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On higher order analogues of de Rham cohomology

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Abstract

In [A.M. Vinogradov, Some homological systems associated with the differential calculus in commutative algebras, Russian Math. Surveys 34 (6) (1979) 250–255] for any commutative K-algebra A, K being a commutative ring, any sequence σ of positive integers and any differentially closed (see Section 3) subcategory $\mathcal{D}$ of $A-\mathbf{Mod}$, higher analogues $\mathbf{dR}_\sigma^{\mathcal{D}}$ of the standard de Rham complex $\mathbf{dR}^{\mathcal{D}} \equiv \mathbf{dR}_{(1,\dots,1,\dots)}^{\mathcal{D}}$ and Spencer complexes were defined. In this paper a detailed exposition of all related functors of differential calculus over general commutative algebras is given for the first time together with some useful working techniques.

In the second part of the paper, these techniques are then applied to prove that all complexes $\mathbf{dR}_\sigma^{\mathcal{D}}$ are quasi-isomorphic under a smoothness assumption on the differentially closed subcategory $\mathcal{D}$. This extends to arbitrary smooth categories of modules the quasi-isomorphism theorem for smooth manifolds and "regular" $\mathbf{dR}_\sigma$ complexes proved in [G. Vezzosi, A.M. Vinogradov, Infinitesimal Stokes' formula for higher-order de Rham complexes, Acta Appl. Math. 49 (3) (1997) 311–329].

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1. Introduction

If K is any commutative ring, given a commutative K-algebra A, a differentially closed category $\mathcal{D}$ of A-modules (see Section 3) and a sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ of positive integers, one can associate to these data some new natural complexes, the most important of which are higher analogues of the

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standard de Rham and Spencer complexes (see [17]), denoted further on by $\mathbf{dR}_\sigma$ and $\mathbf{Sp}_\sigma$, respectively. If M is a smooth manifold, $A = C^\infty(M)$ is the algebra of smooth functions on M, $\mathcal{D}$ is the category of "geometric" A-modules (see Section 3) and $\sigma = (1, 1, \ldots)$, then these complexes coincide with the standard differential geometrical partners. It is in particular remarkable that higher order analogues of the de Rham differential are higher order differential operators which are natural in the category of smooth manifolds. This is a new class of natural differential operators which could hardly been discovered and studied with the traditional differential geometric methods (see, e.g., [8]).

These higher order differential complexes being natural, they deserve by themselves to be investigated. Their specific role is revealed by the fact that they are, in a sense, natural prolongations of the corresponding classical complexes and, moreover, were looked for and found exactly in this perspective. In the formal theory of partial differential equations and in many parts of differential geometry (and implicitly also in algebraic geometry) the prolongation procedure plays a fundamental role. While Gauss' "Theorema Egregium" is a direct consequence of the Gauss–Peterson–Codazzi derivation (i.e., prolongation) formulas, the complexity of problems, say, in modern differential geometry requires a structural organization of the procedure of taking consecutive derivations of initial data of the problem. For instance, Cartan's exterior forms' methods (see, e.g., [3] for a modern treatment) and Spencer's cohomology approach to formal integrability of partial differential equations (see [6,12]) are to be mentioned as milestones for progress in that direction. The formal integrability problem for Euler–Lagrange equations arising from degenerate actions, which still resists a mathematically satisfactory solution in spite of long time efforts (see, for instance, [4,5]) gives a very instructive example of the situation where the lack of such an organization prevents the solution. One of the hopes we relate with higher order analogues of the de Rham complex is to put forward a solution of this and similar problems by combining them with the $\mathcal{C}$-spectral sequence techniques [18].

The aforementioned and any other possible applications of these new complexes require sufficiently elaborated techniques to work with higher differential forms and with many related objects as well. In this paper we continue this kind of foundational work started in [14] by paying more attention to the related cohomological features. This allows us, in particular, to prove (Theorem 5.1) in a general smooth algebraic setting, that higher de Rham complexes are quasi-isomorphic to the standard one. This generalizes much our previous quasi-isomorphism result in [14] which dealed with "regular" de Rham complexes on smooth manifolds and confirms the intuitive idea that higher de Rham complexes are in fact, not only formally, prolongations of the standard one and suggests that this result should remain true when only "mild" singularities occur.

Recently A. Verbovetsky [13] announced an interesting connection between higher de Rham complexes and the compatibility complexes techniques in formal integrability theory of partial differential equations, which gives new insights on Theorem 5.1.

The paper is essentially divided in two parts; in the first one (Sections 2–4) we develop the necessary elements of the theory of differential calculus on a commutative K-algebra A and in the second (Section 5 and Appendix A) we prove the smooth rigidity theorem for higher order de Rham cohomologies.

More precisely, Section 2 recalls the definitions of differential operators and higher derivations in the category $A - \mathbf{Mod}$ of A-modules.

Section 3 introduces the notion of a differentially closed subcategory $\mathcal{D}$ of $A - \mathbf{Mod}$ and studies representative objects in $\mathcal{D}$ (e.g., jets, higher differential forms etc.) for basic differential functors. The category $\mathcal{D} = A - \mathbf{Mod}_{\mathrm{geom}}$ of *geometric* modules over the algebra $A = C^\infty(M)$, M being a differentiable manifold, gives an example of a differentially closed subcategory which is of fundamental importance for differential geometry (recall that a $C^\infty(M)$-module P is geometric if $\bigcap_{x\in M} \mathfrak{m}_x P = 0$, $\mathfrak{m}_x$ being the ideal of functions vanishing at x). We also introduce the notion of smoothness of a differentially closed category which reduces to the known algebraic geometrical notion if $\mathcal{D} = A - \mathbf{Mod}$.

In Section 4 higher de Rham complexes $\mathbf{dR}_\sigma$ and $\mathbf{dR}_\infty$ associated with a given differentially closed category are defined and studied. In the case $\sigma = (1, 1, \ldots)$ and $\mathcal{D} = A - \mathbf{Mod}$ or $\mathcal{D} = A - \mathbf{Mod}_{\mathrm{geom}}$ we obtain the usual algebraic Kähler–de Rham or smooth de Rham complexes, respectively.

In Section 5 we prove (Theorem 5.1) that if $\mathcal{D}$ is a smooth differentially closed subcategory of modules over a characteristic zero K-algebra A, then all the higher de Rham complexes are quasi isomorphic. The idea of the proof is to show that for any k the kernel of the natural epimorphism $\mathbf{dR}_{(\sigma_1,\ldots,\sigma_k+1,1,\ldots,1)} \to \mathbf{dR}_{(\sigma_1,\ldots,\sigma_k,1,\ldots,1)}$ is acyclic; this is done by exhibiting an explicit acyclic resolution via the so-called holonomy complexes (Definition 5.4). Finally, Appendix A contains the definition of the maps building the above resolution which was too long to be included in Section 5 without losing the logical path of the proof of Theorem 5.1.

Notations and conventions

K: a commutative ring with unit;

A: a commutative K-algebra with unit;

$R - \mathbf{Mod}$: the category of R-modules for a commutative ring R;

$\mathbf{DIFF}_A$: the category whose objects are the A-modules and the morphisms are differential operators (Section 1) between them;

$A - \mathbf{BiMod}$: the category of A-bimodules, whose objects are understood as *ordered couples* (P, P^+) of A-modules and whose morphisms are the usual morphisms of bimodules. Note that P and P^+ coincide as K-modules;

$\mathbf{K}$ $(A - \mathbf{Mod})$ (resp. $\mathbf{K}$ $(K - \mathbf{Mod})$, resp. $\mathbf{K}$ $(\mathbf{DIFF}_A)$): the category of complexes in $A - \mathbf{Mod}$ (resp. $K - \mathbf{Mod}$, resp. $\mathbf{DIFF}_A$);

If $\mathcal{D}$ is a full subcategory of $A - \mathbf{Mod}$, $[\mathcal{D}, \mathcal{D}]$ will denote the category of functors $\mathcal{D} \to \mathcal{D}$; a functor $T : \mathcal{D} \to \mathcal{D}$ will be said *strictly representable* in $\mathcal{D}$ if it exists $\tau \in Ob(\mathcal{D})$ and a functorial isomorphism $T \simeq Hom_A(\tau, \cdot)$ in $[\mathcal{D}, \mathcal{D}]$.

$A - \mathbf{BiMod}_{\mathcal{D}}$ (resp. $\mathbf{K}$ $(\mathbf{DIFF}_{A,\mathcal{D}})$) will be the subcategory of $A - \mathbf{BiMod}$ whose objects are couples of objects in $\mathcal{D}$ (resp. the subcategory of $\mathbf{K}$ $(\mathbf{DIFF}_A)$ whose objects are complexes of objects in $\mathcal{D}$).

A sequence $T_1 \to T_2 \to T_3$ of functors $T_i : \mathcal{D} \to \mathcal{D}$, $i = 1, 2, 3$ (and functorial morphisms) $\mathcal{D}$ being an abelian subcategory of $A - \mathbf{Mod}$, is said *exact in* $[\mathcal{D}, \mathcal{D}]$ if it is exact in $\mathcal{D}$ when applied to any object of $\mathcal{D}$.

Let $\mathbb{N}_+^\infty = \varprojlim \mathbb{N}_+^k$ be the set of infinite sequences of positive integers. If $\sigma \in \mathbb{N}_+^n$ (or $\sigma \in \mathbb{N}_+^\infty$) then $\sigma(r) \doteq (\sigma_1, \ldots, \sigma_r)$ for $r \leqslant n$ (or any $r \in \mathbb{N}_+$). We denote by $\mathbf{1}$ the element $(1, \ldots, 1, 1, \ldots, 1, \ldots) \in \mathbb{N}_+^\infty$.

2. Absolute and relative functors

We recall here in a slightly different way the necessary definitions from [9,16,17] (see also [10]).

If P and Q are A-modules and $a \in A$ we define:

$$\delta_a : Hom_K(P, Q) \to Hom_K(P, Q), \qquad \Phi \mapsto \{\delta_a\Phi : p \mapsto \Phi(ap) - a\Phi(p)\}, \quad p \in P$$

(where we use juxtaposition to indicate both A-module multiplications in P and Q). For each $a \in A$, δ_a is a homomorphism of K-modules, and commutativity of A implies that $\delta_{a_1} \circ \delta_{a_2} = \delta_{a_2} \circ \delta_{a_1}$, for any $a_1, a_2 \in A$.

Definition 2.1. A *K-differential operator* (a *DO* for short) *of order* $\leqslant s$ from an A-module P to an A-module Q is an element $\Delta \in Hom_K(P, Q)$ such that:

$$[\delta_{a_0} \circ \delta_{a_1} \circ \cdots \circ \delta_{a_s}](\Delta) = 0, \quad \forall \{a_0, a_1, \dots, a_s\} \subset A.$$

We will write synthetically $\delta_{a_0,\dots,a_s}$ for $\delta_{a_0} \circ \delta_{a_1} \circ \cdots \circ \delta_{a_s}$.

The set $Diff_k(P, Q)$ of differential operators of order $\leqslant k$ from P to Q is endowed naturally with two different A-module structures:

(i) $(Diff_k(P, Q), \tau) \doteq \mathrm{Diff}_k(P, Q)$ (left),
$\tau : A \times Diff_k(P, Q) \to Diff_k(P, Q) : (a, \Delta) \mapsto \tau(a, \Delta) : p \mapsto a\Delta(p)$,
(ii) $(Diff_k(P, Q), \tau^+) \doteq \mathrm{Diff}_k^+(P, Q)$ (right),
$\tau^+ : A \times Diff_k(P, Q) \to Diff_k(P, Q) : (a, \Delta) \mapsto \tau^+(a, \Delta) : p \mapsto \Delta(ap)$.

We will often write, to be concise, $\tau(a, \Delta) \equiv a\Delta$ and $\tau^+(a, \Delta) \equiv a^+\Delta$. It is easy to see that $(Diff_k(P, Q), (\tau, \tau^+)) \equiv (\mathrm{Diff}_k(P, Q), \mathrm{Diff}_k^+(P, Q)) \doteq \mathrm{Diff}_k^{(+)}(P, Q)$ is an A-bimodule.

Remark 2.2. Since for any $a_0 \in A$ and $p \in P$, $\delta_{a_0}(\Delta) \equiv 0 \Leftrightarrow \Delta(a_0 p) = a_0\Delta(p)$, $\mathrm{Diff}_0(P, Q)$ and $Hom_A(P, Q)$ are identified as A-(bi)modules: $Hom_A(P, Q) \simeq \mathrm{Diff}_0(P, Q) \simeq \mathrm{Diff}_0^+(P, Q)$.

For any $k \leqslant l$, we have an induced monomorphism of A-bimodules:

$$\mathrm{Diff}_k^{(+)}(P, Q) \hookrightarrow \mathrm{Diff}_l^{(+)}(P, Q), \quad k \leqslant l;$$

the direct limit of the system in $A - \mathbf{BiMod}$:

$$\mathrm{Diff}_0^{(+)}(P, Q) \hookrightarrow \mathrm{Diff}_1^{(+)}(P, Q) \hookrightarrow \cdots \hookrightarrow \mathrm{Diff}_n^{(+)}(P, Q) \hookrightarrow \cdots$$

is denoted by $\mathrm{Diff}^{(+)}(P, Q) = (\mathrm{Diff}(P, Q), \mathrm{Diff}^+(P, Q))$.

To a given A-module P we can associate the following three functors

$$\mathrm{Diff}_k : Q \mapsto \mathrm{Diff}_k(P, Q),$$
$$\mathrm{Diff}_k^+ : Q \mapsto \mathrm{Diff}_k^+(P, Q),$$
$$\mathrm{Diff}_k^{(+)} : Q \mapsto \mathrm{Diff}_k^{(+)}(P, Q).$$

Let us put $\mathrm{Diff}_k^{(+)}(A, Q) \equiv \mathrm{Diff}_k^{(+)} Q$. By Remark 2.2, $\mathrm{Diff}_0^+ = \mathrm{Diff}_0 = \mathrm{Id}_{A-\mathbf{Mod}}$. To simplify notations we will write $\mathrm{Diff}^+_{\sigma_1,\dots,\sigma_n}$ instead of $\mathrm{Diff}^+_{\sigma_1} \circ \cdots \circ \mathrm{Diff}^+_{\sigma_n}$.

Definition 2.3. For any $s, t \geqslant 0$, define an A-module homomorphism

$$\mathrm{C}_{s,t}(P) : \mathrm{Diff}_s^+(\mathrm{Diff}_t^+ P) \to \mathrm{Diff}_{s+t}^+ P,$$
$$\mathrm{C}_{s,t}(P)(\Delta) : a \mapsto \Delta(a)(1), \quad \Delta \in \mathrm{Diff}_s^+(\mathrm{Diff}_t^+ P).$$

Then $P \mapsto \mathrm{C}_{s,t}(P)$ defines a morphism $\mathrm{Diff}^+_{s,t} \to \mathrm{Diff}^+_{s+t}$ of functors called the *composition* or "*gluing*" *morphism*.

Note that $\mathrm{D}_{(k)}(Q) \doteq \{\Delta \in \mathrm{Diff}_k Q \mid \Delta(1) = 0\}$ is an A-submodule of $\mathrm{Diff}_k Q$ but not of $\mathrm{Diff}_k^+ Q$! The functor $\mathrm{D}_{(k)}: Q \mapsto D_{(k)}(Q)$ allows to form the following short exact sequence:

$$0 \to \mathrm{D}_{(k)} \xrightarrow{i_k} \mathrm{Diff}_k \xrightarrow{p_k} \mathrm{Id}_{A-\mathbf{Mod}} \to 0 \tag{2.1}$$

of functors $A - \mathbf{Mod} \to A - \mathbf{Mod}$, where i_k is the canonical inclusion and p_k is defined by:

$$p_k(Q): \mathrm{Diff}_k Q \to Q : \Delta \mapsto \Delta(1),$$

for any A-module Q. The functor monomorphism $\mathrm{Id}_{A-\mathbf{Mod}} \equiv \mathrm{Diff}_0 \hookrightarrow \mathrm{Diff}_k$ splits (1), so that $\mathrm{Diff}_k = \mathrm{D}_{(k)} \oplus \mathrm{Id}_{A-\mathbf{Mod}}$. Note that $\mathrm{D}_{(1)}(Q)$ is nothing but the A-module of all Q-valued K-linear *derivations* on A, denoted in the literature usually by $\mathrm{Der}_{A/K}(Q)$ (see for example [2]).

Let Q be an A-module and P, P^+ be the left and right A-modules corresponding to an A-bimodule $P^{(+)} \equiv (P, P^+)$. Let's denote by $\mathrm{Diff}_k^\bullet(Q, P^+)$ (resp. $\mathrm{D}_{(k)}^\bullet(P^+)$) the A-module which coincides with $\mathrm{Diff}_k(Q, P^+)$ (resp. $\mathrm{D}_{(k)}(P^+)$) as a K-module and whose A-module structure is inherited by that of P[1]:

$$\big(\text{mult. by } a \text{ in } \mathrm{Diff}_k^\bullet(Q, P^+)\big) \qquad (a^\bullet \Delta)(q) \doteq a\Delta(q),$$
$$\big(\text{mult. by } a \text{ in } D_k^\bullet(P^+)\big) \quad (a^\bullet \delta)(q) \doteq a\delta(q),$$

where both $a\Delta(q)$ and $a\delta(q)$ denote the multiplication by a in P. The correspondence

$$\mathrm{D}_{(k)}^{(\bullet)}: P^{(+)} \mapsto \big(D_{(k)}(P^+), D_{(k)}^\bullet(P^+)\big)$$

defines an endofunctor of $A - \mathbf{BiMod}$ in the obvious way.

If $Q = A$ we write $\mathrm{Diff}_k^\bullet(P^+)$ for $\mathrm{Diff}_k^\bullet(A, P^+)$. Obviously, $\mathrm{D}_{(k)}^\bullet(P^+)$ is an A-submodule of $\mathrm{Diff}_k^\bullet(P^+)$.

Let us consider the following *ordered "special" triples* of A-modules:

$$(P, P^+; Q)$$

with $P^{(+)} \doteq (P, P^+)$ being an A-bimodule and Q an A-submodule of P. The corresponding morphisms are those of underlying A-bimodules "respecting" the distinguished submodules, i.e.:

$$f: (P, P^+) \to (\overline{P}, \overline{P}^+) \text{ such that } f(Q) \subset \overline{Q}.$$

Example 2.1. (i) If P is an A-module and $s \leqslant k$, then $(\mathrm{Diff}_k P, \mathrm{Diff}_k^+ P; \mathrm{D}_{(s)}(P))$ is a special triple, for each $s \leqslant k$.

(ii) If $P^{(+)} \doteq (P, P^+)$ is an A- bimodule, then we have the following special triples:

$$\big(\mathrm{Diff}_k^\bullet P^+, \mathrm{Diff}_k^+ P^+; \mathrm{D}_{(k)}^\bullet(P^+)\big),$$
$$\big(\mathrm{Diff}_k P, \mathrm{Diff}_k^+ P; \mathrm{D}_{(k)}(P)\big),$$
$$\big(\mathrm{Diff}_k P^+, \mathrm{Diff}_k^+ P^+; \mathrm{D}_{(k)}(P^+)\big).$$

[1] These A-module structures are well defined due to the fact that $(P, P^+) \equiv P^{(+)}$ is a bimodule. Obviously one can give similar definitions with P^+ replaced by P.

We associate to any such triple $(P^{(+)}; Q)$ the following K-modules:

$$\mathrm{Diff}_k(Q \subset P^+) \doteq \{\Delta \in \mathrm{Diff}_k P^+ \mid im(\Delta) \subset Q\},$$
$$\mathrm{D}_{(k)}(Q \subset P^+) \doteq \{\Delta \in \mathrm{D}_{(k)}(P^+) \mid im(\Delta) \subset Q\}.$$

The A-module structure of $\mathrm{Diff}_k^\bullet(P^+)$ (resp., of $\mathrm{Diff}_k^+(P^+)$ or of $\mathrm{D}_{(k)}^\bullet(P^+)$) induces an A-module structure on $\mathrm{Diff}_k(Q \subset P^+)$ (resp., on $\mathrm{Diff}_k^+(Q \subset P^+)$, resp. on $\mathrm{D}_{(k)}(Q \subset P^+)$) that will be denoted by $\mathrm{Diff}_k^\bullet(Q \subset P^+)$ (resp. $\mathrm{Diff}_k^+(Q \subset P^+)$ or $\mathrm{D}_{(k)}(Q \subset P^+)$). Therefore we have natural inclusions of A-modules

$$\mathrm{Diff}_k^\bullet(Q \subset P^+) \subset \mathrm{Diff}_k^\bullet(P^+),$$
$$\mathrm{Diff}_k^+(Q \subset P^+) \subset \mathrm{Diff}_k^+(P^+),$$
$$\mathrm{D}_{(k)}(Q \subset P^+) \subset \mathrm{D}_{(k)}^\bullet(P^+),$$

and a special triple $(\mathrm{Diff}_k^\bullet(Q \subset P^+),\ \mathrm{Diff}_k^+(Q \subset P^+),\ \mathrm{D}_{(k)}(Q \subset P^+))$. The following result is straightforward

Lemma 2.4. *If $\overline{P}^{(+)}$ is a sub-bimodule of $P^{(+)}$ and $Q \subset \overline{P}$, then we have:*

$$\mathrm{D}_{(k)}(Q \subset \overline{P}^+) = \mathrm{D}_{(k)}(Q \subset P^+),$$
$$\mathrm{Diff}_k^+(Q \subset \overline{P}^+) = \mathrm{Diff}_k^+(Q \subset P^+),$$
$$\mathrm{Diff}_k^\bullet(Q \subset \overline{P}^+) = \mathrm{Diff}_k^\bullet(Q \subset P^+).$$

We define now some *absolute functors* we will need in the following (see Definition 2.8 for their "relative" version).

Definition 2.5. For $k \geqslant 0$ and $(P^{(+)}; Q)$ a special triple, define the special triple

$$\underline{\mathcal{P}}_{(k)}(P^{(+)}; Q) \doteq \big(\mathrm{Diff}_k^\bullet(Q \subset P^+),\ \mathrm{Diff}_k^+(Q \subset P^+);\ \mathrm{D}_{(k)}(Q \subset P^+)\big).$$

For $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbf{N}_+^n$, define $\underline{\mathcal{P}}_\sigma(P^{(+)}; Q) \equiv \underline{\mathcal{P}}_{(\sigma_1,\ldots,\sigma_n)}(P^{(+)}; Q)$ as:

$$\underline{\mathcal{P}}_{(\sigma_1,\ldots,\sigma_n)}(P^{(+)}; Q) \doteq \underline{\mathcal{P}}_{(\sigma_1)}\big(\ldots\big(\underline{\mathcal{P}}_{(\sigma_{n-1})}\big(\underline{\mathcal{P}}_{(\sigma_n)}(P^{(+)}; Q)\big)\big)\big)$$

and we write the special triple $\underline{\mathcal{P}}_{(\sigma_1,\ldots,\sigma_n)}(P^{(+)}; Q)$ as

$$\big(\mathcal{P}^\bullet_{(\sigma_1,\ldots,\sigma_n)}(P^{(+)}; Q), \mathcal{P}^+_{(\sigma_1,\ldots,\sigma_n)}(P^{(+)}; Q), \mathrm{D}^\bullet_{(\sigma_1,\ldots,\sigma_n)}(P^{(+)}; Q)\big).$$

Any $\underline{\mathcal{P}}_{(\sigma_1,\ldots,\sigma_n)}$ can be viewed as a functor from special triples to special triples, with the obvious meaning.

If P is an A-module we write $\mathrm{D}_{(\sigma_1,\ldots,\sigma_n)}(P)$ for $\mathrm{D}^\bullet_{(\sigma_1,\ldots,\sigma_n)}(P, P; P)$. It is not difficult to see that

$$\mathrm{D}_{(\sigma_1,\ldots,\sigma_n)}(P) = \mathrm{D}_{(\sigma_1)}\big(\mathrm{D}_{(\sigma_2,\ldots,\sigma_n)}(P) \subset \mathrm{Diff}^+_{\sigma_2,\ldots,\sigma_n}(P)\big).$$

Furthermore,

$$\mathrm{D}_{(\sigma_1,\ldots,\sigma_n)}(P) \hookrightarrow \mathrm{D}^\bullet_{(\sigma_1)}\big(\mathrm{Diff}^+_{\sigma_2,\ldots,\sigma_n}(P)\big) \hookrightarrow \mathrm{Diff}^\bullet_{\sigma_1}\big(\mathrm{Diff}^+_{\sigma_2,\ldots,\sigma_n}(P)\big)$$

are injective A-homomorphisms, while the inclusion $\mathrm{D}_{(\sigma_1,\ldots,\sigma_n)}(P) \hookrightarrow \mathrm{Diff}^+_{\sigma_1,\ldots,\sigma_n}(P)$ is a DO of order $\leqslant \sigma_1$. So, the functors D_σ's can be defined also inductively as:

Definition 2.6. Let $\sigma = (\sigma_1, \sigma_2, \ldots \sigma_n, \ldots) \in \mathbf{N}_+^\infty$ and $\sigma(n) = (\sigma_1, \ldots, \sigma_n)$. Then the functors $\mathrm{D}_{\sigma(n)} : A-\mathbf{Mod} \to A-\mathbf{Mod}$ are defined by induction:

$$\mathrm{D}_{\sigma(1)} \doteq \mathrm{D}_{(\sigma_1)},$$
$$\mathrm{D}_{\sigma(n)} : P \mapsto \mathrm{D}_{(\sigma_1)}\big(\mathrm{D}_{(\sigma_2,\ldots,\sigma_n)}(P) \subset \mathrm{Diff}^+_{\sigma_2,\ldots,\sigma_n}(P)\big).$$

If $\sigma = (1, \ldots 1, 1, \ldots)$ we also write D_n for $\mathrm{D}_{\sigma(n)}$.

For any $\sigma \in \mathbf{N}_+^\infty$ and $n \in \mathbf{N}_+$, we have an exact sequence of endofunctors of $A-\mathbf{Mod}$:

$$0 \to \mathrm{D}_{\sigma(n)} \overset{\mathrm{I}_{\sigma(n)}}{\to} \mathrm{D}^\bullet_{\sigma(n-1)} \circ \mathrm{Diff}^{(+)}_{\sigma_n} \overset{\pi_{\sigma(n)}}{\to} \mathrm{D}_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}, \tag{2.2}$$

where $\mathrm{I}_{\sigma(n)}$ is the natural inclusion and $\pi_{\sigma(n)}$ is the composition (see Definition 2.3)

$$\mathrm{D}^\bullet_{\sigma(n-1)} \circ \mathrm{Diff}^{(+)}_{\sigma_n} \hookrightarrow \mathrm{D}^\bullet_{\sigma(n-2)} \circ \mathrm{Diff}^{(+)}_{\sigma_{n-1}} \circ \mathrm{Diff}^+_{\sigma_n} \xrightarrow{\mathrm{D}^\bullet_{\sigma(n-2)}(\mathrm{C}_{\sigma_{n-1},\sigma_n})} \mathrm{D}^\bullet_{\sigma(n-2)} \circ \mathrm{Diff}^{(+)}_{\sigma_{n-1}+\sigma_n}.$$

Remark 2.7. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbf{N}_+^n$. We have a canonical split exact short exact sequence of endofunctors of $A-\mathbf{Mod}$

$$0 \longrightarrow \mathrm{D}_\sigma \underset{\rho_\sigma}{\overset{l_\sigma}{\rightleftarrows}} \mathcal{P}^\bullet_\sigma \overset{\psi_\sigma}{\longrightarrow} \mathrm{D}_{(\sigma_2,\ldots,\sigma_n)} \longrightarrow 0$$

where l_σ is the canonical inclusion, ψ_σ is given by (P is an A-module) $\psi_\sigma(P)(\Delta) = \Delta(1)$ and ρ_σ by $\rho_\sigma(P)(\Delta) = \Delta - \Delta(1)$.

Hence $\mathcal{P}^\bullet_\sigma \simeq \mathrm{D}_{(\sigma_2,\ldots,\sigma_n)} \oplus \mathrm{D}_\sigma$.

Now we define the *relative* (i.e., relative to an arbitrary A-module P) *functors*.

Definition 2.8. If $k \geqslant 0$ and P is an A-module, define $\underline{\mathcal{P}}_{(k)}[P]$ to be the functor assigning to a special triple $(Q, Q^+; S)$ the A-bimodule

$$\big(\mathrm{Diff}^\bullet_k(P, S \subset Q^+), \mathrm{Diff}^+_k(P, S \subset Q^+)\big).$$

If $\sigma(n) = (\sigma_1, \ldots, \sigma_n) \in \mathbf{N}_+^n$, $n > 1$, define the functor $\underline{\mathcal{P}}_{\sigma(n)}[P]$ as the composition

$$\underline{\mathcal{P}}_{\sigma(n)}[P] \doteq \underline{\mathcal{P}}_{(\sigma_1)}[P] \circ \underline{\mathcal{P}}_{(\sigma_2,\ldots,\sigma_n)}.$$

As in the absolute case (Definition 2.5), we call $\underline{\mathcal{P}}^\bullet_{\sigma(n)}[P]$ (resp. $\underline{\mathcal{P}}^+_{\sigma(n)}[P]$) the underlying left (resp. right) A-module of $\underline{\mathcal{P}}_{\sigma(n)}[P]$.

By Lemma 2.4, we have $(\mathcal{P}^\bullet_{\sigma(n)}, \mathcal{P}^+_{\sigma(n)}) \equiv (\mathcal{P}^\bullet_\sigma[A], \mathcal{P}^+_\sigma[A])$. Moreover, $\underline{\mathcal{P}}_{\sigma(n)}[P]$ is (controvariantly) functorial in P and we have

Lemma 2.9. *If* $0 \to P_1 \overset{f}{\to} P_2 \overset{g}{\to} P_3 \to 0$ *is exact (resp. split exact) in* $A-\mathbf{Mod}$, *then*

$$0 \to \underline{\mathcal{P}}^\bullet_{\sigma(n)}[P_3] \overset{g^\vee}{\to} \underline{\mathcal{P}}^\bullet_{\sigma(n)}[P_2] \overset{f^\vee}{\to} \underline{\mathcal{P}}^\bullet_{\sigma(n)}[P_1],$$
$$0 \to \underline{\mathcal{P}}^+_{\sigma(n)}[P_3] \overset{g^\vee}{\to} \underline{\mathcal{P}}^+_{\sigma(n)}[P_2] \overset{f^\vee}{\to} \underline{\mathcal{P}}^+_{\sigma(n)}[P_1]$$

are exact (*resp.*

$$0 \to \mathcal{P}^{\bullet}_{\sigma(n)}[P_3] \xrightarrow{g^{\vee}} \mathcal{P}^{\bullet}_{\sigma(n)}[P_2] \xrightarrow{f^{\vee}} \mathcal{P}^{\bullet}_{\sigma(n)}[P_1] \to 0,$$

$$0 \to \mathcal{P}^{+}_{\sigma(n)}[P_3] \xrightarrow{g^{\vee}} \mathcal{P}^{+}_{\sigma(n)}[P_2] \xrightarrow{f^{\vee}} \mathcal{P}^{+}_{\sigma(n)}[P_1] \to 0$$

are exact).

Proof. Straightforward, by induction on n. □

If P is an A-module, there are exact sequences of functors $A - \mathbf{Mod} \to A - \mathbf{Mod}$

$$0 \to \mathcal{P}^{\bullet}_{\sigma(n)}[P] \hookrightarrow \mathcal{P}^{\bullet}_{\sigma(n-1)}[P] \circ \mathrm{Diff}^{(+)}_{\sigma_n} \xrightarrow{q_{\sigma(n)}} \mathcal{P}^{\bullet}_{(\sigma_1,\dots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}[P], \tag{2.3}$$

where the monomorphism is the natural inclusion while $q_{\sigma(n)}$ is induced by the "gluing" morphism with respect to the pair of indexes (σ_{n-1}, σ_n), i.e.:

$$\mathcal{P}^{\bullet}_{(\sigma_1,\dots,\sigma_{n-1})}[P] \circ \mathrm{Diff}^{(+)}_{\sigma_n}(Q) \ni \Delta \mapsto q_{\sigma(n)}(\Delta) = \overline{\Delta} \in \mathcal{P}^{\bullet}_{(\sigma_1,\dots\sigma_{n-2},\sigma_{n-1}+\sigma_n)}[P](Q),$$

$$\big(\dots\big((\overline{\Delta}(p))(a_1)\big)\dots\big)(a_{n-2}) \doteq \big(\big(\dots\big((\Delta(p))(a_1)\big)\dots\big)(a_{n-2})\big)(1),$$

where $p \in P$ and $a_1, \dots, a_{n-2} \in A$. We have analogous exact sequences:

$$0 \to \mathcal{P}^{\bullet}_{\sigma(n)}[P] \hookrightarrow \mathrm{Diff}^{\bullet}_{\sigma_1}(P, \cdot) \circ \mathcal{P}^{+}_{(\sigma_2,\dots,\sigma_n)}[P] \xrightarrow{g_{\sigma(n)}} \mathcal{P}^{\bullet}_{(\sigma_1+\sigma_2,\sigma_3,\dots,\sigma_n)}[P], \tag{2.4}$$

where $g_{\sigma(n)} : \Delta \mapsto \widehat{\Delta}$ with $\widehat{\Delta}(p) \doteq \Delta(p)(1)$, $p \in P$ (i.e., we "glue" with respect to the first two indexes); the upper boldface dot in $\mathrm{Diff}^{\bullet}_{\sigma_1}(P, \cdot)$ denotes the A-module structure induced by $\mathcal{P}^{\bullet}_{(\sigma_2,\dots,\sigma_n)}[P]$.

The following definition will allow us to be concise in the next section:

Definition 2.10. For any $n > 0$ and any $\sigma \in \mathbf{N}^n_+$, the functors (in $[A - \mathbf{Mod}, A - \mathbf{Mod}]$) $\mathcal{P}^{\bullet}_{\sigma}$, $D^{\bullet}_{\sigma}$ are called the *relevant absolute functors* while, if P is an A -module, the functors $\mathcal{P}^{\bullet}_{\sigma}[P]$, are called the *relevant functors relative* to the A-module P.

3. Absolute and relative representative objects

In this section we consider (strict) representative objects of the functors introduced in the previous section. We obtain, as particular cases, the standard modules of Kähler differential forms of Algebraic Geometry and the de Rham forms of Differential Geometry. We emphasize that in our approach all these (and not only those of degree one) are obtained as representative objects of suitable functors. One of the major advantages of this approach is to allow natural generalizations.

Let $\mathcal{D}$ be a full subcategory of $A - \mathbf{Mod}$. We denote by $A - \mathbf{BiMod}_{\mathcal{D}}$ the subcategory of $A - \mathbf{BiMod}$ whose objects are couples of objects of $\mathcal{D}$; a special triple $(P, P^+; Q)$ (i.e., a triple of A-modules such that (P, P^+) is an A-bimodule and Q is a submodule of P) with P, P^+ and Q belonging to $\mathcal{D}$ will be called a special triple in $\mathcal{D}$ (Section 1).

Definition 3.1. A full *abelian* subcategory $\mathcal{D}$ of $A - \mathbf{Mod}$ is said to be *differentially closed* if the following properties are satisfied:

(a) each functor defined in the previous section, when restricted to $\mathcal{D}$ (or $A - \mathbf{BiMod}_{\mathcal{D}}$, or to special triples in $\mathcal{D}$) has values in $\mathcal{D}$ (or in $A - \mathbf{BiMod}_{\mathcal{D}}$, or in special triples in $\mathcal{D}$);

(b) if $T : A - \mathbf{Mod} \to A - \mathbf{Mod}$ is a *relevant absolute* functor or a *relevant relative* functor, *relative to an object of* $\mathcal{D}$, then $T_{|\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is strictly representable in $\mathcal{D}$;

(c) $A \in Ob(\mathcal{D})$;

(d) $\mathcal{D}$ is closed under tensor products (over A);

(e) $\mathcal{D}$ is closed under taking subobjects (i.e., if $P \subseteq Q$ in $A - \mathbf{Mod}$ and Q is in $\mathcal{D}$ then P is in $\mathcal{D}$).

Condition (a) is needed to have an ambient category which is “closed” with respect to functorial differential calculus; as it will be clear in the following, since among the functors of Section 1 there are also compositions of relevant “elementary” ones, we would like that representative objects of these nonelementary functors[2] (for example $\mathrm{D}^{\bullet}_{(s)} \circ \mathrm{Diff}^{(+)}_{t}$), if existing, *could be expressed in terms* of representative objects of the relevant “elementary” ones ($\mathrm{D}_{(s)}$ and Diff_t in the example). Condition (d) makes it possible.

$\mathcal{D}$ being abelian and satisfying (b), exactness of sequences of strictly representable functors yields exactness of the “dual” sequences of representative objects in $\mathcal{D}$. Condition (e) is related to the existence of canonical generators for some representative objects and will become clear in the sequel. Note also that (e) implies that if f is a morphism in $\mathcal{D}$, $im(f)$ (resp. $\ker(f)$) is the same when considered in $\mathcal{D}$ or in $A - \mathbf{Mod}$.

Let us recall some elementary facts about bimodules, mainly to fix our notations.

If $P^{(+)} = (P, P^{+})$ is an A-bimodule, $a \in A$ and $p \in P^{(+)}$, we write ap for the multiplication in P and $a^{+}p$ for the multiplication in P^{+}. If Q is an A-module we denote by:

(I) $P^{+} \otimes^{\bullet}_{A} Q$ the A-module obtained from the abelian group $P^{+} \otimes_{A} Q$ with multiplication by elements of A defined as

$$a^{\bullet}(p \otimes q) \doteq (ap) \otimes q, \quad a \in A,\ p \in P^{+},\ q \in Q$$

(note that $a^{\bullet}(p \otimes q) \neq p \otimes aq$). Then $P^{+} \otimes^{(\bullet)}_{A} Q \doteq (P^{+} \otimes^{\bullet}_{A} Q, P^{+} \otimes_{A} Q)$ is an A-bimodule;

(II) $Hom^{\bullet}_{A}(Q, P^{+})$ the A-module obtained from the abelian group $Hom_{A}(Q, P^{+})$ with multiplication by elements of A defined as:

$$[a^{\bullet} f](p) \doteq a \cdot \big(f(p)\big), \quad a \in A,\ p \in P,\ f \in Hom_{A}(Q, P^{+}).$$

Denote by $Hom^{(\bullet)}_{A}(Q, P^{+}) \doteq (Hom^{\bullet}_{A}(Q, P^{+}), Hom_{A}(Q, P^{+}))$ the corresponding A-bimodule;

(III) $Hom^{+}_{A}(P, Q)$ the A-module obtained from the abelian group $Hom_{A}(P, Q)$ with multiplication by elements of A:

$$[a^{+} f](p) \doteq f(a^{+}p).$$

Then

$$Hom^{(+)}_{A}(P, Q) \doteq \big(Hom_{A}(P, Q), Hom^{+}_{A}(P, Q)\big)$$

is an A-bimodule.

In a similar way we can define the A-modules $P \otimes^{+}_{A} Q$, $Hom^{+}_{A}(P, Q)$ and $Hom^{\bullet}_{A}(P^{+}, Q)$.

[2] Relevant or not.

Example 3.1. If P and Q are A-modules, we have an isomorphism in $[A-\mathbf{Mod},\ A-\mathbf{Mod}]$

$$\mathrm{Diff}_s(\cdot, Q) \simeq Hom_A^{\bullet}(\cdot, \mathrm{Diff}_s^{+} Q).$$

We leave to the reader the verification of the following result:

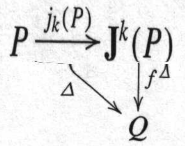
Lemma 3.2. *Let R and P be A-modules and (Q, Q^+) an A-bimodule. Then we have a canonical isomorphism in $A-\mathbf{BiMod}$:*

$$Hom_A^{(\bullet)}\big(R, Hom_A^{+}(Q, P)\big) \overset{\varphi}{\simeq} Hom_A^{(+)}(Q^+ \otimes_A^{\bullet} R, P).$$

Proposition 3.3. *Let P be an A-module and $k \in \mathbf{N}_+$. Then $\mathrm{Diff}_k(P,\cdot): A-\mathbf{Mod} \to A-\mathbf{Mod}$ is strictly representable by the so-called k-jet module $\mathbf{J}^k(P)$.*

Proof. See [10, p. 12]. □

In other words, there exists a universal DO $j_k(P): P \to \mathbf{J}^k(P)$, of order $\leqslant k$ (usually denoted simply by j_k), such that for any DO $\Delta: P \to Q$ of order $\leqslant k$, there is a unique A-homomorphism f^{Δ} and a commutative diagram



The A-module $\mathbf{J}^k(P)$ is generated by $\{j_k(p) \mid p \in P\}$. Moreover, $\mathbf{J}^k(P)$ has a bimodule structure $\mathbf{J}^k_{(+)}(P) = (\mathbf{J}^k(P), \mathbf{J}^k_{+}(P))$ which can be described in the following, purely functorial, way. Suppose $\mathcal{D} \subseteq A-\mathbf{Mod}$ is a subcategory such that $\forall P \in Ob(\mathcal{D})$ the functor $\mathrm{Diff}_k(P,\cdot)$ when restricted to $\mathcal{D}$ has values in $\mathcal{D}$ and is strictly representable in $\mathcal{D}$ (e.g., $\mathcal{D} = A-\mathbf{Mod}$ by Proposition 3.2). Let $\mathbf{J}^k_{\mathcal{D}}(P)$ be the corresponding representative object, i.e., $Hom_A(\mathbf{J}^k_{\mathcal{D}}(P),\cdot) \simeq \mathrm{Diff}_k(P,\cdot)$, isomorphism of functors $\mathcal{D} \to \mathcal{D}$. If $j_k^{\mathcal{D}}: P \to \mathbf{J}^k_{\mathcal{D}}(P)$ corresponds to the identity morphism of $\mathbf{J}^k_{\mathcal{D}}(P)$, we can define, for each $a \in A$, the DO

$$a^+: P \to \mathbf{J}^k_{\mathcal{D}}(P): p \mapsto j_k^{\mathcal{D}}(ap).$$

The corresponding A-endomorphism of $\mathbf{J}^k_{\mathcal{D}}$ is still denoted by a^+ and gives the required second A-module structure $\mathbf{J}^k_{\mathcal{D}+}$ on the abelian group $\mathbf{J}^k_{\mathcal{D}}$.

Using the bimodule $\mathbf{J}^k_{(+)}$ and Proposition 3.3, we get an isomorphism of functors $A-\mathbf{Mod} \to A-\mathbf{Mod}$

$$\mathrm{Diff}_k^{+} \simeq Hom_A^{+}(\mathbf{J}^k,\cdot). \tag{3.1}$$

If P, Q are A-modules, then:

$$Hom_A\big(\mathbf{J}^k_{+} \otimes_A^{\bullet} P, Q\big) \simeq Hom_A^{\bullet}\big(P, Hom_A^{+}(\mathbf{J}^k, Q)\big)$$

by Lemma 3.2; therefore $Hom_A^{\bullet}(P, Hom_A^{+}(\mathbf{J}^s, Q)) \simeq Hom_A^{\bullet}(P, \mathrm{Diff}_s^{+} Q)$ and, by (3.1) and Example 3.1, we finally get $Hom_A^{\bullet}(P, \mathrm{Diff}_s^{+} Q) \simeq \mathrm{Diff}_s(P, Q)$. Since representative objects of the same functor are canonically isomorphic, we have proved:

Lemma 3.4. *There are canonical isomorphisms in* $[A-\mathbf{Mod},\ A-\mathbf{Mod}]$:

$$\mathbf{J}^k(\cdot)\simeq \mathbf{J}^k_+\otimes^\bullet_A(\cdot),$$
$$\mathbf{J}^k_+(\cdot)\simeq \mathbf{J}^k_+\otimes_A(\cdot).$$

We are now able to prove a basic result

Lemma 3.5. (a) *If* $\tau\in\mathbf{N}^\infty_+$, $n>0$, $t\geqslant 0$ *and* $\mathrm{D}_{\tau(n)}$ *is strictly representable in* $A-\mathbf{Mod}$ *by* $\Lambda^{\tau(n)}$, *then* $\mathrm{D}^\bullet_{\tau(n)}\circ\mathrm{Diff}^{(+)}_t$ *is strictly representable in* $A-\mathbf{Mod}$ *by* $\mathbf{J}^t(\Lambda^{\tau(n)})$.

(b) *If* $s,t\geqslant 0$, *then*

$$\mathcal{P}^\bullet_{(s)}\circ\mathrm{Diff}^{(+)}_t\equiv\mathrm{Diff}^\bullet_s\circ\mathrm{Diff}^{(+)}_t : A-\mathbf{Mod}\to A-\mathbf{Mod}$$

is strictly representable by $\mathbf{J}^t(\mathbf{J}^s)$.

(c) *If* $s,t\geqslant 0$ *and* P *is an* A-*module, then*

$$\mathcal{P}^\bullet_{(s)}[P]\circ\mathrm{Diff}^{(+)}_t\equiv\mathrm{Diff}^\bullet_s\big(P,\mathrm{Diff}^{(+)}_t(\cdot)\big) : A-\mathbf{Mod}\to A-\mathbf{Mod}$$

is strictly representable by $\mathbf{J}^t(\mathbf{J}^s(P))$.

(d) *If* $\mathcal{P}^\bullet_{\sigma(n)}$ *is strictly representable by* $\mathrm{Hol}^{\sigma(n)}$, *then*

$$\mathcal{P}^\bullet_{\sigma(n)}\circ\mathrm{Diff}^{(+)}_k : A-\mathbf{Mod}\to A-\mathbf{Mod}$$

is strictly representable by $\mathbf{J}^k(\mathrm{Hol}^{\sigma(n)})$.

(e) *If* P *is an* A-*module and* $\mathcal{P}^\bullet_{\sigma(n)}[P]$ *is strictly representable by* $\mathrm{Hol}^{\sigma(n)}[P]$, *then*

$$\mathcal{P}^\bullet_{\sigma(n)}[P]\circ\mathrm{Diff}^{(+)}_k : A-\mathbf{Mod}\to A-\mathbf{Mod}$$

is strictly representable by $\mathbf{J}^k(\mathrm{Hol}^{\sigma(n)}[P])$.

Proof. The proofs are very similar. We prove only (b) and (d).

(b)

$$\mathrm{Diff}^\bullet_s(\mathrm{Diff}^+_t P)\simeq Hom^\bullet_A(\mathbf{J}^s,\mathrm{Diff}^+_t P)\simeq Hom^\bullet_A(\mathbf{J}^s, Hom^+_A(\mathbf{J}^t,P));$$

by (3.1) this is isomorphic to $Hom_A(\mathbf{J}^t_+\otimes^\bullet\mathbf{J}^s,P)$ and, finally by Proposition 3.4, to $Hom_A(\mathbf{J}^t(\mathbf{J}^s),P)$.

(d)

$$\begin{aligned}\mathcal{P}^\bullet_{\sigma(n)}\big(\mathrm{Diff}^{(+)}_k(P)\big)&\simeq Hom^\bullet_A\big(\mathrm{Hol}^{\sigma(n)},\mathrm{Diff}^+_t P\big)\simeq Hom_A\big(\mathrm{Hol}^{\sigma(n)}, Hom^+_A(\mathbf{J}^k,P)\big)\\&\simeq Hom_A\big(\mathbf{J}^k_+\otimes^\bullet\mathrm{Hol}^{\sigma(n)},P\big)\simeq Hom_A\big(\mathbf{J}^k\big(\mathrm{Hol}^{\sigma(n)}\big),P\big).\end{aligned}$$ □

Note that if D_σ is representable for any $\sigma\in\mathbf{N}^n_+$, $n>0$, then $\mathcal{P}^\bullet_\sigma$ is representable for any $\sigma\in\mathbf{N}^n_+$, $n>0$, by Remark 2.7.

Remark 3.6. Note that for any A-module P we have:

$$\begin{aligned}\mathrm{Diff}^\bullet_k\big(\mathrm{Diff}^+_l(\mathrm{Diff}^+_m P)\big)&\simeq Hom_A\big(\mathbf{J}^l(\mathbf{J}^k),\mathrm{Diff}^+_m P\big)\\&\simeq Hom_A\big(\mathbf{J}^l(\mathbf{J}^k), Hom^+_A(\mathbf{J}^m,P)\big)\simeq Hom^+_A\big(\mathbf{J}^m_+\otimes^\bullet_A\mathbf{J}^l(\mathbf{J}^k),P\big)\\&\simeq Hom^+_A\big(\mathbf{J}^m(\mathbf{J}^l(\mathbf{J}^k)),P\big).\end{aligned}$$

In this case $\mathbf{J}^m(\mathbf{J}^l(\mathbf{J}^k))$ represents the functor $P \mapsto \mathrm{Diff}^{\bullet}_k(\mathrm{Diff}^+_l(\mathrm{Diff}^+_m P))$, but not strictly.

We conclude these preliminaries with the following elementary result

Lemma 3.7. *Let $0 \to T_1 \xrightarrow{i} T_2 \xrightarrow{\varphi} T_3$ be an exact sequence in $[A - \mathbf{Mod},\ A - \mathbf{Mod}]$, with T_2 and T_3 strictly representable by τ_2 and τ_3, respectively; then T_1 is also strictly representable by the quotient $\tau_2/\varphi^{\vee}(\tau_3)$, where $\varphi^{\vee}: \tau_3 \to \tau_2$ is the dual-representative of φ.*

Proof. If P is an A-module, the morphism $\chi_P : T_1(P) \to Hom_A(\frac{\tau_2}{\varphi^{\vee}(\tau_3)}, P)$:

$$q \mapsto \chi_P(q) : [t_2]_{\mathbf{Mod}\varphi^{\vee}(\tau_3)} \mapsto \hat{q}\,(t_2)$$

where $\hat{q} \doteq i(P)(q)$, is well defined since $\hat{q} \circ \varphi^{\vee} = \varphi(P)(\hat{q}) = 0$ and is an isomorphism, natural in P. □

The next theorem, collecting some of the results above, asserts that $A - \mathbf{Mod}$ *is itself differentially closed* (see Definition 3.1).

Theorem 3.8. *Let P be an A-module, $\sigma \in \mathbf{N}^{\infty}_{+}$ and $k \in \mathbf{N}$. Then:*

(i) *$\mathrm{Diff}_k(P, \cdot)$ is strictly representable in $A - \mathbf{Mod}$ by the k-jet module $\mathbf{J}^k(P)$;*
(ii) *for each $n > 0$, the functor $\mathrm{D}_{\sigma(n)}$ is strictly representable in $A - \mathbf{Mod}$ by the so-called* higher de Rham forms' module of type $\sigma(n)$, $\Lambda^{\sigma(n)}$;
(iii) *$\mathcal{P}^{\bullet}_{\sigma(n)}$ and $\mathcal{P}^{\bullet}_{\sigma(n)}[P] : A - \mathbf{Mod} \to A - \mathbf{Mod}$ are strictly representable by the so-called* absolute holonomy module *of type $\sigma(n)$*, $\mathrm{Hol}^{\sigma(n)}$ *and* relative holonomy module *of type $\sigma(n)$*, $\mathrm{Hol}^{\sigma(n)}[P]$.

Proof. (i) is Proposition 3.3.

(ii) The strict representability of $\mathrm{D}_{\sigma(n)}$ in $A - \mathbf{Mod}$ may be proved by induction on n. The case $n = 1$ follows from the exact sequence (2.1), Lemma 3.7 and (i). Now, suppose we have proved strict representability of $\mathrm{D}_{\tau(k)}$ for each $\tau \in \mathbf{N}^{\infty}_{+}$ and each $k \leqslant n - 1$. From the exact sequence in $[A - \mathbf{Mod}, A - \mathbf{Mod}]$

$$0 \to \mathrm{D}_{\sigma(n)} \hookrightarrow \mathrm{D}^{\bullet}_{\sigma(n-1)} \circ \mathrm{Diff}^{+}_{\sigma_n} \to \mathrm{D}_{(\sigma(n-2),\sigma_{n-1}+\sigma_n)}, \tag{3.2}$$

the last morphism being $\Delta \mapsto \widehat{\Delta}$ where:

$$((\ldots((\widehat{\Delta}(a_1))(a_2))\ldots)(a_{n-2}))(a_{n-1}) \doteq (((\ldots((\Delta(a_1))(a_2))\ldots)(a_{n-2}))(a_{n-1}))(1)$$

(i.e., we use the “gluing” morphism of Definition 2.3 with respect to the last two indexes), Lemma 3.5 (a) and Lemma 3.7, we obtain strict representability for $\mathrm{D}_{\sigma(n)}$.

(iii) the case of $\mathcal{P}^{\bullet}_{\sigma(n)}$ follows, as for (ii), by induction via Lemma 3.5, Lemma 3.7 and by any of the following two exact sequences in $[A - \mathbf{Mod},\ A - \mathbf{Mod}]$:

$$0 \to \mathcal{P}^{\bullet}_{\sigma(n)} \hookrightarrow \mathcal{P}^{\bullet}_{\sigma(n-1)} \circ \mathrm{Diff}^{(+)}_{\sigma_n} \to \mathcal{P}^{\bullet}_{(\sigma(n-2),\sigma_{n-1}+\sigma_n)}, \tag{3.3}$$

$$0 \to \mathcal{P}^{\bullet}_{\sigma(n)} \hookrightarrow \mathrm{Diff}^{\bullet}_{\sigma_1} \circ \mathcal{P}^{+}_{(\sigma_2,\ldots,\sigma_n)} \xrightarrow{\mathrm{C}_{\sigma_1,\sigma_2}(\mathrm{Diff}^{+}_{\sigma_3,\ldots,\sigma_n})} \mathcal{P}^{\bullet}_{(\sigma_1+\sigma_2,\sigma_3,\ldots,\sigma_n)} \tag{3.4}$$

where:

(a) the upper boldface dot in $\mathrm{Diff}^{\bullet}_{\sigma_1}$ in (3.4) refers to the A-bimodule structure $(\mathcal{P}^{\bullet}_{(\sigma_2,\dots,\sigma_n)}, \mathcal{P}^{+}_{(\sigma_2,\dots,\sigma_n)})$;

(b) the morphisms on the right are defined in the only natural way by using the "gluing" morphism of Definition 2.3: for (3.3) we "glue" with respect to the last two indexes while in (3.4) we "glue" with respect to the first two. The case of $\mathcal{P}^{\bullet}_{\sigma(n)}[P]$ is proved analogously, using (2.3) instead of (3.3) or (2.4) in place of (3.4). □

Remark 3.9. For any $k > 0$, we have $\Lambda^{(k)} \simeq \mathrm{I}/\mathrm{I}^{k+1}$ where I is the kernel of the ring multiplication $A \otimes_K A \to A$; hence $\Lambda^{(1)} \simeq \Omega^1_{A/K}$ is just the A-module of *Kähler differentials* (relative to K). Moreover it is not difficult to show [10, p. 17] that $\Lambda^{(1,\dots,1)} \simeq \Lambda^n \doteq \Lambda^1 \wedge \cdots \wedge \Lambda^1$ (n times) $\simeq \Omega^n_{A/K}$ and that for each $k, l \in \mathbf{N}_+$ the map $\Delta \mapsto \Delta$ induces a monomorphism $[A-\mathbf{Mod},\ A-\mathbf{Mod}]$:

$$\mathrm{D}_{\mathbf{1}(k+l)} \hookrightarrow \mathrm{D}_{\mathbf{1}(k)} \circ \mathrm{D}_{\mathbf{1}(l)},$$

whose dual representative A-homomorphism is just the wedge product $\wedge : \Lambda^k(A) \otimes_A \Lambda^l(A) \to \Lambda^{k+l}(A)$.

If $\mathcal{D}$ is a differentially closed subcategory, we will denote the strict representatives in $\mathcal{D}$ of the relevant functors by adding $\mathcal{D}$ as a subscript to the symbol used to denote the corresponding representative object in $A-\mathbf{Mod}$; for example, we write $\Lambda^{\sigma(n)}_{\mathcal{D}}$ for the representative object in $\mathcal{D}$ of the functor $\mathrm{D}_{\sigma(n)} : \mathcal{D} \to \mathcal{D}$.

Remark 3.10. As we did for $\mathbf{J}^{\sigma_1}_{\mathcal{D}} = \mathrm{Hol}^{(\sigma_1)}_{\mathcal{D}}$, we can exhibit another compatible A-module structure on $\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}$, $\forall n > 0$. Let $a \in A$ and $\widehat{\mathrm{id}} \in \mathcal{P}^{\bullet}_{\sigma(n)}(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}})$ correspond to the identity of $\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}$ under the representability isomorphism. Since $\mathcal{P}^{\bullet}_{\sigma(n)}(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}})$ and $\mathcal{P}^{+}_{\sigma(n)}(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}})$ coincide as sets, we can consider $a^+\widehat{\mathrm{id}}$ (multiplication in $\mathcal{P}^{+}_{\sigma(n)}(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}})$) as an A-endomorphism of $\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}$. It is easy to verify that this choice defines another A-module structure on $\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}$, denoted by $\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D},+}$ and that $(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}, \mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D},+})$ is an A-bimodule.

Lemma 3.11. *If $P \in Ob(\mathcal{D})$ we have a canonical isomorphism $\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}[P] \simeq \mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D},+} \otimes^{\bullet} P$ in $\mathcal{D}$.*

Proof. Let Q be an object in $\mathcal{D}$. By Lemma 3.2, we have

$$Hom\big(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D},+} \otimes^{\bullet} P, Q\big) \simeq Hom^{\bullet}\big(P, Hom^{+}\big(\mathrm{Hol}^{(\sigma_1,\dots,\sigma_n)}_{\mathcal{D}}, Q\big)\big) \simeq Hom^{\bullet}\big(P, \mathcal{P}^{+}_{\sigma(n)}(Q)\big)$$

and by definition of $\mathcal{P}^{+}_{\sigma(n)}$ and Example 3.1

$$Hom^{\bullet}\big(P, \mathcal{P}^{+}_{\sigma(n)}(Q)\big) \simeq \mathcal{P}^{\bullet}_{(\sigma_1)}[P]\big(\mathcal{P}_{(\sigma_2,\dots,\sigma_n)}(Q)\big) = \mathcal{P}^{\bullet}_{(\sigma_1,\dots,\sigma_n)}[P](Q). \qquad \square$$

Remark 3.12. For any differentiable closed subcategory $\mathcal{D} \subseteq A-\mathbf{Mod}$ it is still true, as in the case $\mathcal{D} = A-\mathbf{Mod}$, that $\mathbf{J}^k_{\mathcal{D}}(P)$ is generated as an A-module by $\{j_k(p) \mid p \in P\}$. In fact, let $\mathbf{J}^k_{\mathcal{D}}(P)^{\sim}$ denote the A-submodule of $\mathbf{J}^k_{\mathcal{D}}(P)$ generated by $\{j_k(p) \mid p \in P\}$; this is still an object of $\mathcal{D}$ by Definition 3.1(e). Now, the composition

$$P \xrightarrow{j^{\mathcal{D}}_k} \mathbf{J}^k_{\mathcal{D}}(P) \overset{\pi}{\twoheadrightarrow} \frac{\mathbf{J}^k_{\mathcal{D}}(P)}{\mathbf{J}^k_{\mathcal{D}}(P)^{\sim}}$$

is the DO of order $\leqslant k$ corresponding to π under the isomorphism

$$Hom_A\big(\mathbf{J}^k_{\mathcal{D}}(P), \mathbf{J}^k_{\mathcal{D}}(P)/\mathbf{J}^k_{\mathcal{D}}(P)^{\sim}\big) \simeq \mathrm{Diff}_k\big(P, \mathbf{J}^k_{\mathcal{D}}(P)/\mathbf{J}^k_{\mathcal{D}}(P)^{\sim}\big).$$

But $\pi \circ j_k^{\mathcal{D}}$ is zero hence $\pi = 0$ and we conclude. Note that this also shows that the canonical morphism $\mathbf{J}^k(P) \to \mathbf{J}^k_{\mathcal{D}}(P)$ is an A-epimorphism.

If $P \in Ob(\mathcal{D})$ and $t \geqslant s$, the monomorphism $\mathrm{Diff}_s(P, \cdot) \subset \mathrm{Diff}_t(P, \cdot)$ in $[\mathcal{D}, \mathcal{D}]$ gives rise to a $\mathcal{D}$-epimorphism (also an A-epimorphism by Remark 3.12) between representative objects:

$$\pi_{t,s}(P) : \mathbf{J}^t_{\mathcal{D}}(P) \to \mathbf{J}^s_{\mathcal{D}}(P)$$

which fits in the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{j_s(P)} & \mathbf{J}^s_{\mathcal{D}}(P) \\ & {\scriptstyle j_t(P)} \searrow & \uparrow {\scriptstyle \pi_{t,s}(P)} \\ & & \mathbf{J}^t_{\mathcal{D}}(P) \end{array}$$

The rule $P \mapsto \mathbf{J}^t_{\mathcal{D}}(P)$ defines in the obvious way a (covariant) functor $\mathcal{D} \to \mathcal{D}$ ([9] or [10]).

The following example shows the importance of the appropriate choice of the differentially closed subcategory of $A - \mathbf{Mod}$ in determining the "geometrical effectiveness" and size of the representative objects of the relevant functors.

Example 3.2. Let M be a smooth real manifold (which we assume Haussdorf and with a countable basis), $K = \mathbb{R}$ and $A = C^{\infty}(M; \mathbb{R})$. Then [19] $\Lambda^{\sigma(n)}$ is in general neither projective nor of finite type over A: in particular, when $\sigma = (1, \ldots, 1, \ldots)$, *it does not* coincide with the A-module of differential n-forms on the manifold M. To obtain these "geometrical" objects we must choose an appropriate subcategory of $A - \mathbf{Mod}$: in our approach, choosing a "geometry" is equivalent to select a differentially closed subcategory $\mathcal{D}$. For finite dimensional (real) differential geometry we may choose $\mathcal{D} \doteq A - \mathbf{Mod}_{\mathrm{geom}}$, the full subcategory of geometric A-modules, i.e., of A-modules P such that $\bigcap_{x \in M} I_x P = (0)$, I_x being the maximal ideal of smooth functions on M vanishing at $x \in M$.

Note that $A - \mathbf{Mod}_{\mathrm{geom}} \supseteq A - \mathbf{Mod}_{\mathrm{pr,\,f.t.}}$, the full subcategory of projective A-modules of finite type, since A itself is a geometric A-module; however, $A - \mathbf{Mod}_{\mathrm{pr,\,f.t.}}$ is not differentially closed because it is not abelian (and does not satisfy (e)). Another reason that makes us prefer working with the bigger $A - \mathbf{Mod}_{\mathrm{geom}}$ is its better functoriality with respect to change of algebras induced by pull backs of smooth mappings of manifolds.[3] $A - \mathbf{Mod}_{\mathrm{geom}}$ is differentially closed due to the fact that the "geometrization" functor

$$(\cdot)_{\mathrm{geom}} : A - \mathbf{Mod} \to A - \mathbf{Mod}_{\mathrm{geom}}$$

$$P \mapsto P_{\mathrm{geom}} \doteq \frac{P}{\bigcap_{x \in M} I_x P}$$

sends representative objects in $A - \mathbf{Mod}$ to representative objects in $A - \mathbf{Mod}_{\mathrm{geom}}$ for all the relevant functors [10]. "Geometrical" objects are obtained as representative objects; for example $\Lambda^{(1,\ldots,1)}_{\mathrm{geom}}$, with

[3] If $f : M \to N$ is a smooth map and P is a geometric $C^{\infty}(M)$-module then P is still geometric when viewed as a $C^{\infty}(N)$-module via the pull back $f^* : C^{\infty}(N) \to C^{\infty}(M)$. Projectivity is not preserved, instead.

$(1,\ldots,1) \in \mathbf{N}^k$, is isomorphic to $\Gamma(\bigwedge^k T^*M)$, the module of sections of the kth exterior power of the cotangent bundle of M, i.e., the module of k-differential forms on M.

It is possible to encode the "*smoothness*" of the geometry we want to describe, completely in the choice of the differentially closed subcategory:

Definition 3.13. A differentially closed subcategory $\mathcal{D}$ of $A - \mathbf{Mod}$ is called *smooth* if $\Lambda_{\mathcal{D}}^{(1)}$ is a projective A-module of finite type.

Example 3.3. (i) If A is a smooth K-algebra, in the usual sense of commutative algebra, then $\mathcal{D} = A - \mathbf{Mod}$ is smooth (e.g., [7, II.8]).

(ii) If M is a smooth manifold and $A = C^\infty(M; \mathbf{R})$ then $A - \mathbf{Mod}_{\text{geom}}$ is smooth while $A - \mathbf{Mod}$ is not.

It can be proved (as in the proof of Theorem 3.8) that if $\mathcal{D}$ is smooth then *all the representative objects of relevant functors are indeed projective and of finite type as A-modules*. However, we want to stress that since representative objects may be constructed also in non-smooth cases, our approach works also in describing singular and even infinite dimensional geometrical situations. However, to resort with useful objects one has to make in each situation an adequate choice of $\mathcal{D}$.

The following proposition will be useful in the next sections:

Proposition 3.14. *Let $\mathcal{D}$ be smooth. Then*

(i) $\Lambda_{\mathcal{D}}^{\mathbf{1}_n} = (0)$ *for* $n \gg 0$;
(ii) *If* $P \in Ob(\mathcal{D})$, $\mathrm{Hol}_{\mathcal{D}}^{\mathbf{1}_n}[P] = (0)$ *for* $n \gg 0$.

Proof. By Lemma 3.11 and Remark 2.7, (ii) follows from (i). Remark 3.9 together with the fact that $\Lambda_{\mathcal{D}}^{(1)}$ is of finite type proves (i). □

4. Higher de Rham complexes

In this section we use the functors introduced in Section 2 and their representative objects (Section 3) to build higher order analogs of the de Rham complex. Their cohomology will be studied in Section 5.

Let $\mathcal{D}$ be a differentially closed subcategory of $A - \mathbf{Mod}$. The dual representative of the monomorphism in $[\mathcal{D}, \mathcal{D}]$:

$$\mathrm{D}_{(\sigma(n),k)} \hookrightarrow \mathrm{D}^{\bullet}_{\sigma(n)} \circ \mathrm{Diff}_k^{(+)}, \quad \sigma(n) \in \mathbf{N}_+^n, \ k \in \mathbf{N}_+,$$

is a $\mathcal{D}$-epimorphism:

$$\mathbf{J}_{\mathcal{D}}^k\big(\Lambda_{\mathcal{D}}^{\sigma(n)}\big) \to \Lambda_{\mathcal{D}}^{(\sigma(n),k)};$$

define $d^{\mathcal{D}}_{(\sigma(n),k)}$ to be the composition

$$\Lambda_{\mathcal{D}}^{\sigma(n)} \xrightarrow{j_k} \mathbf{J}_{\mathcal{D}}^k\big(\Lambda_{\mathcal{D}}^{\sigma(n)}\big) \to \Lambda_{\mathcal{D}}^{(\sigma(n),k)}. \tag{4.1}$$

Obviously, $d^{\mathcal{D}}_{(\sigma(n),k)}$ is a DO of order $\leqslant k$.

Definition 4.1. If $\sigma \in \mathbf{N}^{\infty}_{+}$, the sequence in $\mathbf{DIFF}_A$

$$0 \to A \xrightarrow{d^{\mathcal{D}}_{(\sigma_1)}} \Lambda^{(\sigma_1)}_{\mathcal{D}} \xrightarrow{d^{\mathcal{D}}_{(\sigma_1,\sigma_2)}} \Lambda^{\sigma(2)}_{\mathcal{D}} \to \cdots \xrightarrow{d^{\mathcal{D}}_{\sigma(k)}} \Lambda^{\sigma(k)}_{\mathcal{D}} \to \cdots \tag{4.2}$$

is called *higher de Rham sequence* of type σ of the K -algebra A and is denoted by $\mathbf{dR}^{\mathcal{D}}_{\sigma}(A)$ or simply by $\mathbf{dR}^{\mathcal{D}}_{\sigma}$; each $d^{\mathcal{D}}_{\sigma(k)}$, $k > 0$, is called *higher de Rham differential* and is a DO of order $\leqslant \sigma_k$.

Remark 4.2. When $\sigma = \mathbf{1} \in \mathbf{N}^{\infty}_{+}$, the corresponding de Rham sequence is called *ordinary*. In this case we write $\Lambda^{k}_{\mathcal{D}}$ for $\Lambda^{(1,\dots,1)}_{\mathcal{D}}$, $(1,\dots,1) \in \mathbf{N}^{k}_{+}$, $\forall k > 0$, so that:

$$\mathbf{dR}^{\mathcal{D}}_{(\underline{1})} \equiv \mathbf{dR}^{\mathcal{D}}\colon 0 \to A \xrightarrow{d} \Lambda^{1}_{\mathcal{D}} \xrightarrow{d} \Lambda^{2}_{\mathcal{D}} \to \cdots \xrightarrow{d} \Lambda^{k}_{\mathcal{D}} \to \cdots \tag{4.3}$$

and each differential is a DO of order $\leqslant 1$.

Each $d^{\mathcal{D}}$ in (4.2) is in fact a differential according to the following:

Proposition 4.3. *$\forall \sigma \in \mathbf{N}^{\infty}_{+}$ the higher de Rham sequence $\mathbf{dR}^{\mathcal{D}}_{\sigma}$ is a complex.*

Proof. Let $n \geqslant 0$ and consider the diagram defining two consecutive higher de Rham differentials:

$$\begin{array}{ccccc} \Lambda^{\sigma(n)}_{\mathcal{D}} & \xrightarrow{\doteq d^{\mathcal{D}}_{\sigma(n+1)}} & \Lambda^{\sigma(n+1)}_{\mathcal{D}} & \xrightarrow{\doteq d^{\mathcal{D}}_{\sigma(n+2)}} & \Lambda^{\sigma(n+2)}_{\mathcal{D}} . \\ {\scriptstyle j_{\sigma_{n+1}}}\searrow & \nearrow{\scriptstyle \pi_1} & {\scriptstyle j_{\sigma_{n+2}}}\searrow & \nearrow{\scriptstyle \pi_2} & \\ & \mathbf{J}^{\sigma_{n+1}}_{\mathcal{D}}(\Lambda^{\sigma(n)}_{\mathcal{D}}) & & \mathbf{J}^{\sigma_{n+2}}_{\mathcal{D}}(\Lambda^{\sigma(n+1)}_{\mathcal{D}}) & \end{array}$$

Since $d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)} \equiv \pi_2 \circ j_{\sigma_{n+2}} \circ \pi_1 \circ j_{\sigma_{n+1}}$ is a DO of order $\leqslant k + l$, there exists a unique A-homomorphism

$$\varphi_{d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)}} : \mathbf{J}^{\sigma_{n+2}+\sigma_{n+1}}_{\mathcal{D}}\left(\Lambda^{\sigma(n)}_{\mathcal{D}}\right) \to \Lambda^{\sigma(n+2)}_{\mathcal{D}}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \Lambda^{\sigma(n)}_{\mathcal{D}} & \xrightarrow{d_{\sigma(n+2)} \circ d_{\sigma(n+2)}} & \Lambda^{\sigma(n+2)}_{\mathcal{D}} \\ & {\scriptstyle j_{\sigma_{n+2}+\sigma_{n+1}}}\searrow & \uparrow{\scriptstyle \varphi^{d_{\sigma(n+2)} \circ d_{\sigma(n+2)}}} \\ & & \mathbf{J}^{\sigma_{n+2}+\sigma_{n+1}}_{\mathcal{D}}(\Lambda^{\sigma(n)}_{\mathcal{D}}) \end{array}$$

It is not difficult to check that $\varphi_{d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)}}$ is just the dual representative of the composition:

$$\mathrm{D}_{\sigma(n+2)} \hookrightarrow \mathrm{D}^{\bullet}_{\sigma(n+1)} \circ \mathrm{Diff}^{(+)}_{\sigma_{n+2}} \to \mathrm{D}^{\bullet}_{\sigma(n)} \circ \mathrm{Diff}^{(+)}_{\sigma_{n+1}+\sigma_{n+2}}$$

which is immediately checked to be zero; therefore $\varphi_{d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)}} = 0$ and $d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)} = 0$ as well. $\square$

Remark 4.4. (i) Let $\mathcal{D} = A - \mathbf{Mod}$. By induction on n we can prove (using for $n = 1$ the explicit description of $\Lambda^{(\sigma_1)}$ given in Section 2) that formula (4.2) implies that $\Lambda^{\sigma(n)}$ is generated by the set

$$\{d_{\sigma(n)}(a_1 d_{\sigma(n-1)}(a_2 \ldots d_{\sigma(1)}(a_n)\ldots)) \mid a_1, a_2, \ldots, a_n \in A\}.$$

In fact, $\mathbf{J}^{\sigma_n}(\Lambda^{\sigma(n-1)})$ is known to be generated over A by the elements $j_{\sigma_n}(\omega)$, $\omega \in \Lambda^{\sigma(n-1)}$ and $\mathbf{J}^{\sigma_n}(\Lambda^{\sigma(n-1)}) \to \Lambda^{\sigma(n)}$ is an epimorphism. This result still holds for $\Lambda_{\mathcal{D}}^{\sigma(n)}$ with d replaced by $d^{\mathcal{D}}$, $\mathcal{D} \subseteq A - \mathbf{Mod}$ being any differentiable closed subcategory: the proof is analogous to the argument used in Remark 3.12(a). This also shows that the canonical morphisms $\Lambda^{\sigma(n)} \to \Lambda_{\mathcal{D}}^{\sigma(n)}$ are A-epimorphisms.

(ii) In the "ordinary" case $(\sigma_1, \ldots, \sigma_n) = (1, \ldots, 1) \equiv \mathbf{1}(n)$, the A-module structure of $\mathrm{Hol}_{\mathcal{D},+}^{\mathbf{1}_n}$ (Remark 3.10) can be expressed via the isomorphism (Remark 2.7)

$$\mathrm{Hol}_{\mathcal{D}}^{(\sigma_1,\ldots,\sigma_n)} \simeq \Lambda_{\mathcal{D}}^{(\sigma_2,\ldots,\sigma_n)} \oplus \Lambda_{\mathcal{D}}^{(\sigma_1,\ldots,\sigma_n)}$$

as $a^+(\rho, \omega) = (a\rho, a\omega + (d_{(1)}a) \wedge \rho)$.

(iii) If $\mathcal{D} = A - \mathbf{Mod}$, $\mathbf{dR_1}$ coincides with the usual algebraic de Rham complex of the K-algebra A ([1] and [2]).

(iv) If $K = \mathbb{R}$, M is a smooth manifold, $A = C^\infty(M;\mathbb{R})$ and $\mathcal{D}$ is the category of geometric A-modules (see Section 2) then $\mathbf{dR}_1^{\mathcal{D}}$ is the geometric de Rham complex on M. It turns out that any natural differential operator occurring in differential geometry can be recovered functorially using our approach: see [14] for the case of the Lie derivative and the corresponding homotopy formula.

If $\sigma, \tau \in \mathbf{N}_+^\infty$ with $\sigma \geqslant \tau$ (i.e., $\sigma_i \geqslant \tau_i$, $\forall i \geqslant 1$), then for each $n > 0$ we have a monomorphism $\mathrm{D}_{\tau(n)} \hookrightarrow \mathrm{D}_{\sigma(n)}$ in $[\mathcal{D}, \mathcal{D}]$; this induces a $\mathcal{D}$-epimorphism on representatives $\Lambda_{\mathcal{D}}^{\sigma(n)} \to \Lambda_{\mathcal{D}}^{\tau(n)}$, $\forall n > 0$. By Remark 4.4, this is also an A-epimorphism. All these epimorphisms commute with higher de Rham differentials and therefore define a morphism of complexes

$$\mathbf{dR}_\sigma^{\mathcal{D}} \to \mathbf{dR}_\tau^{\mathcal{D}} \tag{4.4}$$

(if $\sigma \geqslant \tau$). So we can consider the (A-epimorphic) inverse system $\{\mathbf{dR}_\sigma^{\mathcal{D}}\}_{\sigma \in \mathbf{N}_+^\infty}$ and give the following:

Definition 4.5. The *infinitely prolonged* (or, simply, *infinite*) *de Rham complex* of the K-algebra A, is the complex in $\mathbf{K}$ ($K - \mathbf{Mod}$)

$$\mathbf{dR}_\infty^{\mathcal{D}}(A) \doteq \varprojlim_{\sigma \in \mathbf{N}_+^\infty} \mathbf{dR}_\sigma^{\mathcal{D}}(A),$$

$$\mathbf{dR}_\infty^{\mathcal{D}}(A)\colon 0 \to A \overset{d_{(\infty)}}{\to} \Lambda_{\mathcal{D}}^{(\infty)} \overset{d_{(\infty,\infty)}}{\to} \Lambda_{\mathcal{D}}^{(\infty,\infty)_2} \to \cdots \to \Lambda_{\mathcal{D}}^{(\infty,\ldots,\infty)_n} \to \cdots, \tag{4.5}$$

where $\Lambda_{\mathcal{D}}^{(\infty,\ldots,\infty)_n} \doteq \varprojlim_{\sigma(n) \in \mathbf{N}_+^n} \Lambda_{\mathcal{D}}^{\sigma(n)}$, $\forall n > 0$.

Remark 4.6 (*Two descriptions of DO's between strict representative objects*). We work in a fixed differentially closed subcategory $\mathcal{D}$ of $A - \mathbf{Mod}$ and all representative objects will be in $\mathcal{D}$.

Let F_1 and F_2 be representative objects of differential functors $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. As seen in the previous section there are many instances of differential functors $\mathcal{F}_1$ having a so to say "associated functor" $\mathcal{F}_1^\bullet$ with domain $A - \mathbf{BiMod}_{\mathcal{D}}$ such that $\mathcal{F}_1^\bullet(\mathrm{Diff}_k^{(+)})$ is strictly representable by $\mathbf{J}^k(F_1)$: the easiest examples are $\mathcal{F}_1 = \mathrm{D}_{\sigma(n)}$ or Diff_l. Now, let

$$\Delta\colon F_1 \to F_2 \tag{4.6}$$

be a DO of order $\leqslant k$. Then, there exists a unique A-homomorphism ([9]: jet-associated to Δ)

$$f_\Delta : \mathbf{J}^k(F_1) \to F_2 \tag{4.7}$$

which represents Δ by duality: $\Delta = f_\Delta \circ j_k(F_1)$. Since $\mathbf{J}^k(F_1)$ is the representative object of $\mathcal{F}_1^\bullet(\mathrm{Diff}_k^{(+)})$, f_Δ defines a unique morphism in $[\mathcal{D}, \mathcal{D}]$:

$$f^\Delta : \mathcal{F}_2 \to \mathcal{F}_1^\bullet\big(\mathrm{Diff}_k^{(+)}\big), \tag{4.8}$$

called *generator morphism* of Δ.

Formulas (4.7) and (4.8) give two different descriptions of a DO between representative objects. Formula (4.8) allows one to identify it with a functorial morphism which, as a rule, may be established in a straightforward way and can then be used to define the corresponding natural DO (4.6). The following examples show this procedure at work in two canonical cases; we assume for simplicity $\mathcal{D} = A - \mathbf{Mod}$.

(i) *Higher de Rham differential* $d_{\sigma(n)}$.

If $\mathcal{F}_2 = \mathrm{D}_{\sigma(n)}$, $\mathcal{F}_1 = \mathrm{D}_{\sigma(n-1)}$, $k = \sigma_n$ and we take for (4.8) the natural inclusion

$$\mathrm{D}_{\sigma(n)} \hookrightarrow \mathrm{D}^\bullet_{\sigma(n-1)}\big(\mathrm{Diff}_{\sigma_n}^{(+)}\big),$$

then $d_{\sigma(n)} : \Lambda^{\sigma(n-1)} \to \Lambda^{\sigma(n)}$ is the corresponding DO (4.6).

(ii) *"Absolute" jet-operator* j_k.

In this almost tautological case, $\mathcal{F}_1 = Hom_A(A, \cdot) \equiv \mathrm{Diff}_0$ and $\mathcal{F}_2 = \mathrm{Diff}_k \equiv Hom_A^\bullet(A, \cdot)\ (\mathrm{Diff}_k^{(+)})$; if we take (4.8) to be the identity

$$\mathrm{Id} : \mathrm{Diff}_k \to Hom_A^\bullet(A, \cdot)\big(\mathrm{Diff}_k^{(+)}\big) \equiv \mathrm{Diff}_k,$$

then (4.6) is just $j_k : A \to \mathbf{J}^k$.

5. Smooth rigidity of higher de Rham cohomology

In this section we prove the main result of this paper, i.e., that in the smooth case the higher-order de Rham cohomologies coincide with the ordinary (i.e., lowest order) one. Essentially this amounts to a fairly intuitive assert: raising the order of the natural DOs involved in the **dR** -complexes does not change the cohomological information, provided the situation in which we are working is smooth.

In this section (and in Appendix A), A is a K-algebra of zero characteristic, containing K as a subring and $\mathcal{D}$ a differentially closed *smooth* subcategory of $A - \mathbf{Mod}$. As in the previous section, all representative objects, unless otherwise stated, will be considered in $\mathcal{D}$.

Smoothness of $\mathcal{D}$ implies that for any $k, l \geqslant 0$, the gluing morphism in $[\mathcal{D}, \mathcal{D}]$

$$\mathrm{Diff}_k^\bullet \circ \mathrm{Diff}_l^{(+)} \xrightarrow{c_{k,l}} \mathrm{Diff}_{k+l}$$

is surjective, i.e., that any DO can be expressed as composition of lower order ones. This can be seen as follows. Let us fix k and proceed by induction on l. The case $l = 0$ is trivial since $\mathrm{Diff}_0 = \mathrm{id}_\mathcal{D}$. To prove the inductive step let us consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}_k^\bullet \circ \mathrm{Diff}_{l+1}^{(+)} & \xrightarrow{c_{k,l+1}} & \mathrm{Diff}_{k+l+1} \\ \uparrow & & \uparrow \\ \mathrm{Diff}_k^\bullet \circ \mathrm{Diff}_l^{(+)} & \xrightarrow[c_{k,l}]{} & \mathrm{Diff}_{k+l} \end{array}$$

(where the vertical arrows are natural inclusions) and suppose $C_{k,l}$ is epic. Passing to the corresponding diagram of representative objects and completing it with kernels, we get a commutative diagram [10, p. 52] with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{S}^{k+l+1}(\Lambda^1) & \xrightarrow{\rho} & \mathrm{S}^{l+1}(\Lambda^1) \otimes \mathbf{J}^k \\
\downarrow & & \downarrow \\
\mathbf{J}^{k+l+1} & \xrightarrow{C^{k,l+1}} & \mathbf{J}^{l+1}(\mathbf{J}^k) \\
\downarrow & & \downarrow \\
\mathbf{J}^{k+l} & \xrightarrow[C^{k,l}]{} & \mathbf{J}^{l}(\mathbf{J}^k) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where S^r denotes the rth symmetric power and $C^{s,t}$ is the dual-representative of $C_{s,t}$. By duality it is enough to prove that $C^{k,l+1}$ is monic. By induction hypothesis $C^{k,l}$ is monic so we are reduced to showing that ρ is monic. It is not difficult to prove (e.g., again by induction on l) that ρ is just the composition

$$\mathrm{S}^{k+l+1}(\Lambda^1) \xrightarrow{\alpha} \mathrm{S}^{l+1}(\Lambda^1) \otimes \mathrm{S}^k(\Lambda^1) \xrightarrow{\beta} \mathrm{S}^{l+1}(\Lambda^1) \otimes \mathbf{J}^k,$$

where $\alpha : \omega_1 \cdots \omega_{k+l+1} \mapsto \sum(\omega_{i_1} \cdots \omega_{i_{l+1}}) \otimes (\omega_{j_1} \cdots \omega_{j_k})$ where the sum is extended to all partitions $((i_1, \ldots, i_{l+1}), (j_1, \ldots, j_k))$ of $\{1, \ldots, k+l+1\}$ of (ordered) length $(l+1, k)$ and $\beta = id_{\mathrm{S}^{l+1}(\Lambda^1)} \otimes i$ with

$$i : \mathrm{S}^k(\Lambda^1) \hookrightarrow \mathbf{J}^k$$

the inclusion of the kernel of $\mathbf{J}^k \to \mathbf{J}^{k-1}$ [10, p. 52]. A is of zero characteristic hence α is well defined and monic; Λ^1 is projective hence β is monic too. Thus ρ is monic and we conclude.

As a consequence $\forall n \geqslant 1$, we have the following short exact sequence in $[\mathcal{D}, \mathcal{D}]$:

$$0 \to \mathrm{D}_{\sigma(n)} \hookrightarrow \mathrm{D}^{\bullet}_{\sigma(n-1)} \circ \mathrm{Diff}^{(+)}_{\sigma_n} \to \mathrm{D}_{(\sigma_1, \ldots, \sigma_{n-2}, \sigma_{n-1}+\sigma_n)} \to 0 \tag{5.1}$$

(the new fact is that the last arrow of the sequence is epic since it is induced by the gluing morphism).

The nth cohomology K-module of the complex

$$\mathbf{dR}_\sigma : 0 \to A \overset{d_{\sigma(1)}}{\to} \Lambda^{\sigma(1)} \to \cdots \to \Lambda^{\sigma(n)} \overset{d_{\sigma(n+1)}}{\to} \Lambda^{\sigma(n+1)} \to \cdots$$

is denoted by:

$$H^n_\sigma \doteq \frac{\ker(d_{\sigma(n+1)})}{im(d_{\sigma(n)})} \equiv H^n(\mathbf{dR}_\sigma).$$

Since H^n_σ only depends on $\sigma(n+1)$, we will write also $H^n_{\sigma(n+1)}$ in place of H^n_σ.

Note that in the situation of Remark 4.4(iv), $H^n_{\sigma(n+1)}$ is the nth de Rham cohomology $\mathbb{R}$-vector space of the smooth manifold M.

The rest of this section will be devoted to proving the following result:

Theorem 5.1 ("Smooth" rigidity of higher de Rham cohomologies). *If $\mathcal{D}$ is a smooth subcategory of $A-\mathbf{Mod}$, then, for each $\tau,\sigma \in \mathbf{N}_+^\infty$ with $\tau \geqslant \sigma$, the canonical $\mathcal{D}$-epimorphism* (4.4):

$$\mathbf{dR}_\tau \to \mathbf{dR}_\sigma$$

is a quasi-isomorphism; so:

$$H_\sigma^n \simeq H_\tau^n, \quad \forall n \geqslant 0. \tag{5.2}$$

Corollary 5.2. (i) *If M is a* smooth *manifold, $A = C^\infty(M;\mathbb{R})$ and $\mathcal{D} = C^\infty(M;\mathbb{R}) - \mathbf{Mod}_{\mathrm{geom}}$, then the higher de Rham cohomologies coincide with the standard de Rham cohomology of M.*

(ii) *If K is a field of zero characteristic and A is a* smooth *K-algebra, then the higher de Rham cohomologies coincide with the standard algebraic one.*

Note that the previous corollary is false, in general, for a singular manifold or a non-regular K-algebra A.

The strategy of the proof of Theorem 5.1 is the following.

Keeping $n \geqslant 0$ fixed, we prove the thesis by reducing, step by step, each entry of $\sigma(n+1)$ to 1, starting from σ_{n+1}, i.e., we prove the chain of isomorphisms

$$H^n_{\sigma(n+1)} \simeq H^n_{(\sigma(n),1)} \simeq H^n_{(\sigma(n-1),1,1)} \simeq \cdots \simeq H^n_{(\sigma_1,1,\ldots,1)} \simeq H^n_{dR}, \tag{5.3}$$

where H^n_{dR} stands for $H^n_{(1,\ldots,1)}$, $(1,\ldots,1) \in \mathbf{N}_+^{n+1}$ (the nth ordinary de Rham cohomology).

The first step in the chain (5.3) is obtained via the following:[4]

Lemma 5.3. *Let $n \in \mathbf{N}_+$. If $\sigma,\tau \in \mathbf{N}_+^\infty$ are such that $\sigma(n) = \tau(n)$, then:*

(i) $\ker d_{\sigma(n+1)} = \ker d_{\tau(n+1)}$;

(ii) $im(d_{\sigma(n+1)}) \simeq im(d_{\tau(n+1)})$ *(where $\simeq$ means $K-\mathbf{Mod}$-isomorphism).*

Proof. (ii) follows trivially from (i). Let $\sigma \in \mathbf{N}_+^n$ and $k > 1$. Consider the short exact sequence:

$$0 \to \mathrm{D}_{(\sigma,k-1,1)} \hookrightarrow \mathrm{D}^\bullet_{(\sigma,k-1)} \circ \mathrm{Diff}_1^{(+)} \xrightarrow{i} \mathrm{D}_{(\sigma,k)} \to 0$$

whose dual-representative:

$$0 \to \Lambda^{(\sigma,k)} \xrightarrow{i^\vee} \mathbf{J}^1\big(\Lambda^{(\sigma,k-1)}\big) \to \Lambda^{(\sigma,k-1,1)} \to 0 \tag{5.4}$$

is likewise exact (in $\mathcal{D}$). We embed the latter in the commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \Lambda^{(\sigma,k)} & \xrightarrow{\;i^\vee\;} & \mathbf{J}^1(\Lambda^{(\sigma,k-1)}) & \longrightarrow & \Lambda^{(\sigma,k-1,1)} & \longrightarrow 0 \\
& {\scriptstyle d_{(\sigma,k)}}\uparrow & & \uparrow{\scriptstyle j_1} & & & \\
& \Lambda^\sigma & \xrightarrow[d_{(\sigma,k-1)}]{} & \Lambda^{(\sigma,k-1)} & & &
\end{array}$$

Now, if $\omega \in \Lambda^\sigma$ is such that $d_{(\sigma,k-1)}(\omega) = 0$, then $j_1(d_{(\sigma,k-1)}(\omega)) = 0$ and, by commutativity, $(i^\vee \circ d_{(\sigma,k)})(\omega) = 0$. But $i^\vee$ is a monomorphism, so $\ker d_{(\sigma,k-1)} \subseteq \ker d_{(\sigma,k)}$, $\forall k > 1$. Since the inverse inclusion is obvious, (i) is proved. □

[4] This lemma has been proved, independently, also by Yu. Torkhov.

To prove the "kth step" of chain (5.3), it is enough to show that:

$$H^n_{(\sigma(k-1),\sigma_k+1,1,\dots,1)_{n+1}} \simeq H^n_{(\sigma(k-1),\sigma_k,1,\dots,1)_{n+1}} \tag{5.5}$$

where we write $(\rho)_r$ if $\rho \in \mathbf{N}^r_+$.

To prove (5.5) we construct an auxiliary complex.

Let P be an object in $\mathcal{D}$ and $\tau \in \mathbf{N}^\infty_+$. As shown at the beginning of this section, smoothness of $\mathcal{D}$ implies that $\forall n > 0$ the "relative" sequence (2.3)

$$0 \to \mathcal{P}^\bullet_{\tau(n)}[P] \to \mathcal{P}^\bullet_{\tau(n-1)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_n} \to \mathcal{P}^\bullet_{(\tau(n-2),\tau_{n-1}+\tau_n)}[P] \to 0 \tag{5.6}$$

is exact also in the last term; hence, its dual representative:

$$0 \to \mathrm{Hol}^{(\tau(n-2),\tau_{n-1}+\tau_n)}[P] \to \mathbf{J}^{\tau_n}\left(\mathrm{Hol}^{\tau(n-1)}[P]\right) \to \mathrm{Hol}^{\tau(n)}[P] \to 0 \tag{5.7}$$

is exact also in the first term. Furthermore, when P varies in $Ob(\mathcal{D})$, (5.7) gives rise to a short exact sequence in $[\mathcal{D},\mathcal{D}]$. We will refer to $\mathrm{Hol}^{\tau(n)}[P]$ as the Hol-*object of type* $\tau(n)$ of the A-module P; we have

$$\mathrm{Hol}^{\tau(n)}[P] \simeq \frac{\mathbf{J}^{\tau_n}(\mathrm{Hol}^{\tau(n-1)}[P])}{\mathrm{Hol}^{(\tau(n-2),\tau_{n-1}+\tau_n)}[P]} \tag{5.8}$$

as A-modules. This allows us to give the following:

Proposition 5.4. *Let $P \in Ob(\mathcal{D})$ and $\tau \in \mathbf{N}^\infty_+$. We define the sequence in $\mathbf{DIFF}_{A,\mathcal{D}}$:*

$$\mathbf{Hol}^\tau[P]: 0 \to \mathrm{Hol}^{\emptyset}[P] \doteq P \xrightarrow{\delta_{\tau(1)}(P)} \mathrm{Hol}^{\tau(1)}[P] \equiv \mathbf{J}^{\tau_1}(P) \to \cdots$$
$$\to \mathrm{Hol}^{\tau(n)}[P] \xrightarrow{\delta_{\tau(n+1)}(P)} \mathrm{Hol}^{\tau(n+1)}[P] \to \cdots,$$

where, for each $n \geqslant 0$,

$$\delta_{\tau(n+1)}[P]: \mathrm{Hol}^{\tau(n)}[P] \to \mathrm{Hol}^{\tau(n+1)}[P] \tag{5.9}$$

is defined to be the DO whose description (4.8) *of Remark* 4.6 *is the canonical inclusion*

$$\mathcal{P}^\bullet_{\tau(n+1)}[P] \hookrightarrow \mathcal{P}^\bullet_{\tau(n)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_{n+1}};$$

equivalently, $\delta_{\tau(n+1)}[P]$ is the composition:

$$\mathrm{Hol}^{\tau(n)}[P] \xrightarrow{j_{\tau_{n+1}}(\mathrm{Hol}^{\tau(n)}[P])} \mathbf{J}^{\tau_{n+1}}\left(\mathrm{Hol}^{\tau(n)}[P]\right) \xrightarrow{p_{\tau(n+1)}(P)} \frac{\mathbf{J}^{\tau_{n+1}}(\mathrm{Hol}^{\tau(n)}[P])}{\mathrm{Hol}^{(\tau(n-1),\tau_n+\tau_{n+1})}[P]} \simeq \mathrm{Hol}^{\tau(n+1)}[P]$$

where $p_{\tau(n+1)}(P)$ is the canonical quotient projection. $\mathbf{Hol}^\tau[P]$ *is a complex in* $\mathbf{DIFF}_{A,\mathcal{D}}$, *called* Hol^τ-complex of P; *moreover,* $\mathbf{Hol}^\tau[P]$ *is natural in P and defines a functor* $\mathbf{Hol}^\tau: \mathcal{D} \to \mathbf{K}\,(\mathbf{DIFF}_{A,\mathcal{D}})$.[5]

---

[5] We recall that $\mathbf{K}\,(\mathbf{DIFF}_{A,\mathcal{D}})$ denotes the category of complexes of differential operators formed by objects in $\mathcal{D}$.

Proof. As always, it is better to work with functors (i.e., differential operators) than with representative objects. In the notations of Remark 4.6, we have that $\varphi^{\delta_{\tau(n+1)}\circ\delta_{\tau(n)}}$[6] coincides with the composition:

$$\begin{array}{cc}
\mathcal{P}^{\bullet}_{\tau(n+1)}[P] & \\
\downarrow & \varphi^{\delta_{\tau(n+1)}} \\
\mathcal{P}^{\bullet}_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_{n+1}} & \\
\downarrow & \varphi^{\delta_{\tau(n)}}(\mathrm{Diff}^{+}_{\tau_{n+1}}) \\
\mathcal{P}^{\bullet}_{\tau(n-1)}[P]\circ (\mathrm{D}^{\bullet}_{(\tau_n,\tau_{n+1})}, \mathrm{D}^{+}_{(\tau_n,\tau_{n+1})}) & \\
\downarrow & \mathcal{P}^{\bullet}_{\tau(n-1)}[P](\mathrm{C}_{\tau_n,\tau_{n+1}}) \\
\mathcal{P}^{\bullet}_{\tau(n-1)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_{n+1}+\tau_n} &
\end{array}$$

where the first two arrows are monomorphisms and the last is the "gluing" morphism with respect to the indices (τ_n, τ_{n+1}). This composition is zero. In fact, if $Q \in Ob(\mathcal{D})$ and $\Delta \in [\mathcal{P}^{\bullet}_{\tau(n+1)}[P]](Q)$, then the image $\overline{\Delta}$ of Δ via this composition, is defined by:

$$\big(\overline{\Delta}(p)\big)(a_1)\dots(a_{n-1}) \doteq \big(\big(\Delta(p)\big)(a_1)\dots(a_{n-1})\big)(1),$$

and is zero because $(\Delta(p))(a_1)\dots(a_{n-1}) \in \mathrm{D}_{\tau_{n+1}}(Q)$, for each $p \in P$, $a_1,\dots,a_{n-1} \in A$. □

Now we show that if $\tau \in \mathbf{N}^{\infty}_{+}$ is *regular*, then, for any object P in $\mathcal{D}$, $\mathbf{Hol}^{\tau}[P]$ is *acyclic*. In order to do this, we will exhibit (functorially in P) a *trivializing homotopy*.

Define:

$$\varphi_{\emptyset}(P) : \mathcal{P}^{\bullet}_{\tau(-1)}[P] \doteq 0 \to \mathcal{P}^{\bullet}_{\tau(0)}[P] \doteq Hom_A(P,\cdot),$$
$$\varphi_{\tau(1)}(P) : \mathcal{P}^{\bullet}_{\tau(0)}[P] \doteq Hom_A(P,\cdot) \hookrightarrow \mathcal{P}^{\bullet}_{\tau(1)}[P] \equiv \mathrm{Diff}_{\tau_1}(P,\cdot)$$

which are morphisms in $[\mathcal{D},\mathcal{D}]$; then define, by induction on n, $\varphi_{\tau(n+1)}(P) \doteq i_{\tau(n+1)}(P) - \hat{\varphi}_{\tau(n)}(P)$, where

$$i_{\tau(n+1)}(P) : \mathcal{P}^{\bullet}_{\tau(n)}[P] \simeq \mathcal{P}^{\bullet}_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{0} \hookrightarrow \mathcal{P}^{\bullet}_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_{n+1}},$$

$$\hat{\varphi}_{\tau(n)}(P) : \mathcal{P}^{\bullet}_{\tau(n)}[P] \hookrightarrow \mathcal{P}^{\bullet}_{\tau(n-1)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_n} \xrightarrow{\varphi_{\tau(n+1)}(P)(\mathrm{Diff}^{+}_{\tau_n})} \mathcal{P}^{\bullet}_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_n} \hookrightarrow \mathcal{P}^{\bullet}_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_{n+1}}.$$

With this definition, $\varphi_{\tau(n+1)}(P) : \mathcal{P}^{\bullet}_{\tau(n)}[P] \to \mathcal{P}^{\bullet}_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_{n+1}}$, but it is easy to resolve the inductive definition in the following one:

$$\begin{aligned}
\big\{[\varphi_{\tau(n+1)}(P)(Q)](\Delta)\big\}(p)(a_1)\dots(a_n) &= a_n\Delta(p)(a_1)\dots(a_{n-1}) \\
&\quad + \sum_{k=1}^{n-1}(-1)^{n-k}\Delta(p)(a_1)\dots(a_k a_{k+1})\dots(a_n) \\
&\quad + (-1)^n\Delta(a_1 p)(a_2)\dots(a_n)
\end{aligned} \tag{5.10}$$

($Q \in Ob(\mathcal{D})$, $p \in P$, $a_1,\dots,a_n \in A$ and $\Delta \in \mathcal{P}^{\bullet}_{\tau(n)}[P](Q)$). This shows that actually

$$\varphi_{\tau(n+1)}(P) : \mathcal{P}^{\bullet}_{\tau(n)}[P] \to \mathcal{P}^{\bullet}_{\tau(n+1)}[P].$$

---

[6] We write shortly $\delta_{\tau(k)}$ instead of $\delta_{\tau(k)}[P]$, for any $k \geqslant 0$.

Therefore, we get a family $\{\varphi_{\tau(n)}(P): \mathcal{P}^{\bullet}_{\tau(n-1)}[P] \to \mathcal{P}^{\bullet}_{\tau(n)}[P]\}_{n>0}$ of morphisms in $[\mathcal{D}, \mathcal{D}]$. Of course, formula (5.10) can equally be taken as the definition of the family $\{\varphi_{\tau(n)}(P)\}_{n>0}$ but the inductive definition can be "dualized", to representative objects, to give the following (keeping the notations of Proposition 5.4):

$$\begin{aligned}
&\varphi^{\emptyset}(P): \mathrm{Hol}^{\tau(0)}[P] \doteq P \to \mathrm{Hol}^{\tau(-1)}[P] \doteq 0,\\
&\varphi^{\tau(1)}(P): \mathrm{Hol}^{\tau(1)}(P) \doteq \mathbf{J}^{\tau_1}(P) \to \mathrm{Hol}^{\tau(0)}[P] \doteq P \quad \text{(natural projection)},\\
&\varphi^{\tau(n+1)}(P): \mathrm{Hol}^{\tau(n+1)}[P] \to \mathrm{Hol}^{\tau(n)}[P],
\end{aligned}$$

where $\varphi^{\tau(n+1)}(P)$ is the only $\mathcal{D}$-morphism corresponding to the DO of order $\leqslant \tau_{n+1}$

$$\Delta_{\tau(n)} \doteq \mathrm{id}_{\mathrm{Hol}^{\tau(n)}[P]} - \delta_{\tau(n)}(P) \circ \varphi^{\tau(n)}(P): \mathrm{Hol}^{\tau(n)}[P] \to \mathrm{Hol}^{\tau(n)}[P].$$

As before, but dually,[7] this definition gives apparently a $\mathcal{D}$-morphism:

$$\hat{\varphi}^{\tau(n+1)}(P): \mathbf{J}^{\tau_{n+1}}\big(\mathrm{Hol}^{\tau(n)}[P]\big) \to \mathrm{Hol}^{\tau(n)}[P]$$

(τ being regular) but formula (5.10) shows that actually $\ker(\hat{\varphi}^{\tau(n+1)}(P)) \supseteq \mathrm{Hol}^{(\tau(n-1), \tau_n + \tau_{n+1})}[P]$, so that, by formula (5.8), $\hat{\varphi}^{\tau(n+1)}(P)$ induces, by passing to the quotient, the morphism $\varphi^{\tau(n+1)}(P)$ we wanted.

Now we have a family of $\mathcal{D}$-morphisms $\{\varphi^{\tau(n)}(P): \mathrm{Hol}^{\tau(n)}[P] \to \mathrm{Hol}^{\tau(n-1)}[P]\}_{n\geqslant 0}$ dual to $\{\varphi_{\tau(n)}(P): \mathcal{P}^{\bullet}_{\tau(n)}[P] \to \mathcal{P}^{\bullet}_{\tau(n-1)}[P]\}_{n\geqslant 0}$.

Proposition 5.5. *For each object P in $\mathcal{D}$ and for each regular $\tau \in \mathbf{N}^{\infty}_{+}$, $\{\varphi^{\tau(n)}(P)\}_{n\geqslant 0}$ is a trivializing homotopy for $\mathbf{Hol}^{\tau}(P)$. Furthermore, $\{\varphi^{\tau(n)}(P)\}_{n\geqslant 0}$ is natural in P.*

Proof. We must show that the *sum* $L + R$ of the two compositions:

$$\begin{aligned}
&L: \mathcal{P}^{\bullet}_{\tau(n)}[P] \hookrightarrow \mathcal{P}^{\bullet}_{\tau(n-1)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_n} \xrightarrow{\varphi_{\tau(n)}(P)(\mathrm{Diff}^{+}_{\tau_n})} \mathcal{P}^{\bullet}_{\tau(n)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_n} \hookrightarrow \mathcal{P}^{\bullet}_{\tau(n)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_{n+1}},\\
&R: \mathcal{P}^{\bullet}_{\tau(n)}[P] \xrightarrow{\varphi_{\tau(n+1)}(P)} \mathcal{P}^{\bullet}_{\tau(n+1)}[P] \hookrightarrow \mathcal{P}^{\bullet}_{\tau(n)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_{n+1}}
\end{aligned}$$

equals $\mathrm{id}_{\mathcal{P}^{\bullet}_{\tau(n)}[P]}$ (which is then homotopic to the zero map) or, equivalently, that the diagram:

$$\begin{array}{ccc}
\mathcal{P}^{\bullet}_{\tau(n)}[P] & \xrightarrow{L+R} & \mathcal{P}^{\bullet}_{\tau(n)}[P] \circ \mathrm{Diff}^{(+)}_{\tau_{n+1}}\\
\mathrm{id} & \searrow & \cup\\
 & & \mathcal{P}^{\bullet}_{\tau(n)}[P] \circ \mathrm{Diff}^{(+)}_{0}
\end{array} \tag{5.11}$$

is commutative. For $Q \in Ob(\mathcal{D})$ and $\Delta \in \mathcal{P}^{\bullet}_{\tau(n)}[P](Q)$, we have by (5.10):

$$\begin{aligned}
&L(\Delta)(p)(a_1)\cdots(a_n)\\
&\quad = \Bigg[a^{+}_{n-1}\Delta(p)(a_1)\ldots(a_{n-2}) + \sum_{s=1}^{n-2}(-1)^{n-1-s}\Delta(p)(a_1)\ldots(a_s a_{s+1})\ldots(a_{n-1})\\
&\qquad + (-1)^{n-1}\Delta(a_1 p)(a_2)\ldots(a_{n-1})\Bigg](a_n)
\end{aligned}$$

[7] Subfunctors of strictly representable functors correspond to quotient objects of the representatives.

$$= -\sum_{s=1}^{n-1}(-1)^{n-s}\Delta(p)(a_1)\dots(a_s a_{s+1})\dots(a_n) + +(-1)^{n-1}\Delta(a_1 p)(a_2)\dots(a_n)$$

while

$$R(\Delta)(p)(a_1)\dots(a_n) = a_n\Delta(p)(a_1)\dots(a_{n-1}) + \sum_{k=1}^{n-1}(-1)^{n-k}\Delta(p)(a_1)\dots(a_k a_{k+1})\dots(a_n)$$
$$+ (-1)^n\Delta(a_1 p)(a_2)\dots(a_n)$$

so that $(L+R)(\Delta)(p)(a_1)\dots(a_n) = a_n\Delta(p)(a_1)\dots(a_{n-1})$, i.e., $(L+R)(\Delta)$ coincides with the image of Δ via the inclusion

$$\mathcal{P}^\bullet_{\tau(n)}[P](Q) \simeq \mathcal{P}^\bullet_{\tau(n)}[P]\circ \mathrm{Diff}_0^{(+)}(Q) \hookrightarrow \mathcal{P}^\bullet_{\tau(n)}[P]\circ \mathrm{Diff}^{(+)}_{\tau_{n+1}}(Q). \qquad \square$$

We now use acyclicity of the $\mathbf{Hol^1}$-complex, $\mathbf{1} = (1,\dots,1,1,\dots,1,\dots) \in \mathbf{N}_+^\infty$, to prove the "$k$th step", i.e., formula (5.5). Let $\sigma = (\sigma_1,\dots,\sigma_k+1) \in \mathbf{N}_+^k$, $(\sigma,\mathbf{1}) = (\sigma_1,\dots,\sigma_k+1,1,1,\dots,1,\dots) \in \mathbf{N}_+^\infty$ and

$$\mathbf{K}^{(k)}_{(\sigma,\mathbf{1})} \doteq \ker(\mathbf{dR}_{(\sigma,\mathbf{1})} \to \mathbf{dR}_{(\sigma_1,\dots,\sigma_k,1,1,\dots,1,\dots)}).$$

For each $(\mu)_s = (\mu_1,\dots,\mu_s) \in \mathbf{N}_+^s$, $1 \leqslant r \leqslant s$, $r,s \in \mathbf{N}_+$, we put:

$$K^{(r)}_{(\mu)_s} \doteq \ker\big(\Lambda^{(\mu_1,\dots,\mu_r,\dots,\mu_s)} \to \Lambda^{(\mu_1,\dots,\mu_r-1,\dots,\mu_s)}\big).$$

To prove the "kth step" it is enough to show acyclicity of $\mathbf{K}^{(k)}_{(\sigma,\mathbf{1})}$. We claim that there exists a resolution of $\mathbf{K}^{(k)}_{(\sigma,\mathbf{1})}$ of the form:

$$\cdots \to \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\big][-k-l] \overset{\psi_l[-k+l]}{\longrightarrow} \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\big][-k-l+1] \to \cdots$$
$$\overset{\psi_2[-(k+2)]}{\longrightarrow} \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+2)}\big][-k-1] \overset{\psi_1[-k+1]}{\longrightarrow} \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}\big][-k] \overset{\rho}{\to} \mathbf{K}^{(k)}_{(\sigma,\mathbf{1})} \to 0 \tag{5.12}$$

where if $r \in \mathbf{N}_+$, $(\cdot)[r]$ denotes, as usual, the r-shift both for complexes and morphisms of complexes. We postpone in Appendix A the definition of the maps of complexes

$$\psi_l : \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\big] \to \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\big][1], \quad l \in \mathbf{N}_+,$$
$$\rho : \mathbf{Hol^1}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}\big][-k] \to \mathbf{K}^{(k)}_{(\sigma,\mathbf{1})}$$

and the proof that (5.12) is actually a resolution.

Assuming the existence of resolution (5.12), the acyclicity of $\mathbf{K}^{(k)}_{(\sigma,\mathbf{1})}$ is then an immediate consequence of acyclicity of $\mathbf{Hol^1}$-complexes together with the following elementary fact

Lemma 5.6. *Let $C^\cdot$, $P_i^\cdot$, $i > 0$, be cochain complexes in $A-\mathbf{Mod}$ and*

$$\cdots \to P_n^\cdot \to P_{n-1}^\cdot \to \cdots \to P_1^\cdot \to C^\cdot \to 0$$

be a resolution of $C^\cdot$. Suppose that $\forall i \geqslant 1$, $P_i^k = (0)$ $\forall k < 0$ (so that $C^k = (0)$ $\forall k < 0$ too). If each $P_i^\cdot$ is acyclic then so is $C^\cdot$.

Proof. It follows from the hypotheses that $C^{\cdot}$ is isomorphic in the derived category $D^+(A-\mathbf{Mod})$ to the total complex associated to the double complex induced by

$$\cdots \to P_n^{\cdot} \to P_{n-1}^{\cdot} \to \cdots \to P_1^{\cdot}$$

which is acyclic. □

Corollary 5.7. *Let $\mathcal{D} \subseteq A-\mathbf{Mod}$ be a differentially closed smooth subcategory. If we define the stable infinite de Rham complex to be*

$$\mathbf{dR}_\infty^{\mathrm{st}} \doteq \varprojlim_{k>0} \mathbf{dR}_{\mathbf{k}}$$

(where $\mathbf{k} \doteq (k,\dots,k,\dots,k,\dots)$) then the canonical morphism $\mathbf{dR}_\infty^{\mathrm{st}} \to \mathbf{dR}_1$ is a quasi-isomorphism.

Proof. We use the following facts:

(i) the index category of the inverse system which defines $\mathbf{dR}_\infty^{\mathrm{st}}$ is countable;

(ii) the canonical $\mathcal{D}$-morphisms $\mathbf{dR}_{\underline{k'}} \to \mathbf{dR}_{\underline{k}}$, $k' \geqslant k$, in the inverse system are epimorphisms.

If we denote by $\varprojlim_k^1$ the first right derived functor of $\varprojlim_k$, (i) and (ii) imply, via standard spectral sequence's arguments (e.g., [11, Corollary 1.1, p. 535]), that there is a short exact sequence

$$0 \to \varprojlim_{k>0}^1\big[H^{n-1}(\mathbf{dR_K})\big] \to H^n\big(\mathbf{dR}_\infty^{\mathrm{st}}\big) \to \varprojlim_{k>0}\big[H^n(\mathbf{dR_k})\big] \to 0.$$

By Theorem 5.1, the term on the right is isomorphic to H^n_{dR}, so we are left to prove that $\varprojlim_{k>0}[H^{n-1}(\mathbf{dR}_{\underline{k}})] = (0)$. But $\varprojlim_{k>0}^1$ is right exact and

$$H^{n-1}(\mathbf{dR_k}) \equiv H^{n-1}_{\mathbf{k}} \cong H^{n-1}_{dR} \doteq H^{n-1}_{\mathbf{1}}, \quad \forall n \geqslant 1,$$

by Theorem 5.1, therefore it will be enough to prove the vanishing of $\varprojlim_{k>0}^1$ for the constant inverse system

$$\cdots \to H^{n-1}_{dR} \overset{id}{\to} H^{n-1}_{dR} \overset{id}{\to} H^{n-1}_{dR} \to \cdots.$$

But this is an easy consequence of the following description (due to Eilenberg) of $\varprojlim_{k>0}^1$ for constant systems.

If we define

$$D_0 : \prod_{k\in\mathbf{N}_+} H^{n-1}_{dR} \to \prod_{k\in\mathbf{N}_+} H^{n-1}_{dR} : (\alpha_k)_{k\in\mathbf{N}_+} \mapsto (\alpha_{k+1}-\alpha_k)_{k\in\mathbf{N}_+};$$

then $\mathrm{coker}(D_0) = \varprojlim_{k>0}^1[H^{n-1}_{dR}]$. Let $(\omega_k)_{k>0} \in \prod_{k\in\mathbf{N}_+} H^{n-1}_{dR}$ and define $(\overline{\omega}_k)_{k>0}$ as $\overline{\omega}_k \doteq \sum_{l=1}^{k-1}\omega_l$; then

$$D_0\big((\overline{\omega}_k)_{k>0}\big) = (\overline{\omega}_{k+1}-\overline{\omega}_k)_{k>0} = (\omega_k)_{k>0}.$$

Therefore D_0 is surjective and we conclude. □

Corollary 5.8. *Let $\mathcal{D} \subseteq A-\mathbf{Mod}$ be a differentially closed smooth subcategory such that $\forall\sigma \in \mathbf{N}_+^\infty$, $\exists n_\sigma \in \mathbf{N}_+$ such that $\Lambda^{\sigma(r)} = (0)$ for any $r > n_\sigma$. Then the canonical morphism (Definition 4.5) $\mathbf{dR}_\infty \to \mathbf{dR}_\sigma$ is a quasi-isomorphism $\forall\sigma \in \mathbf{N}_+^\infty$.*

Proof. Under our hypotheses

$$\widehat{\mathbf{N}}_+ \doteq \bigsqcup_{k\in\mathbf{N}_+} \{\mathbf{k}\in\mathbf{N}_+^\infty \mid \mathbf{k}\,(n)\equiv(k,\ldots,k)\in\mathbf{N}_+^n,\ \forall n\in\mathbf{N}_+\}$$

is cofinal in the index category of the system $\{\mathbf{dR}_\sigma\}$; hence $\mathbf{dR}_\infty \simeq \underleftarrow{\lim}_{k>0}\mathbf{dR}_\mathbf{K}$ and the thesis follows from Corollary 5.7. □

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Appendix A

This appendix is devoted to defining the maps in the sequence (5.12) and to showing that (5.12) is exact. There are two kinds of maps of complexes to be defined:

$$\rho\equiv\left(\rho^n:\mathbf{Hol}^{\mathbf{1}_{n-k}}\left[K^{(k)}_{(\sigma_1,\ldots,\sigma_k+1)}\right]\to K^{(k)}_{(\sigma_1,\ldots,\sigma_k+1,1,\ldots,1)_n}\right)_{n\geqslant 0},$$
$$\psi_l\equiv\left(\psi_l^n\right)_{n\geqslant 0}:\mathbf{Hol}^{\mathbf{1}}\left[K^{(k)}_{(\sigma_1,\ldots,\sigma_k+l+1)}\right]\to\mathbf{Hol}^{\mathbf{1}}\left[K^{(k)}_{(\sigma_1,\ldots,\sigma_k+l)}\right][1].$$

Let us first define ρ. We will define a functorial morphism

$$\Theta:\mathrm{D}_{(\sigma_1,\ldots,\sigma_k+1,1,\ldots,1)_n}\to\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\left[K^{(k)}_{(\sigma_1,\ldots,\sigma_k+1)}\right]$$

and show that the sequence

$$0\to\mathrm{D}_{(\sigma_1,\ldots,\sigma_k,1,\ldots,1)_n}\overset{\Phi}{\hookrightarrow}\mathrm{D}_{(\sigma_1,\ldots,\sigma_k+1,1,\ldots,1)_n}\overset{\Theta}{\to}\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\left[K^{(k)}_{(\sigma_1,\ldots,\sigma_k+1)}\right]$$

is exact so that the dual representative of Θ will pass to the quotient defining our surjective ρ.

Define Θ to be the following composition:

$$\begin{aligned}\mathrm{D}_{(\sigma_1,\ldots,\sigma_k+1,1,\ldots,1)_n}&\simeq\mathrm{D}^\bullet_{(\sigma_1,\ldots,\sigma_k+1,1)_{k+1}}\left(\mathrm{D}_{\mathbf{1}_{n-k-1}}\subset\mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}\right)\\&\simeq Hom^\bullet_A\left(\Lambda^{(\sigma_1,\ldots,\sigma_k+1,1)},\mathrm{D}_{\mathbf{1}_{n-k-1}}\subset\mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}\right)\xrightarrow{\circ d_{(\sigma_1,\ldots,\sigma_k+1,1)}}\\&\to\mathrm{Diff}^\bullet_1\left(\Lambda^{(\sigma_1,\ldots,\sigma_k+1)},\mathrm{D}_{\mathbf{1}_{n-k-1}}\subset\mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}\right)\\&\simeq\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\left[\Lambda^{(\sigma_1,\ldots,\sigma_k+1)}\right]\overset{\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}[j]}{\longrightarrow}\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\left[K^{(k)}_{(\sigma_1,\ldots,\sigma_k+1)}\right],\end{aligned}$$

where

$$0\to K^{(k)}_{(\sigma_1,\ldots,\sigma_k+1)}\overset{j}{\to}\Lambda^{(\sigma_1,\ldots,\sigma_k+1)}\overset{\pi}{\to}\Lambda^{(\sigma_1,\ldots,\sigma_k)}\to 0.$$

Then, $\Phi \circ \Theta$ coincides with the following composition:

$$\begin{aligned}
D_{(\sigma_1,\dots,\sigma_k,1,\dots,1)_n} &\simeq D^\bullet_{(\sigma_1,\dots,\sigma_k,1)_{k+1}}\big(D_{\mathbf{1}_{n-k-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}\big)\\
&\simeq Hom^\bullet_A\big(\Lambda^{(\sigma_1,\dots,\sigma_k,1)}, D_{\mathbf{1}_{n-k-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}\big) \xrightarrow{\circ d_{(\sigma_1,\dots,\sigma_k,1)}}\\
&\to \mathrm{Diff}^\bullet_1\big(\Lambda^{(\sigma_1,\dots,\sigma_k)}, D_{\mathbf{1}_{n-k-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}\big)\\
&\simeq \mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\big[\Lambda^{(\sigma_1,\dots,\sigma_k)}\big] \xrightarrow{\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}[\pi]} \mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\big[\Lambda^{(\sigma_1,\dots,\sigma_k+1)}\big] \xrightarrow{\mathcal{P}^\bullet_{\mathbf{1}_{n-k}}[j]} \mathcal{P}^\bullet_{\mathbf{1}_{n-k}}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}\big];
\end{aligned}$$

but $\pi \circ j = 0$ hence $im(\Phi) \subseteq \ker(\Theta)$. We prove the reverse inclusion.

Let P be an object in $\mathcal{D}$ and

$$h \in Hom^\bullet_A\big(\Lambda^{(\sigma_1,\dots,\sigma_k+1,1)}, D_{\mathbf{1}_{n-k-1}}(P) \subset \mathrm{Diff}^+_{\mathbf{1}_{n-k-1}}(P)\big) \simeq D_{(\sigma_1,\dots,\sigma_k+1,1,\dots,1)_n}(P)$$

be such that $\Theta(h) = h \circ d_{(\sigma_1,\dots,\sigma_k+1,1)} \circ j = 0$; we claim that $h \in im(\Phi)$. Now

$$h \circ d_{(\sigma_1,\dots,\sigma_k+1,1)} \circ j = h \circ j' \circ d_{(\sigma_1,\dots,\sigma_k+1,1)}|_{K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}}$$

where

$$0 \to K^{(k)}_{(\sigma_1,\dots,\sigma_k+1,1)} \xrightarrow{j'} \Lambda^{(\sigma_1,\dots,\sigma_k+1,1)} \xrightarrow{\pi'} \Lambda^{(\sigma_1,\dots,\sigma_k,1)} \to 0.$$

But $h \in im(\Phi)$ iff $h \circ j' = 0$ so it is enough to show that $im(d_{(\sigma_1,\dots,\sigma_k+1,1)}|_{K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}})$ generates $K^{(k)}_{(\sigma_1,\dots,\sigma_k+1,1)}$ over A (since both h and j' are A-homomorphisms). We know that $im(j_1 : Q \to \mathbf{J}^1(Q))$ generates $\mathbf{J}^1(Q)$ over A for any object Q in $\mathcal{D}$ (Section 2). Moreover, the 3×3 lemma gives us an exact[8] commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K^{(k)}_{(\sigma_1,\dots,\sigma_k+2)} & \longrightarrow & \mathbf{J}^1(K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}) & \xrightarrow{t} & K^{(k)}_{(\sigma_1,\dots,\sigma_k+1,1)} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+2)} & \longrightarrow & \mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+1)}) & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+1,1)} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+1)} & \longrightarrow & \mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k)}) & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k,1)} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(where we used the fact that the functor $\mathbf{J}^k(\cdot)$ is exact if $\mathcal{D}$ is smooth: this follows from Lemma 3.4 since $\mathbf{J}^k_+$ is projective); but $d_{(\sigma_1,\dots,\sigma_k+1,1)}|_{K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}} = t \circ j_1$ (by definition of d), hence $im(d_{(\sigma_1,\dots,\sigma_k+1,1)}|_{K^{(k)}_{(\sigma_1,\dots,\sigma_k+1)}})$ generates $K^{(k)}_{(\sigma_1,\dots,\sigma_k+1,1)}$ over A and we have finished.

[8] Smoothness of $\mathcal{D}$ enters here.

Now let's turn to the definition of

$$\psi_l \equiv \left(\psi_l^n : \mathbf{Hol}^{\mathbf{1}_n}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right] \to \mathbf{Hol}^{\mathbf{1}_{n+1}}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\right]\right)_{n\geqslant 0}.$$

First of all

$$\psi_l^0 = 0 : \left(\mathbf{Hol}^{\mathbf{1}}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right]\right)^0 = (0) \to \left(\mathbf{Hol}^{\mathbf{1}}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\right]\right)^1 = \mathbf{J}^1\left(K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\right).$$

For $n > 0$

$$\psi_l^n : \mathrm{Hol}^{\mathbf{1}_n}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right] \to \mathrm{Hol}^{\mathbf{1}_{n+1}}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\right]$$

will be defined as the dual representative of a functorial morphism

$$\psi_n^l : \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\right] \to \mathcal{P}^\bullet_{\mathbf{1}_n}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right].$$

From the exact sequence

$$0 \to K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)} \xrightarrow{i} \Lambda^{(\sigma_1,\dots,\sigma_k+l)} \xrightarrow{p} \Lambda^{(\sigma_1,\dots,\sigma_k+l-1)} \to 0$$

$$(\text{resp. } 0 \to K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)} \xrightarrow{i'} \Lambda^{(\sigma_1,\dots,\sigma_k+l+1)} \xrightarrow{p'} \Lambda^{(\sigma_1,\dots,\sigma_k+l)} \to 0)$$

and the fact that $\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}$ (resp. $\Lambda^{(\sigma_1,\dots,\sigma_k+l)}$) is projective we get (Proposition 2.9) an exact sequence of functors $\mathcal{D} \to \mathcal{D}$

$$0 \to \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}\right] \xrightarrow{\varepsilon} \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right] \xrightarrow{\eta} \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\right] \to 0$$

$$(\text{resp. } 0 \to \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right] \xrightarrow{\varepsilon'} \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}\right] \xrightarrow{\eta'} \mathcal{P}^\bullet_{\mathbf{1}_n}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right] \to 0)$$

with $\varepsilon = \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}[p]$ and $\eta = \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}[i]$ (resp. $\varepsilon' = \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}[p']$ and $\eta' = \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}[i']$). To define ψ_n^l it will be then enough to define

$$\bar{\psi}_n^l : \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right] \to \mathcal{P}^\bullet_{\mathbf{1}_n}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right]$$

and show that $\bar{\psi}_n^l \circ \varepsilon = 0$. We know (Section 2) that there is an exact sequence

$$0 \to \Lambda^{(\sigma_1,\dots,\sigma_k+l+1)} \xrightarrow{s} \mathbf{J}^1\left(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right) \xrightarrow{q} \Lambda^{(\sigma_1,\dots,\sigma_k+l,1)} \to 0$$

and this gives by Proposition 2.9 the exact sequence

$$0 \to \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l,1)}\right] \xrightarrow{\alpha} \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})\right] \xrightarrow{\beta} \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}\right] \to 0$$

(with $\alpha = \mathcal{P}^\bullet_{\mathbf{1}_n}[q]$ and $\beta = \mathcal{P}^\bullet_{\mathbf{1}_n}[s]$). Then we take $\overline{\psi}_n^l$ to be the composition

$$\begin{aligned}
&\mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right] \simeq \mathcal{P}^\bullet_{(1,1)}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right]\left(\mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}\right) \\
&\simeq \mathit{Hom}^\bullet_A\left(\mathrm{Hol}^{(1,1)}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right], \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}\right) \xrightarrow{\circ\delta_{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})} \\
&\to \mathrm{Diff}^\bullet_1\left(\mathrm{Hol}^{(1)}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right], \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}\right) \\
&\simeq \mathcal{P}^\bullet_{(1)}\left[\mathbf{J}^1\left(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right)\right]\left(\mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}\right) \\
&\simeq \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\mathbf{J}^1\left(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\right)\right] \xrightarrow{\beta} \mathcal{P}^\bullet_{\mathbf{1}_n}\left[\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}\right] \xrightarrow{\eta'} \mathcal{P}^\bullet_{\mathbf{1}_n}\left[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\right].
\end{aligned}$$

Now we show that $\bar{\psi}^l_n \circ \varepsilon = 0$.

Note that using the identifications

$$\mathcal{P}^\bullet_{\mathbf{1}_n}[\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})] \simeq \mathrm{Diff}^\bullet_1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \simeq \mathit{Hom}^\bullet_A(\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}})$$

and

$$\mathit{Hom}^\bullet_A(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \simeq \mathcal{P}^\bullet_{\mathbf{1}_n}[\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)})]$$

(resp. the identifications

$$\mathcal{P}^\bullet_{\mathbf{1}_n}[\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}] \simeq \mathit{Hom}^\bullet_A(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}})$$

and

$$\mathit{Hom}^\bullet_A(\mathbf{J}^1(K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \simeq \mathcal{P}^\bullet_{\mathbf{1}_n}[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}])$$

β (resp. η') is given by

$$\mathit{Hom}^\bullet_A(\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \xrightarrow{\circ \mathbf{J}^1(s)} \mathit{Hom}^\bullet_A(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}})$$

(resp. by

$$\mathit{Hom}^\bullet_A(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \xrightarrow{\circ \mathbf{J}^1(i')} \mathit{Hom}^\bullet_A(\mathbf{J}^1(K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}})).$$

With a similar analysis we see that ε, viewed as a morphism

$$\mathit{Hom}^\bullet_A(\mathrm{Hol}^{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \xrightarrow{\varepsilon} \mathit{Hom}^\bullet_A(\mathrm{Hol}^{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}), \mathrm{D}_{\mathbf{1}_n} \subset \mathrm{Diff}^+_{\mathbf{1}_n})$$

is given by taking the composition with $[\mathbf{J}^1(\mathbf{J}^1(p))]$ where

$$[\mathbf{J}^1(\mathbf{J}^1(p))]: \frac{\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}))}{\mathbf{J}^2(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)})} \simeq \mathrm{Hol}^{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}) \to \mathrm{Hol}^{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}) \simeq \frac{\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}))}{\mathbf{J}^2(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})}$$

is the quotient map of $\mathbf{J}^1(\mathbf{J}^1(p)) : \mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)})) \to \mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}))$.

Therefore $\bar{\psi}^l_n \circ \varepsilon$, viewed as a morphism

$$\mathit{Hom}^\bullet_A(\mathrm{Hol}^{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}}) \to \mathit{Hom}^\bullet_A(\mathbf{J}^1(K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}), \mathrm{D}_{\mathbf{1}_{n-1}} \subset \mathrm{Diff}^+_{\mathbf{1}_{n-1}})$$

is given by

$$f \mapsto f \circ [\mathbf{J}^1(\mathbf{J}^1(p))] \circ \xi_{(\sigma_1,\dots,\sigma_k+l)} \circ \mathbf{J}^1(s) \circ \mathbf{J}^1(i')$$

where

$$\xi_{(\sigma_1,\dots,\sigma_k+l)} : \mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})) \to \frac{\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}))}{\mathbf{J}^2(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})}$$

is the natural projection. (Recall from Section 4 that $\delta_{(1,1)}(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})$ is given by the composition

$$\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}) \xrightarrow{j_1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})} \mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})) \xrightarrow{\xi_{(\sigma_1,\dots,\sigma_k+l)}} \frac{\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}))}{\mathbf{J}^2(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})}.)$$

But $[\mathbf{J}^1(\mathbf{J}^1(p))] \circ \xi$ coincides with

$$\mathbf{J}^1\big(\mathbf{J}^1\big(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}\big)\big) \xrightarrow{\mathbf{J}^1(\mathbf{J}^1(p))} \mathbf{J}^1\big(\mathbf{J}^1\big(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}\big)\big) \xrightarrow{\xi_{(\sigma_1,\dots,\sigma_k+l-1)}} \frac{\mathbf{J}^1(\mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}))}{\mathbf{J}^2(\Lambda^{(\sigma_1,\dots,\sigma_k+l)})}$$

($\xi_{(\sigma_1,\dots,\sigma_k+l-1)}$ being again the natural projection), so that

$$\bar{\psi}_n^l \circ \varepsilon : f \mapsto f \circ \xi_{(\sigma_1,\dots,\sigma_k+l)} \circ \mathbf{J}^1\big(\mathbf{J}^1(p)\big) \circ \mathbf{J}^1(s) \circ \mathbf{J}^1(i') = f \circ \xi_{(\sigma_1,\dots,\sigma_k+l)} \circ \mathbf{J}^1\big(\mathbf{J}^1(p) \circ s \circ i'\big).$$

Again as above, the 3×3 lemma gives us an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)} & \longrightarrow & \mathbf{J}^1(K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}) & \longrightarrow & K^{(k)}_{(\sigma_1,\dots,\sigma_k+l,1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i'} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+l+1)} & \xrightarrow{s} & \mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l)}) & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+l,1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+l)} & \longrightarrow & \mathbf{J}^1(\Lambda^{(\sigma_1,\dots,\sigma_k+l-1)}) & \longrightarrow & \Lambda^{(\sigma_1,\dots,\sigma_k+l-1,1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which finally shows that $\mathbf{J}^1(p) \circ s \circ i' = 0$ and hence $\bar{\psi}_n^l \circ \varepsilon = 0$.

Therefore

$$\psi_n^l : \mathcal{P}^\bullet_{\mathbf{1}_{n+1}}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\big] \to \mathcal{P}^\bullet_{\mathbf{1}_n}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\big]$$

is well defined as well as its dual representative

$$\psi_l^n : \mathrm{Hol}^{\mathbf{1}_n}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l+1)}\big] \to \mathrm{Hol}^{\mathbf{1}_{n+1}}\big[K^{(k)}_{(\sigma_1,\dots,\sigma_k+l)}\big]$$

as we wanted.

Just as in the case of ρ, an easy application of the 3×3 lemma proves that $im(\psi_{n+1}^{l-1}) = \ker(\psi_n^l)$.

It is immediate to verify that ρ and ψ_l so defined are maps of complexes; therefore (5.12) is a resolution of $\mathbf{K}^{(k)}_{(\sigma_1,\dots,\sigma_k+1,\mathbf{1})}$ as desired.

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