^g_{\text{pre}} \) is smooth, and any derived stack quasi-smooth over a smooth base is quasi-smooth (by the corresponding exact triangle of tangent complexes). Therefore the derived stack \( \mathbb{R}M^g(X; \beta) \) is quasi-smooth.

Proposition \( \text{(1.2)} \) then recovers the standard (absolute) perfect obstruction theory (yielding trivial Gromov-Witten invariants for \( X = S \) a \( K3 \) surface; see e.g. \( [O-P1, 5.3.5] \)) on \( \mathbb{M}^g(X; \beta) \) via the canonical map \( j^*([\mathbb{R}M^g(X; \beta)]) \to \mathbb{M}^g(X; \beta) \) induced by the closed immersion \( j : \mathbb{M}^g(X; \beta) \hookrightarrow \mathbb{R}M^g(X; \beta) \).

### 3 The derived determinant morphism

In this section we start by defining a quite general perfect determinant map of derived stacks

\[ \text{det}_{\text{Perf}} : \text{Perf} \longrightarrow \text{Pic} = BG_m \]

whose construction requires a small detour into Waldhausen \( K \)-theory. We think this perfect determinant might play an important role in other contexts as well, e.g. in a general GW/DT correspondence.

Using the perfect determinant together with a natural perfect complex on \( \mathbb{R}M^g(X; \beta) \), we will be able to define a map
\[
\delta_1(X) : \mathbb{R}M^g(X) \longrightarrow \mathbb{R}\text{Pic}(X)
\]
which will be one of the main ingredients in the construction of the reduced derived stack of stable maps \( \mathbb{R}M^g_{\text{red}}(S; \beta) \), for a \( K3 \)-surface \( S \), given in the next section.

### 3.1 The perfect determinant map

The aim of this subsection is to produce a determinant morphism \( \text{det}_{\text{Perf}} : \text{Perf} \longrightarrow \text{Pic} \) in \( \text{Ho}(\text{dSt}_{\mathbb{C}}) \) extending the natural determinant morphism \( \text{Vect} \longrightarrow \text{Pic} \). To do this, we will have to pass through Waldhausen \( K \)-theory.

We start with the classical determinant map in \( \text{Ho}(\text{St}_{\mathbb{C}}) \), \( \text{det} : \text{Vect} \longrightarrow \text{Pic} \), induced by the map sending a vector bundle to its top exterior power. Consider the following simplicial stacks
\[
B_* \text{Pic} : \Delta^{op} \ni [n] \longmapsto (\text{Pic})^n
\]

\(^2\)The \([1]\) shift in [CF-K] Thm. 5.4.8] is clearly a typo: their proof is correct and yields no shift.
(with the simplicial structure maps given by tensor products of line bundles, or equivalently, induced by the product in the group structure of $BG_m \simeq \text{Pic}$), and

$$B_\bullet \text{Vect} : \Delta^{op} \ni [n] \mapsto wS_n \text{Vect},$$

where, for any commutative $C$-algebra $R$, $wS_n \text{Vect}(R)$ is the nerve of the category of sequences of split monomorphisms

$$0 \to M_1 \to M_2 \to \ldots \to M_n \to 0$$

with morphisms the obvious equivalences, and the simplicial structure maps are the natural ones described in [Wal, 1.3]. Similarly, we define the simplicial object in stacks

$$B_\bullet \text{Perf} : \Delta^{op} \ni [n] \mapsto wS_n \text{Perf}$$

(see [Wal, 1.3] for the definition of $wS_n$ in this case). Now, $B_\bullet \text{Pic}$ and $B_\bullet \text{Vect}$, and $B_\bullet \text{Perf}$ are pre-$\Delta^{op}$-stacks according to Def. 1.4.1 of [To-1], and the map $\det$ extends to a morphism

$$\det : B_\bullet \text{Vect} \to B_\bullet \text{Pic}$$

in the homotopy category of pre-$\Delta^{op}$-stacks. By applying the functor $i : \text{Ho}(\text{St}_C) \to \text{Ho}(\text{dSt}_C)$ (that will be, according to our conventions, omitted from notations), we get a determinant morphism (denoted in the same way)

$$\det : B_\bullet \text{Vect} \to B_\bullet \text{Pic}$$

in homotopy category of pre-$\Delta^{op}$-derived stacks. We now pass to Waldhausen $K$–theory, i.e. apply $K := \Omega \circ | - |$ (see [To-1, Thm 1.4.3], where the loop functor $\Omega$ is denoted by $\mathbb{R}\Omega_*$, and the realization functor $| - |$ by $B$), and observe that, by [To-1, Thm 1.4.3 (2)], there is a canonical isomorphism in $\text{Ho}(\text{dSt}_C)$

$$K(B_\bullet \text{Pic}) \simeq \text{Pic}$$

since $\text{Pic}$ is group-like (i.e. an $H_\infty$-stack in the parlance of [To-1, Thm 1.4.3]). This gives us a map in $\text{Ho}(\text{dSt}_C)$

$$K(\det) : K(B_\bullet \text{Vect}) \to \text{Pic}.$$  

Now, consider the map $u : K^{\text{Vect}} := K(B_\bullet \text{Vect}) \to K(B_\bullet \text{Perf}) := K^{\text{Perf}}$ in $\text{Ho}(\text{dSt}_C)$, induced by the inclusion $\text{Vect} \hookrightarrow \text{Perf}$. By [Wal, Thm. 1.7.1], $u$ is an isomorphism in $\text{Ho}(\text{dSt}_C)$. Therefore, we get a diagram in $\text{Ho}(\text{dSt}_C)$

$$\begin{array}{ccc}
K^{\text{Vect}} & \xrightarrow{K(\det)} & \text{Pic} \\
\downarrow u & & \\
\text{Perf} & \xrightarrow{\text{1st-level}} & K^{\text{Perf}}
\end{array}$$

where $u$ is an isomorphism. This allows us to give the following
Definition 3.1 The induced map in $\text{Ho}(\text{dSt}_\mathbb{C})$

$$\det_{\text{Perf}} : \text{Perf} \rightarrow \text{Pic}$$

is called the perfect determinant morphism.

For any smooth complex projective scheme $X$, the perfect determinant morphism $\det_{\text{Perf}} : \text{Perf} \rightarrow \text{Pic}$ induces a map in $\text{Ho}(\text{dSt}_\mathbb{C})$

$$\det_{\text{Perf}}(X) : \mathbb{R}\text{Perf}(X) := \mathbb{R}\text{HOM}_{\text{dSt}_\mathbb{C}}(X, \text{Perf}) \rightarrow \mathbb{R}\text{HOM}_{\text{dSt}_\mathbb{C}}(X, \text{Pic}) =: \mathbb{R}\text{Pic}(X).$$

As perhaps not totally unexpected (e.g. [Ill, Rem. 5.3.3]), the tangent morphism to the perfect determinant map is given by the trace for perfect complexes

Proposition 3.2 Let $X$ be a smooth complex projective scheme, and $\det_{\text{Perf}}(X) : \mathbb{R}\text{Perf}(X) \rightarrow \mathbb{R}\text{Pic}(X)$ the induced perfect determinant map. For any complex point $x_E : \text{Spec} \mathbb{C} \rightarrow \mathbb{R}\text{Perf}(X)$, corresponding to a perfect complex $E$ over $X$, the tangent map

$$T_{x_E}\det_{\text{Perf}}(X) : T_{x_E}\mathbb{R}\text{Perf}(X) \simeq \mathbb{R}\text{Hom}(E, E)[1] \rightarrow \mathbb{R}\text{Hom}(\mathcal{O}_S, \mathcal{O}_S)[1] \simeq T_{x_E}\mathbb{R}\text{Pic}(X)$$

is given by the trace map for perfect complexes of [Ill, Ch. 5, 3.7.3].

3.2 The map $\mathbb{R}\overline{M}_g(X) \rightarrow \mathbb{R}\text{Perf}(X)$

A map

$$\mathbb{R}\overline{M}_g(X) \rightarrow \mathbb{R}\text{Perf}(X) = \mathbb{R}\text{HOM}_{\text{dSt}_\mathbb{C}}(X, \text{Perf})$$

in $\text{Ho}(\text{dSt}_\mathbb{C})$ is, by adjunction, the same thing as a map

$$\mathbb{R}\overline{M}_g(X) \times X \rightarrow \text{Perf}$$

i.e. a derived perfect complex on $\mathbb{R}\overline{M}_g(X) \times X$; so, it is enough to find such an appropriate perfect complex.

Let

$$\pi : \mathbb{R}C_{g, X} \rightarrow \mathbb{R}\overline{M}_g(X) \times X$$

be the derived stable universal family (§2.2).

Proposition 3.3 $\mathbb{R}\pi_*(\mathcal{O}_{\mathbb{R}C_{g, X}})$ is a perfect complex on $\mathbb{R}\overline{M}_g(X) \times X$.

Proof. The truncation of $\pi$ is proper, hence it is enough to prove that $\pi$ is also quasi-smooth. To see this, observe that both $\mathbb{R}C_{g, X}$ and $\mathbb{R}\overline{M}_g(X) \times X$ are smooth over $\mathbb{R}\overline{M}_g(X)$. Then conclude, since any map between derived stacks smooth over a base is quasi-smooth. □

Remark 3.4 If we fix a class $\beta \in H_2(X, \mathbb{Z})$, the corresponding $\beta$-decorated version of Proposition 3.3 obviously holds.

We may therefore give the following
**Definition 3.5** We will denote by

\[ A_X : \mathbb{R}\overline{M}_g(X) \to \mathbb{R}\text{Perf}(X) \]

the map induced by the perfect complex \( \mathbb{R}\pi_*(\mathcal{O}_{\mathcal{R}Cg,X}) \).

Note that, by definition, \( A_X \) sends a complex point of \( \mathbb{R}\overline{M}_g(X) \), corresponding to a stable map \( f : C \to X \) to the perfect complex \( \mathbb{R}f_*\mathcal{O}_C \) on \( X \).

**The tangent morphism of \( A_X \).** Let us describe the tangent morphism of \( A_X \) at a complex point of \( \mathbb{R}\overline{M}_g(X) \) corresponding to a stable map \( f : C \to X \). We have

\[ T_{(f:C \to X)} \mathbb{R}\overline{M}_g(X) \simeq \Gamma(C, \text{Cone}(T_C \to f^*TX)) \simeq \Gamma(C, T_{C/X}[1]) \simeq \mathbb{R}\text{Hom}(\mathbb{L}_{C/X}, \mathcal{O}_C[1]). \]

By definition of the cotangent complex, we have

\[ \mathbb{R}\text{Hom}(\mathbb{L}_{C/X}, \mathcal{O}_C[1]) \simeq \mathbb{D}\text{er}_{f^{-1}\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C[1]) \to \mathbb{R}\text{Hom}_{f^{-1}\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C[1]). \]

By decomposing the scheme map \( f : C \to X \) into the composition of morphisms of ringed spaces

\[ (C, \mathcal{O}_C) \xrightarrow{\varphi} (C, f^{-1}\mathcal{O}_C) \xrightarrow{f^0} (X, \mathcal{O}_X) \]

where \( \varphi \) (resp. \( f^0 \)) is the identity (resp. coincides with \( f \)) on the underlying spaces, and is given by \( f^{-1}\mathcal{O}_X \to \mathcal{O}_C \) (resp. by the identity) at the level of structure sheaves, we obtain ([Ha-RD, Prop. 5.2]) natural isomorphisms

\[ \mathbb{R}\text{Hom}_{f^{-1}\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C[1]) \simeq \Gamma(C, \mathbb{R}\text{Hom}_{f^{-1}\mathcal{O}_X}(\mathbb{R}\varphi_*\mathcal{O}_C, \mathbb{R}\varphi_*\mathcal{O}_C[1])) \simeq \Gamma(X, \mathbb{R}f_*\mathbb{R}\text{Hom}_{f^{-1}\mathcal{O}_X}(\mathbb{R}\varphi_*\mathcal{O}_C, \mathbb{R}\varphi_*\mathcal{O}_C[1])), \]

and a map (by ([Ha-RD, Prop. 5.5]))

\[ \Gamma(X, \mathbb{R}f_*\mathbb{R}\text{Hom}_{f^{-1}\mathcal{O}_X}(\mathbb{R}\varphi_*\mathcal{O}_C, \mathbb{R}\varphi_*\mathcal{O}_C[1])) \to \Gamma(X, \mathbb{R}\text{Hom}_{\mathcal{O}_X}(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C[1])). \]

The tangent morphism of \( A_X : \mathbb{R}\overline{M}_g(X) \to \mathbb{R}\text{Perf}(X) \) at a complex point of \( \mathbb{R}\overline{M}_g(X) \) corresponding to a stable map \( f : C \to X \) is then obtained as the composition

\[ \tau_f := T_fA_X : T_{(f:C \to X)} \mathbb{R}\overline{M}_g(X) \simeq \mathbb{R}\text{Hom}(\mathbb{L}_{C/X}, \mathcal{O}_C[1]) \to \mathbb{R}\text{Hom}_{f^{-1}\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C[1]) \to \mathbb{R}\Gamma(X, \mathbb{R}\text{Hom}_{\mathcal{O}_X}(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C))[1] = T_{\mathbb{R}f_*\mathcal{O}_C}\mathbb{R}\text{Perf}(X). \]

**Remark 3.6** The map \( A_X \) gives therefore a global derived geometrical interpretation of the canonical maps in ([Bu-Fl], Prop. 7.21).

**Remark 3.7** We relate the tangent map of \( A_X \) to the Atiyah class of \( \mathbb{R}f_*\mathcal{O}_C \). Take a complex point of \( \mathbb{R}\overline{M}_g(X) \) corresponding to a stable map \( f : C \to X \), and let \( p : C \to \text{Spec} \mathbb{C} \) and \( q : S \to \text{Spec} \mathbb{C} \) denote the structural morphisms, so that \( p = q \circ f \). Let us consider the perfect complex \( E := \mathbb{R}f_*\mathcal{O}_C \) on \( X \). The dual complex \( E^\vee \) has an Atiyah class (relative to \( X/\mathbb{C} \); see ([III], Ch. 4, 2.3.7])

\[ at_{S/\mathbb{C}}(E^\vee) : E^\vee \to E^\vee \otimes^L \Omega^1_{X}[1] \]
and taking its dual, we get (the biduality map $E \to E^\vee$ is an isomorphism in $D(X)$ since $E$ is perfect, [SGA6 I.7.2])

$$\text{at}(E^\vee) : \mathbb{R}f_*\mathcal{O}_C \otimes^L T_X[-1] \longrightarrow \mathbb{R}f_*\mathcal{O}_C.$$ 

By adjunction we get a map

$$\text{at}^\vee : T_X \longrightarrow \mathbb{R}\text{Hom}_X(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C)[1]$$

fitting into the following commutative diagram (adapt [B-F] Lemma 4.9 (2))

$$\begin{array}{ccc}
\mathbb{R}q_*T_X & \xrightarrow{\mathbb{R}q_*(\text{at}^\vee)} & \mathbb{R}q_*\mathbb{R}\text{Hom}_X(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C)[1] \\
& & \downarrow \tau_f \\
\mathbb{R}q_*\mathbb{R}f_*f^*T_X & \simeq & \mathbb{R}p_*f^*T_X \longrightarrow \mathbb{R}p_*\text{Cone}(T_C \to f^*T_X)
\end{array}$$

We end this section by the following

**Definition 3.8** We denote by $\delta_1(X)$ the composition

$$\mathbb{R}\mathcal{M}_g(X) \xrightarrow{\Lambda_X} \mathbb{R}\text{Perf}(X) \xrightarrow{\text{det}_{\text{perf}(X)}} \mathbb{R}\text{Pic}(X),$$

and, for a complex point of $\mathbb{R}\mathcal{M}_g(X)$ corresponding to a stable map $f : C \to X$, by

$$\Theta_f := T_f\delta_1(X) : T_{(f:C\to X)}\mathbb{R}\mathcal{M}_g(X) \xrightarrow{\tau_f} \mathbb{R}\text{Hom}_X(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C)[1] \xrightarrow{\text{tr}_X} \mathbb{R}\Gamma(X, \mathcal{O}_X)[1].$$

Note that, as a map of explicit complexes, we have

$$\Theta_f : \mathbb{R}\Gamma(C, \text{Cone}(T_C \to f^*T_X)) \xrightarrow{\tau_f} \mathbb{R}\text{Hom}_X(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C)[1] \xrightarrow{\text{tr}_X} \mathbb{R}\Gamma(X, \mathcal{O}_X)[1].$$

**Remark 3.9 - First Chern class of $\mathbb{R}f_*\mathcal{O}_C$ and the map $\Theta_f$.** Using Remark 3.7, we can relate the map $\Theta_f$ above to the *first Chern class* of the perfect complex $\mathbb{R}f_*\mathcal{O}_C$ ([III Ch. V]). With the same notations as those in Remark 3.7, the following diagram is commutative

$$\begin{array}{ccc}
\mathbb{R}q_*T_X & \xrightarrow{\mathbb{R}q_*(\text{at}^\vee)} & \mathbb{R}q_*\mathbb{R}\text{Hom}_X(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C)[1] \\
& & \downarrow \tau_f \\
\mathbb{R}q_*\mathbb{R}f_*f^*T_X & \simeq & \mathbb{R}p_*f^*T_X \longrightarrow \mathbb{R}p_*\text{Cone}(T_C \to f^*T_X) \xrightarrow{\Theta_f} \mathbb{R}q_*\mathcal{O}_X[1]
\end{array}$$

In this diagram, the composite upper row is the image under $\mathbb{R}q_*$ of the first Chern class $c_1(\mathbb{R}f_*\mathcal{O}_C) \in \text{Ext}_X^1(T_X, \mathcal{O}_X) \simeq H^1(X, \Omega_X^1)$.

There is a small abuse here: we are implicitely using the canonical map $T_X \to \mathbb{R}\text{Hom}_X(\Omega_X^1, \mathcal{O}_X)$ in $D(X)$ (\text{Hom} always maps to $\mathbb{R}\text{Hom}$).
4 The reduced derived stack of stable maps to a $K3$-surface

In this section we specialize to the case of a $K3$-surface $S$, with a fixed nonzero curve class $\beta \in H_2(S;\mathbb{Z}) \simeq H^2(S;\mathbb{Z})$. After recalling in some detail the reduced obstruction theory of O-M-P-T, we first identify canonically the derived Picard stack $\mathbb{R}\text{Pic}(S)$ with $\text{Pic}(S) \times \mathbb{R}\text{Spec}(\text{Sym}(H^0(S,K_S)[1]))$ where $K_S$ is the canonical sheaf of $S$. This result is then used to define the reduced version $\mathbb{M}_{g,\beta}^{\text{red}}(S)$ of the derived stack of stable maps of type $(g,\beta)$ to $S$, and to show that this induces, via the canonical procedure available for any algebraic derived stack, a modified obstruction theory on its truncation $\mathbb{M}_g(S;\beta)$ whose deformation and obstruction spaces are then compared with those of the reduced theory of O-M-P-T. As a terminological remark, given an obstruction theory, we will call deformation space what is usually called its tangent space (while we keep the terminology obstruction space). We do this to avoid confusion with tangent spaces, tangent complexes or tangent cohomologies of possibly related (derived) stacks.

4.1 Review of reduced obstruction theory

For a $K3$-surface $S$, the moduli of stable maps of genus $g$ curves to $S$ with non-zero effective class $\beta \in H^{1,1}(S,\mathbb{C}) \cap H^2(S,\mathbb{Z})$ (note that Poincaré duality yields a canonical isomorphism $H_2(S;\mathbb{Z}) \simeq H^2(S;\mathbb{Z})$ between singular (co)homologies) carries a relative perfect obstruction theory. This obstruction theory is given by

$$(R\pi_*F^*TS)^{\vee} \to \mathbb{L}\mathbb{M}_g(S;\beta)/\mathbb{M}_g^{\text{pre}}.$$  

Here $\pi: \mathcal{C}_{g,\beta, S} \to \mathbb{M}_g(S;\beta)$ is the universal curve, $F: \mathcal{C}_{g,\beta, S} \to S$ is the universal morphism from the universal curve to $S$, and $\mathbb{M}_g^{\text{pre}}$ denotes the Artin stack of prestable curves. A Riemann-Roch argument along with the fact that a $K3$-surface has trivial canonical bundle yields the expected dimension of $\mathbb{M}_g(S;\beta)$:

$$\exp \dim \mathbb{M}_g(S;\beta) = g - 1.$$  

We thus expect no rational curves on a $K3$-surface. This result stems from the deformation invariance of Gromov-Witten invariants. A $K3$-surface admits deformations such that the homology class $\beta$ is no longer of type $(1,1)$, and thus can not be the class of a curve.

This is unfortunate, given the rich literature on enumerative geometry of $K3$-surfaces, and is in stark contrast to the well-known conjecture that a projective $K3$-surface over an algebraically closed field contains infinitely many rational curves. Further evidence that there should be an interesting Gromov-Witten theory of $K3$-surfaces are the results of Bloch, Ran and Voisin that rational curves deform in a family of $K3$-surfaces provided their homology classes remain of type $(1,1)$. The key ingredients in the proof is the semi-regularity map. We thus seek a new kind of obstruction theory for $\mathbb{M}_g(S;\beta)$ which is deformation invariant only for such deformations of $S$ which keep $\beta$ of type $(1,1)$.

Such a new obstruction theory, called the reduced obstruction theory, was introduced in [O-P2] [M-P] [M-P-T]. Sticking to the case of moduli of morphisms from a fixed curve $C$ to $S$, the obstruction space at a fixed morphism $f$ is $H^1(C,f^*TS)$.

This obstruction space admits a map

$$H^1(C,f^*TS) \xrightarrow{\sim} H^1(C,f^*\Omega_S) \xrightarrow{H^1(df)} H^1(C,\Omega^1_C) \xrightarrow{H^1(\omega_C)} H^1(C,\omega_C) \simeq \mathbb{C},$$
where the first isomorphism is induced by the choice of a holomorphic symplectic form on \( S \). The difficult part is to prove that all obstructions for all types of deformations of \( f \) (and not only curvilinear ones) lie in the kernel of this map. Once this is proven, \( \mathcal{M}_g(S; \beta) \) carries a reduced obstruction theory which yields a virtual class, called the reduced class. This reduced class is one dimension larger that the one obtained from the standard perfect obstruction theory and leads to many interesting enumerative results (see [P, M-P, M-P-T]).

We will review below the construction of the reduced deformation and obstruction spaces giving all details since they will be needed in our comparison result (Thm. 4.8).

4.1.1 Deformation and obstruction spaces of the reduced theory according to O-M-P-T

For further reference, we give here a self-contained treatment of the reduced deformation and reduced obstruction spaces on \( \mathcal{M}_g(S; \beta) \) according to Okounkov-Maulik-Pandharipande-Thomas.

Let us fix a stable map \( f : C \to S \) of class \( \beta \neq 0 \) and genus \( g \); \( p : C \to \text{Spec} \mathbb{C} \) and \( q : S \to \text{Spec} \mathbb{C} \) will denote the structural morphisms. Let \( \omega_C \simeq p^* \mathcal{O}_{\text{Spec} \mathbb{C}} \) be the dualizing complex of \( C \), and \( \omega_C = \omega_C[-1] \) the corresponding dualizing sheaf.

First of all, the deformation spaces of the standard (i.e. unreduced) and reduced theory, at the stable map \( f \), coincide with

\[
H^0(C, \text{Cone}(\mathcal{T}_C \to f^*T_S))
\]

where \( \mathcal{T}_C \) is the cotangent complex of the curve \( C \).

Let’s recall now ([P] §3.1) the construction of the reduced obstruction space. We give here a canonical version, independent of the choice of a holomorphic symplectic form on \( S \).

Consider the canonical isomorphism\(^4\)

\[
\varphi : T_S \otimes H^0(S, K_S) \longrightarrow \Omega_S^1.
\]

By tensoring this by \( H^0(S, K_S)^\vee \simeq H^2(S, \mathcal{O}_S) \) (this isomorphism is canonical by Serre duality) which is of dimension 1 over \( \mathbb{C} \), we get a canonical sequence of isomorphisms of \( \mathcal{O}_S \)-Modules

\[
T_S \longrightarrow T_S \otimes H^0(S, K_S) \otimes H^2(S, \mathcal{O}_S) \longrightarrow \Omega_S^1 \otimes H^2(S, \mathcal{O}_S).
\]

We denote by \( \psi : T_S \to \Omega_S^1 \otimes H^2(S, \mathcal{O}_S) \) the induced, canonical isomorphism. Form this, we get a canonical isomorphism of \( \mathcal{O}_C \)-Modules

\[
f^* \psi : f^* T_S \longrightarrow f^* (\Omega_S^1) \otimes H^2(S, \mathcal{O}_S).
\]

\(^4\)We use throughout the standard abuse of writing \( F \otimes V \) for \( F \otimes_{\mathcal{O}_X} p^* V \), for any scheme \( p : X \to \text{Spec} \mathbb{C} \), any \( \mathcal{O}_X \)-Module \( F \), and any \( \mathbb{C} \)-vector space \( V \).
Now consider the canonical maps

\[
f^*\Omega^1_S \xrightarrow{s} \Omega^1_C \xrightarrow{t} \omega_C \simeq p^!\mathcal{O}_{\text{Spec} C}[-1]
\]

where \(\omega_C \simeq \omega_C[-1]\) is the dualizing sheaf of \(C\) and \(\omega_C = p^!\mathcal{O}_{\text{Spec} C}\) the dualizing complex of \(C\) (see [Ha-RD] Ch. V). We thus obtain a map

\[
\bar{\theta} : f^*T_S \to \omega_C \otimes H^2(S,\mathcal{O}_S)[-1] \simeq \omega_C \otimes H^2(S,\mathcal{O}_S).
\]

By the properties of dualizing complexes, we have

\[
\omega_C \otimes H^2(S,\mathcal{O}_S)[-1] = \omega_C \otimes p^*(H^2(S,\mathcal{O}_S))[-1] \simeq p^!(H^2(S,\mathcal{O}_S)[-1]),
\]

so we get a canonical morphism

\[
f^*T_S \to p^!(H^2(S,\mathcal{O}_S)[-1])
\]

which induces, by applying \(\mathbb{R}p_*\) and composing with the adjunction map \(\mathbb{R}p_*p^! \to \text{Id}\), a canonical map

\[
\bar{\alpha} : \mathbb{R}\Gamma(C, f^*T_S) \simeq \mathbb{R}p_*(f^*T_S) \xrightarrow{\mathbb{R}p_*(\bar{\theta})} \mathbb{R}p_*(\omega_C \otimes H^2(S,\mathcal{O}_S)) \simeq \mathbb{R}p_*p^!(H^2(S,\mathcal{O}_S)[-1]) \to H^2(S,\mathcal{O}_S)[-1].
\]

Since \(\mathbb{R}\Gamma\) is a triangulated functor, to get a unique induced map

\[
\alpha : \mathbb{R}\Gamma(C, \text{Cone}(\Lambda_C \to f^*T_S)) \to H^2(S,\mathcal{O}_S)[-1]
\]

it will be enough to observe that \(\text{Hom}_{D(C)}(\mathbb{R}p_*\Lambda_C[1], \mathbb{H}^2(S,\mathcal{O}_S)[-1]) = 0\) (which is obvious since \(\mathbb{R}p_*\Lambda_C[1]\) lives in degrees \([-1, 0]\), while \(\mathbb{H}^2(S,\mathcal{O}_S)[-1]\) in degree 1), and to prove the following

**Lemma 4.1** The composition

\[
\mathbb{R}p_*\Lambda_C \xrightarrow{\mathbb{R}p_*f^*T_S} \mathbb{R}p_*\omega_C \otimes H^2(S,\mathcal{O}_S)
\]

vanishes in the derived category \(D(C)\).

**Proof.** If \(C\) is smooth, the composition

\[
\Lambda_C \xrightarrow{f^*T_S} f^*\Omega^1_S \otimes H^2(S,\mathcal{O}_S) \xrightarrow{s \otimes \text{id}} \Omega_C \otimes H^2(S,\mathcal{O}_S)
\]

is obviously zero, since \(\Lambda_C \simeq T_C\) in this case, and a curve has no 2-forms. For a general prestable \(C\), we proceed as follows. Let’s consider the composition

\[
\theta : \Lambda_C \xrightarrow{f^*T_S} f^*\Omega^1_S \otimes H^2(S,\mathcal{O}_S) \xrightarrow{s \otimes \text{id}} \Omega_C \otimes H^2(S,\mathcal{O}_S) \xrightarrow{t \otimes \text{id}} \omega_C \otimes H^2(S,\mathcal{O}_S) := \mathcal{L}.
\]

On the smooth locus of \(C\), \(\mathcal{H}^0(\theta)\) is zero (by the same argument used in the case \(C\) smooth), hence the image of \(\mathcal{H}^0(\theta) : \mathcal{H}^0(\Lambda_C) \simeq T_C \to \mathcal{L}\) is a torsion subsheaf of the line bundle \(\mathcal{L}\). But \(C\) is Cohen-Macaulay, therefore this image is 0, i.e. \(\mathcal{H}^0(\theta) = 0\); and, obviously, \(\mathcal{H}^i(\theta) = 0\) for any \(i\) (i.e. for \(i = 1\)). Now we use the hypercohomology spectral sequences

\[
\mathcal{H}^p(C, \mathcal{H}^q(\Lambda_C)) \Rightarrow \mathbb{H}^{p+q}(C, \Lambda_C) \simeq \mathcal{H}^{p+q}(\mathbb{R}\Gamma(C, \Lambda_C)),
\]

which vanishes in degree 1. Now applying the hypercohomology spectral sequence to \(\omega_C \otimes H^2(S,\mathcal{O}_S) := \mathcal{L}\), we have
Proof of Lemma. 

Lemma 4.3 Since the curve class \( \beta \neq 0 \), the map

\[
H^1(t \circ s) : H^1(C, f^*\Omega^1_S) \to H^1(C, \omega_C)
\]

is nonzero (hence surjective).

Proof of Lemma. By \( \beta \neq 0 \) implies nontriviality of the map \( df : f^*\Omega^1_S \to \Omega^1_C \). But \( S \) is a smooth surface and \( C \) a prestable curve, hence in the short exact sequence

\[
f^*\Omega^1_S \to \Omega^1_C \to \Omega^1_{C/S} \to 0
\]

the sheaf of relative differentials \( \Omega^1_{C/S} \) is concentrated at the (isolated, closed) singular points and thus its \( H^1 \) vanishes. Therefore the map

\[
H^1(s) : H^1(C, f^*\Omega^1_S) \to H^1(C, \Omega^1_C)
\]

is surjective. The same argument yields surjectivity, hence nontriviality (since \( H^1(C, \omega_C) \) has dimension 1 over \( \mathbb{C} \)), of the map \( H^1(t) : H^1(C, \Omega^1_C) \to H^1(C, \omega_C) \), by observing that, on the smooth locus of \( C, \Omega^1_C \simeq \omega_C \) and \( H^1(t) \) is the induced isomorphism. In particular, \( H^1(C, \Omega^1_C) \neq 0 \). Therefore both \( H^1(s) \) and \( H^1(t) \) are non zero and surjective, so the same is true of their composition. \( \Box \)

\[\text{We thank R. Pandharipande for pointing out this statement, of which we give here our proof.}\]
4.2 The canonical projection $\mathbb{R}\text{Pic}(S) \to \mathbb{R}\text{Spec} (\text{Sym}(H^0(S, K_S)[1]))$

In this subsection we identify canonically the derived Picard stack $\mathbb{R}\text{Pic}(S)$ of a $K3$-surface with $\text{Pic}(S) \times \mathbb{R}\text{Spec} (\text{Sym}(H^0(S, K_S)[1]))$, where $K_S := \Omega^2_S$ is the canonical sheaf of $S$; this allows us to define the canonical map $\text{pr}_{\text{der}} : \mathbb{R}\text{Pic}(S) \to \mathbb{R}\text{Spec} (\text{Sym}(H^0(S, K_S)[1]))$ which is the last ingredient we will need to define the reduced derived stack $\mathbb{RM}_{g}^{\text{red}}(S; \beta)$ of stable maps of genus $g$ and class $\beta$ to $S$ in the next subsection.

In the proof of the next Proposition, we will need the following elementary result (which holds true for $k$ replaced by any semisimple ring, or $k$ replaced by a hereditary commutative ring and $E$ by a bounded above complex of free modules)

**Lemma 4.4** Let $k$ be a field and $E$ be a bounded above complex of $k$-vector spaces. Then there is a canonical map $E \to E_{<0}$ in the derived category $D(k)$, such that the obvious composition

$$E_{<0} \to E \to E_{<0}$$

is the identity.

**Proof.** Any splitting $p$ of the map of $k$-vector spaces

$$\ker(d_0 : E_{-1} \to E_0) \to E_{-1}$$

yields a map $\overline{\pi} : E \to E_{<0}$ in the category $\text{Ch}(k)$ of complexes of $k$-vector spaces. To see that different splittings $p$ and $q$ gives the same map in the derived category $D(k)$, we consider the canonical exact sequences of complexes

$$0 \to E_{<0} \to E \to E_{\geq 0} \to 0$$

and apply $\text{Ext}^0(-, E_{<0})$, to get an exact sequence

$$\text{Ext}^0(E_{\geq 0}, E_{<0}) \to \text{Ext}^0(E, E_{<0}) \to \text{Ext}^0(E_{<0}, E_{<0}).$$

Now, the the class of the difference $(\overline{\pi} - \overline{\pi})$ in $\text{Hom}_{D(k)}(E, E_{<0}) = \text{Ext}^0(E, E_{<0})$ is in the kernel of $b$, so it is enough to show that $\text{Ext}^0(E_{\geq 0}, E_{<0}) = 0$. But $E_{\geq 0}$ is a bounded above complex of projectives, therefore (e.g. [Wei, Cor. 10.4.7]) $\text{Ext}^0(E_{\geq 0}, E_{<0}) = 0$ is a quotient of $\text{Hom}_{\text{Ch}(k)}(E_{\geq 0}, E_{<0})$ which obviously consists of the zero morphism alone. \(\square\)

**Proposition 4.5** Let $G$ be a derived group stack locally of finite presentation over a field $k$, $e : \text{Spec} k \to G$ its identity section, and $\mathfrak{g} := T_{e}G$. Then there is a canonical map in $\text{Ho}(\text{dSt}_k)$

$$\gamma(G) : t_0(G) \times \mathbb{R}\text{Spec}(A) \to G$$

where $A := k \oplus (\mathfrak{g}^\vee)_{<0}$ is the commutative differential non-positively graded $k$-algebra which is the trivial square zero extension of $k$ by the complex of $k$-vector spaces $(\mathfrak{g}^\vee)_{<0}$.

**Proof.** First observe that $\mathbb{R}\text{Spec}(A)$ has a canonical $k$-point $x_0 : \text{Spec} k \to \mathbb{R}\text{Spec}(A)$, corresponding to the canonical projection $A \to k$. By definition of the derived cotangent
complex of a derived stack (\cite[1.4.1]{HAG-II}), giving a map $\alpha$ such that

$$
\begin{array}{c}
\mathbb{R}\text{Spec}(A) \\
\downarrow x_0 \\
\text{Spec } k
\end{array}
\xrightarrow{\alpha}
\begin{array}{c}
\mathbb{R}\text{Spec}(A) \\
e \\
\text{Spec } k
\end{array}
$$

is equivalent to giving a morphism in the derived category of complex of $k$-vector spaces

$$
\alpha' : \mathcal{L}_{G,e} \cong \mathfrak{g}^\vee \longrightarrow (\mathfrak{g}^\vee)_{<0}.
$$

Since $k$ is a field, we may take as $\alpha'$ the canonical map provided by Lemma 4.4, and define $\gamma(G)$ as the composition

$$
t_0(G) \times \mathbb{R}\text{Spec}(A) \xrightarrow{j \times \text{id}} G \times \mathbb{R}\text{Spec}(A) \xrightarrow{\text{id} \times \alpha'} G \times G \xrightarrow{\mu} G
$$

where $\mu$ is the product in $G$. \hfill \Box

**Proposition 4.6** Let $S$ be a $K3$ surface over $k = \mathbb{C}$, and $G := \mathbb{R}\text{Pic}(S)$ its derived Picard group stack. Then the map $\gamma(G)$ defined in (the proof of) Proposition 4.5 is an isomorphism

$$
\gamma_S := \gamma(\mathbb{R}\text{Pic}(S)) : \text{Pic}(S) \times \mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1])) \longrightarrow \mathbb{R}\text{Pic}(S)
$$

in $\text{Ho}(\text{dSt}_{\mathbb{C}})$, where $K_S$ denotes the canonical bundle on $S$.

**Proof.** Since $G := \mathbb{R}\text{Pic}(S)$ is a derived group stack, $\gamma(G)$ is an isomorphism if and only if it induces an isomorphism on truncations, and it is étale at $e$, i.e. the induced map

$$
T_{(t_0(e), x_0)}(\gamma(G)) : T_{(t_0(e), x_0)}(t_0(G) \times \mathbb{R}\text{Spec}(A)) \longrightarrow T_e(G)
$$

is an isomorphism in the derived category $\text{D}(k)$, where $x_0$ is the canonical canonical $k$-point $\text{Spec } k \rightarrow \mathbb{R}\text{Spec}(A)$, corresponding to the canonical projection $A \rightarrow \mathbb{C}$. Since $\pi_0(A) \simeq \mathbb{C}$, $t_0(\gamma(G))$ is an isomorphism of stacks. So we are left to showing that $\gamma(G)$ induces an isomorphism between tangent spaces. Now,

$$
\mathfrak{g} \equiv T_e(G) = T_e(\mathbb{R}\text{Pic}(S)) \simeq \mathbb{R}\Gamma(S, \mathcal{O}_S)[1],
$$

and, $S$ being a K3-surface, we have

$$
\mathfrak{g} \simeq \mathbb{R}\Gamma(S, \mathcal{O}_S)[1] \simeq H^0(S, \mathcal{O}_S)[1] \oplus H^2(S, \mathcal{O}_S)[-1]
$$

so that

$$
(\mathfrak{g}^\vee)_{<0} \simeq H^2(S, \mathcal{O}_S)^\vee[1] \simeq H^0(S, K_S)[1]
$$

(where we have used Serre duality in the last isomorphism). But $H^0(S, K_S)$ is free of dimension 1, so we have a canonical isomorphism

$$
\mathbb{C} \oplus (\mathfrak{g}^\vee)_{<0} \simeq \mathbb{C} \oplus H^0(S, K_S)[1] \simeq \text{Sym}(H^0(S, K_S)[1])
$$
in the homotopy category of commutative simplicial C-algebras. Therefore
\[ T_{(t_0,e,x_0)}(\text{Pic}(S) \times \mathbb{R}\text{Spec}(A)) \simeq g_{\leq 0} \oplus g_{> 0} \simeq H^0(S, O_S)[1] \oplus H^2(S, O_S)[-1] \]
and \( T_{(t_0,e,x_0)}(\gamma(G)) \) is obviously an isomorphism (given, in the notations of the proof of Prop. 4.5, by the sum of the dual of \( \alpha' \) and the canonical map \( g_{\leq 0} \to g \)).

Using Prop. 4.6, we are now able to define the projection \( \text{pr}_{\text{der}} \) of \( \mathbb{R}\text{Pic}(S) \) onto its full derived factor as the composite
\[
\mathbb{R}\text{Pic}(S) \xrightarrow{\gamma(S)^{-1}} \text{Pic}(S) \times \mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1])) \xrightarrow{\text{pr}_2} \mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1])).
\]

Note that \( \text{pr}_{\text{der}} \) yields on tangent spaces the canonical projection
\[
\text{Te}(\mathbb{R}\text{Pic}(S; \beta)) = g \to g_{> 0} = T_{x_0}(\mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1]))) \simeq H^2(S, O_S)[-1],
\]
where \( x_0 \) is the canonical canonical k-point \( \text{Spec } \mathbb{C} \to \text{Spec}(\text{Sym}(H^0(S, K_S)[1])), \) and
\[ g \simeq H^0(S, O_S)[1] \oplus H^2(S, O_S)[-1]. \]

### 4.3 The reduced derived stack of stable maps \( \mathbb{R}\tilde{M}_g^{\text{red}}(S; \beta) \)

In this subsection we define the reduced version of the derived stack of stable maps of type \((g, \beta)\) to \( S \) and describe the obstruction theory it induces on its truncation \( \tilde{M}_g(S; \beta) \).

Let us define \( \delta_1^{\text{der}}(S, \beta) \) as the composition (see Def. 3.8)
\[
\mathbb{R}\tilde{M}_g(S; \beta) \xrightarrow{\delta_1(S)} \mathbb{R}\text{Pic}(S) \xrightarrow{\text{pr}_{\text{der}}} \mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1])).
\]

**Definition 4.7** The reduced derived stack of stable maps of genus \( g \) and class \( \beta \) to \( S \) \( \mathbb{R}\tilde{M}_g^{\text{red}}(S; \beta) \) is defined by the following homotopy-cartesian square in \( \text{dSt}_C \)
\[
\begin{array}{ccc}
\mathbb{R}\tilde{M}_g^{\text{red}}(S; \beta) & \to & \mathbb{R}\tilde{M}_g(S; \beta) \\
\downarrow \delta_1^{\text{der}}(S, \beta) & & \downarrow \delta_1^{\text{der}}(S, \beta) \\
\text{Spec } \mathbb{C} & \to & \mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1]))
\end{array}
\]

Since the truncation functor \( t_0 \) commutes with homotopy fiber products and
\[ t_0(\mathbb{R}\text{Spec}(\text{Sym}(H^0(S, K_S)[1]))) \simeq \text{Spec } \mathbb{C}, \]
we get
\[ t_0(\mathbb{R}\tilde{M}_g^{\text{red}}(S; \beta)) \simeq \tilde{M}_g(S; \beta) \]
i.e. \( \mathbb{R}\tilde{M}_g^{\text{red}}(S; \beta) \) is a derived extension (Def. 1.1) of the usual stack of stable maps of type \((g, \beta)\) to \( S \), different from \( \mathbb{R}\tilde{M}_g(S; \beta) \).

\^Recall that, if \( M \) is a \( \mathbb{C} \)-vector space, \( T_{x_0}(\mathbb{R}\text{Spec}(\text{Sym}(M)[1])) \simeq M^\vee [-1] \).
We are now able to compute the obstruction theory induced, according to §1, by the closed immersion $j_{\text{red}} : \M_g(S; \beta) \hookrightarrow \M_g^{\text{red}}(S; \beta)$.

By applying Proposition 1.2 to the derived extension $\M_g^{\text{red}}(S; \beta)$ of $\M_g(S; \beta)$, we get an obstruction theory

$$j_{\text{red}}^* \mathbb{L}_{\M_g^{\text{red}}(S; \beta)} \rightarrow \mathbb{L}_{\M_g(S; \beta)}$$

that we are now going to describe.

Let $\rho : \M_g^{\text{red}}(S; \beta) \rightarrow \M_g(S; \beta)$ be the canonical map. Since $\M_g^{\text{red}}(S; \beta)$ is defined by the homotopy pullback diagram in Def. 4.7, we get an isomorphism in the derived category of $\M_g^{\text{red}}(S; \beta)$

$$\rho^*(\mathbb{L}_{\M_g(S; \beta)} / R\text{Spec}(\text{Sym}(H^0(S,K_S)[1]))) \cong \mathbb{L}_{\M_g^{\text{red}}(S; \beta)}.$$

We will show below that $\M_g^{\text{red}}(S; \beta)$ is quasi-smooth so that, by Corollary 1.3, $j_{\text{red}}^* \mathbb{L}_{\M_g^{\text{red}}(S; \beta)} \rightarrow \mathbb{L}_{\M_g(S; \beta)}$ is indeed a perfect obstruction theory on $\M_g(S; \beta)$. Now, for any $C$-point $\text{Spec} C \rightarrow \M_g^{\text{red}}(S; \beta)$, corresponding to a stable map $(f : C \rightarrow S)$ of type $(g, \beta)$, we get a distinguished triangle

$$\mathbb{L}_{\text{RSpec}(\text{Sym}(H^0(S,K_S)[1])), x_0} \rightarrow \mathbb{L}_{\M_g(S; \beta), (f : C \rightarrow S)} \rightarrow \mathbb{L}_{\M_g^{\text{red}}(S; \beta), (f : C \rightarrow S)}$$

(where we have denoted by $(f : C \rightarrow S)$ also the induced $C$-point of $\M_g^{\text{red}}(S; \beta)$: recall that a derived stack and its truncation have the same classical points, i.e. points with values in usual commutative $C$-algebras) in the derived category of complexes of $C$-vector spaces. By dualizing, we get that the tangent complex

$$T_{(f:C\rightarrow S)}^{\text{red}} := T_{(f:C\rightarrow S)}(\M_g^{\text{red}}(S; \beta))$$

of $\M_g^{\text{red}}(S; \beta)$ at the $C$-point $(f : C \rightarrow S)$ of type $(g, \beta)$, sits into a distinguished triangle

$$T_{(f:C\rightarrow S)}^{\text{red}} \rightarrow \Gamma(C, \text{Cone}(T_C \rightarrow f^*T_S)) \xrightarrow{\Theta_f} \Gamma(S, \mathcal{O}_S)[1] \xrightarrow{\text{pr}} H^2(S, \mathcal{O}_S)[-1],$$

where $\Theta_f$ is the composite

$$\Theta_f : \Gamma(C, \text{Cone}(T_C \rightarrow f^*T_S)) \xrightarrow{\tau_f} \text{RHom}_S(\mathcal{O}_C, \text{Rf}_*\mathcal{O}_C)[1] \xrightarrow{\text{tr}_S} \Gamma(S, \mathcal{O}_S)[1],$$

and $\text{pr}$ denotes the tangent map of $\text{pr}_{\text{der}}$ taken at the point $\delta_1(S)(f : C \rightarrow S)$. Note that the map $\text{pr}$ obviously induces an isomorphism on $H^1$. 

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4.4 Quasi-smoothness of $\mathbb{M}^\text{red}_g(S;\beta)$ and comparison with O-M-P-T reduced obstruction theory.

In the case $\beta \neq 0$ is a curve class in $H^2(S,\mathbb{Z})$, we will prove quasi-smoothness of the derived stack $\mathbb{M}^\text{red}_g(S;\beta)$, and compare the induced obstruction theory with that of Okounkov-Maulik-Pandharipande-Thomas (see §4.1.1 and §2.2 or [M-P] §2.2 and [P]).

**Theorem 4.8** Let $\beta \neq 0$ be a curve class in $H^2(S,\mathbb{Z}) \simeq H_2(S,\mathbb{Z})$, $f : C \to S$ a stable map of type $(g,\beta)$, and

$$T^\text{red}_{(f:C\to S)} := T_{(f:C\to S)}(\mathbb{M}^\text{red}_g(S;\beta)) \to \Gamma(C, \text{Cone}(T_C \to f^*T_S)) \to H^2(S,\mathcal{O}_S)[-1]$$

the corresponding distinguished triangle. Then,

1. the rightmost arrow in the triangle above induces on $H^1$ a map

$$H^1(\Theta_f) : H^1(C, \text{Cone}(T_C \to f^*T_S)) \to H^2(S,\mathcal{O}_S)$$

which is nonzero (hence surjective, since $H^2(S,\mathcal{O}_S)$ has dimension 1 over $\mathbb{C}$). Therefore the derived stack $\mathbb{M}^\text{red}_g(S;\beta)$ is everywhere quasi-smooth;

2. $H^0(T^\text{red}_{(f:C\to S)})$ (resp. $H^1(T^\text{red}_{(f:C\to S)})$) coincides with the reduced deformation space (resp. the reduced obstruction space) of O-M-P-T.

**Proof.**

First Proof of quasi-smoothness – Let us prove quasi-smoothness first. It is clearly enough to prove that the composite

$$H^1(C, f^*T_S) \to H^1(C, \text{Cone}(T_C \to f^*T_S)) \xrightarrow{H^1(\tau_f)} \text{Ext}^2_S(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C) \xrightarrow{H^2(\text{tr}_S)} H^2(S,\mathcal{O}_S)$$

is non zero (hence surjective). Recall that $p : C \to \text{Spec} \mathbb{C}$ and $q : S \to \text{Spec} \mathbb{C}$ denote the structural morphisms, so that $p = q \circ f$. Now, the map

$$\mathbb{R}q_*T_S \to \mathbb{R}q_*\mathbb{R}f_*f^*T_S$$

induces a map $H^1(S, T_S) \to H^1(C, f^*T_S)$, and by Remark 3.9 the following diagram commutes

$$\begin{array}{ccc}
H^1(S, T_S) & \xrightarrow{(-,\mathbb{at}(\mathbb{R}f_*\mathcal{O}_C))} & \text{Ext}^2_S(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C) \\
\downarrow & & \uparrow H^1(\tau_f) \\
H^1(C, f^*T_S) & \to & H^1(C, \text{Cone}(T_C \to f^*T_S))
\end{array}$$

So, we are reduced to proving that the composition

$$a : H^1(S, T_S) \xrightarrow{(-,\mathbb{at}(\mathbb{R}f_*\mathcal{O}_C))} \text{Ext}^2_S(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C) \xrightarrow{H^2(\text{tr}_S)} H^2(S,\mathcal{O}_S)$$

does not vanish. But, since the first Chern class is the trace of the Atiyah class, this composition acts as follows (on maps in the derived category of $S$)

$$(\xi : \mathcal{O}_S \to T_S[1]) \xrightarrow{a(\xi)} (\mathcal{O}_S \xrightarrow{\xi} \Omega^1_S \otimes T_S[2] \xrightarrow{-\xi^2} \mathcal{O}_S[2])$$

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where
\[ c_1 := c_1(\mathbb{R}f_*\mathcal{O}_C) : \mathcal{O}_S \to \Omega^1_S[1] \]

is the first Chern class of the perfect complex \( \mathbb{R}f_*\mathcal{O}_C \). What we have said so far, is true for any smooth complex projective scheme \( X \) in place of \( S \). We now use the fact that \( S \) is a K3-surface. Choose a non zero section \( \sigma : \mathcal{O}_S \to \Omega^2_S \) of the canonical bundle, and denote by \( \varphi_\sigma : \Omega^1_S \xrightarrow{\sim} T_S \) the induced isomorphism. A linear algebra computation shows then that the composition
\[ \mathcal{O}_S \xrightarrow{((\varphi_\sigma \circ c_1) \wedge \xi) \oslash \sigma} (T_S \wedge T_S \otimes \Omega^2_S)[2] \xrightarrow{\langle -, -, [2]\rangle} \mathcal{O}_S[2] \]

coincides with \( a(\xi) \). But, since \( \beta \neq 0 \), we have that \( c_1 \neq 0 \). \( \sigma \) is nondegenerate, so this composition cannot vanish for all \( \xi \), and we conclude.

Second Proof of quasi-smoothness – Let us prove quasi-smoothness first. By Serre duality, passing to dual vector spaces and maps, we are left to proving that the composite
\[ H^0(S, \Omega^2_S) \xrightarrow{\text{tr}^\vee} \text{Ext}^0_S(\mathbb{R}\text{Hom}(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C), \Omega^2_S) \xrightarrow{\text{tr}^\vee} \text{Ext}^0(\mathbb{R}f_*f^*T_S[-1], \Omega^2_S) \]

is non-zero. So it is enough to prove that the map obtained by further composing to the left with the adjunction map
\[ \text{Ext}^0(\mathbb{R}f_*f^*T_S[-1], \Omega^2_S) \to \text{Ext}^0(T_S[-1], \Omega^2_S) \]

is nonzero. But this new composition acts as follows
\[ H^0(S, \Omega^2_S) \ni (\sigma : \mathcal{O}_S \to \Omega^2_S) \mapsto (\sigma \circ \text{tr}) \mapsto (\sigma \circ \text{tr} \circ \text{at}) = (\sigma \circ c_1(\mathbb{R}f_*\mathcal{O}_C)) \in \text{Ext}^0(T_S[-1], \Omega^2_S) \]

where \( \text{at} : T_S[-1] \to \mathbb{R}\text{Hom}(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C) \) is the dual of the Atiyah class of \( (\mathbb{R}f_*\mathcal{O}_C)^\vee \) (see Remark 3.9). But, since \( \beta \neq 0 \), we have \( c_1(\mathbb{R}f_*\mathcal{O}_C) \neq 0 \) and we conclude.

Sketch of proof of the comparison – Let us move now to the second point of Thm. 4.8, i.e. the comparison statement about deformations and obstructions spaces. First of all it is clear that, for any \( \beta \),
\[ H^0(\mathbb{P}_{\text{red,}(f:C\to S)}) \simeq H^0(\mathbb{P}_{\mathbb{R}\mathbb{M}_q(S,\beta),(f:C\to S)}) \simeq H^0(C, \text{Cone}(\mathbb{T}_C \to f^*T_S)) \]

therefore our deformation space is the same as O-M-P-T’s one. Let us then concentrate on obstruction spaces.

We begin by noticing the following fact

**Lemma 4.9** There is a canonical morphism in \( \mathbb{D}(\mathbb{C}) \)
\[ \nu : \mathbb{R}p_*\omega_C \otimes^L H^2(S, \mathcal{O}_S) \to \mathbb{R}q_*\mathcal{O}_S[1] \]

inducing an isomorphism on \( H^1 \).
Proof of Lemma. To ease notation we will simply write $\otimes$ for $\otimes^L$. Recall that $p : C \to \text{Spec} \mathbb{C}$ and $q : S \to \text{Spec} \mathbb{C}$ denote the structural morphisms, so that $p = q \circ f$. Since $S$ is a $K3$-surface, the canonical map

$$\mathcal{O}_S \otimes H^0(S, \Omega^2_S) \to \Omega^2_S$$

is an isomorphism. Since $f^!$ preserves dualizing complexes, $\omega_S \simeq \Omega^2_S[2]$ and $\omega_C \simeq \omega_C[1]$, we have

$$\omega_C \simeq f^!(\mathcal{O}_S \otimes H^0(S, \Omega^2_S))[1].$$

By applying $\mathbb{R}p_*$ and using the adjunction map $\mathbb{R}f_! f^! \to \text{Id}$, we get a map

$$\mathbb{R}p_* \omega_C \simeq \mathbb{R}q_* \mathbb{R}f_* f^!(\mathcal{O}_S[1] \otimes H^0(S, \Omega^2_S)) \to \mathbb{R}q_* (\mathcal{O}_S[1] \otimes H^0(S, \Omega^2_S)) \simeq \mathbb{R}q_* \mathcal{O}_S[1] \otimes H^0(S, \Omega^2_S)$$

(the last isomorphism being given by projection formula). Tensoring this map by $H^0(S, \Omega^2_S)^\vee \simeq H^2(S, \mathcal{O}_S)$ (a canonical isomorphism by Serre duality), and using the canonical evaluation map $V \otimes V^\vee \to \mathbb{C}$ for a $\mathbb{C}$-vector space $V$, we get the desired canonical map

$$\nu : \mathbb{R}p_* \omega_C \otimes H^2(S, \mathcal{O}_S) \to \mathbb{R}q_* \mathcal{O}_S[1].$$

The isomorphism on $H^1$ is obvious since the trace map $\mathbb{R}^1 p_* \omega_C \to \mathbb{C}$ is an isomorphism ($\mathbb{C}$ is geometrically connected).

If $\sigma : \mathcal{O}_S \sim \Omega^2_S$ is a nonzero element in $H^0(S, \Omega^2_S)$, and $\varphi : T_S \simeq \Omega^1_S$ the induced isomorphism, the previous Lemma gives us an induced map

$$\nu(\sigma) : \mathbb{R}p_* \omega_C \to \mathbb{R}q_* \mathcal{O}_S[1],$$

and an induced isomorphism

$$H^1(\nu(\sigma)) =: \nu_\sigma : H^1(C, \omega_C) \sim \to H^2(S, \mathcal{O}_S).$$

Using the same notations as in §4.1.1 to prove that our reduced obstruction space

$$\ker(H^1(\Theta_f) : H^1(C, \text{Cone}(\mathbb{T}_C \to f^*T_S)) \to H^2(S, \mathcal{O}_S))$$

coincides with O-M-P-T’s one, it will be enough to show that the following diagram is commutative

$$\begin{array}{ccc}
H^1(C, f^*T_S) & \xrightarrow{\text{can}} & H^1(\mathbb{R}\Gamma(C, \text{Cone}(\mathbb{T}_C \to f^*T_S))) \\
\downarrow \text{can} & & \downarrow \text{can} \\
H^1(\mathbb{R}\Gamma(C, \text{Cone}(\mathbb{T}_C \to f^*T_S))) & \xrightarrow{\sim} & H^1(\Theta_f) \\
\downarrow \text{can} & & \downarrow \text{can} \\
H^1(C, \omega_C) & \xrightarrow{\nu_\sigma} & H^2(S, \mathcal{O}_S). \\
\end{array}$$

But this follows from the commutativity of

$$\begin{array}{ccccccccc}
\mathbb{R}p_* f^* T_S[-1] & \xrightarrow{\mathbb{R}p_* (\nu(\sigma))} & \mathbb{R}p_* f^* \Omega^1_S[-1] & \xrightarrow{\mathbb{R}p_* (s)} & \mathbb{R}p_* \Omega_C[1][-1] & \xrightarrow{\mathbb{R}p_* (t)} & \mathbb{R}p_* \omega_C[-1] \\
\downarrow \text{id} & & \downarrow \text{tr} & & \downarrow \text{id} & & \downarrow \nu(\sigma)[-1] \\
\mathbb{R}p_* f^* T_S[-1] & \xrightarrow{\nu(\sigma)} & \mathbb{R}q_* \mathbb{R}\text{Hom}_S(\mathbb{R}f_* \mathcal{O}_C, \mathbb{R}f_* \mathcal{O}_C) & \xrightarrow{\text{tr}} & \mathbb{R}q_* \mathcal{O}_S \\
\end{array}$$
whose verification is left to the reader.

\[\square\]

**Remark 4.10** Note that by Lemma 4.2, the second assertion of Theorem 4.8 implies the first one. Nonetheless, we have preferred to give an independent proof of the quasismoothness of $\mathbb{R}^{\text{red}}_{\overline{M}_g}(S; \beta)$ because we find it conceptually more relevant than the comparison with O-M-P-T, meaning that quasismoothness alone would in any case imply the existence of some perfect reduced obstruction theory on $\overline{M}_g(S; \beta)$, regardless of its comparison with the one introduced and studied by O-M-P-T.

Moreover, we could only find in the literature a definition of O-M-P-T global reduced obstruction theory (relative to $\overline{M}^{\text{pre}}_g$) with values in the $\tau_{\geq -1}$-truncation of the cotangent complex of the stack of stable maps\(^7\) ([M-P 2.2, formula (14)]), while there is a complete description of the corresponding pointwise tangent and obstruction spaces. Therefore, our comparison is necessarily limited to these spaces. And our construction might also be seen as establishing such a reduced global obstruction theory (in the usual sense, i.e. with values in the full cotangent complex).

Theorem 4.8 shows that the distinguished triangle

$$T^\text{red}_{(f:C \to S)} := T_{(f:C \to S)}(\mathbb{R}^{\text{red}}_{\overline{M}_g}(S; \beta)) \to \mathbb{R}\Gamma(C, \text{Cone}(T_C \to f^*T_S)) \to H^2(S, \mathcal{O}_S)[-1]$$

induces isomorphisms

$$H^i(T^\text{red}_{(f:C \to S)}) \simeq H^i(C, \text{Cone}(T_C \to f^*T_S)),$$

for any $i \neq 1$, while in degree 1, it yields a short exact sequence

$$0 \to H^1(T^\text{red}_{(f:C \to S)}) \to H^1(C, \text{Cone}(T_C \to f^*T_S)) \to H^2(S, \mathcal{O}_S) \to 0.$$

So, the tangent complexes of $\mathbb{R}^{\text{red}}_{\overline{M}_g}(S; \beta)$ and $\mathbb{R}^{\text{pre}}_{\overline{M}_g}(S; \beta)$ (hence our induced reduced and the standard obstruction theories) only differ at the level of $H^1$ where the former is the kernel of a 1-dimensional quotient of the latter.

**References**


\(^7\)The reason being that uniqueness of the obstruction theory is a only assured a priori, by factoring through the cone, if one consider such a truncation.


