Trace and Künneth formulas for singularity categories and applications

Bertrand Toën* and Gabriele Vezzosi

January 2019

Abstract

We present an ℓ-adic trace formula for saturated and admissible dg-categories over a base monoidal dg-category. Moreover, we prove Künneth formulas for dg-category of singularities, and for inertia-invariant vanishing cycles. As an application, we prove a version of Bloch's Conductor Conjecture (stated by Spencer Bloch in 1985), under the additional hypothesis that the monodromy action of the inertia group is unipotent.

Contents

*Partially supported by ERC-2016-ADG-741501 and ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02.

1 Introduction

Motivated by applications to arithmetic geometry, the first one being presented in this paper, we develop methods of derived and non-commutative geometry on a mixed-characteristic base. Though both derived and non-commutative geometry have been mainly used so far over a base field (possibly of positive characteristic), their flexibility is definitely wider, as we try to demonstrate in this paper. More precisely, we prove and investigate a trace formula for dg-categories, and Künneth formulas for dg-categories of singularities and for inertia-invariant vanishing cycles, and we give an application of these results to a categorical version of Bloch's Conductor Conjecture (BCC) in the presence of unipotent monodromy. We are convinced that the tools developed in this paper will have other applications to arithmetic geometry, some of which are already under investigation by the authors. Another seemingly more flexible tool, that of *integration maps* from K-theory of dg-categories, will be developed in a forthcoming paper and applied to more general cases of the original BCC.

In this paper, we present four main results. As a first step, in Section 2, we prove a quite general *trace formula for dg-categories* (or non-commutative schemes). This is done by using the non-commutative ℓ-adic realization functor r_ℓ recently introduced in [BRTV] as a functor from dg-categories over an excellent discrete valuation ring A to ℓ-adic sheaves on $Spec\, A$. Via r_ℓ, we first introduce a ℓ-adic version of the Chern character for dg-categories over A, as a lax-monoidal natural transformation $Ch_\ell : \mathsf{HK} \to |r_\ell|$, where HK is the non-connective, homotopy invariant K-theory functor on dg-categories, and $|r_\ell|$ is the Eilenberg-Mac Lane construction applied to the derived global sections of the ℓ-adic realization functor r_ℓ. We prove the following result (Theorem 2.4.9 in the paper).

Theorem A (Trace formula for dg-categories). *Let $\mathcal{B}$ be a monoidal dg-category over A. For T a dg-category which is a (left) $\mathcal{B}$-module, satisfying both a version of smoothness and properness* over $\mathcal{B}$ *and an admissibility property*[1] *with respect to r_ℓ, a trace formula holds*

$$Ch_\ell([\mathsf{HH}(T/\mathcal{B}; f)]) = tr_{r_\ell(\mathcal{B})}(r_\ell(f)) \tag{1}$$

for any endomorphism $f : T \to T$ of $\mathcal{B}$-modules.

Here $[\mathsf{HH}(T/\mathcal{B}; f)]$ is the class in $\mathsf{HK}_0(\mathsf{HH}(\mathcal{B}/A))$ induced by the endomorphism f, and $\mathsf{HK}_0(\mathsf{HH}(\mathcal{B}/A))$ is the 0-th K-theory of Hochschild homology of $\mathcal{B}$ over A. The r.h.s. $tr_{r_\ell(\mathcal{B})}(r_\ell(f))$ is the trace[2] of the endomorphism $r_\ell(f)$, and the trace formula (1) is an equality in $\pi_0(|r_\ell(\mathsf{HH}(B/A))|) \simeq H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(B/A)))$. Note that the extra generality of working over $\mathcal{B}$ (instead of over A, or over an E_∞- algebra over A) is actually needed for applications, as shown in Section 5: the ℓ-adic realization of a dg-category is

[1] The ℓ-adic realization functor is only lax-monoidal, and we say that T is r_ℓ-admissible over $\mathcal{B}$ if r_ℓ behaves like a symmetric monoidal functor for T over $\mathcal{B}$ (see Definition 2.4.8).

[2] Or rather its image under the canonical map $\alpha : H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \to H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$.

always 2-periodic, i.e. it is a module over $\mathbb{Q}_\ell(\beta) := \oplus_{n\in\mathbb{Z}}\mathbb{Q}_\ell(2n)$, so it has no chance of being smooth and proper over A itself, unless it is trivial; in particular, no reasonable trace formula is available over A.

Our second main result (Section 3) is a *Künneth type formula for inertia-invariant vanishing cycles.* Here we push forward the investigation begun in [BRTV] about the relation between vanishing cycles and the ℓ-adic realization of the dg-category of singularities. For X and Y regular schemes, endowed with a flat, proper and generically smooth map to the strictly henselian excellent trait $S = Spec\, A$, we define an inertia-invariant convolution product $(E \circledast F)^{\mathrm{I}}$, for E an ℓ-adic sheaf on the special fiber X_s, and F an ℓ-adic sheaf on the special fiber Y_s ($\mathrm{I} := \mathrm{Gal}(\bar{K}/K)$ denoting the inertia group in this situation). If we denote by ν_X (respectively, ν_Y) the complex of vanishing cycles for X/S (respectively, Y/S), we prove the following result (Theorem 3.4.2 in the paper).

Theorem B (Künneth formula for inertia-invariant vanishing cycles). *We have equivalences*

$$(\nu_X \circledast \nu_Y)^{\mathrm{I}} \simeq r_\ell(\mathsf{Sing}(X \times_S Y)) \simeq \mathit{Cofib}(\eta_{X\times_S Y} : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{X\times_S Y}(\beta)).$$

where $\mathsf{Sing}(X\times_S Y) = \mathsf{Coh}^b(X\times_S Y)/\mathsf{Perf}(X\times_S Y)$ *is the dg-category of singularities of the fiber product* $X \times_S Y$*, and* $\eta_{X\times_S Y}$ *is the (2-periodized) ℓ-adic fundamental class.*

This is our Künneth formula for inertia-invariant vanishing cycles, and it seems to be a new result in the theory of vanishing cycles, especially in the mixed characteristic case. Note that it might also be viewed as a Thom-Sebastiani formula for inertia-invariant vanishing cycles but we would like to stress that it is not a consequence of the usual Thom-Sebastiani formula for vanishing cycles (see [Il2]), and that it holds only if inertia invariants are taken into account in defining the convolution $(E \circledast F)^{\mathrm{I}}$[3]. We also discuss appropriate conditions ensuring that $(\nu_X \circledast \nu_Y)^{\mathrm{I}}$ is equivalent to $(\nu_X \boxtimes \nu_Y)^{\mathrm{I}}$ (Corollary 3.4.5).

The third main result of this paper is a *Künneth type formula for dg-categories of singularities* (Section 4). For X and Y regular schemes, endowed with a flat, proper and generically smooth map to the strictly henselian excellent trait $S = Spec\, A$, we may consider the dg-categories of singularities $\mathsf{Sing}(X_s) := \mathsf{Coh}^b(X_s)/\mathsf{Perf}(X_s)$ and $\mathsf{Sing}(Y_s) := \mathsf{Coh}^b(X_s)/\mathsf{Perf}(X_s)$ of the corresponding special fibers. Consider $\mathcal{B} := \mathsf{Sing}(s \times_S s)$ (s being the closed point in S, and $s \times_S s$ being the derived fiber product); then $\mathcal{B}$ is a monoidal dg-category for the convolution product coming from the derived groupoid structure of $s \times_S s$. Moreover, $\mathcal{B}$ acts on both $\mathsf{Sing}(X_s)$ and $\mathsf{Sing}(Y_s)$, in such a way that the tensor product $\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s)$ makes sense as a dg-category over A (Proposition 4.1.7). Our Künneth formula for dg-categories of singularities is then the following result (Theorem 4.2.1 in the paper).

Theorem C (Künneth formula for dg-categories of singularities). *There is a canonical equivalence*

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \mathsf{Sing}(X \times_S Y)$$

as dg-categories over A.

Note that this result is peculiar to singularity categories: it is false if we replace Sing by Coh^b. Also notice that $\mathcal{B}$ is defined as the convolution dg-category of a derived groupoid, and is an object in derived algebraic geometry that *cannot* be described within classical algebraic geometry. Finally, we prove that

[3]In particular, only $(E \circledast F)^{\mathrm{I}}$, and not $(E \circledast F)$, makes sense in our context.

$\mathsf{Sing}(X_s)$ is smooth and proper (i.e. saturated) over $\mathcal{B}$, for any X regular scheme, endowed with a flat, proper and generically smooth map to S. This fact is well-known in characteristic zero (see, for instance, [Pr]) but is, in our opinion, a deep and surprising result in our general setting.
Smoothness and properness over $\mathcal{B}$ is one half of the properties needed to apply the trace formula to $\mathsf{Sing}(X_s)$ over $\mathcal{B}$. Note that, without further hypothesis, it is not true that $\mathsf{Sing}(X_s)$ is also admissible (with respect to r_ℓ).

Our fourth and final main result is a *categorical version of Bloch's Conductor formula for unipotent monodromy* (section 5).
In his seminal 1987 paper [Bl], S. Bloch introduced what is now called Bloch's intersection number $[\Delta_X, \Delta_X]_S$, for a flat, proper map of schemes $X \longrightarrow S$ where X is regular, and S is a henselian trait S. This number can be defined as the degree of the localized top Chern class of the coherent sheaf $\Omega^1_{X/S}$ and measures the "relative" singularities of X over S. In the same paper, Bloch introduced his famous *conductor formula*, which can be seen as a conjectural computation of the Bloch's intersection number in terms of the arithmetic geometry of X/S. It reads as follows.

Bloch's Conductor Conjecture (BCC)
We have an equality

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) - Sw(X_{\bar{K}}),$$

where $X_{\bar{k}}$ and $X_{\bar{K}}$ denotes the special and generic geometric fibers of X over S, $\chi(-)$ denotes $\mathbb{Q}_\ell$-adic Euler characteristic, and $Sw(-)$ is the Swan conductor.

In [Bl] the above formula is proven in relative dimension 1. Further results implying special cases of BCC have been obtained since then by Kato-Saito [Ka-Sa] and others, the most recent one being a full proof in the geometric case by T. Saito [Sai] (see section §5 for a more detailed discussion about the status of the art about BCC). In the mixed characteristic case, the conjecture is open in general outside the cases covered in [Ka-Sa]. In particular, for isolated singularities the above conjecture already appeared in Deligne's exposé [SGA7-I, Exp. XVI], and remains open.
In Section 5 we propose a first step towards a new understanding of Bloch's conductor formula using the results developed in the previous sections. We start by defining an analog of Bloch's intersection number, that we call the *categorical Bloch's intersection class* (Definition 5.2.1), and denote it by $[\Delta_X, \Delta_X]^{\mathrm{cat}}_S$. It is an element in $H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$, where $\mathcal{B} = \mathsf{Sing}(s \times_S s)$ (as in Section 4, see above), and it is basically defined as an intersection class in the setting of non-commutative algebraic geometry. The precise comparison with the original Bloch's number is not covered in this work and will appear in a forthcoming paper.
It is easy to see that there are canonical inclusions $\mathbb{Z} \hookrightarrow \mathbb{Q}_\ell \hookrightarrow H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A)))$, and for any $\lambda \in \mathbb{Z}$, we will write $\lambda^\wedge$ its image under the canonical map $\alpha : H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \to H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$.
Our fourth main result is then the following theorem (Theorem 5.2.2 in the paper).

Theorem D (Categorical BCC for unipotent monodromy) *With the notations of BCC, if the monodromy action of the inertia on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unipotent, then we have an equality*

$$[\Delta_X, \Delta_X]^{cat}_S = \chi(X_{\bar{k}})^\wedge - \chi(X_{\bar{K}})^\wedge.$$

Since unipotent monodromy action implies tame action, the Swan conductor vanishes for unipotent monodromy, so that Theorem D is completely analogous to BCC under this hypothesis.
We believe the main ideas in our proof of Theorem D are new in the subject, and might be also useful to answer other related questions in algebraic and arithmetic geometry. The key point in the proof of Theorem D is that it is a *direct consequence* of the trace formula for dg-categories (Theorem A). By the main result of [BRTV], the ℓ-adic realization of the dg-category of singularities $\mathsf{Sing}(X_s)$ of the special fiber is the inertia invariant part of vanishing cohomology of X/S (suitably 2-periodized). Moreover, our Künneth formulas for inertia-invariant vanishing cycles (Section 3), and for dg-categories of singularities (Section 4), imply that $\mathsf{Sing}(X_s)$ is smooth and proper over $\mathcal{B}$. Finally, again our Künneth formulas *together with the hypothesis of unipotent monodromy* show that $\mathsf{Sing}(X_s)$ is also r_ℓ-admissible, so that we may apply our trace formula to the identity endomorphism of $\mathsf{Sing}(X_s)$ over $\mathcal{B}$: Theorem D is then an immediate corollary of the trace formula applied to the identity map of $\mathsf{Sing}(X_s)$.

We conclude this introduction by remarking that the hypothesis of unipotent inertia action in Theorem D is, of course, a bit restrictive: unipotent monodromy implies tame monodromy and our theorem does not deal with the interesting arithmetic aspects encoded by the Swan conductor. However we are convinced that we can go beyond the unipotent case, and prove some new cases of BCC conjecture (possibly the whole unrestricted conjecture), by considering $\mathsf{Sing}(X \times_S S')$ as a module over $\mathcal{B}' := \mathsf{Sing}(S' \times_S S')$, where $S' = Spec\, A' \to S$ is a totally ramified Galois extension of strictly henselian traits such that the inertia for S' acts unipotently on $H^*((X \times_S S')_{\bar{K}'}, \mathbb{Q}_\ell)$. Here K' is the fraction field of A', and the existence of such an extension $S' \to S$ is guaranteed by Grothendieck local monodromy theorem ([SGA7-I, Exp. I, Théorème 1.2]). This strategy for BCC, as well as the comparison between the categorical and the classical Bloch numbers, will be investigated in a future work.

Acknowledgments. We thank A. Abbès, A. Blanc, M. Robalo and T. Saito, for various helpful conversations on the topics of this paper. We also thank all the french spiders and the south-italy sea waves in the 2017 summer nights, for keeping the authors company while they were, separately, writing down a first version of this manuscript.

This work is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (Grant Agreement No 741501).

Notations.

- Throughout the text, A will denote a (discrete) commutative noetherian ring. When needed, A will be required to satisfy further properties (such as being excellent, local and henselian) that will be made precise in due course.

- $L(A)$ will denote the A-linear dg-category of (fibrant and) cofibrant A-dg-modules localized with respect to quasi-isomorphisms.

- $\mathbf{dgCat}_A$ will denote the Morita ∞-category of dg-categories over A (see §2.1).

- **Top** denotes the ∞-category of spaces (obtained, e.g. as the coherent nerve of the Dwyer-Kan localization of the category of simplicial sets along weak homotopy equivalences). **Sp** denotes the ∞-category of spectra.

- If R is a an associative and unital monoid in a symmetric monoidal ∞-category $\mathcal{C}$, we will write $\mathbf{Mod}(R)$ or $\mathbf{Mod}_{\mathcal{C}}(R)$ for the ∞-category of left R-modules in $\mathcal{C}$ ([Lu-HA]).

2 A non-commutative trace formalism

2.1 ∞-Categories of dg-categories

We denote by A a commutative ring. We remind here some basic facts about the ∞-category of dg-categories, its monoidal structure and its theory of monoids and modules.

We consider the category $dgCat_A$ of small A-linear dg-categories and A-linear dg-functors. We remind that an A-linear dg-functor $T \longrightarrow T'$ is a Morita equivalence if the induced functor of the corresponding derived categories of dg-modules $f^* : D(T') \longrightarrow D(T)$ is an equivalence of categories (see [To1] for details). The ∞-category of dg-categories over S is defined to be the localisation of $dgCat_A$ along these Morita equivalences, and will be denoted by $\mathbf{dgCat}_S$ or $\mathbf{dgCat}_A$. Being the ∞-category associated to a model category, $\mathbf{dgCat}_A$ is a presentable ∞-category. As in [To1, § 4], the tensor product of A-linear dg-categories can be derived to a symmetric monoidal structure on the ∞-category $\mathbf{dgCat}_A$. This symmetric monoidal structure moreover distributes over colimits making $\mathbf{dgCat}_A$ into a presentable symmetric monoidal ∞-category. We have a notion of rigid, or, equivalently, dualizable, object in $\mathbf{dgCat}_A$. It is a well known fact that dualizable objects in $\mathbf{dgCat}_A$ are precisely smooth and proper dg-categories over A (see [To2, Prop. 2.5]).

The compact objects in $\mathbf{dgCat}_A$ are the dg-categories of finite type over A in the sense of [To-Va]. We denote their full sub-∞-category by $\mathbf{dgCat}_A^{ft} \subset \mathbf{dgCat}_A$. The full sub-category $\mathbf{dgCat}_S^{ft}$ is preserved by the monoidal structure, and moreover any dg-category is a filtered colimit of dg-categories of finite type. We thus have a natural equivalence of symmetric monoidal ∞-categories

$$\mathbf{dgCat}_A \simeq \mathbf{Ind}(\mathbf{dgCat}_S^{ft}).$$

We will from time to time have to work in a bigger ∞-category, denoted by $\mathbf{dgCAT}_A$, that contains $\mathbf{dgCat}_A$ as a non-full sub-∞-category. By [To2], we have a symmetric monoidal ∞-category $\mathbf{dgCat}_A^{lp}$ of *presentable* dg-categories over A. We define $\mathbf{dgCAT}_A$ the full sub-∞-category of $\mathbf{dgCat}_A^{lp}$ consisting of all compactly generated dg-categories. The ∞-category $\mathbf{dgCat}_A$ can be identified with the non-full sub-∞-category of $\mathbf{dgCAT}_A$ which consists of compact objects preserving dg-functors. This provides a faithful embedding of symmetric monoidal ∞-categories

$$\mathbf{dgCat}_A \hookrightarrow \mathbf{dgCAT}_A.$$

At the level of objects, this embedding sends a small dg-category T to the compactly generated dg-category $\hat{T}$ of dg-modules over T^o. An equivalent description of $\mathbf{dgCAT}_A$ is as the ∞-category of small dg-categories together with the mapping spaces given by the classifying space of *all* bi-dg-modules between small dg-categories.

Definition 2.1.1 *A* monoidal A-dg-category *is a unital and associative monoid in the symmetric monoidal ∞-category* $\mathbf{dgCat}_A$*. A* module over a monoidal A-dg-category B *will, by definition, mean a left B-module in* $\mathbf{dgCat}_A$ *in the sense of [Lu-HA], and the ∞-category of (left) B-modules will be denoted by* $\mathbf{dgCat}_B$*.*

For a B-module T, we have a morphism $\mu : B \otimes_A T \to T$ in $\mathbf{dgCat}_A$, that will be simply denoted by $(b, x) \mapsto b \otimes x$. For a monoidal A-dg-category B, we will denote by $B^{\otimes\text{-op}}$ the monoidal A-dg-category where the monoid structure is the opposite to the one of B, i.e. $b \otimes^{op} b' := b' \otimes b$. Note that $B^{\otimes\text{-op}}$ should not be confused with B^o (which is still a monoidal A-dg-category), where the "arrows" and not the monoid structure have been reversed, i.e. $B^o(b, b') := B(b', b)$. By definition, a *right* B-module is a (left) $B^{\otimes\text{-op}}$-module. The ∞-category of right B-modules will be denoted by $\mathbf{dgCat}_{B^{\otimes\text{-op}}}$, or simply by $\mathbf{dgCat}^B$. If B is a monoidal A-dg-category, then $B^{\otimes\text{-op}} \otimes_A B$ is again a monoidal A-dg-category, and B can be considered either as a left $B^{\otimes\text{-op}} \otimes_A B$ (denoted by B^L), or as a right $B^{\otimes\text{-op}} \otimes_A B$-module (denoted by B^R). For T a B-module, and T' a right B-module, then $T' \otimes_A T$ is naturally a right $B^{\otimes\text{-op}} \otimes_A B$-module, and we define

$$T' \otimes_B T := (T' \otimes_A T) \otimes_{B^{\otimes\text{-op}} \otimes_A B} B^L$$

which is an object in $\mathbf{dgCat}_A$.

Let B be a monoidal A-dg-category. We can consider the symmetric monoidal embedding $\mathbf{dgCat}_A \hookrightarrow \mathbf{dgCAT}_A$, so that the image $\widehat{B}$ of B is a monoid in $\mathbf{dgCAT}_A$. The ∞-category of $\widehat{B}$-modules in $\mathbf{dgCAT}_A$ is denoted by $\mathbf{dgCAT}_B$, and its objects are called *big B-modules*. The natural ∞-functor $\mathbf{dgCat}_B \longrightarrow \mathbf{dgCAT}_B$ is faithful and its image consists of all big B-modules $\widehat{T}$ such that the morphism $\widehat{B} \hat{\otimes}_A \widehat{T} \longrightarrow \widehat{T}$ is a small morphism.

It is known (see [To2]) that the symmetric monoidal ∞-category $\mathbf{dgCAT}_A$ is rigid, that for any $\widehat{T}$ its dual is given by $\widehat{T^o}$, and that the evaluation and coevaluation morphisms are defined by T considered as $T^o \otimes_A T$-module. This formally implies that if $\widehat{T}$ is a big B-module, then its dual $\widehat{T^o}$ is naturally a *right* big B-module. We thus have two big morphisms

$$\mu : \widehat{B} \hat{\otimes}_A \widehat{T} \longrightarrow \widehat{T}\ , \quad \mu^o : \widehat{T^o} \hat{\otimes}_A \widehat{B} \longrightarrow \widehat{T^o}.$$

These morphisms also provide a third big morphism

$$h : \widehat{T^o} \hat{\otimes}_A \widehat{T} \longrightarrow \widehat{B}.$$

The big morphism h is obtained by duality from

$$\mu^* : \widehat{T} \longrightarrow \widehat{B} \hat{\otimes}_A \widehat{T}$$

the right adjoint to μ. We now make the following definitions.

Definition 2.1.2 *Let B be a monoidal dg-category, and T a B-module. We say that:*

1. *T is* cotensored *(over B) if the big morphism μ^o defined above is a small morphism (i.e. a morphism in $\mathbf{dgCat}_A$);*

2. *T is* proper *or* enriched *(over B) if the big morphism h defined above is a small morphism (i.e. a morphism in $\mathbf{dgCat}_A$).*

We can make the above definition more explicit as follows. Let B and T be as above; for two objects $b \in B$ and $x \in T$, we can consider the dg-functor

$$x^b : T^o \longrightarrow L(A)$$

sending $y \in T$ to $T(b \otimes y), x)$ where $(b, x) \mapsto b \otimes x : B \otimes_A T \longrightarrow T$ is the B-module structure on T. Then, T is cotensored over B if and only if for all b and x the above dg-module $x^b : T^o \longrightarrow L(A)$ is compact in the derived category $D(T^o)$ of all T^o-dg-modules. When T is furthermore assumed to be triangulated, then this is equivalent to ask for the dg-module to be representable by an object $x^b \in T$. In a similar manner, we can phrase properness of T over B by saying that, for any $x \in T$ and $y \in T$, the dg-module

$$B^o \longrightarrow L(A)$$

sending b to $T(b \otimes x, y)$ is compact (or representable, if B is furthermore assumed to be triangulated).

Remark 2.1.3 It is important to notice that, by definition, when T is cotensored, then the big right B-module $\widehat{T^o}$ is in fact a small right B-module. Equivalently, when T is cotensored, then T^o is naturally a right B-module inside $\mathbf{dgCat}_A$, with the right module structure given by the morphism in $\mathbf{dgCat}_A$

$$\mu^o : T^o \otimes_A B \longrightarrow T^o$$

sending $(x, b) \in T^o \otimes_A B$ to the cotensor $x^b \in T^o$.

We end this section by recalling some facts about existence of tensor products of modules over monoidal dg-categories. As a general fact, since $\mathbf{dgCat}_A$ is a presentable symmetric monoidal ∞-category, for any monoidal dg-category B there exists a tensor product ∞-functor

$$\otimes_B : \mathbf{dgCat}_B \times \mathbf{dgCat}^B \longrightarrow \mathbf{dgCat}_A,$$

sending a left B-module T and a right B-module T' to $T' \otimes_B T$ (see [Lu-HA]). Now, if T is a B-module which is also cotensored (over B) in the sense of Definition 2.1.2, we have that T^o is a right B-module, and we can thus form

$$T^o \otimes_B T \in \mathbf{dgCat}_A.$$

When T is not cotensored, the object $T \otimes_B T^o$ does not exist anymore. However, we can always consider the presentable dg-categories $\widehat{T}$ and $\widehat{T^o}$ as left and right modules over $\widehat{B}$, respectively, and their tensor product $\widehat{T^o} \widehat{\otimes}_{\widehat{B}} \widehat{T}$ now only makes sense as a presentable dg-category which has no reason to be compactly generated, in general. Of course, when T is cotensored, this presentable dg-category is compactly generated and we have

$$\widehat{T^o \otimes_B T} \simeq \widehat{T^o} \widehat{\otimes}_{\widehat{B}} \widehat{T}.$$

Remark 2.1.4 The following, easy observation, will be useful in the sequel. Let B be a monoidal dg-category, and assume that B is generated, as a triangulated dg-category, by its unit object $1 \in B$. Then, any big B-module is small (i.e. in the image of $\mathbf{dgCat}_B \longrightarrow \mathbf{dgCAT}_B$), and also cotensored.

2.2 The ℓ-adic realization of dg-categories

We denote by $\mathcal{SH}_S$ the stable $\mathbb{A}^1$-homotopy ∞-category of schemes over S (see [Vo, Def. 5.7] and [Ro, § 2]). It is a presentable symmetric monoidal ∞-category whose monoidal structure will be denoted by $\wedge_S$. Homotopy invariant algebraic K-theory defines an $\mathbb{E}_\infty$-ring object in $\mathcal{SH}_S$ that we denote by $\mathbf{B}\mathbb{U}_S$ (a more standard notation is KGL_S). We denote by $\mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{B}\mathbb{U}_S)$ the ∞-category of $\mathbf{B}\mathbb{U}_S$-modules in $\mathcal{SH}_S$. It is a presentable symmetric monoidal ∞-category whose monoidal structure will be denoted by $\wedge_{\mathbf{B}\mathbb{U}_S}$.

As proved in [BRTV], there exists a lax symmetric monoidal ∞-functor

$$\mathsf{M}^{-} : \mathbf{dgCat}_S \longrightarrow \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{B}\mathbb{U}_S),$$

which will be denoted by $T \mapsto \mathsf{M}^T$ (while it is denoted by $T \mapsto \mathcal{M}^{\vee}_S(T)$ in [BRTV]). The precise construction of the ∞-functor M^{-} is rather involved and uses in an essential manner the theory of non-commutative motives of [Ro] as well as the comparison with the stable homotopy theory of schemes. Intuitively, the ∞-functor M^{-} sends a dg-category T to the homotopy invariant K-theory functor $S' \mapsto \mathsf{HK}(S' \otimes_S T)$. To be more precise, there is an obvious forgetful ∞-functor

$$\mathsf{U} : \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{B}\mathbb{U}_S) \longrightarrow \mathbf{Fun}^{\infty}(Sm_S^{op}, \mathbf{Sp}),$$

to the ∞-category of presheaves of spectra on the category Sm_S of smooth S-schemes. For a given dg-category T over S, the presheaf $\mathsf{U}(\mathsf{M}^T)$ is defined by sending a smooth S-scheme $S' = \mathbf{Spec}\, A' \to \mathbf{Spec}\, A = S$ to $\mathsf{HK}(A' \otimes_A T)$, the homotopy invariant non-connective K-theory spectrum of $A' \otimes_A T$ (see [Ro, 4.2.3]).

The ∞-functor M^{-} satisfies some basic properties which we recall here.

1. The ∞-functor M^{-} is a localizing invariant, i.e. for any short exact sequence $T_0 \hookrightarrow T \longrightarrow T/T_0$ of dg-categories over A, the induced sequence

$$\mathsf{M}^{T_0} \longrightarrow \mathsf{M}^{T} \longrightarrow \mathsf{M}^{T/T_0}$$

exhibits M^{T_0} has the fiber of the morphism $\mathsf{M}^T \to \mathsf{M}^{T/T_0}$ in $\mathbf{Mod}(\mathbf{B}\mathbb{U}_S)$.

2. The natural morphism $\mathbf{B}\mathbb{U}_S \longrightarrow \mathsf{M}^A$, induced by the lax monoidal structure of M^{-}, is an equivalence of $\mathbf{B}\mathbb{U}_S$-modules.

3. The ∞-functor $T \mapsto \mathsf{M}^T$ commutes with filtered colimits.

4. For any quasi-compact and quasi-separated scheme X, and any morphism $p : X \longrightarrow S$, we have a natural equivalence of $\mathbf{B}\mathbb{U}_S$-modules

$$\mathsf{M}^{\mathsf{Perf}(X)} \simeq p_*(\mathbf{B}\mathbb{U}_X),$$

where $p_* : \mathbf{Mod}_{\mathcal{SH}_X}(\mathbf{B}\mathbb{U}_X) \longrightarrow \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{B}\mathbb{U}_S)$ is the direct image of $\mathbf{B}\mathbb{U}$-modules, and $\mathsf{Perf}(X)$ is the dg-category of perfect complexes on X.

We now let ℓ be a prime number invertible in A. We denote by $Sh^{\mathsf{ct}}_{\mathbb{Q}_\ell}(S)$ the ∞-category of *constructible* $\mathbb{Q}_\ell$-adic complexes on the étale site $S_{\acute{e}t}$ of S. It is a symmetric monoidal ∞-category, and we denote by

$$Sh_{\mathbb{Q}_\ell}(S) := \mathbf{Ind}(Sh^{\mathsf{ct}}_{\mathbb{Q}_\ell}(S))$$

its completion under filtered colimits (see [Ga-Lu, Def. 4.3.26]). According to [Ro, Cor. 2.3.9], there exists an ℓ-adic realization ∞-functor

$$R_\ell : \mathcal{SH}_S \longrightarrow Sh_{\mathbb{Q}_\ell}(S).$$

By construction, R_ℓ is a symmetric monoidal ∞-functor sending a smooth scheme $p : X \longrightarrow S$ to $p_! p^!(\mathbb{Q}_\ell)$, or, in other words, to the relative ℓ-adic homology of X over S.

We let $\mathbb{T} := \mathbb{Q}_\ell[2](1)$, and we consider the $\mathbb{E}_\infty$-ring object in $Sh_{\mathbb{Q}_\ell}(S)$

$$\mathbb{Q}_\ell(\beta) := \oplus_{n \in \mathbb{Z}} \mathbb{T}^{\otimes n}.$$

In this notation, β stands for $\mathbb{T}$, and $\mathbb{Q}_\ell(\beta)$ for the algebra of Laurent polynomials in β, so we might as well write

$$\mathbb{Q}_\ell(\beta) = \mathbb{Q}_\ell[\beta, \beta^{-1}].$$

As shown in [BRTV], there exists a canonical equivalence $R_\ell(\mathbf{B}\mathbb{U}_S) \simeq \mathbb{Q}_\ell(\beta)$ of $\mathbb{E}_\infty$-ring objects in $Sh_{\mathbb{Q}_\ell}(S)$, that is induced by the Chern character from algebraic K-theory to étale cohomology. We thus obtain a well-defined symmetric monoidal ∞-functor

$$\mathsf{R}_\ell : \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{B}\mathbb{U}_S) \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta)),$$

from $\mathbf{B}\mathbb{U}_S$-modules in $\mathcal{SH}_S$ to $\mathbb{Q}_\ell(\beta)$-modules in $Sh_{\mathbb{Q}_\ell}(S)$. By pre-composing with the functor $T \mapsto \mathsf{M}^T$, we obtain a lax monoidal ∞-functor

$$r_\ell := \mathsf{R}_\ell \circ \mathsf{M}^- : \mathbf{dgCat}_S \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta)).$$

Definition 2.2.1 *The ∞-functor defined above*

$$r_\ell : \mathbf{dgCat}_S \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))$$

is called the ℓ-adic realization functor for dg-categories over S.

From the standard properties of the functor $T \mapsto \mathsf{M}^T$, recalled above, we obtain the following properties for the ℓ-adic realization functor $T \mapsto r_\ell(T)$.

1. The ∞-functor r_ℓ is a *localizing invariant*, i.e. for any short exact sequence $T_0 \hookrightarrow T \longrightarrow T/T_0$ of dg-categories over A, the induced sequence

$$r_\ell(T_0) \longrightarrow r_\ell(T) \longrightarrow r_\ell(T/T_0)$$

   is a fibration sequence in $\mathbf{Mod}(\mathbb{Q}_\ell(\beta))$.

2. The natural morphism

$$\mathbb{Q}_\ell(\beta) \longrightarrow r_\ell(A),$$

   induced by the lax monoidal structure, is an equivalence in $\mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))$.

3. The ∞-functor r_ℓ commutes with filtered colimits.

4. For any separated morphism of finite type $p : X \longrightarrow S$, we have a natural morphism of $\mathbb{Q}_\ell(\beta)$-modules

$$r_\ell(\mathsf{Perf}(X)) \longrightarrow p_*(\mathbb{Q}_\ell(\beta)),$$

   where $p_* : \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_{\ell,X}(\beta)) \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))$ is induced by the direct image $Sh^{\mathsf{ct}}_{\mathbb{Q}_\ell}(X) \longrightarrow Sh^{\mathsf{ct}}_{\mathbb{Q}_\ell}(S)$ of constructible $\mathbb{Q}_\ell$-complexes. If either p is proper, or A is a field, this morphism is an equivalence.

Remark 2.2.2 The ℓ-adic realization functor of Definition 2.2.1 can be easily "sheafified", as follows. For X a separated, excellent scheme, and T a sheaf of dg-categories on X_{Zar} (i.e. $T \in \mathbf{dgCat}_X := \lim \mathbf{dgCat}_{\mathcal{U}^\bullet}$ where $\mathcal{U}^\bullet$ is the Cech-nerve of any Zariski open cover $\mathcal{U}$ of X), by taking the limit along the Cech-nerve of a Zariski open cover of X, we define $\mathsf{M}^-_X : \mathbf{dgCat}_X \to \mathbf{Mod}_{\mathcal{SH}_X}(\mathbf{B}\mathbb{U}_X)$ and $\mathsf{R}_{\ell,X} : \mathbf{Mod}_{\mathcal{SH}_X}(\mathbf{B}\mathbb{U}_X) \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}(\beta))$. The composite

$$r_{\ell,X} := \mathsf{R}_{\ell,X} \circ \mathsf{M}^-_X : \mathbf{dgCat}_X \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}(\beta))$$

is called the *ℓ-adic realization functor for dg-categories over X*. Obviously we have $r_\ell = r_{\ell,S}$, for r_ℓ as in Definition 2.2.1. Moreover, for X as above, any separated morphism of finite type $p : X \longrightarrow S$ induces a commutative diagram[4]

$$\begin{tikzcd}
\mathbf{dgCat}_X \arrow[r, "r_{\ell,X}"] \arrow[d, "p_*"'] & \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}(\beta)) \arrow[d, "p_*"] \\
\mathbf{dgCat}_S \arrow[r, "r_\ell"'] & \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))
\end{tikzcd}$$

2.3 The ℓ-adic Chern character

As explained in [BRTV], there is a symmetric monoidal ∞-category $\mathcal{SH}^{nc}_A$ of non-commutative motives over A. As an ∞-category it is the full sub-∞-category of ∞-functors of (co)presheaves of spectra

$$\mathbf{dgCat}^{ft}_A \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance.

We consider $\Gamma : Sh_{\mathbb{Q}_\ell}(S) \longrightarrow \mathbf{dg}_{\mathbb{Q}_\ell}$, the global section ∞-functor, taking an ℓ-adic complex on $S_{\acute{e}t}$ to its hyper-cohomology. Composing this with the Dold-Kan construction $\mathbb{R}\mathbf{Map}_{\mathbf{dg}_{\mathbb{Q}_\ell}}(\mathbb{Q}_\ell, -) : \mathbf{dg}_{\mathbb{Q}_\ell} \longrightarrow \mathbf{Sp}$ we obtain an ∞-functor

$$|-| : Sh_{\mathbb{Q}_\ell}(S) \longrightarrow \mathbf{Sp},$$

which computes hyper-cohomology of $S_{\acute{e}t}$ with ℓ-adic coefficients, i.e. for any $E \in Sh_{\mathbb{Q}_\ell}(S)$, we have natural isomorphisms

$$H^i(S_{\acute{e}t}, E) \simeq \pi_{-i}(|E|)\,, i \in \mathbb{Z}.$$

By what we have seen in our last paragraph, the composite functor $T \mapsto |r_\ell(T)|$ provides a (co)presheaf of spectra

$$\mathbf{dgCat}^{ft}_A \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance. It thus defines an object $|r_\ell| \in \mathcal{SH}^{nc}_A$. The fact that r_ℓ is lax symmetric monoidal implies moreover that $|r_\ell|$ is endowed with a natural structure of a $\mathbb{E}_\infty$-ring object in $\mathcal{SH}^{nc}_A$.

Each $T \in \mathbf{dgCat}^{ft}_A$ defines a corepresentable object $h^T \in \mathcal{SH}^{nc}_A$, characterized by the ($\infty$-)functorial equivalence

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}^{nc}_A}(h^T, F) \simeq F(T),$$

[4] Note that the functor $p_* : \mathbf{dgCat}_X \to \mathbf{dgCat}_S$ consists simply in viewing a dg-category over X as an $A-$dg-category via the morphism p.

for any $F \in \mathcal{SH}_A^{nc}$. The existence of h^T is a formal statement, however the main theorem of [Ro] implies that we have a natural equivalence of spectra

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_A^{nc}}(h^T, h^A) \simeq \mathsf{HK}(T),$$

where $\mathsf{HK}(T)$ stands for non-connective homotopy invariant algebraic K-theory of the dg-category T. In other words, $T \mapsto \mathsf{HK}(T)$ defines an object in $\mathcal{SH}_A^{nc}$ which is isomorphic to h^B. By Yoneda lemma, we thus obtain an equivalence of spaces

$$\mathbb{R}\mathbf{Map}^{lax-\otimes}(\mathsf{HK}, |r_\ell|) \simeq \mathbb{R}\mathbf{Map}_{\mathbb{E}_\infty-\mathbf{Sp}}(\mathbb{S}, |r_\ell(A)|) \simeq *.$$

In other words, there exists a unique (up to a contractible space of choices) lax symmetric monoidal natural transformation

$$\mathsf{HK} \longrightarrow |r_\ell|,$$

between lax monoidal ∞-functors from $\mathbf{dgCat}_A^{ft}$ to $\mathbf{Sp}$. We extend this to all dg-categories over A, as usual, by passing to Ind-completion $\mathbf{dgCat}_A \simeq \mathbf{Ind}(\mathbf{dgCat}_A^{ft})$.

Definition 2.3.1 *The natural transformation defined above is called the* ℓ-adic Chern character. *It is denoted by*

$$Ch_\ell : \mathsf{HK}(-) \longrightarrow |r_\ell(-)|.$$

Remark 2.3.2 Definition 2.3.1 contains a built-in, formal Grothendieck-Riemann-Roch formula. Indeed, for any B-linear dg-functor $f : T \longrightarrow T'$, the square of spectra

$$\begin{tikzcd}
\mathsf{HK}(T) \arrow[r, "f_!"] \arrow[d, "Ch_{\ell,T}"'] & \mathsf{HK}(T') \arrow[d, "Ch_{\ell,T'}"] \\
|r_\ell(T)| \arrow[r, "f_!"'] & |r_\ell(T')|
\end{tikzcd}$$

commutes up to a natural equivalence.

2.4 Trace formula for dg-categories

Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category ([To-Ve-1], [Lu-HA, Definition 2.0.0.7]).

Hypothesis 2.4.1 *The underlying ∞-category $\mathcal{C}$ has small sifted colimits, and the tensor product preserves small colimits in each variable.*

Definition 2.4.2 *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category satisfying Hypothesis 2.4.1. We denote by $\underline{\mathbf{Alg}}(\mathcal{C})$ the $(\infty, 2)$-category of algebras in $\mathcal{C}^\otimes$ denoted by $\mathbf{Alg}_{(1)}(\mathcal{C}^\otimes)$ in* [Lu-COB, Definition 4.1.11].

Informally, one can describe $\underline{\mathbf{Alg}}(\mathcal{C})$ as the $(\infty, 2)$-category with:

- objects: associative unital monoids (=: E_1-algebras) in $\mathcal{C}$.
- $Map_{\underline{\mathbf{Alg}}(\mathcal{C})}(B, B') := \mathbf{Bimod}_{B',B}(\mathcal{C}^\otimes)$, the ∞-category of (B', B)-bimodules.

- The composition of 1-morphisms (i.e. of bimodules) is given by tensor product.
- The composition of 2-morphisms (i.e. of morphisms between bimodules) is the usual composition.

Definition 2.4.3 *Let B be an algebra in $\mathcal{C}$ and X a left B-module. Let us identify X with a 1-morphism $\underline{X} : 1_{\mathcal{C}} \to B$ in $\underline{\mathbf{Alg}}(\mathcal{C})$. A* right B-dual *of X is defined as a right adjoint $\underline{Y} : B \to 1_{\mathcal{C}}$ to $\underline{X}$.*

Unraveling the definition, we get that a right dual of X is a left $B^{\otimes\text{-op}}$-module Y, the unit u of adjunction (or *coevaluation*) is a map $coev : 1_{\mathcal{C}} \to Y \otimes_B X$ in $\mathcal{C}$, the counit v of adjunction (or *evaluation*) is a map $ev : X \otimes Y \to B$ of (B,B)-bimodules; u and v satisfy usual compatibilities.
Note that, if a right B-dual of X exists then it is "unique" (i.e. unique up to a contractible space of choices).

If the right B-module Y is the right B-dual of the left B-module X, then we can define the trace of any map $f : X \to X$ of left B-modules, as follows.
Recall that we have a coevaluation map in $\mathcal{C}$ $coev : 1_{\mathcal{C}} \to Y \otimes_B X$ in $\mathcal{C}$ and an evaluation map of (B,B)-bimodules $ev : X \otimes Y \to B$.
Consider the graph Γ_f defined as the composite

$$1_{\mathcal{C}} \xrightarrow{coev} Y \otimes_B X \xrightarrow{id \otimes f} Y \otimes_B X.$$

We now elaborate on the evaluation map. Observe that

- $B \in \mathcal{C}$ has a left $B \otimes B^{\otimes\text{-op}}$-module structure that we will denote by B^L.
- $B \in \mathcal{C}$ has a right $B \otimes B^{\otimes\text{-op}}$-module structure (i.e. a left $B^{\otimes\text{-op}} \otimes B$-module), that we will denote by B^R.
- $ev : X \otimes Y \to B^L$ is a map of left $B \otimes B^{\otimes\text{-op}}$- modules.
- the composite $ev' : Y \otimes X \xrightarrow{\sigma} X \otimes Y \xrightarrow{ev} B^R$ is a map of left $(B^{\otimes\text{-op}} \otimes B)$-modules

Apply $(-) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L$ to the composite

$$ev' : Y \otimes X \xrightarrow{\sigma} X \otimes Y \xrightarrow{ev} B^R$$

to get

$$ev_{HH} : \ (Y \otimes X) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L \longrightarrow B^R \otimes_{B \otimes B^{\otimes\text{-op}}} B^L =: \mathsf{HH}_{\mathcal{C}}(B) .$$

Now observe that $(Y \otimes X) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L \simeq Y \otimes_B X$ in $\mathcal{C}$.
Note that, by definition, $\mathsf{HH}_{\mathcal{C}}(B)$ is (only) an object in $\mathcal{C}$, called the *Hochschild homology object* of B.

Definition 2.4.4 *The* non-commutative trace of $f : X \to X$ over B *is defined as the composite*

$$Tr_B(f) : \ 1_{\mathcal{C}} \xrightarrow{\Gamma_f} Y \otimes_B X \simeq (Y \otimes X) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L \xrightarrow{ev_{HH}} B^R \otimes_{B \otimes B^{\otimes\text{-op}}} B^L =: \mathsf{HH}_{\mathcal{C}}(B) .$$

$Tr_B(f)$ is a morphism in $\mathcal{C}$.

Remark 2.4.5 Let $B \in \mathbf{CAlg}(\mathcal{C}^{\otimes})$, and let us still denote by B its image via the canonical map $\mathbf{CAlg}(\mathcal{C}^{\otimes}) \to \mathbf{Alg}_{E_1}(\mathcal{C}^{\otimes})$. In this case, $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$ is a symmetric monoidal ∞-category, and if $X \in \mathbf{Mod}_B(\mathcal{C}^{\otimes})$ is a dualizable object (in the usual sense), then its (left and right) dual in $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$ is also a right-dual of X according to Definition 2.4.3. In this case, any $f : X \to X$ in $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$, has therefore *two possible traces*, a non-commutative one (as in Definition 2.4.4)

$$Tr_B(f) : 1_{\mathcal{C}} \longrightarrow \mathsf{HH}_{\mathcal{C}}(B)$$

which is a morphism in $\mathcal{C}$, and a more standard, commutative one

$$Tr_B^{\mathrm{c}}(f) : B \longrightarrow B$$

which is a morphism on $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$. The two traces are related by the following commutative diagram

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "u_B"] \arrow[d, "Tr_B(f)"'] & B \arrow[d, "Tr_B^{\mathrm{c}}(f)"] \\ \mathsf{HH}_{\mathcal{C}}(B) \arrow[r, "a"'] & B \end{tikzcd}$$

where $a : \mathsf{HH}_{\mathcal{C}}(B) \to B$ is the canonical augmentation (which exists since B is commutative), and $u_B : 1_{\mathcal{C}} \to B$ is the unit map of the algebra B in $\mathcal{C}$.

The case of dg-categories. Let us specialize the previous discussion to the case $\mathcal{C}^{\otimes} = \mathbf{dgCat}_A$.
Let B be a monoidal dg-category, i.e. an associative and unital monoid in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$.

Proposition 2.4.6 *For any B-module T which is cotensored in the sense of Definition 2.1.2, the big B-module $\widehat{T} \in \mathbf{dgCAT}_B$ has a right dual in the symmetric monoidal ∞-category $\mathbf{dgCAT}_A$ whose underlying big dg-category is $\widehat{T^o}$.*

Proof. This is very similar to the argument used in [To2, Prop. 2.5 (1)]. We consider $\widehat{T^o}$, and we define evaluation and coevaluation maps as follows.

The big morphism h introduced right before Definition 2.1.2

$$h : \widehat{T^o} \widehat{\otimes}_A \widehat{T} \longrightarrow \widehat{B},$$

whose domain is naturally a $\widehat{B}$-bimodule, can be canonically lifted to a morphism of bimodules. We choose this as our evaluation morphism.

The coevaluation is then obtained by duality. We start by the diagonal bimodule

$$T : (T^o \otimes_A T)^o \longrightarrow L(A) = \widehat{A}.$$

sending (x, y) to $T(y, x)$. This morphism naturally descends to $(T^o \otimes_B T)^o$, providing a dg-functor

$$(T^o \otimes_B T)^o \longrightarrow \widehat{A}.$$

Note that T^o is naturally a right B-module since T is assumed to be cotensored (and note that, otherwise, $T^o \otimes_B T$ would not make sense). This dg-functor is an object in $\widehat{T^o \otimes_B T} \simeq \widehat{T^o} \widehat{\otimes}_A \widehat{T}$, and thus defines a coevaluation morphism

$$\widehat{A} \longrightarrow \widehat{T^o} \widehat{\otimes}_A \widehat{T}.$$

These two evaluation and coevaluation morphisms satisfy the required triangular identities, and thus make $\widehat{T^o}$ a right dual to $\widehat{T}$. $\square$

According to the previous Proposition, any cotensored B-module T has a big right dual, so it comes equipped with big evaluation and coevaluation maps.

Definition 2.4.7 *For a monoidal dg-category B, and a B-module $T \in \mathbf{dgCat}_B$, we say that T is* saturated over B *if*

1. *T is cotensored (over B), and*
2. *the evaluation and coevaluation morphisms are small (i.e. are maps in $\mathbf{dgCat}_A$).*

In particular, if T saturated over B, and $f : T \to T$ is a morphism in $\mathbf{dgCat}_{\mathcal{B}}$, then the trace

$$Tr_B(f : T \to T) : A \to \mathsf{HH}(B/A) = B^R \otimes_{B^{\otimes\text{-op}} \otimes_A B} B^L$$

is also small, i.e. it is a morphism *inside* $\mathbf{dgCat}_A$.

Since our ℓ-adic realization functor r_ℓ is only lax-monoidal, in order to establish our trace formula for an endomorphism $f : T \to T$ of a saturated B-module, we need to restrict to those saturated B-modules on which r_ℓ is in fact symmetric monoidal.

Definition 2.4.8 *A saturated $T \in \mathbf{dgCat}_B$ is called* $\ell^{\otimes}$-admissible *if the canonical map*

$$r_\ell(T^{op}) \otimes_{r_\ell(B)} r_\ell(T) \to r_\ell(T^{op} \otimes_B T)$$

is an equivalence in $Sh_{\mathbb{Q}_\ell}(S)$.

The trace $Tr_B(f)$ is a map $A \to \mathsf{HH}(B/A)$, hence it induces a map in $\mathbf{Sp}$

$$K(Tr_B(f)) : K(A) \to K(\mathsf{HH}(B/A))$$

which is actually a map of $K(A)$-modules (in spectra), since K is lax-monoidal. Hence it corresponds to an element denoted as

$$[\mathsf{HH}(T/B, f)] \equiv tr_B(f) \in K_0(\mathsf{HH}(B/A)).$$

Therefore, its image by the ℓ-adic Chern character $Ch_{\ell,0} := \pi_0(Ch_\ell)$

$$Ch_{\ell,0} : K_0(\mathsf{HH}(B/A)) \to Hom_{D(r_\ell(A))}(r_\ell(A), r_\ell(\mathsf{HH}(B/A))) \simeq H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(B/A))),$$

is an element $Ch_{\ell,0}([\mathsf{HH}(T/B; f)]) \in H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(B/A)))$.

On the other hand, the trace $Tr_{r_\ell(B)}(r_\ell(f))$ of $r_\ell(f)$ over $r_\ell(B)$ is, by definition, a morphism

$$r_\ell(A) \simeq \mathbb{Q}_\ell(\beta) \to \mathsf{HH}(r_\ell(B)/r_\ell(A))$$

in $\mathbf{Mod}_{r_\ell(A)}(Sh_{\mathbb{Q}_\ell}(S))$.
We may further compose this with the canonical map

$$\mathsf{HH}(r_\ell(B)/r_\ell(A)) \to r_\ell(\mathsf{HH}(B/A))$$

(given by lax-monoidality of $r_\ell(-)$), to get a map

$$r_\ell(A) \to r_\ell(\mathsf{HH}(B/A))$$

in $\mathbf{Mod}_{r_\ell(A)}(Sh_{\mathbb{Q}_\ell}(S))$. This is the same thing as an element denoted as

$$tr_{r_\ell(B)}(r_\ell(f)) \in \pi_0(|r_\ell(\mathsf{HH}(B/A))|) \simeq H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(B/A))).$$

Theorem 2.4.9 *Let B a monoidal dg-category over A, $T \in \mathbf{dgCat}_B$ a saturated and $\ell^{\otimes}$-admissible B-module, and $f : T \to T$ map in $\mathbf{dgCat}_B$. Then, we have an equality*

$$Ch_{\ell,0}([\mathsf{HH}(T/B, f)]) = tr_{r_\ell(B)}(r_\ell(f))$$

in $H^0(S_{ét}, r_\ell(\mathsf{HH}(B/A))$.

Proof. This is a formal consequence of uniqueness of right duals, and of the resulting fact that traces are preserved by symmetric or lax symmetric monoidal ∞-functors under our admissibility condition. The key statement is the following lemma, left as an exercise to the reader, and applied to the case where F is our ℓ-adic realization functor.

Lemma 2.4.10 *Let*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

be a lax symmetric monoidal ∞-functor between presentable symmetric monoidal ∞-categories. Let B be a monoid in $\mathcal{C}$, M a left B-module, and $f : M \longrightarrow M$ a morphism of B-modules. We assume that M has a right dual M^o, and that the natural morphism

$$F(M^o) \otimes_{F(B)} F(M) \longrightarrow F(M^o \otimes_B M)$$

is an equivalence. Then, $F(M)$ has a right dual, and we have

$$F(Tr(f)) = i(Tr(F(f)))$$

as elements in $\pi_0(Hom_{\mathcal{D}}(\mathbf{1}, F(\mathsf{HH}(B)))$, where i is induced by the natural morphism $\mathsf{HH}(F(B)) \longrightarrow F(\mathsf{HH}(B))$.

□

3 Invariant vanishing cycles

This section gathers general results about inertia-invariant vanishing cycles (*I-vanishing cycles*, for short), their relations with dg-categories of singularities, and their behaviour under products. These results are partially taken from [BRTV], and the only original result is Proposition 3.4.2 that can be seen as a version of Thom-Sebastiani formula in the mixed-characteristic setting.

All along this section, A will be a strictly henselian excellent dvr with fraction field $K = \mathsf{Frac}(A)$, and perfect (hence algebraically closed) residue field k. We let $S = Spec\, A$. All schemes over S are assumed to be separated and of finite type over S. We denoted by $i : s := Spec\, k \longrightarrow S$ the closed point of S, and $j : \eta := Spec\, K \longrightarrow S$ its generic point. For an S-scheme X, we denote by $X_s := X \times_S s$ its special fiber, and $X_\eta = X \times_S \eta$ its generic fiber. Accordingly, we write $X_{\bar{\eta}} := X \times_S Spec\, K^{sp}$ for the *geometric* generic fiber.

3.1 Trivializing the Tate twist

We let ℓ be a prime invertible in k, and we denote by p the characteristic exponent of k. As k is algebraically closed, we may, and will, choose once for all a group isomorphism

$$\mu_\infty(k) \simeq \mu_\infty(K) \simeq (\mathbb{Q}/\mathbb{Z})[p^{-1}]$$

between the group of roots of unity in k and the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$. Equivalently, we have chosen a given group isomorphism

$$\lim_{(n,p)=1} \mu_n(k) \simeq \hat{\mathbb{Z}}',$$

where $\hat{\mathbb{Z}}' := \lim_{(n,p)=1} \mathbb{Z}/n$. In particular, we have selected a topological generator of $\lim_{(n,p)=1} \mu_n(k)$, corresponding to the image of $1 \in \mathbb{Z}$ inside $\hat{\mathbb{Z}}'$. The choice of the isomorphism above also provides a specific induced isomorphism $\mathbb{Q}_\ell(1) \simeq \mathbb{Q}_\ell$ of $\mathbb{Q}_\ell$-sheaves on S, where (1) denotes, as usual, the Tate twist. By taking tensor powers of this isomorphism, we get various induced isomorphisms $\mathbb{Q}_\ell(i) \simeq \mathbb{Q}_\ell$ for all $i \in \mathbb{Z}$.

We remind that the absolute Galois group I of K (which coincides with the *inertia group* in our case) sits in an extension of pro-finite groups[5]

$$1 \longrightarrow P \longrightarrow \mathrm{I} \longrightarrow \mathrm{I}_t \simeq \hat{\mathbb{Z}}' \longrightarrow 1\,,$$

where P is a pro-p-group (the *wild inertia* subgroup). For any continuous finite dimensional $\mathbb{Q}_\ell$-representation V of I, the group P acts by a finite quotient G_V on V. Moreover, the Galois cohomology of V can then be explicitly identified with the two-terms complex

$$V^G \xrightarrow{1-T} V^G$$

where T is the action of the chosen topological generator of I_t. This easily implies that for any $\mathbb{Q}_\ell$-representation V of I, the natural pairing on Galois cohomology

$$H^i(\mathrm{I}, V) \otimes H^{1-i}(\mathrm{I}, V^\vee) \longrightarrow H^1(\mathrm{I}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$$

is non-degenerate. In other words, if we denote by V^{I} the complex of cohomology of I with coefficients in V, we have a natural quasi-isomorphism $(V^{\mathrm{I}})^\vee \simeq (V^\vee)^{\mathrm{I}}[1]$.

3.2 Reminders on actions of the inertia group

Let $X \longrightarrow S$ be an S-scheme (separated and of finite type, according to our conventions). We recall from [SGA7-II, Exp. XIII, 1.2] that we can associate to X a *vanishing topos* $(X/S)^\nu_{et}$ which is defined as (a 2-)fiber product of toposes

$$(X/S)^\nu_{et} := (X_s)^\sim_{et} \times_{s^\sim_{et}} \eta^\sim_{et}.$$

Since S is strictly henselian, $s^\sim_{et}$ is in fact the punctual topos, and the fiber product above is in fact a product of topos. The topos $\eta^\sim_{et}$ is equivalent to the topos of sets with continuous action of $\mathrm{I} =$

[5]Note that the tame inertia quotient I_t is canonically isomorphic to $\hat{\mathbb{Z}}'(1)$, and it becomes isomorphic to $\hat{\mathbb{Z}}'$ through our choice.

$\mathrm{Gal}(K^{sp}/k)$, where K^{sp} denotes a seprable closure of K. Morally, $(X/S)^{\nu}_{et}$ is the topos of étale sheaves on X_s, endowed with a continuous action of I (see [SGA7-II, Exp. XIII, 1.2.4]).

As explained in [BRTV] we have an ℓ-adic ∞-category $\mathcal{D}((X/S)^{\nu}_{et}, \mathbb{Z}_\ell)$. Morally speaking, objects of this ∞-category consist of the data of an object $E \in \mathcal{D}(\bar{X}_s, \mathbb{Z}_\ell) = Sh_{\mathbb{Z}_\ell}(\bar{X}_s)$ together with a continuous action of I. We say that such an object is *constructible* if E is a constructible object in $\mathcal{D}(\bar{X}_s, \mathbb{Z}_\ell) = Sh_{\mathbb{Z}_\ell}(\bar{X}_s)$, and we denote by $\mathcal{D}_c((X/S)^{\nu}_{et}, \mathbb{Z}_\ell)$ the full sub-∞-category of constructible objects.

Definition 3.2.1 *The* ∞-category of ind-constructible I-equivariant ℓ-adic complexes on X_s *is defined by*

$$\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell) := \mathsf{Ind}(\mathcal{D}_c((X/S)^{\nu}_{et}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

The full sub-∞-category of constructible objects is $\mathcal{D}^{\mathrm{I}}_{c}(X_s, \mathbb{Q}_\ell) := \mathcal{D}_c((X/S)^{\nu}_{et}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Note that since we have chosen trivialisations of the Tate twists, $\mathbb{Q}_\ell(\beta)$ is identified with $\mathbb{Q}_\ell[\beta, \beta^{-1}]$ where β is a free variable in degree 2. This is a graded algebra object in $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$, and we define the ∞-category $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell(\beta))$ as the ∞-category of $\mathbb{Q}_\ell(\beta)$-modules in $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$, or equivalently, the 2-periodic ∞-category of ind-constructible $\mathbb{Q}_\ell$-adic complexes on $(X/S)^{\nu}_{et}$.

Definition 3.2.2 *An object $E \in \mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell(\beta))$ is* constructible *if it belongs to the thick triangulated sub-∞-category generated by objects of the form $E_0(\beta) = E_0 \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\beta)$ for E_0 a constructible object in $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$.*

The full sub-∞-category of $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell(\beta))$ consisting of constructible objects is denoted $\mathcal{D}^{\mathrm{I}}_{c}(X_s, \mathbb{Q}_\ell(\beta))$. Similarly, for any S-scheme X, we define $\mathcal{D}_c(X, \mathbb{Q}_\ell(\beta))$ as the full sub-∞-category of objects in $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell(\beta))$ generated by $E_0(\beta)$ for E_0 constructible.

Note that strictly speaking an object of $\mathcal{D}^{\mathrm{I}}_{c}(X_s, \mathbb{Q}_\ell(\beta))$ is not constructible in the usual sense, as its underlying object in $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ is 2-periodic.

The topos $(X/S)^{\nu}_{et}$ comes with a natural projection $(X/S)^{\nu}_{et} \longrightarrow (X_s)^{\sim}_{et}$ whose direct image is an ∞-functor denoted by

$$(-)^{\mathrm{I}} : \mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$$

called the I-*invariants* functor. This ∞-functor preserves constructibility. The ∞-categories $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$ and $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ carries natural symmetric monoidal structures and the ∞-functor $(-)^{\mathrm{I}}$ comes equipped with a natural lax symmetric monoidal structure (being induced by the direct image of a morphism of toposes). Moreover, $(-)^{\mathrm{I}}$ is the right adjoint of the symmetric monoidal ∞-functor $U : \mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$ endowing objects in $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ with the trivial action of I. This gives $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$ the structure of a $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$-module via $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell) \times \mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell) \to \mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell) : (E, F) \mapsto U(E) \otimes F$. This bi-functor distributes over colimits, thus by the adjoint theorem, we get an enrichment of $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$ over $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$. Note that $\mathcal{D}^{\mathrm{I}}_{ic}(X_s, \mathbb{Q}_\ell)$ is also enriched over itself.

It is important to notice that the I-invariants functor commutes with base change in the following sense. Let $f : \operatorname{Spec} k \longrightarrow X_s$ be a geometric point. The morphism f defines a geometric point of $(X_s)^{\sim}_{et}$ and thus induces a geometric morphism of toposes

$$\eta^{\sim}_{et} \longrightarrow (X/S)^{\nu}_{et}.$$

We thus have an inverse image functor

$$f^* : \mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c(\eta, \mathbb{Q}_\ell) = D_c(\mathrm{I}, \mathbb{Q}_\ell).$$

where $\mathcal{D}_c(\mathrm{I}, \mathbb{Q}_\ell)$ is the ∞-category of finite dimensional complexes of ℓ-adic representations of I. As usual, the square of ∞-functors

$$\begin{tikzcd} \mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell) \arrow[r, "(-)^{\mathrm{I}}"] \arrow[d, "f^*"'] & \mathcal{D}_c(X_s, \mathbb{Q}_\ell) = Sh^{\mathsf{ct}}_{\mathbb{Q}_\ell}(X_s) \arrow[d, "f^*"] \\ \mathcal{D}_c(\mathrm{I}, \mathbb{Q}_\ell) \arrow[r, "(-)^{\mathrm{I}}"'] & \mathcal{D}_c(\mathbb{Q}_\ell) \end{tikzcd}$$

comes equipped with a natural transformation

$$f^*((-)^{\mathrm{I}}) \Rightarrow (f^*(-))^{\mathrm{I}}.$$

It can be checked that this natural transformation is always an equivalence. In particular, for any geometric point x in X_s, we have a natural equivalence of ℓ-adic complexes $(E)_x^{\mathrm{I}} \simeq (E_x)^{\mathrm{I}}$, for any $E \in \mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$.

The dualizing complex ω of the scheme X_s can be used in order to obtain a dualizing object in $\mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$ as follows. We consider ω as an object in $\mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$ endowed with the trivial I-action. We then have an equivalence of ∞-categories

$$\mathbb{D}_I : \mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)^{op}$$

sending E to $\mathbb{R}\underline{Hom}(E, \omega)$, where $\mathbb{R}\underline{Hom}$ denotes the natural enrichment of $\mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$ over itself. We obviously have a canonical biduality equivalence $\mathbb{D}_{\mathrm{I}}^2 \simeq id$. The duality functor $\mathbb{D}_I$ is compatible with the usual Grothendieck duality functor $\mathbb{D}$ for the scheme X_s up to a shift, as explained by the following lemma.

Lemma 3.2.3 *For any object $E \in \mathcal{D}_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$, there is a functorial equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$*

$$d : \mathbb{D}(E^{\mathrm{I}})[-1] \simeq (\mathbb{D}_I(E))^{\mathrm{I}}.$$

Proof. Taking I-invariants is a lax monoidal ∞-functor, so we have a natural map $E^{\mathrm{I}} \otimes \mathbb{D}_I(E)^{\mathrm{I}} \longrightarrow (E \otimes \mathbb{D}_I(E))^{\mathrm{I}}$, that can be composed with the evaluation morphism $E \otimes \mathbb{D}_I(E) \longrightarrow \omega$ to obtain $E^{\mathrm{I}} \otimes \mathbb{D}_I(E)^{\mathrm{I}} \longrightarrow \omega^{\mathrm{I}}$. As the action of I on ω is trivial, we have a canonical equivalence $\omega^{\mathrm{I}} \simeq \omega \otimes \mathbb{Q}_\ell^{\mathrm{I}} \simeq \omega \oplus \omega[-1]$. By projection on the second factor we get a pairing $E^{\mathrm{I}} \otimes \mathbb{D}_I(E)^{\mathrm{I}} \longrightarrow \omega[-1]$, and thus a map

$$\mathbb{D}_I(E)^{\mathrm{I}} \longrightarrow \mathbb{D}(E^{\mathrm{I}})[-1].$$

We claim that the above morphism is an equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$. For this it is enough to check that the above morphism is a stalkwise equivalence. Now, the stalk of the above morphism at a geometric point x of X_s can be written as

$$(E(x)^\vee)^{\mathrm{I}} \longrightarrow (E(x)^{\mathrm{I}})^\vee[-1]$$

where $E(x) := H^*_x(X, E) \in D_c(\mathbb{Q}_\ell)$ is the local cohomology of E at x, and $(-)^\vee$ is now the standard linear duality over $\mathbb{Q}_\ell$. The result now follows from the following well-known duality for $\mathbb{Q}_\ell$-representations of I: for any finite dimensional Ql-representation V of I, the fundamental class in $H^1(\mathrm{I}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$, induces a canonical isomorphism of Galois cohomologies

$$H^*(\mathrm{I}, V^\vee) \simeq H^{1-*}(\mathrm{I}, V)^\vee.$$

□

3.3 Invariant vanishing cycles and dg-categories

From [SGA7-II, Exp. XIII] and [BRTV, 4.1], the vanishing cycles construction provides an ∞-functor

$$\phi : \mathcal{D}_c(X, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}^{\mathrm{I}}_c(X_s, \mathbb{Q}_\ell).$$

Applied to the constant sheaf $\mathbb{Q}_\ell$, we get this way an object denoted by $\nu_{X/S}$ (or simply ν_X if S is clear) in $\mathcal{D}^{\mathrm{I}}_c(X_s, \mathbb{Q}_\ell)$.

Definition 3.3.1 *The* I-invariant vanishing cycles *of X relative to S (or* I-vanishing cycles*, for short) is the object*

$$\nu^{\mathrm{I}}_X := (\nu_X)^{\mathrm{I}} \in \mathcal{D}_c(X_s, \mathbb{Q}_\ell).$$

There are several possible descriptions of invariant vanishing cycles. First of all, by its very definition, ν^{I}_X is related to the I-invariant nearby cycles $\psi^{\mathrm{I}}_X := (\psi_X)^{\mathrm{I}}$ by means of an exact triangle in $D_c(X_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}^{\mathrm{I}}_\ell \longrightarrow \psi^{\mathrm{I}}_X \longrightarrow \nu^{\mathrm{I}}_X. \tag{2}$$

Another description, in terms of *local cohomology*, is the following. We let $U = X_K$ be the open complement of X_s inside X, and $j_X : U \hookrightarrow X$ and $i_X : X_s \hookrightarrow X$ the corresponding immersions. Then, the I-vanishing cycles enters in an exact triangle in $D_c(X_s, \mathbb{Q}_\ell)$

$$\nu^{\mathrm{I}}_X \longrightarrow \mathbb{Q}_\ell \longrightarrow i^!_X(\mathbb{Q}_\ell)[2]. \tag{3}$$

Triangle (2) follows from the octahedral axiom applied to the triangles (1) and

$$\mathbb{Q}_\ell \longrightarrow \mathbb{Q}^{\mathrm{I}}_\ell \simeq \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-1] \longrightarrow \mathbb{Q}_\ell[-1],$$

taking also into account the triangle

$$i^!_X\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow i^*_X(j_X)_*j^*_X\mathbb{Q}_\ell \simeq \psi^{\mathrm{I}}_X.$$

We get one more description of ν^{I}_X (or rather, of $\nu^{\mathrm{I}}_X(\beta) := \nu^{\mathrm{I}}_X \otimes \mathbb{Q}_\ell(\beta)$) using the *$\ell$-adic realization of the dg-category of singularities* studied in [BRTV], at least when X is a regular scheme with smooth generic fiber. Let $\mathsf{Sing}(X_s) = \mathsf{Coh}^b(X_s)/\mathsf{Perf}(X_s)$ be the dg-category of singularities of the scheme X_s. This dg-category is naturally linear over the dg-categories $\mathsf{Perf}(X_s)$ and $\mathsf{Perf}(X)$, and thus we can take its ℓ-adic realization $r_{\ell,X}(\mathsf{Sing}(X_s))$ over X (see Remark 2.2.2) which is a $\mathbb{Q}_{\ell,X}(\beta)$-module in $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell)$

supported on X_s, hence can be identified with a $\mathbb{Q}_{\ell,X_s}(\beta)$-module in $\mathcal{D}_{ic}(X_s,\mathbb{Q}_\ell)$. When X is a regular scheme and X_K is smooth over K, we have from [BRTV] a canonical equivalence in $\mathcal{D}_c(X_s,\mathbb{Q}_\ell(\beta))$ [6]

$$\nu_X^{\mathrm{I}}(\beta)[1] \simeq r_{\ell,X}(\mathsf{Sing}(X_s)) \tag{4}$$

where $\nu_X^{\mathrm{I}}(\beta)$ stands for $\nu_X^{\mathrm{I}} \otimes \mathbb{Q}_\ell(\beta)$.

We conclude this section with another description of $\nu_X^{\mathrm{I}}(\beta)$, see equivalence (6), this time in terms of *sheaves of singularities*. We need a preliminary result.

Lemma 3.3.2 *We assume that S is excellent. Let $p : X \to S$ be a separated morphism of finite type. We write $r_{\ell,X} : \mathcal{SH}_X \to Sh_{\mathbb{Q}_\ell}(X)$, for the ℓ-adic realization* over X*, as in Remark 2.2.2.Then, we have*

1. *$r_{\ell,X}(\mathsf{Perf}(X)) \simeq \mathbb{Q}_{\ell,X}(\beta)$ in $\mathcal{D}_{ic}(X,\mathbb{Q}_\ell)$.*
2. *$r_{\ell,X}(\mathsf{Coh}^b(X)) \simeq \omega_X(\beta)$ in $\mathcal{D}_{ic}(X,\mathbb{Q}_\ell)$, where $\omega_X \simeq p^!(\mathbb{Q}_\ell)$ is the ℓ-adic dualizing complex of X[7].*
3. *There exists a canonical map $\eta_X : \mathbb{Q}_{\ell,X}(\beta) \longrightarrow \omega_X(\beta)$ in $\mathcal{D}_c(X,\mathbb{Q}_\ell(\beta))$, called the* 2-periodic ℓ-adic fundamental class *of X.*

Proof. First of all, separated finite type morphisms of noetherian schemes are compactifiable (by Nagata's theorem), thus we can assume that $p : X \to S$ is proper.

(1) follows immediately from [BRTV, Prop. 3.9 and formula (3.7.13)]. In order to prove (2) we first produce a map $\alpha : r_\ell(\mathsf{Coh}^b(X)) \to \omega_X(\beta)$. In the notations of [BRTV, §3] (note that [BRTV]'s notation for M_X^T is $\mathcal{M}_X^\vee(T)$), we first construct a map $\alpha^{\mathrm{mot}} : \mathcal{M}_X^\vee(\mathsf{Coh}^b(X)) \to p^!(\mathbf{B}\mathbb{U}_S) =: \omega_X^{\mathrm{mot}}$ in $\mathrm{SH}(X)$, whose étale ℓ-adic realization will be α. Since p is proper, α^{mot} is the same thing, by adjunction, as a map $p_*(\mathcal{M}_X^\vee(\mathsf{Coh}^b(X))) \to \mathbf{B}\mathbb{U}_S$ in $\mathrm{SH}(S)$. Now, $p_*(\mathcal{M}_X^\vee(\mathsf{Coh}^b(X)))$ is just $\mathcal{M}_S^\vee(\mathsf{Coh}^b(X)))$, where $\mathsf{Coh}^b(X)$ is viewed as a dg-category over S, via p. If Y is smooth over S, we have by [Pr, Prop. B.4.1], an equivalence of S-dg-categories

$$\mathsf{Coh}^b(X) \otimes_S \mathsf{Coh}^b(Y) \simeq \mathsf{Coh}^b(X \times_S Y) \tag{5}$$

Through this identification, $\mathcal{M}_S^\vee(\mathsf{Coh}^b(X)) \in \mathrm{SH}(S)$ is the ∞-functor $Y \mapsto \mathrm{KH}(\mathsf{Coh}^b(X \times_S Y))$, and $\mathrm{KH}(\mathsf{Coh}^b(X \times_S Y))$ is equivalent to the G-theory spectrum $\mathrm{G}(X \times_S Y)$ of $X \times_S Y$, by $\mathbb{A}^1$-invariance of G-theory. Since S is regular, $\mathbf{B}\mathbb{U}_S \simeq \mathrm{G}_S := \mathrm{G}(-/S)$ canonically in $\mathrm{SH}(S)$, and we can take the map $\mathcal{M}_S^\vee(\mathsf{Coh}^b(X))) \to \mathbf{B}\mathbb{U}_S \simeq \mathrm{G}_S$ to be the push forward p_* on G-theories $\mathrm{G}(X \times_S -) \to G(-/S)$. This gives us a map $\alpha^{\mathrm{mot}} : \mathcal{M}_X^\vee(\mathsf{Coh}^b(X)) \to p^!(\mathbf{B}\mathbb{U}_S) =: \omega_X^{\mathrm{mot}}$. Now observe that by [BRTV, formula (3.7.13)], the étale ℓ-adic realization of $p^!(\mathbf{B}\mathbb{U}_S)$ is canonically equivalent to $p^!(\mathbb{Q}_\ell(\beta)) \simeq \omega_X(\beta)$ (since étale ℓ-adic realization commutes with six operations, [BRTV, Rmk. 3.23]). Therefore, we get our map $\alpha : r_\ell(\mathsf{Coh}^b(X)) \to \omega_X(\beta)$. Checking that α is an equivalence is a local statement, i.e. it is enough to show that if $j : V = \mathrm{Spec}, A \hookrightarrow X$ is an open affine subscheme, then $j^*(\alpha)$ is an equivalence. Now, $j^* r_\ell(\mathsf{Coh}^b(X)) \simeq r_{\ell,V}(j^*\mathsf{Coh}^b(X)) \simeq r_{\ell,V}(\mathsf{Coh}^b(V))$ (where $r_{\ell,V}$ denotes the ℓ-adic realization over V), and $j^*\omega_X \simeq j^!\omega_X \simeq \omega_V$, so $j^*\alpha$ identifies with a map $r_{\ell,V}(\mathsf{Coh}^b(V)) \to \omega_V(\beta)$. Since V is affine and of finite type over S, we can choose a closed immersion $i : V \hookrightarrow V'$, with V' affine and smooth (hence

[6]Strictly speaking, in [BRTV] the equivalence (4) is proved only after push-forward to S but the very same proof shows also the equivalence (4).

[7]Note that since S is excellent, X is excellent so that ω_X exists by a theorem of Gabber ([ILO, Exp. XVII, Th. 0.2]).

regular) over S. Let $h : V' \setminus V \hookrightarrow V'$ be the complementary open immersion. Since V' and $V' \setminus V$ are regular, by Quillen localization and the properties of the nc realization functor $\mathcal{M}^\vee$ (see [BRTV]), we get a cofiber sequence

$$\mathcal{M}^\vee_{V'}(\mathsf{Coh}^b(V)/V') \to \mathbf{B}\mathbb{U}_{X'} \to h_*\mathbf{B}\mathbb{U}_{V'\setminus V},$$

where the notation $\mathsf{Coh}^b(V)/V'$ means that $\mathsf{Coh}^b(V)$ is viewed as a dg-catgeory over V', via i. In other words, $\mathcal{M}^\vee_{V'}(\mathsf{Coh}^b(V)/V') \simeq i_*\mathcal{M}^\vee_V(\mathsf{Coh}^b(V))$. If we apply i^* to this cofiber sequence, and compare what we obtain to the application of i^* to the standard localization sequence

$$i_*i^!\mathbf{B}\mathbb{U}_{V'} \to \mathbf{B}\mathbb{U}_{V'} \to h_*h^*\mathbf{B}\mathbb{U}_{V'} = h_*\mathbf{B}\mathbb{U}_{V'\setminus V},$$

we finally get, after étale ℓ-adic realization, that $\omega_V(\beta) \simeq i^!\mathbb{Q}_\ell(\beta) \simeq r_{\ell,V}(\mathsf{Coh}^b(V))$. This implies that $j^*\alpha$ is also an equivalence.
By (1) and (2), the map in (3) is finally obtained by applying $r_{\ell,X}$ to the inclusion $\mathsf{Perf}(X) \to \mathsf{Coh}^b(X)$. □

Remark 3.3.3 Note that Lemma 3.3.2 applies, in particular, to give a 2-periodic ℓ-adic fundamental class map $\eta_U : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_U(\beta)$ for any open subscheme $U \hookrightarrow X$ over S, whenever X is proper over S.

Definition 3.3.4 *Let X/S be a separated and finite type S-scheme. The* sheaf of singularities of X *is defined to be the cofiber of the 2-periodic ℓ-adic fundamental class morphism η_X (Lemma 3.3.2 (3))*

$$\omega^o_X := \mathit{Cofib}(\eta_X : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_X(\beta)).$$

Remark 3.3.5 By construction of η_X in Lemma 3.3.2, ω^o_X is then the ℓ-adic realization $r_{\ell,X}(\mathsf{Sing}(X))$ of the dg-category $\mathsf{Sing}(X)$, considered as a dg-category over X, and this is true without any regularity hypothesis on X.

Remark 3.3.6 If $p : \mathcal{X} \to S$ is a proper lci map from a *derived* scheme $\mathcal{X}$, we can still define a 2-periodic ℓ-adic fundamental class map $\eta_{\mathcal{X}}$, as in Lemma 3.3.2. This can be done by observing that the pushforward on G-theories along the inclusion of the truncation $\mathrm{t}_0\mathcal{X} \to \mathcal{X}$ is an equivalence, and that p being lci we have a natural inclusion $\mathsf{Perf}(\mathcal{X}) \to \mathsf{Coh}^b(\mathcal{X})$. We further observe that in this case, while the ∞-category $\mathcal{D}_c(\mathcal{X}, \mathbb{Q}_\ell(\beta))$ only depends on the reduced subscheme $(\mathrm{t}_0\mathcal{X})_{red}$, and the same is true for the objects $\mathbb{Q}_{\ell,\mathcal{X}}(\beta)$ and $\omega_{\mathcal{X}}(\beta)$, in contrast, the morphism $\eta_{\mathcal{X}}$ *does* depend on the derived structure on $\mathcal{X}$, and thus it is not a purely topological invariant.

Let us come back to X a regular scheme, proper over S. We have a canonical equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell(\beta))$

$$\nu^{\mathrm{I}}_X(\beta)[1] \simeq \omega^o_{X_s}. \tag{6}$$

This is a reformulation of the equivalence (4), in view of Lemma 3.3.2.

Remark 3.3.7 When X is not regular anymore, but still proper and lci[8] over S, there is nonetheless a natural morphism $\nu^{\mathrm{I}}_X(\beta)[1] \longrightarrow \omega^o_{X_s}$, constructed as follows. Consider again the triangle (3)

$$\nu^{\mathrm{I}}_X \longrightarrow \mathbb{Q}_\ell \longrightarrow i^!_X(\mathbb{Q}_\ell)[2].$$

[8]Note that a morphism of finite type between regular schemes is lci, since we can check that its relative cotangent complex has perfect amplitude in $[-1, 0]$.

On X, we do have the 2-periodic ℓ-adic fundamental class $\eta_X : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_X(\beta)$, and by taking its !-pullback by $i_X^!$, we get a morphism $i_X^!(\mathbb{Q}_\ell)(\beta) \longrightarrow \omega_{X_s}(\beta)$. This produces a sequence of morphisms

$$\nu_X^{\mathrm{I}}(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \longrightarrow i_X^!(\mathbb{Q}_\ell)[2](\beta) = i_X^!(\mathbb{Q}_\ell)(\beta) \longrightarrow \omega_{X_s}(\beta).$$

The resulting composite morphism $\mathbb{Q}_\ell(\beta) \to \omega_{X_s}(\beta)$ is the 2-periodic ℓ-adic fundamental class of X_s. Moreover, by construction, the composition $\nu_X^{\mathrm{I}}(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{X_s}(\beta)$ is canonically the zero map, and this induces the natural morphism

$$\alpha_X : \nu_X^{\mathrm{I}}(\beta)[1] \longrightarrow \omega_{X_s}^o$$

we were looking for. Summing up, the morphism α_X always exists for any proper, lci scheme X over S, and is an equivalence whenever X is regular.

3.4 A Künneth theorem for invariant vanishing cycles

In this section, we consider two separated, finite type S-schemes X and Y, such that X_K and Y_K are smooth over K, and both X and Y are regular and connected. For simplicity[9] we also assume that X and Y are flat over S. We set $Z := X \times_S Y$, and consider this as a scheme over S. We have $Z_s \simeq X_s \times_s Y_s$, and the ∞-category $D_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell)$ comes equipped with pull-back functors

$$p^* : D_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell) \longrightarrow D_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell) \longleftarrow D_c^{\mathrm{I}}(Y_s, \mathbb{Q}_\ell) : q^*.$$

By taking their tensor product, we get an external product functor

$$\boxtimes := p^*(-) \otimes q^*(-) : D_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell) \times D_c(Y_s, \mathbb{Q}_\ell) \longrightarrow D_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell).$$

For two objects $E \in D_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$ and $F \in D_c^{\mathrm{I}}(Y_s, \mathbb{Q}_\ell)$, we can define the Künneth morphism in $D_c(Z_s, \mathbb{Q}_\ell)$

$$\mathsf{k} : (E \boxtimes F)^{\mathrm{I}}[-1] \longrightarrow E^{\mathrm{I}} \boxtimes F^{\mathrm{I}}$$

as follows. Since Grothendieck duality on Z_s is compatible with external products, in order to define k it is enough to define its dual

$$\mathbb{D}(E^{\mathrm{I}}) \boxtimes \mathbb{D}(F^{\mathrm{I}}) \longrightarrow \mathbb{D}((E \boxtimes F)^{\mathrm{I}})[1].$$

By Lemma 3.2.3, the datum of such a morphism is equivalent to that of a morphism

$$\mathbb{D}_I(E)^{\mathrm{I}} \boxtimes \mathbb{D}_I(F)^{\mathrm{I}} \longrightarrow \mathbb{D}_I(E \boxtimes F)^{\mathrm{I}}.$$

We now define k as the map induced by the composite

$$\mathbb{D}_I(E)^{\mathrm{I}} \boxtimes \mathbb{D}_I(F)^{\mathrm{I}} \xrightarrow{\mu_{(-)^{\mathrm{I}}}} (\mathbb{D}_I(E) \boxtimes \mathbb{D}_I(F))^{\mathrm{I}} \xrightarrow{(\mu_{D_I})^{\mathrm{I}}} \mathbb{D}_I(E \boxtimes F)^{\mathrm{I}}$$

where $\mu_{(-)^{\mathrm{I}}}$ is the lax monoidal structure on $(-)^{\mathrm{I}}$, and μ_{D_I} the one on D_I[10].

[9] In the non-flat case, the fiber product of X and Y over S, to be considered below, should be replaced by the derived fiber product.

[10] Note that μ_{D_I} is in fact an equivalence.

Definition 3.4.1 *With the above notations, the* I-invariant convolution *of the two objects* $E \in D_c^{\mathrm{I}}(X_s, \mathbb{Q}_\ell)$ *and* $F \in D_c^{\mathrm{I}}(Y_s, \mathbb{Q}_\ell)$ *is defined to be the cone of the Künneth morphism, and denoted by* $(E \circledast F)^{\mathrm{I}}$*. By definition it sits in a triangle*

$$(E \boxtimes F)^{\mathrm{I}}[-1] \xrightarrow{\mathsf{k}} E^{\mathrm{I}} \boxtimes F^{\mathrm{I}} \longrightarrow (E \circledast F)^{\mathrm{I}}.$$

The main result of this section is the following proposition, relating the I-invariant convolution of vanishing cycles on X and Y to the dualizing complex of Z. It can be also considered as a computation of the ℓ-adic realization of the ∞-category $\mathsf{Sing}(Z)$ of singularities of Z.
Note that, as X and Y are generically smooth over S, so is Z, and thus the 2-periodic ℓ-adic fundamental class map $\eta_Z : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_Z(\beta)$ of Lemma 3.3.2 is an equivalence over the generic fiber. Therefore, ω_Z^o is supported on Z_s, so that it can (and will) be considered canonically as an object in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell)$.

Theorem 3.4.2 *With the above notations and assumptions, there is a canonical equivalence*

$$\omega_Z^o \simeq (\nu_X \circledast \nu_Y)^{\mathrm{I}}(\beta)$$

in $D_c(Z_s, \mathbb{Q}_\ell(\beta))$.

Proof. The proof of this theorem will combine various exact triangles together with an application of Gabber's Künneth formula for nearby cycles.

To start with, the vanishing cycles ν_Z of Z sits in an exact triangle in $\mathcal{D}_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell \longrightarrow \psi_Z \longrightarrow \nu_Z\ ,$$

where $\psi_Z = \psi_Z(\mathbb{Q}_\ell)$ is the complex of nearby cycles of Z over S. According to [Be-Be, Lemma 5.1.1] or [Il1, 4.7]), we have a natural equivalence in $\mathcal{D}_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell)$, induced by external product

$$\psi_Z \simeq \psi_X \boxtimes \psi_Y.$$

The object ν_Z then becomes the cone of the tensor product of the two morphisms in $\mathcal{D}_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell \longrightarrow p^*(\psi_X) \qquad \mathbb{Q}_\ell \longrightarrow q^*(\psi_Y)$$

where p and q are the two projections from Z down to X and Y, respectively. Now, cones of tensor products are computed via the following well known lemma.

Lemma 3.4.3 *Let* $\mathcal{C}$ *be a stable symmetric monoidal* ∞*-category, and*

$$u : x \to y \qquad v : x' \to y'$$

two morphisms. Let $C(u)$ *be the cone of* u*,* $C(v)$ *be the cone of* v*, and* $C(u \otimes v)$ *the cone of the tensor product* $u \otimes v : x \otimes x' \to y \otimes y'$*. Then, there exists a natural exact triangle*

$$C(u) \otimes x' \bigoplus x \otimes C(v) \longrightarrow C(u \otimes v) \longrightarrow C(u) \otimes C(v).$$

Proof of lemma. Factor $u \otimes v$ as $x \otimes x' \xrightarrow{u \otimes id} y \otimes x' \xrightarrow{id \otimes v} y \otimes y'$, and apply the octahedral axiom to the triangles

$$x \otimes x' \xrightarrow{u \otimes id} y \otimes x' \xrightarrow{f} C(u) \otimes x'$$

$$y \otimes x' \xrightarrow{id \otimes v} y \otimes y' \longrightarrow y \otimes C(v) \xrightarrow[{[1]}]{d'} y \otimes x'[1] \ ,$$

to get a triangle

$$C(u) \otimes x' \longrightarrow C(u \otimes v) \longrightarrow y \otimes C(v) \xrightarrow[{[1]}]{\theta} C(u) \otimes x'[1]$$

together with the compatibility $\theta = f[1] \circ d'$. Now observe that $\theta \circ (u \otimes \mathrm{id}_{C(v)}) = 0$, and apply the octahedral axiom to the triangles

$$x \otimes C(v) \xrightarrow{u \otimes id} y \otimes C(v) \longrightarrow C(u) \otimes C(v) \ ,$$

$$y \otimes C(v) \xrightarrow{\theta} C(u) \otimes x'[1] \xrightarrow{f} C(u \otimes v)[1]$$

to conclude. ♢

Lemma 3.4.3 implies the existence of a natural exact triangle in $\mathcal{D}_c^{\mathrm{I}}(Z_s, \mathbb{Q}_\ell)$

$$\nu_X \boxplus \nu_Y \longrightarrow \nu_Z \longrightarrow \nu_X \boxtimes \nu_Y,$$

which, by taking I-invariants, yields an exact triangle in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell)$

$$(\mathbf{T1}) \qquad \nu_X^{\mathrm{I}} \boxplus \nu_Y^{\mathrm{I}} \longrightarrow \nu_Z^{\mathrm{I}} \longrightarrow (\nu_X \boxtimes \nu_Y)^{\mathrm{I}}.$$

Lemma 3.4.4 *Let k be an algebraically closed field, $s := \operatorname{Spec} k$, $p_X : X \to s$, $p_Y : Y \to s$ be proper morphisms of schemes, and $p_1 : Z := X \times_s Y \to X$, $p_2 : Z := X \times_s Y \to Y$ the natural projections. If ω_Z, ω_X, ω_Y denote the $\mathbb{Q}_\ell$-adic dualizing complexes of Z, X, and Y, respectively, there is a canonical equivalence*

$$a : p_1^* \omega_X \otimes p_2^* \omega_Y \longrightarrow \omega_Z.$$

Proof of lemma. We first exhibit the map a. We denote simply by $\underline{\mathrm{Hom}}_T(-,-)$ the derived internal hom in $\mathcal{D}_c(T, \mathbb{Q}_\ell)$ (so that $D := \underline{\mathrm{Hom}}_Z(-, \omega_Z)$ is the $\mathbb{Q}_\ell$-adic duality on Z). By adjunction, giving a map a is the same thing as giving a map $\omega_X \to (p_1)_* \underline{\mathrm{Hom}}_Z(p_2^* \omega_Y, p_2^! \omega_Y)$. Since p_X is proper, by [SGA4-III, Exp XVIII, 3.1.12], we have a canonical equivalence

$$(p_1)_* \underline{\mathrm{Hom}}_Z(p_2^* \omega_Y, p_2^! \omega_Y) \longrightarrow p_X^! (p_Y)_* \underline{\mathrm{Hom}}_Y(\omega_Y, \omega_Y).$$

Therefore, we are left to define a map

$$\omega_X \simeq p_X^! \mathbb{Q}_\ell \to p_X^! (p_Y)_* \underline{\mathrm{Hom}}_Y(\omega_Y, \omega_Y),$$

and we take $p_X^!(\alpha)$ for this map, where α is the adjoint to the canonical map $\mathbb{Q}_\ell \simeq p_Y^* \mathbb{Q}_\ell \to \underline{\mathrm{Hom}}_Y(\omega_Y, \omega_Y)$. One can then prove that a is an equivalence, by checking it stalkwise. □

By Lemma 3.4.4, the dualizing complex $\omega_{Z_s} = (Z_s \to s)^! \mathbb{Q}_\ell$ of $Z_s \simeq X_s \times_s Y_s$ is canonically equivalent to $\omega_{X_s} \boxtimes \omega_{Y_s}$, and, through this equivalence, the virtual fundamental class of Z_s

$$\eta_{Z_s} : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{Z_s}(\beta) \simeq \omega_{X_s} \boxtimes \omega_{Y_s}(\beta)$$

is simply given by the external tensor product of the virtual fundamental classes of X_s and Y_s. By Lemma 3.4.3 we thus get a second exact triangle in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell(\beta))$

$$(\mathbf{T2}) \qquad \omega^o_{X_s} \boxplus \omega^o_{Y_s} \longrightarrow \omega^o_{Z_s} \longrightarrow \omega^o_{X_s} \boxtimes \omega^o_{Y_s}.$$

There is a morphism from the triangle (T1) to the triangle (T2) which is defined using the natural morphism

$$\alpha_Z : \nu^{\mathrm{I}}_Z(\beta)[1] \longrightarrow \omega^o_{Z_s}$$

introduced in Remark 3.3.7. In fact, Z is proper and lci over S (since X/S is flat and lci[11], and being lci is stable under flat base change and composition), and the map α_Z is defined for any proper, lci scheme Z over S, being an equivalence when Z is regular with smooth generic fiber. Using the compatible maps α_X, α_Y and α_Z we get a commutative square

$$\begin{tikzcd}
\nu^{\mathrm{I}}_X(\beta)[1] \boxplus \nu^{\mathrm{I}}_Y(\beta)[1] \arrow[r] \arrow[d, "\alpha_Z \oplus \alpha_Y"'] & \nu^{\mathrm{I}}_Z(\beta)[1] \arrow[d, "\alpha_Z"] \\
\omega^o_{X_s} \boxplus \omega^o_{Y_s} \arrow[r] & \omega^o_{Z_s}.
\end{tikzcd}$$

This produces a morphism from triangle $(T1)$ (tensored by $\mathbb{Q}_\ell(\beta)[1]$) to triangle $(T2)$

$$\begin{tikzcd}
\nu^{\mathrm{I}}_X(\beta)[1] \boxplus \nu^{\mathrm{I}}_Y(\beta)[1] \arrow[r] \arrow[d] & \nu^{\mathrm{I}}_Z(\beta)[1] \arrow[r] \arrow[d, "\alpha_Z"] & (\nu_X \boxtimes \nu_Y)^{\mathrm{I}}(\beta)[1] \arrow[d] \\
\omega^o_{X_s} \boxplus \omega^o_{Y_s} \arrow[r] & \omega^o_{Z_s} \arrow[r] & \omega^o_{X_s} \boxtimes \omega^o_{Y_s}.
\end{tikzcd} \tag{7}$$

Since X and Y are regular with smooth generic fibers, the maps α_X and α_Y are equivalences, therefore the leftmost vertical morphism is also an equivalence. Thus the right hand square is a cartesian square.

Now, the rightmost vertical morphism can be written, again using the equivalences α_X and α_Y, as

$$(\nu_X \boxtimes \nu_Y)^{\mathrm{I}}(\beta)[1] \longrightarrow (\nu^{\mathrm{I}}_X(\beta)[1]) \boxtimes_{\mathbb{Q}_\ell(\beta)} (\nu^{\mathrm{I}}_Y(\beta)[1]) \simeq (\nu^{\mathrm{I}}_X \boxtimes \nu^{\mathrm{I}}_Y)[2](\beta)$$

This morphism is the Künneth map k of Definition 3.4.1 tensored by $\mathbb{Q}_\ell[2](\beta) \simeq \mathbb{Q}_\ell(\beta)$, and thus its cone is $(\nu_X \circledast \nu_Y)^{\mathrm{I}}(\beta)$. In order to finish the proof of the proposition it then remains to show that the cone of the middle vertical morphism in (7)

$$\alpha_Z : \nu^{\mathrm{I}}_Z(\beta)[1] \longrightarrow \omega^o_{Z_s}$$

can be canonically identified with ω^o_Z.

[11] Again, note that a morphism of finite type between regular schemes is lci, since we can check that its relative cotangent complex has perfect amplitude in $[-1, 0]$.

For this, we remind the exact triangle (3) tensored by $\mathbb{Q}_\ell(\beta)$

$$\mathbb{Q}_\ell(\beta) \longrightarrow i_Z^!(\mathbb{Q}_\ell(\beta)) \longrightarrow \nu_Z^{\mathrm{I}}(\beta)[1].$$

Using the fundamental class $\eta_Z : \mathbb{Q}_\ell(\beta) \to \omega_Z(\beta)$, we get a morphism of triangles

$$\begin{tikzcd} \mathbb{Q}_\ell(\beta) \arrow[r] \arrow[d, "id"'] & i_Z^!(\mathbb{Q}_\ell(\beta)) \arrow[r] \arrow[d, "i_Z^!(\eta_Z)"] & \nu_Z^{\mathrm{I}}(\beta)[1] \arrow[d] \\ \mathbb{Q}_\ell(\beta) \arrow[r] & i_Z^!(\omega_Z(\beta)) \arrow[r] & \omega_{Z_s}^o. \end{tikzcd}$$

The right hand square is thus cartesian, so that the cone of the vertical morphism on the right is canonically identified with the cone of the vertical morphism in the middle. By definition, this cone is $i_Z^!(\omega_Z^o)$. Since Z_K is smooth over K, the ℓ-adic complex ω_Z^o is supported on Z_s, and thus $i_Z^!(\omega_Z^o)$ is canonically equivalent to ω_Z^o, and we conclude. $\square$

Corollary 3.4.5 *We keep the same notations and assumptions as in Proposition 3.4.2, and we further assume one of the following conditions:*

1. *the* I*-action on ν_X and on ν_Y is tame, or*
2. *the reduced scheme $(X_s)_{red}$ is smooth over k.*

Then, there is a canonical equivalence

$$\omega_Z^o \simeq (\nu_X \boxtimes \nu_Y)^{\mathrm{I}}(\beta)$$

in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell(\beta))$.

Proof. It is enough to prove that under any one of the two assumptions, $(\nu_X \circledast \nu_Y)^{\mathrm{I}}$ is canonically equivalent to $(\nu_X \boxtimes \nu_Y)^{\mathrm{I}}$. If the scheme $(X_s)_{red}$ is smooth over k, then we have $\nu_X^{\mathrm{I}}(\beta) = 0$. Indeed, triangle (3) can be then re-written

$$\nu_X^{\mathrm{I}} \longrightarrow \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[2n+2]$$

where n is the dimension of X_s. By tensoring by $\mathbb{Q}_\ell(\beta)$, we get a triangle

$$\nu_X^{\mathrm{I}}(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \xrightarrow{b} \mathbb{Q}_\ell[2n+2](\beta) \simeq \mathbb{Q}_\ell(\beta)$$

where b is an equivalence. Therefore, $\nu_X^{\mathrm{I}}(\beta) = 0$, and, by definition of I-invariant convolution, this implies that $(\nu_X \circledast \nu_Y)^{\mathrm{I}} \simeq (\nu_X \boxtimes \nu_Y)^{\mathrm{I}}$.

Assume now that the action of I on ν_X and ν_Y is tame. This means that the action of I factors through the natural quotient $\mathrm{I} \longrightarrow \mathrm{I}_t$, where I_t is the tame inertia group, which is canonically isomorphic to $\hat{\mathbb{Z}}'$, the prime-to-p part of the profinite completion of $\mathbb{Z}$. As we have chosen a topological generator T of I_t (see Section 3.1), the actions of I are then completely caracterized by the automorphisms T on

ν_X and ν_Y. Moreover, ν_X^{I} is then naturally equivalent to the homotopy fiber of $(1-T) : \nu_X \to \nu_X$, and similarly for ν_Y^{I}. From this it is easy to see that the Künneth map

$$(\nu_X \boxtimes \nu_Y)^{\mathrm{I}}[-1] \longrightarrow \nu_X^{\mathrm{I}} \boxtimes \nu_Y^{\mathrm{I}}$$

fits in an exact triangle

$$(\nu_X \boxtimes \nu_Y)^{\mathrm{I}}[-1] \longrightarrow \nu_X^{\mathrm{I}} \boxtimes \nu_Y^{\mathrm{I}} \longrightarrow (\nu_X \boxtimes \nu_Y)^{\mathrm{I}}$$

where the second morphism is induced by the lax monoidal structure on $(-)^{\mathrm{I}}$. We conclude that there is a natural equivalence $(\nu_X \circledast \nu_Y)^{\mathrm{I}} \simeq (\nu_X \boxtimes \nu_Y)^{\mathrm{I}}$. $\square$

4 Künneth formula for dg-categories of singularities

4.1 The monoidal dg-category $\mathcal{B}$ and its action

We keep our standing assumptions: A is an excellent strictly henselian dvr with perfect residue field k and fraction field K. We let $S = \mathrm{Spec}, A$ and $s = \mathrm{Spec}\, k$, as usual, and choose an uniformizer π of A.

We let $G := s \times_S^h s$ (derived fiber product), considered as a derived scheme over S. The derived scheme G has a canonical structure of groupoid in derived schemes acting on s. The composition in the groupoid G induces a convolution monoidal structure on the dg-category of coherent complexes on G

$$\odot : \mathsf{Coh}^b(G) \otimes_A \mathsf{Coh}^b(G) \longrightarrow \mathsf{Coh}^b(G).$$

More explicitly, we have a map of derived schemes

$$G \times_s G \xrightarrow{q} G,$$

defined as the projection on the first and third components $s \times_S s \times_S s \to s \times_S s$. We then define $\odot$ by the formula

$$E \odot F := q_*(E \boxtimes_s F)$$

for two coherent complexes E and F on G. More generally, if $X \longrightarrow S$ is any scheme, with special fiber X_s (possibly a derived scheme, by taking the derived fiber at s), the groupoid G acts naturally on X_s via the natural projection

$$q_X : G \times_s X_s \simeq (s \times_S s) \times_s (s \times_S X) \simeq s \times_S X_s \longrightarrow X_s.$$

This defines an external action

$$\odot : \mathsf{Coh}^b(G) \otimes_A \mathsf{Coh}^b(X_s) \longrightarrow \mathsf{Coh}^b(X_s)$$

by $E \odot M := (q_X)_*(E \boxtimes_s M)$.

The homotopy coherences issues for the above $\odot$-structures can be handled using the fact that the construction $Y \mapsto \mathsf{Coh}^b(Y)$ is in fact a symmetric lax monoidal ∞-functor from a certain ∞-category of correspondences between derived schemes to the ∞-category of dg-categories over A. As a result, $\mathsf{Coh}^b(G)$ is endowed with a natural structure of a monoid in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$, and that, for any X/S, $\mathsf{Coh}^b(X_s)$ is naturally a module over $\mathsf{Coh}^b(G)$ in $\mathbf{dgCat}_A$. However, for our purposes it will be easier and more efficient to provide explicit models for both $\mathsf{Coh}^b(G)$ and its action on $\mathsf{Coh}^b(X_s)$. This will be done locally in the Zariski topology in a similar spirit to [BRTV, Section 2]; the global construction will then be obtained by a rather straightforward gluing procedure.

4.1.1 The monoidal dg-categories $\mathcal{B}^+$ and $\mathcal{B}$

Let K_A be the Koszul commutative A-dg-algebra of A with respect to π

$$K_A : \ A \xrightarrow{\pi} A$$

sitting in degrees $[-1, 0]$. The canonical generator of K_A in degree -1 will be denoted by h. In the same way, we define the commutative A-dg-algebra

$$K_A^2 := K_A \otimes_A K_A$$

which is the Koszul dg-algebra of A with respect to the sequence (π, π). As a commutative graded A-algebra, K_A^2 is $Sym_A(A^2[1])$, and it is endowed with the unique multiplicative differential sending the two generators h and h' in degree -1 to π (and hh' to $\pi \cdot h' - \pi \cdot h$).

Moreover, K_A^2 has a canonical structure of *Hopf algebroid* over K_A, in which the source and target map are the two natural inclusions $K_A \longrightarrow K_A^2$, whereas the unit is given by the multiplication $K_A^2 \to K_A$. The composition (or coproduct) in this Hopf algebroid structure is given by

$$\Delta := \mathrm{id} \otimes 1 \otimes \mathrm{id} : K_A \otimes_A K_A = K_A^2 \longrightarrow K_A^2 \otimes_{K_A} K_A^2 = K_A \otimes_A K_A \otimes_A K_A.$$

Finally, the antipode is the automorphism of K_A^2 exchanging the two factors K_A. This structure of Hopf algebroid endows the dg-category $\mathsf{Mod}(K_A^2)$, of $K^2(A)$-dg-modules, with a unital and associative monoidal structure $\odot$. It is explicitely given for two object E and F, by the formula

$$E \odot F := E \otimes_{K_A} F$$

where the $K_A \otimes_A K_A \otimes_A K_A$-module on the rhs is considered as a K_A^2 module via the map Δ. The unit of this monoidal structure is the object K_A, viewed as a K_A^2-module by the multiplication $K_A^2 \to K_A$. It is not hard to see that $\odot$ preserves cofibrant K_A^2-dg-modules; more generally it makes $\mathsf{Mod}(K_A^2)$ into a monoidal model category in the sense of [Ho, Ch. 4]. Note however that the unit K_A is *not* cofibrant in this model structure.

Definition 4.1.1 *The* monoidal dg-category $\mathcal{B}^+_{str}$ *is defined to be* $\mathsf{Mod}^c(K_A^2)$*, the dg-category of all cofibrant dg-modules over* K_A^2 *which are perfect over* A*, together with the unit object* K_A*. It is endowed with the monoidal structure* $\odot$ *described above.*

By Appendix A, the localization of the dg-category $\mathcal{B}^+_{str}$ along all quasi-isomorphisms, defines a monoidal dg-category.

Definition 4.1.2 *The* monoidal dg-category $\mathcal{B}^+$ *is defined to be the localization*

$$W_{eq}^{-1}(\mathcal{B}^+_{str}),$$

where W_{eq} *is the set of quasi-isomorphisms. It is naturally a unital and associative monoid in the symmetric monoidal* ∞*-category* $\mathbf{dgCat}_A$*.*

Remark 4.1.3 Note that $\mathcal{B}^+$ defined above is a model for $\mathsf{Coh}^b(G)$, for our derived groupoid $G = s \times_S s$ above. Indeed, the commutative dga K_A^2 is quasi-isomorphic to the normalization of the simplicial algebra $k \otimes_A^{\mathbb{L}} k$. Now, since $G \to S$ is a closed immersion, a quasi-coherent complex E on G is coherent iff its direct image on S is coherent, hence perfect, S being regular. In particular, we have that $\mathsf{Coh}^b(G)$ is equivalent to the dg-category of all cofibrant K_A^2-dg-modules which are perfect over A The latter dg-category is also naturally equivalent to the localization of $\mathcal{B}^+_{str}$ along quasi-isomorphisms[12]

We now introduce the weak monoidal dg-category $\mathcal{B}$, defined as a further localization of $\mathcal{B}^+$. This will be our main "base monoid" for the module dg-categories we will be interested in.

Definition 4.1.4 *The* weak monoidal dg-category $\mathcal{B}$ *is defined to be the localization*

$$\mathcal{B} := L_W(\mathcal{B}^+),$$

where W is the set of morphisms in $\mathcal{B}^+$ whose cone is perfect as a K_A^2-dg-module.

As $\mathcal{B}^+$ it itself defined as a localization of $\mathcal{B}^+_{str}$, if W_{pe} denotes the morphisms in $\mathcal{B}^+_{str}$ whose cones are perfect over K_A^2, we have $W_{\mathrm{eq}} \subseteq W_{\mathrm{pe}}$, then $\mathcal{B}$ can also be realized as localization of $\mathcal{B}^+_{str}$ directly, as

$$\mathcal{B} \simeq L_{W_{\mathrm{pe}}} \mathcal{B}^+_{str}.$$

Since the monoidal structure $\odot$ is compatible with W_{pe}, this presentation implies that $\mathcal{B}$ comes equipped with a natural structure of an associative and unital monoid in $\mathbf{dgCat}_A$. Note, moreover, that $\mathcal{B}$ comes equipped with a natural morphism of monoids

$$\mathcal{B}^+ \longrightarrow \mathcal{B}$$

given by the localization map.

4.1.2 The local actions

Let now $X = \operatorname{Spec} R$ be a regular scheme flat over A. As done above for the monoid structure on $\mathcal{B}^+$, we will define a *strict model* for $\mathsf{Coh}^b(X_s)$, together with a *strict model* for the $\mathsf{Coh}^b(G)$-action on $\mathsf{Coh}^b(X_s)$. In order to do this, let K_R be the Koszul dg-algebra of R with respect to π, which comes equipped with a natural map $K_A \longrightarrow K_R$ of cdga's over A. We consider $\mathsf{Mod}^c(K_R)$, the dg-category of all cofibrant K_R-dg-modules which are perfect as R-modules (note that R is regular, and see Remark 4.1.3). The same argument as in Remark 4.1.3 then shows that this dg-category is naturally equivalent to $\mathsf{Coh}^b(X_s)$. Moreover, $\mathsf{Mod}^c(K_R)$ has a structure of a $\mathcal{B}^+_{str}$-module dg-category defined as follows. For $E \in \mathcal{B}^+_{str}$, and $M \in \mathsf{Mod}^c(K_R)$, we can define

$$E \odot M := E \otimes_{K_A} M,$$

where, in the rhs, we have used the "right" K_A-dg-module structure on E, i.e. the one induced by the composition

$$K_A \xrightarrow{\sim} A \otimes_A K_A \xrightarrow{\mathrm{id} \otimes u} K_A \otimes_A K_A\ ,$$

$u : A \to K_A$ being the canonical map. As E is either the unit or it is cofibrant over K_A^2 (and thus cofibrant over K_A), $E \otimes_{K_A} M$ is again a cofibrant K_R-module, and again perfect over R, i.e. $E \odot M \in \mathsf{Mod}^c(K_R)$. By localization along quasi-isomorphisms, (see Proposition 4.1.5 for details) we

[12] We leave to the reader to check that adding the unit objet K_A does not change the localization.

obtain that $\mathsf{Coh}^b(X_s)$ carries a natural $\mathcal{B}^+$-module structure as an object in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$.

We now apply a similar argument in order to define a *$\mathcal{B}$-action on* $\mathsf{Sing}(X_s)$. Let again $X = Spec\, R$ be a regular scheme over A, and consider $\mathsf{Mod}^c(K_R)$ as a $\mathcal{B}^+_{str}$-module dg-category as above. Let $W_{R,\mathrm{pe}}$ be the set of morphisms in $\mathsf{Mod}^c(K_R)$ whose cones are perfect dg-modules over K_R. By localization we then get a $\mathcal{B}$-module structure on $L_{W_{R,\mathrm{pe}}}\mathsf{Mod}^c(K_R)$. Note that the localization $L_{W_{R,\mathrm{pe}}}\mathsf{Mod}^c(K_R)$ is a model for the dg-category $\mathsf{Sing}(X_s)$, which therefore comes equipped with the structure of a $\mathcal{B}$-module in $\mathbf{dgCat}_A$.

We gather the details of above constructions in the following

Proposition 4.1.5 *Let $X = \mathrm{Spec}\, R$ be a regular scheme, flat over $S = \mathrm{Spec}\, A$, and X_s its special fiber. Then there is a canonical $\mathcal{B}^+$-module structure (resp., $\mathcal{B}$-module structure) on $\mathsf{Coh}^b(X_s)$ (resp., on $\mathsf{Sing}(X_s)$), inside $\mathbf{dgCat}_A$.*

Proof. This is an easy application of the localization results presented in Appendix A.
We first treat the case of $\mathcal{B}^+$ and $T := \mathsf{Coh}^b(X_s)$. If $T^{\mathrm{str}} := \mathsf{Mod}^c(X_r)$ and $W_{T,\mathrm{eq}}$ denotes the quasi-isomorphisms in T^{str}, we have $W^{-1}_{T,\mathrm{eq}}T^{\mathrm{str}} \simeq T$ in $\mathbf{dgCat}_A$. Analogously, $W^{-1}_{\mathrm{eq}}\mathcal{B}^+_{\mathrm{str}} \simeq \mathcal{B}^+$ in $\mathbf{dgCat}_A$. In order to apply the localization result of Appendix A, we need to prove that the tensor product $\odot : \mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to T^{\mathrm{str}}$ (defined in 4.1.2) sends $W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}}$ to $W_{T,\mathrm{eq}}$. If $L, L' \in \mathcal{B}^+_{\mathrm{str}}$, $E, E' \in T^{\mathrm{str}}$, and $w' : L \to L'$, $w : E \to E'$ are quasi-isomorphisms, then $w' \odot id_E$ is again a quasi-isomorphism (because L, and L' are cofibrant over K^2_A, hence over K_A, and thus w' is in fact a homotopy equivalence), and the same is true for $id_L \odot w$ (since L is cofibrant over K_A). Therefore, $\odot$ does send $W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}}$ to $W_{T,\mathrm{eq}}$, and there is an induced canonical map $(W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}})^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W^{-1}_{T,\mathrm{eq}}T^{\mathrm{str}} \simeq \mathsf{Coh}^b(X_s)$. By composing this with the natural equivalence $(W_{\mathrm{eq}}\otimes id \cup id \otimes W_{T,\mathrm{eq}})^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W^{-1}_{\mathrm{eq}}\mathcal{B}^+_{\mathrm{str}} \otimes_A W^{-1}_{T,\mathrm{eq}}T^{\mathrm{str}}$ (Appendix A), we finally get our $\mathcal{B}^+$-module structure on $\mathsf{Coh}^b(X_s)$ inside $\mathbf{dgCat}_A$.
We now treat the case of $\mathcal{B}$ and $T = \mathsf{Sing}(X_s)$. Here, we consider the pairs $(T^{\mathrm{str}} = \mathsf{Mod}^c(X_r), W_T)$ where W_T are the maps in T^{str} whose cones are perfect over K_R, and $(\mathcal{B}^*_{\mathrm{str}}, W)$, where W are the maps in $\mathcal{B}^+_{\mathrm{str}}$ whose cones are perfect over K^2_A. We have $W^{-1}\mathcal{B}^*_{\mathrm{str}} \simeq \mathcal{B}$, and $W^{-1}_T T^{\mathrm{str}} \simeq \mathsf{Sing}(X_s)$, and we need to prove that both $W \odot id$ and $id \odot W_T$ are contained in W_T. Let $u : L \to L' \in W$ and $v : E \to E' \in W_T$, and $C(-)$ denote the cone construction. We have $C(id_L \odot v) \simeq L \otimes_{K_A} C(v)$ and $C(u \odot id_E) \simeq C(u) \otimes_{K_A} E$. By hypothesis, L is perfect over A hence over K_A (since K_A is perfect over A), and $C(v)$ is perfect over K_A, since $X \to S$ is lci (as a map of finite type between regular schemes), and thus $K_A \to K_R$ is derived lci (recall that X/S is flat so that X_s is also the derived fiber), and pushforward along a lci map preserves perfect complexes. So, $C(id_L \odot v) \in W_T$. On the other hand, the "right-hand" map $K_A \to K^2_A$ (with respect to which L and L' are viewed as K_A-dg-modules in the definition of $\odot$) is derived lci, hence $C(u)$ is perfect over K_A, being perfect over K^2_A by hypothesis. Moreover, since $s \to S$ is a closed immersion, E is perfect over K_A iff $(X \to S)_*E$ is perfect ($=$ coherent, S being regular) over S; but $(X_s \to X)_*E$ is perfect by hypothesis, and pushforward along $X \to S$ preserves perfect complexes, since X/S is lci. Therefore $C(u \odot id_E) \in W_T$, and we deduce that $\odot$ does send $W \otimes id \cup id \otimes W_T$ to W_T. This gives us an induced canonical map $(W \otimes id \cup id \otimes W_T)^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W^{-1}_T T^{\mathrm{str}} \simeq \mathsf{Sing}(X_s)$. By composing this with the natural equivalence $(W \otimes id \cup id \otimes W_T)^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A W^{-1}_T T^{\mathrm{str}}$ (Appendix A), we finally get our $\mathcal{B}$-module structure on $\mathsf{Sing}(X_s)$ inside $\mathbf{dgCat}_A$.

$\square$

Lemma 4.1.6 *The natural morphism*

$$\mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathcal{B} \longrightarrow \mathsf{Sing}(X_s)$$

is an equivalence of $\mathcal{B}$*-modules.*

Proof. This is a reformulation of [BRTV, Proposition 2.43]. □

4.1.3 The global actions

We now let X be a regular scheme, not necessarily affine anymore. We have by Zariski descent

$$\mathsf{Coh}^b(X_s) \simeq \lim_{Spec\, R \subset X} \mathsf{Mod}^c(K_R),$$

where the limit is taken over all affine opens $Spec\, R$ of X. The right hand side of the above equivalence is a limit of dg-categories underlying $\mathcal{B}^+$-module structures (Proposition 4.1.5). As the forgetful functor from $\mathcal{B}^+$-modules to dg-categories reflects limits, this endows $\mathsf{Coh}^b(X_s)$ with a unique structure of $\mathcal{B}^+$-module.

In the same way, we have Zariski descent for $\mathsf{Sing}(X_s)$ in the sense that

$$\mathsf{Sing}(X_s) \simeq \lim_{Spec\, R \subset X} L_{W_{R,\mathrm{pe}}}(\mathsf{Mod}^c(K_R)).$$

The right hand side of the above equivalence is a limit of dg-categories underlying $\mathcal{B}$-module structures (Proposition 4.1.5). As the forgetful functor from $\mathcal{B}$-modules to dg-categories reflects limits, this makes the dg-category $\mathsf{Sing}(X_s)$ into a $\mathcal{B}$-module in a natural way.

An important property of these $\mathcal{B}^+$ and $\mathcal{B}$-module structures is given in the following proposition.

Proposition 4.1.7 *Let X be a flat regular scheme over S.*

1. *The $\mathcal{B}^+$-module structure on $\mathsf{Coh}^b(X_s)$ is cotensored.*
2. *The $\mathcal{B}$-module structure on $\mathsf{Sing}(X_s)$ is cotensored.*

Proof. This follows from Remark 2.1.4 since the monoidal triangulated dg-categories $\mathcal{B}^+$ and $\mathcal{B}$ are both generated by their unit objects. □

4.2 Künneth formula for dg-categories of singularities

In the previous section we have seen that, for any regular scheme X over S, the dg-category $\mathsf{Sing}(X_s)$ are equipped with a natural $\mathcal{B}$-module structure. In this section we compute tensor products of dg-categories of singularities over $\mathcal{B}$.

From a general point of view, let $T \in \mathbf{dgCat}_A$ be a $\mathcal{B}$-module, and assume that T is also *co-tensored* over $\mathcal{B}$ (Def. 4.1.7). Then T^o has a natural structure of a $\mathcal{B}^{\otimes-op}$-module given by co-tensorisation. By Proposition 4.1.7, we may take $T = \mathsf{Sing}(X_s)$, so that T^o is a $\mathcal{B}^{\otimes-op}$-module. In particular, if we have another regular scheme Y, we are entitled to consider the tensor product

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s),$$

which is a well defined object in $\mathbf{dgCat}_A$. We further assume, for simplicity, that X and Y are also flat over S. The main result of this section is the following proposition, which is a categorical counterpart of our Künneth formula for vanishing cycles (Proposition 3.4.2).

Theorem 4.2.1 *Let X and Y be two regular schemes, flat over S, with smooth generic geometric fibers. There is a natural equivalence in $\mathbf{dgCat}_A$*

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \mathsf{Sing}(X \times_S Y).$$

Proof. As $\mathcal{B}$ is a localization of $\mathcal{B}^+$, the natural ∞-functor on enriched dg-categories (which is lax symmetric monoidal)

$$\mathbf{dgCat}_{\mathcal{B}} \longrightarrow \mathbf{dgCat}_{\mathcal{B}^+}$$

is fully faithful. Moreover, it commutes with tensor products in the following sense: for two objects T and T' in $\mathbf{dgCat}_{\mathcal{B}}$, there is a natural equivalence $T^o \otimes_{\mathcal{B}} T' \simeq T^o \otimes_{\mathcal{B}^+} T'$. Therefore, in order to prove the theorem it enough to construct an equivalence of dg-categories over A

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Sing}(Y_s) \simeq \mathsf{Sing}(X \times_S Y).$$

Now, we claim that the result is local on $Z := X \times_S Y$. Indeed, we have two prestacks of dg-categories on the small Zariski site Z_{zar}:

$$U \times_S V \mapsto \mathsf{Sing}(U_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(V_s) \qquad U \times_S V \mapsto \mathsf{Sing}(U \times_S V)$$

for any two affine opens $U \subset X$ and $V \subset Y$. These two prestacks are stacks of dg-categories, and thus we have equivalences in $\mathbf{dgCat}_A$

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \lim_{U \subset X, V \subset Y} \mathsf{Sing}(U_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(V_s) \tag{8}$$

$$\mathsf{Sing}(X \times_S Y) \simeq \lim_{U \subset X, V \subset Y} \mathsf{Sing}(U \times_S V). \tag{9}$$

The stack property (9) is proved in [BRTV, 2.3] (where it is moreover shown that this is a stack for the h-topology). The stack property (8) is then a consequence of the same descent argument for dg-categories of singularities. Indeed, we have the following lemma.

Lemma 4.2.2 *Let Z be an S-scheme, and F be a stack of $\mathcal{O}_Z$-linear dg-categories. Assume that F is a $\mathcal{B}^{\otimes-op}$-module stack*[13]*, and let T_0 be a $\mathcal{B}$-module dg-category. Then, the prestack $F \otimes_{\mathcal{B}} T_0$ of dg-categories of Z_{Zar}, sending $W \subset Z$ to $F(W) \otimes_{\mathcal{B}} T_0$ is a stack.*

[13] I.e., F is a stack on Z_{Zar} with values in the ∞-category of $\mathcal{B}^{\otimes-op}$-modules in $\mathbf{dgCat}_A$.

Proof of Lemma 4.2.2. This is an application of the main result of [To2]. We denote, as usual, by $\widehat{T} := \mathbb{R}\underline{\mathcal{H}om}(T, \widehat{A})$ the (non-small) dg-category of all T^o-dg-modules. By the main result of [To2], the dg-category

$$\lim_{W \subset Z} (\widehat{F(W) \otimes_{\mathcal{B}}} T_0)$$

is compactly generated, and its dg-category of compact objects is equivalent in $\mathbf{dgCat}_A$ to $\lim_{W\subset Z} F(W) \otimes_{\mathcal{B}} T_0$. Moreover, we have

$$(\widehat{F(W) \otimes_{\mathcal{B}}} T_0) \simeq \widehat{F(W)} \widehat{\otimes}_{\widehat{\mathcal{B}}} \widehat{T}_0,$$

where $\widehat{\otimes}$ is the symmetric monoidal structure on presentable dg-categories. As $\widehat{\otimes}$ is rigid when restricted to compactly generated dg-categories, we have that $\widehat{\otimes}_{\widehat{\mathcal{B}}}$ distributes over limits on both factors, and we have

$$(\widehat{F(Z) \otimes_{\mathcal{B}}} T_0) \simeq \lim_{W\subset Z} \widehat{F(W)} \widehat{\otimes}_{\widehat{\mathcal{B}}} \widehat{T}_0.$$

Passing to the sub-dg-categories of compact objects, we find that

$$F(Z) \otimes_{\mathcal{B}} T_0 \simeq \lim_{W\subset Z} F(W) \otimes_{\mathcal{B}} T_0$$

which is the statement of the lemma. $\diamond$

Lemma 4.2.2 immediately implies the stack property (8).
We are thus reduced to the case where X and Y are both affine, and we have to produce an equivalence

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \mathsf{Sing}(X \times_S Y)$$

that is compatible with Zariski localization on both X and Y.

In this case, we start by the following.

Lemma 4.2.3 *There is a natural equivalence of dg-categories over A*

$$\mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Coh}^b(Y_s) \simeq \mathsf{Coh}^b_{Z_s}(Z),$$

where $Z := X \times_S Y$, and the right hand side is the dg-category of coherent complexes on Z with cohomology supported on the special fiber Z_s. This equivalence is furthermore functorial in X and Y.

Proof of lemma 4.2.3. We use the strict models introduced in our last section. Let $X := Spec\, B$ and $Y := Spec\, C$, and K_B, K_C the Koszul dg-algebras of B and C with respect to the element π. As in our previous section, we have the Hopf dg-algebroid K_A^2 and its monoidal dg-category of modules $\mathsf{Mod}^c(K_A^2)$. Now, $\mathsf{Mod}^c(K_A^2)$ acts on both $\mathsf{Mod}^c(K_B)$ and $\mathsf{Mod}^c(K_C)$, the dg-categories of cofibrant K_B (resp. K_C) dg-modules which are perfect over B (resp. over C).

We define a dg-functor

$$\theta : (\mathsf{Mod}^c(K_B))^o \otimes_{\mathsf{Mod}^c(K_A^2)} \mathsf{Mod}^c(K_C) \longrightarrow \mathsf{Mod}^c(B \otimes_A C),$$

where $\mathsf{Mod}^c(B \otimes_A C)$ is the category of perfect $B \otimes_A C$-dg-modules. This dg-functor sends a pair of objects (E, F) to the object $\mathbb{D}(E) \otimes_{K_A} F$, where $\mathbb{D}(E) = \underline{Hom}_{K_B}(E, K_B)$ is the K_B-linear dual of E. After localization with respect to quasi-isomorphisms, we get a well defined morphism in $\mathbf{dgCat}_A$

$$\theta : \mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Coh}^b(Y_s) \longrightarrow \mathsf{Coh}^b(Z).$$

In order to finish the proof, we have to check the following two conditions:

1. The image of θ generates (by shifts, sums, cones and retracts) the full sub-dg-category $\mathsf{Coh}^b_{Z_s}(Z)$;

2. The dg-functor θ above is fully faithful.

Now, on the level of objects the dg-functor θ sends a pair (E, F), of coherent complexes on X_s and Y_s, to the coherent complex on Z

$$j_*(\mathbb{D}(E) \boxtimes_k F),$$

where $j : Z_s \hookrightarrow Z$ is the closed embedding, $\boxtimes_k$ denotes the external product on $Z_s = X_s \times_s Y_s$, and $\mathbb{D}(E)$ denotes the Grothendieck dual of E on the Gorenstein scheme X_s. It is known that coherent complexes of the form $E \boxtimes_k F$ generate $\mathsf{Coh}^b(Z_s)$. As the coherent complexes of the form $j_*(G)$, for $G \in \mathsf{Coh}^b(Z_s)$, clearly generate the dg-category $\mathsf{Coh}^b_{Z_s}(Z)$, we see that condition (1) above is satisfied.

It now remains to show that θ is fully faithful. Given two pairs of objects $(E, F), (E', F') \in \mathsf{Coh}^b(X_s) \times \mathsf{Coh}^b(Y_s)$, we have the morphism induced by θ

$$\mathbb{R}\underline{Hom}(E', E) \otimes_{\mathcal{B}^+(1)} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(\mathbb{D}(E) \boxtimes F), j_*(\mathbb{D}(E') \boxtimes F')),$$

where $\mathcal{B}^+(1)$ denotes the algebra of endomorphism of the unit in $\mathcal{B}^+$. As we have already seen, $\mathcal{B}^+ \simeq k[u]$ as an $\mathbb{E}_1$-algebra. As X_s and Y_s are Gorenstein schemes, the structure sheaf $\mathcal{O}$ is a dualizing complex, and $E \mapsto \mathbb{D}(E)$ is an (anti)equivalence. We thus have $\mathbb{R}\underline{Hom}(E', E) \simeq \mathbb{R}\underline{Hom}(\mathbb{D}(E), \mathbb{D}(E'))$, and the above morphism can thus be written in the form

$$\mathbb{R}\underline{Hom}(E, E') \otimes_{k[u]} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')),$$

where it is simply induced by the direct image functor j_*. Both the source and the target of the above morphism enter in a distinguished triangle. On the left hand side, for any two $k[u]$-dg-modules M and N, we have a triangle of A-dg-modules

$$(M \otimes_k N)[-2] \longrightarrow M \otimes_k N \longrightarrow M \otimes_{k[u]} N,$$

where the morphism on the right is the natural projection. This exact triangle follows from the isomorphism $M \otimes_{k[u]} N \simeq (M \otimes_k N) \otimes_{k[u] \otimes_k k[u]} k[u]$, and from the exact triangle of $k[u]$ bi-dg-modules

$$(k[u] \otimes_k k[u])[-2] \xrightarrow{a} k[u] \otimes_k k[u] \xrightarrow{m} k[u],$$

where a is given by multiplication by $(u \otimes 1 - 1 \otimes u)$, and m is the product map.
On the right hand side, we have, by adjunction,

$$\mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')) \simeq \mathbb{R}\underline{Hom}(j^* j_*(E \boxtimes F), E' \boxtimes F').$$

The adjunction map $j^* j_* \to id$, provides an exact triangle of coherent sheaves on Z_s

$$E \boxtimes_k F[1] \longrightarrow j^* j_*(E \boxtimes_k F) \longrightarrow E \boxtimes_k F.$$

The coboundary map of this triangle

$$E \boxtimes_k F \longrightarrow E \boxtimes_k F[2]$$

is precisely given by the action of $k[u]$. We thus obtain another exact triangle

$$(\mathbb{R}\underline{Hom}(E, F) \otimes_k \mathbb{R}\underline{Hom}(E', F'))[-2] \longrightarrow \mathbb{R}\underline{Hom}(E, F) \otimes_k \mathbb{R}\underline{Hom}(E', F') \longrightarrow$$

$$\longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')).$$

By inspection, the morphism θ is compatible with these two triangles, and provides an equivalence

$$\mathbb{R}\underline{Hom}(E, E') \otimes_{k[u]} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')).$$

Therefore θ is fully faithful. $\diamond$

Since X/S and Y/S are generically smooth, also $Z = X \times_S Y$ is generically smooth over S. Therefore $\mathsf{Sing}(Z) \simeq \mathsf{Coh}^b_{Z_s}(Z)/\mathsf{Perf}_{Z_s}(Z)$. Hence, in order to finish the proof of Proposition 4.2.1, we are left to prove that the image of $\mathsf{Perf}_{Z_s}(Z)$ under a quasi-inverse of θ is generated by $\mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Perf}(Y_s)$ and $\mathsf{Perf}(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Coh}^b(Y_s)$. Now, the dg-category $\mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Perf}(Y_s)$ is generated by the image of the natural dg-functor

$$\mathsf{Coh}^b(X_s)^o \otimes_A \mathsf{Perf}(Y_s) \longrightarrow \mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Perf}(Y_s),$$

and the dg-category $\mathsf{Perf}(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Coh}^b(Y_s)$ is generated by the image of the natural dg-functor

$$\mathsf{Perf}(X_s)^o \otimes_A \mathsf{Coh}^b(Y_s) \longrightarrow \mathsf{Perf}(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Coh}^b(Y_s).$$

Therefore, what we have to prove is that $\mathsf{Perf}_{Z_s}(Z)$ is generated by objects of the form $E \boxtimes_A F$, for $E \in \mathsf{Coh}^b(X_s)$, $F \in \mathsf{Coh}^b(Y_s)$, one of them being perfect.

We first notice that, if E or F is perfect, indeed, $E \boxtimes_A F$ is a perfect complex on Z. To see this, we may localize on Z, and write $X = Spec\, B$ and $Y = Spec\, C$. The objects E and F then correspond to bounded coherent complexes over $B_s = B \otimes_A k$ and $C_s = C \otimes_A k$ respectively. Assume that E is perfect (the complementary case being totally analogous). By dévissage we can assume that $E = B_s$, so that $E \boxtimes_A F \simeq (B \boxtimes_A F) \otimes_A k$. In other words, if we write $j : Y_s \hookrightarrow Y$ and $i : Z_s \hookrightarrow Z$ for the closed embeddings, and $p : Z \longrightarrow Y$ for the second projection, we have

$$E \boxtimes_A F \simeq i_* i^* p^*(j_*(F)).$$

But $j_*(F)$ is perfect on Y (because Y is regular), so $p^* j_*(F)$ is perfect on Z. Moreover, since i is an lci morphism, $i_* i^*$ preserves perfect complexes on Z, and we conclude that, indeed, $E \boxtimes_A F$ is a perfect complex on Z.

Finally, the fact that the image of $\mathsf{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Perf}(Y_s)$ and $\mathsf{Perf}(X_s)^o \otimes_{\mathcal{B}^+} \mathsf{Coh}^b(Y_s)$ inside $\mathsf{Coh}^b_{Z_s}(Z)$ generates the whole dg-category $\mathsf{Perf}_{Z_s}(Z)$ follows from the fact that the image of $\mathsf{Perf}(X_s)^o \otimes_A \mathsf{Perf}(Y_s)$ inside $\mathsf{Perf}(Z)$ already generates $\mathsf{Perf}_{Z_s}(Z)$. Indeed, for two perfect complexes E on X_s and F on Y_s, the image of $E \boxtimes_A F$ in $\mathsf{Perf}(Z)$ is of the form $i_*(E \boxtimes_k F) \oplus i_*(E \boxtimes_k F)[1]$. As objects of the form $E \boxtimes_k F$ generate $\mathsf{Perf}(Z_s)$ we are done, and Proposition 4.2.1 is proved. $\square$

4.3 Saturatedness

As a consequence of the Künneth formula for dg-categories of singularities we prove the following result.

Proposition 4.3.1 *Let X be a regular and flat S-scheme.*

1. *If X is proper over S, then the dg-category $\mathsf{Coh}^b(X_s)$ is proper over $\mathcal{B}^+$.*

2. *If X is proper over S, then the dg-category $\mathsf{Sing}(X_s)$ is saturated over $\mathcal{B}$.*

Proof. (1) We have to show that the big morphism

$$h : \mathsf{Coh}^b(X_s)^o \widehat{\otimes_A} \mathsf{Coh}^b(X_s) \longrightarrow \widehat{\mathcal{B}^+} \simeq \widehat{\mathsf{Coh}^b(G)}.$$

is small. Here $G = s \times_S s$, and the morphism h is obtained as follows. The derived scheme G is a derived groupoid acting on X_s by means of the natural projection on the last two factors

$$\mu : G \times_s X_s = s \times_S s \times_S X \longrightarrow X_s.$$

The projection on the first and third factors provides another morphism

$$p : G \times_s X_s \longrightarrow X_s.$$

Thus, the morphisms p and μ together define a morphism of derived schemes

$$q : G \times_s X_s \longrightarrow X_s \times_S X_s.$$

Finally, we denote by $r : G \times_s X_s \longrightarrow G$ the natural projection. Now, for two coherent complexes E and F on X_s, we first form the external Hom complex $\mathcal{H}om_A(E,F)$ which is a coherent complex on $X_s \times_S X_s$, and we have

$$h(E,F) \simeq r_*(q^*\mathcal{H}om_A(E,F)).$$

This is a quasi-coherent complex on G. Both q and r are local complete intersection morphisms of derived schemes, and moreover r is proper. This implies that q^* and r_* preserve coherent complexes, and thus that $h(E,F)$ is coherent on G.

(2) We have $\mathsf{Sing}(X_s) \simeq \mathsf{Coh}^b(X_s) \otimes_{\mathcal{B}^+} \mathcal{B}$, thus (1) implies that $\mathsf{Sing}(X_s)$ is proper over $\mathcal{B}$. To prove it is smooth we need to prove that the coevaluation big morphism $A \longrightarrow \mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(X_s)$ is a small morphism. Using our Künneth formula for dg-category of singularities (Proposition 4.2.1) this morphism corresponds to the data of an ind-object in $\mathsf{Sing}(X \times_S X)$. This object is the structure sheaf of the diagonal Δ_X inside $X \times_S X$ which is an object in $\mathsf{Sing}(X \times_S X)$. This shows that the coevaluation morphism is a small morphism and thus that $\mathsf{Sing}(X_s)$ is indeed saturated over $\mathcal{B}$. $\Box$

Remark 4.3.2 Proposition 4.3.1 (2) remains true if we only suppose that the singular locus of X_s is proper over $s = \mathrm{Spec}\,\mathrm{k}$.

5 Application to Bloch's conductor formula with unipotent monodromy

In this Section we first recall Bloch's Conductor Conjecture and the current state of the art for it, then we prove a version of Bloch's conductor under the hypothesis that the monodromy action is unipotent, where Bloch's number is replaced by a categorical Bloch class (see Definition 5.2.1). We are convinced that Bloch's number always agree with the categorical Bloch class but we will not prove (nor use) this fact in this paper. We are aware of a proof of this comparison in the geometric case (i.e. when the inclusion $s \to S$ has a retraction) but we will defer this to another paper, where we hope to give the comparison also when S has mixed characteristic.

5.1 Bloch's Conductor Conjecture

Our base scheme is a discrete valuation ring $S = \mathbf{Spec}\, A$, with perfect residue field k, and fraction field K. Let $p : X \longrightarrow S$ be proper and flat morphism of finite type, and of relative dimension n. We assume that the generic fiber X_K is smooth over K, and that X is a regular scheme. We write $\bar{K}$ for the separable closure of K (inside a fixed algebraic closure).
In his 1985 paper [Bl], Bloch formulated the following conductor formula conjecture which is a kind of vast arithmetic generalization of Gauss-Bonnet formula, where an intersection theoretic (coherent) term, the *Bloch's number*, is conjectured to be equal to an arithmetic (étale) term, the *Artin conductor*. We address the reader to [Bl] and [Ka-Sa] for more detailed definitions of the various objects involved in the statement.

Conjecture 5.1.1 [Bloch's conductor Conjecture] *Under the above hypotheses on $p : X \to S$, we have an equality*

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell) - \mathsf{Sw}(X_{\bar{K}}),$$

where $\chi(Y, \ell)$ denotes the $\mathbb{Q}_\ell$-adic Euler characteristic *of a variety Y, for ℓ prime to the characteristic of k, $\mathsf{Sw}(X_{\bar{K}})$ is the* Swan conductor *of the $\mathsf{Gal}(\bar{K}/K)$-representation $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$, and $[\Delta_X, \Delta_X]_S$ is* Bloch's number *of X/S, i.e. the degree in $\mathsf{CH}_0(k) \simeq \mathbb{Z}$ of Bloch's localised self-intersection $(\Delta_X, \Delta_X)_S \in \mathsf{CH}_0(X_k)$ of the diagonal in X. The (negative of the) rhs is called the* Artin conductor *of X/S, and denoted by $\mathsf{Art}(X/S)$.*

Conjecture 5.1.1 is known to hold in several special cases that we recall below:

1. When k is of characteristic zero, Conjecture 5.1.1 follows from the work of [Kap]. When furthermore X_s has only isolated singularities the conductor formula was known as the *Milnor formula* stating that the dimension of the space of vanishing cycles equals the dimension of the Jacobian ring.

2. When S is equicharacteristic, Conjecture 5.1.1 has been proved recently in [Sai], based on Beilinson's theory of singular support of ℓ-adic sheaves. The special subcase of isolated singularities already appeared in [SGA7-I, Exp. XVI].

3. When X is semi-stable over S, i.e. the reduced special fiber $(X_s)_{red} \subset X$ is a normal crossing divisor, Conjecture 5.1.1 was proved in [Ka-Sa].

In view of the above list of results, one of the major open cases is that of isolated singularities in *mixed characteristic*, which is the conjecture appearing in Deligne's exposé [SGA7-I, Exp. XVI].

It is classical and easy to see that Conjecture 5.1.1 can be reduced to the case where S is excellent and strictly henselian (so that k algebraically closed)[14]. We will thus assume from now on that k *is algebraically closed.*

[14]We first reduce to the complete dvr (hence henselian and excellent) case by proper base-change, and then we further reduce to the excellent, *strictly* henselian case.

5.2 An analog of Bloch's Conjecture for unipotent monodromy

We start by introducing a *categorical variant* of Bloch's number $[\Delta_X.\Delta_X]_S$ defined in terms of dg-categories of singularities.

The dg-category $\mathsf{Sing}(X_s)$ of singularities of the special fiber comes equipped with its canonical $\mathcal{B}$-module structure described in Proposition 4.1.7. As X is proper over S, we know by Proposition 4.3.1 that $\mathsf{Sing}(X_s)$ is saturated over $\mathcal{B}$. We are thus entitled to take the trace (Definition 2.4.4) of the identity of $\mathsf{Sing}(X_s)$, which is a morphism

$$A \longrightarrow \mathsf{HH}(\mathcal{B}/A)$$

in $\mathbf{dgCat}_A$. This trace morphism is then, by definition, determined by a perfect $\mathsf{HH}(\mathcal{B}/A)^o$-dg-module $\mathsf{HH}(\mathsf{Sing}(X_s)/\mathcal{B})$, and thus provides us with a class in K-theory

$$[\mathsf{HH}(\mathsf{Sing}(X_s)/\mathcal{B})] \in K_0(\mathsf{HH}(\mathcal{B}/A)).$$

The Chern character natural transformation (applied to $\mathsf{HH}(\mathcal{B}/A)$, see Definition 2.3.1) of this class is an element $Ch_{\ell,\mathsf{HH}(\mathcal{B}/A)}([\mathsf{HH}(\mathsf{Sing}(X_s)/\mathcal{B})])$ in $\pi_0|r_\ell(\mathsf{HH}(\mathcal{B}/A))| = H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$.

Definition 5.2.1 *With the above notations and hypotheses on X/S, the* categorical Bloch class *of X/S is defined as*

$$[\Delta_X, \Delta_X]_S^{cat} := Ch_\ell([\mathsf{HH}(\mathsf{Sing}(X_s)/\mathcal{B}]) \in H^0(S_{ét}, r_\ell(\mathsf{HH}(\mathcal{B}/A))).$$

Notice that though $\mathcal{B}$ is just an associative algebra in $\mathbf{dgCat}_A$ (an E_2-algebra over A), its ℓ-adic realization $r_\ell(\mathcal{B})$ is in fact a commutative monoid in $Sh_{\mathbb{Q}_\ell}(S)$, naturally equivalent to $i_*(\mathbb{Q}_\ell(\beta)\oplus\mathbb{Q}_\ell(\beta)[1])$ where $i : s \to S$ is the inclusion of the closed point. Therefore there is a canonical augmentation $\mathsf{a} : \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A)) \to r_\ell(\mathcal{B})$, hence an induced augmentation

$$\mathsf{a} : H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) = \pi_0|\mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))| \longrightarrow \pi_0|r_\ell(\mathcal{B})| = H^0(S_{\text{ét}}, r_\ell(B)) \simeq \mathbb{Q}_\ell$$

that is a left inverse to

$$\sigma : \mathbb{Q}_\ell \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \to H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A)))$$

(induced by $r_\ell(\mathcal{B}) \to \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))$).

For $\lambda \in \mathbb{Q}_\ell$, we will simply write $\lambda^\wedge$ for the image of λ via the composition

$$\mathbb{Q}_\ell \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \xhookrightarrow{\sigma} H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \xrightarrow{\alpha} H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A))).$$

We are now ready to prove our version of Bloch's Conductor Conjecture for unipotent monodromy..

Theorem 5.2.2 *Let X/S be as in Conjecture 5.1.1, and assume further that the inertia subgroup* $\mathrm{I} := \mathsf{Gal}(\bar{K}/K^{unr}) \subseteq \mathsf{Gal}(\bar{K}/K)$ *acts unipotently on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$. Then we have an equality*

$$[\Delta_X, \Delta_X]_S^{cat} = \chi(X_{\bar{k}}, \ell)^\wedge - \chi(X_{\bar{K}}, \ell)^\wedge$$

in $H^0(S_{ét}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$.

Remark 5.2.3 Since we have not proved that the map

$$\alpha : H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \to H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$$

is injective, the equality in Theorem 5.2.2 is not a priori an equality of integers (and it might even be the trivial equality $0 = 0$ of classes in $H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$). However, we conjecture that the canonical map $u : \mathcal{B} \to \mathsf{HH}(\mathcal{B}/A)$ is in fact an $\mathbb{A}^1$-homotopy equivalence; this would imply that $r_\ell(u) : r_\ell(\mathcal{B}) \to r_\ell(\mathsf{HH}(\mathcal{B}/A))$ is an equivalence in $Sh_{\mathbb{Q}_\ell}(S)$, so that $H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A))) \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \simeq \mathbb{Q}_\ell$. In particular α would be injective, and the equality in Theorem 5.2.2 would indeed be an equality of integers.

Remark 5.2.4 Recall that the inertia group I sits in an extension of pro-finite groups

$$1 \longrightarrow P \longrightarrow \mathrm{I} \longrightarrow \mathrm{I}_t \longrightarrow 1\,,$$

where I_t is the *tame inertia* quotient, and the *wild inertia* P is a pro-p-subgroup. For any continuous finite dimensional $\mathbb{Q}_\ell$-representation V of I, the group P acts through a finite quotient on V (see []). Therefore, if I is supposed to act unipotently on V, then P acts necessarily trivially, i.e. the I-action factors through a I_t-action, i.e., by definition, the I action is *tame*. By definition, the Swan conductor $Sw_{\mathrm{I}}(V)$(see e.g. [Ka-Sa, §6.1]) vanishes for a tame I-representation V. As a consequence, granting Remark 5.2.3, under the hypothesis of unipotent action of I on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$, Conjecture 5.1.1 becomes Theorem 5.2.2 *if* Bloch's number $[\Delta_X, \Delta_X]_S$ is replaced by the categorical Bloch class $[\Delta_X, \Delta_X]_S^{cat}$. Though we currently know a proof only in the geometric case, we are actually convinced that the categorical Bloch class is *always*, i.e. regardless any hypothesis on the monodromy action and on the mixed or equal characteristic property of S, an integer equal to Bloch's intersection number $[\Delta_X, \Delta_X]_S$ appearing in Conjecture 5.1.1. This fact will not be used in this paper and will be more closely investigated in a future one.

Proof of Thm. 5.2.2. We may suppose S strictly henselian, so that $\mathrm{I} = \mathsf{Gal}(\bar{K}/K)$. We want to apply our trace formula (Theorem 2.4.9) to $T = \mathsf{Sing}(X_s)$ and $f = id$. In order to do this, we need to check that the conditions of being *saturated* and $\ell^{\otimes}$*-admissible* over $\mathcal{B}$ are met by such T.

By Proposition 4.3.1, we know that T is *saturated over* $\mathcal{B}$.
Let us now show that T is also $\ell^{\otimes}$*-admissible* over $\mathcal{B}$, i.e. that the canonical map

$$\varphi : r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T) \longrightarrow r_\ell(T^o \otimes_{\mathcal{B}} T)$$

is an equivalence.
First of all we notice that, since both the source and the target of φ are ℓ-adic complexes on S, supported at s, it is enough to show that $i^*(\varphi)$ is an equivalence, $i : s \to S$ being the canonical inclusion (note that $i^* i_*$ is equivalent to the identity functor).
Let us first elaborate on the target of $i^*(\varphi)$. By Künneth for dg-categories of singularities, we have the canonical equivalence of Proposition 4.2.1

$$T^o \otimes_{\mathcal{B}} T \simeq \mathsf{Sing}(X \times_S X).$$

Moreover, since a unipotent action of I is also tame, by Cor. 3.4.5 we have

$$r_\ell(\mathsf{Sing}(X \times_S X)) \simeq (p_{Z_s})_*(\nu_X \boxtimes \nu_X)^{\mathrm{I}}(\beta),$$

where $p_{Z_s} : Z_s = X_s \times_s X_s \to S$ is the composite $Z_s \xrightarrow{q_{Z_s}} s \xrightarrow{i} S$, and, as usual, ν_X denotes vanishing cycles for X/S on X_s. Therefore

$$i^* r_\ell(\mathsf{Sing}(X \times_S X)) \simeq (q_{Z_s})_*(\nu_X \boxtimes \nu_X)^{\mathrm{I}}(\beta)$$

in $Sh_{\mathbb{Q}_\ell}(s)$. Since $q_{Z_s} = q_{X_s} \times_s q_{X_s}$, where $q_{X_s} : X_s \to s$ is the canonical map, we have

$$i^* r_\ell(\mathsf{Sing}(X \times_S X)) \simeq (q_{Z_s})_*(\nu_X \boxtimes \nu_X)^{\mathrm{I}}(\beta) \simeq ((q_{X_s})_*(\nu_X) \otimes_{\mathbb{Q}_\ell} (q_{X_s})_*(\nu_X))^{\mathrm{I}}(\beta) \tag{10}$$

in $Sh_{\mathbb{Q}_\ell}(s)$, that we may rewrite as

$$i^* r_\ell(\mathsf{Sing}(X \times_S X)) \simeq (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X))^{\mathrm{I}}(\beta) \tag{11}$$

where we have introduced the hypercohomology $\mathbb{H}(X_s, \mathcal{E}) := (q_{X_s})_*(\mathcal{E})$, for $\mathcal{E} \in Sh_{\mathbb{Q}_\ell}(X_s)$.
Let us now look more carefully at the source of the map $i^*(\varphi)$. By the main theorem of [BRTV] (see also formula (4)), we have

$$r_\ell(T) \simeq (p_{X_s})_*(\nu_X[-1]^{\mathrm{I}})(\beta)$$

where $p_{X_s} : X_s \to S$ is the composite $X_s \xrightarrow{q_{X_s}} s \xrightarrow{i} S$. Therefore,

$$i^*(r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T)) \simeq i^*(r_\ell(T^o)) \otimes_{i^* r_\ell(\mathcal{B})} i^* r_\ell(T) \simeq ((q_{X_s})_*(\nu_X[-1])^{\mathrm{I}} \otimes_{\mathbb{Q}_\ell^{\mathrm{I}}} q_{X_s})_*(\nu_X[-1])^{\mathrm{I}})(\beta) \tag{12}$$

in $Sh_{\mathbb{Q}_\ell}(s)$, that we may rewrite as[15]

$$i^*(r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T)) \simeq (\mathbb{H}(X_s, \nu_X[-1])^{\mathrm{I}} \otimes_{\mathbb{Q}_\ell^{\mathrm{I}}} \mathbb{H}(X_s, \nu_X[-1])^{\mathrm{I}})(\beta). \tag{13}$$

By recalling that $\mathcal{E}(\beta) \simeq \mathcal{E}[-2](\beta)$, for any $\mathcal{E} \in Sh_{\mathbb{Q}_\ell}(X_s)$, and by combining (11) and (13), we see that

$$i^*(\varphi) : i^*(r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T)) \longrightarrow i^* r_\ell(T^o \otimes_{\mathcal{B}} T)$$

is then equivalent to $\psi[-2](\beta)$, where ψ is the lax-monoidal structure morphism for the functor $(-)^{\mathrm{I}}$, applied to the pair of ℓ-adic complexes $(\mathbb{H}(X_s, \nu_X), \mathbb{H}(X_s, \nu_X))$,

$$\psi : \mathbb{H}(X_s, \nu_X)^{\mathrm{I}} \otimes_{\mathbb{Q}_\ell^{\mathrm{I}}} \mathbb{H}(X_s, \nu_X)^{\mathrm{I}} \longrightarrow (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X))^{\mathrm{I}}. \tag{14}$$

The fact that the morphism ψ is an equivalence is a consequence of the following general lemma.

Lemma 5.2.5 *Let $\mathcal{D}_{uni}(\mathrm{I}, \mathbb{Q}_\ell)$ be the full sub-∞-category of $\mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \simeq \mathcal{D}_c^{\mathrm{I}}(Spec\, \bar{K}, \mathbb{Q}_\ell)$ consisting of all objects E for which the action of I on each $H^i(E)$ is unipotent. Then the invariant ∞-functor induces an equivalence of symmetric monoidal ∞-categories*

$$(-)^{\mathrm{I}} : \mathcal{D}_{uni}(\mathrm{I}, \mathbb{Q}_\ell) \simeq \mathcal{D}_{perf}(\mathbb{Q}_\ell[\epsilon_1]),$$

where $\mathbb{Q}_\ell[\epsilon_1]$ is the free commutative $\mathbb{Q}_\ell$-dg-algebra generated by ϵ_1 in degree 1, and $\mathcal{D}_{perf}(\mathbb{Q}_\ell[\epsilon_1])$ is its ∞-category of perfect dg-modules.

[15] Note that $r_\ell(T) \simeq r_\ell(T^o)$ because the K-theory of a dg-category is canonically isomorphic to the K-theory of the opposite dg-category.

Proof of lemma 5.2.5. This is a well known fact. The ∞-functor

$$(-)^{\mathrm{I}} : \mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c(\mathbb{Q}_\ell)$$

is lax symmetric monoidal, so it induces a lax monoidal ∞-functor

$$(-)^{\mathrm{I}} : \mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}(\mathbb{Q}_\ell^{\mathrm{I}}).$$

It is easy to see that $\mathbb{Q}_\ell^{\mathrm{I}}$ is canonically equivalent to $\mathbb{Q}_\ell[\epsilon_1]$, and the choice of such an equivalence only depends on the choice of a generator of $H^1(\mathrm{I}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$. We thus have an induced lax symmetric monoidal ∞-functor

$$(-)^{\mathrm{I}} : \mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}(\mathbb{Q}_\ell[\epsilon_1]).$$

The above ∞-functor is in fact symmetric monoidal, as it preserves unit objects, and moreover $\mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell)$ is generated by the unit object $\mathbb{Q}_\ell$.
The lemma is then a direct consequence of the following fact: let $x \in \mathcal{C}$ be a compact object in a compactly generated stable ∞-category $\mathcal{C}$, then the ∞-functor

$$\mathcal{C}(x, -) : \mathcal{C} \longrightarrow \mathsf{Mod}_{\mathbf{Sp}}(End(x))$$

induces an equivalence of stable ∞-categories

$$\langle x \rangle \simeq \mathsf{Perf}_{\mathbf{Sp}}(End(x)),$$

where $\langle x \rangle \subset \mathcal{C}$ denotes the full stable ∞-category generated by the object x, and $End(x)$ denotes the ring spectrum of endormophisms of x. $\diamondsuit$

The above lemma implies that the map

$$\psi[-2] : \mathbb{H}(X_s, \nu_X[-1])^{\mathrm{I}} \otimes_{\mathbb{Q}_\ell^{\mathrm{I}}} \mathbb{H}(X_s, \nu_X[-1])^{\mathrm{I}} \longrightarrow (\mathbb{H}(X_s, \nu_X[-1]) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X[-1]))^{\mathrm{I}}$$

is an equivalence, and thus, as observed above, the same is true for $i^*(\varphi)$, and hence for the admissibility map

$$\varphi : r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T) \longrightarrow r_\ell(T^o \otimes_{\mathcal{B}} T),$$

so that T is indeed $\ell^{\otimes}$*-admissible over* $\mathcal{B}$ as we wanted.

Now that we have checked all the conditions for the trace formula of Theorem 2.4.9 to hold in our case, we get an equality

$$Ch_\ell([\mathsf{HH}(T/\mathcal{B}, id)]) = tr_{r_\ell(\mathcal{B})}(id : r_\ell(T) \to r_\ell(T)) \tag{15}$$

in $H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$. Recall that

$$tr_{r_\ell(\mathcal{B})}(id : r_\ell(T) \to r_\ell(T)) := \alpha(Tr_{r_\ell(\mathcal{B})}(id : r_\ell(T) \to r_\ell(T))),$$

where $\alpha : H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \to H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$ is the canonical map.

We are left to identify the two sides of (15).
The left hand side is, by definition, our categorical Bloch number $[\Delta_X, \Delta_X]_S^{cat}$.

Let us now investigate the r.h.s. of (15): we need to show that $Tr_{r_\ell(B)}(r_\ell(id_T)) = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell)$. As already noticed, $r_\ell(\mathcal{B})$ is a commutative monoid in $Sh_{\mathbb{Q}_\ell}(S)$, naturally equivalent to $i_*(\mathbb{Q}_\ell(\beta) \oplus \mathbb{Q}_\ell(\beta)[1])$ where $i : s \to S$ is the inclusion of the closed point. Therefore there is a canonical augmentation $\mathsf{a} : \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A)) \to r_\ell(\mathcal{B})$, hence an induced augmentation

$$\mathsf{a} : H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) = \pi_0|\mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))| \longrightarrow \pi_0|r_\ell(\mathcal{B})| = H^0(S_{\text{ét}}, r_\ell(B)).$$

Consider the diagram

$$\begin{tikzcd}
H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \arrow[r, "\alpha"] \arrow[d, bend right, "\mathsf{a}"'] & H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A))) \\
\mathbb{Q}_\ell \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \arrow[u, "\sigma"'] \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

where can is the map induced by the canonical map $u : \mathcal{B} \to \mathsf{HH}(\mathcal{B}/A)$, and σ is the map induced by $r_\ell(u)$, so that $\mathsf{a} \circ \sigma = \mathrm{id}$. By compatibility between the non-commutative trace and the commutative trace (Remark 2.4.5)

$$\mathsf{a}(Tr_{r_\ell(\mathcal{B})}(r_\ell(id_T)) = Tr^{\mathrm{c}}_{r_\ell(\mathcal{B})}(r_\ell(id_T))$$

.
Since $r_\ell(\mathcal{B}) \simeq i_*(\mathbb{Q}_\ell(\beta) \oplus \mathbb{Q}_\ell(\beta)[1])$, we have $K_0(r_\ell(\mathcal{B})) \simeq \mathbb{Z}$ and the following commutative diagram

$$\begin{tikzcd}
\mathbb{Q}_\ell \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \arrow[r, hook, "\sigma"] & H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \arrow[r, "\mathsf{a}"] & H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \\
K_0(r_\ell(\mathcal{B})) \arrow[ur, "Tr(id_-)"] \arrow[urr, "Tr^{\mathrm{c}}(id_-)"'] & & \\
& \mathbb{Z} \arrow[ul, "\mathrm{iso}"] \arrow[uul] \arrow[uu, bend right] \arrow[uur] &
\end{tikzcd}$$

where the maps with source $\mathbb{Z}$ are the unique maps of commutative rings, and $\mathsf{a} \circ Tr(id_-) = Tr^{\mathrm{c}}(id_-)$ again by the functorial compatibility of non-commutative and commutative traces. By considering the class $[r_\ell(T)]$ in $K_0(r_\ell(\mathcal{B}))$, and using that $\mathsf{a} \circ \sigma = \mathrm{id}$, we deduce from the previous commutative diagram the equality of ℓ-adic numbers

$$\sigma(Tr^{\mathrm{c}}_{r_\ell(\mathcal{B})}(r_\ell(id_T)) = Tr_{r_\ell(B)}(r_\ell(id_T)). \tag{16}$$

Now, unfolding the definition of $Tr_{r_\ell(\mathcal{B})}$, and using Lemma 5.2.5, the ℓ-adic number (16) can be described as follows. The dg-algebra $\mathbb{Q}_\ell^{\mathrm{I}}$ is such that $K_0(\mathbb{Q}_\ell^{\mathrm{I}}) \simeq \mathbb{Z}$. Viewing $\mathbb{Z}$ inside $\mathbb{Q}_\ell$, this isomorphism is induced by sending the class of dg-module E to the trace of the identity inside $\mathsf{HH}_0(\mathbb{Q}_\ell^{\mathrm{I}}) \simeq \mathbb{Q}_\ell$. Using the functoriality of the trace for the morphism of commutative dg-algebras $\mathbb{Q}_\ell^{\mathrm{I}} \longrightarrow \mathbb{Q}_\ell^{\mathrm{I}}(\beta)$, we see that (16) is simply the trace of the identity on $\mathbb{H}(X_s, \nu_X[-1])$, as an object in $\mathcal{D}_{uni}(\mathrm{I}, \mathbb{Q}_\ell)$. This trace is easy to compute, as it equals 1 on the unit object $\mathbb{Q}_\ell$. Since the unit object generates $\mathcal{D}_{uni}(\mathrm{I}, \mathbb{Q}_\ell)$, we have that the trace of the identity on any object $E \in \mathcal{D}_{uni}(\mathrm{I}, \mathbb{Q}_\ell)$ equals the Euler characteristic of the underlying complex of $\mathbb{Q}_\ell$-vector spaces.

We thus have shown that (16) can be rewritten as

$$Tr_{r_\ell(B)}(r_\ell(id_T)) = \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^{i-1}(X_s, \nu_X).$$

However, by proper base change, the complex $\mathbb{H}(X_s, \nu_X)$ appears in an exact triangle

$$H(X_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow H(X_{\bar{K}}, \mathbb{Q}_\ell) \longrightarrow \mathbb{H}(X_s, \nu_X)$$

and thus we have the equality

$$Tr_{r_\ell(B)}(r_\ell(id_T)) = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell)$$

in $\mathbb{Z} \hookrightarrow \mathbb{Q}_\ell \hookrightarrow H^0(S_{\text{ét}}, \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A)))$.
Therefore

$$[\Delta_X, \Delta_X]_S^{cat} = \chi(X_{\bar{k}}, \ell)^\wedge - \chi(X_{\bar{K}}, \ell)^\wedge,$$

as required. $\square$

A Localizations of monoidal dg-categories

In this Appendix we remind some basic facts about localizations of dg-categories as introduced in [To1]. The purpose of the section is to explain the multiplicative properties of the localization construction. In particular, we explain how localization of strict monoidal dg-categories gives rise to monoids in $\mathbf{dgCat}_A$, and thus to monoidal dg-categories in the sense of our Definition 2.1.1.

Let T be a dg-category over A, together with W a set of morphisms in $Z^0(T)$, the underlying category of T (this is the category of 0-cycles in T, i.e. $Z^0(T)(x,y) := Z^0(T(x,y))$). For the sake of brevity, we will just say that W *is a set of maps in* T. In other words, we allow W not to be strictly speaking a *subset* of the set of morphisms in T, but just a set together with a map $W \to Mor(T)$ from W to the set of morphisms in T. Recall that a localization of T with respect to W, is a dg-category $L_W T$ together with a morphism in $\mathbf{dgCat}_A$

$$l : T \longrightarrow L_W T$$

such that, for any $U \in \mathbf{dgCat}_A$, map induced by l on mapping spaces

$$Map(L_W T, U) \longrightarrow Map(T, U)$$

is fully faithful and its image consists of all $T \to U$ sending W to equivalences in U (i.e. the induced functor $[T] \to [U]$ sends elements of W to isomorphisms in $[U]$).

As explained in [To1], localization always exists, and are unique up to a contractible space of choices (because they represents an obvious ∞-functor). We will describe here a model for $(T, W) \mapsto L_W T$ which will have nice properties with respect to tensor products of dg-categories. For this, let $dgcat_A^{W,c}$ be the category of pairs (T, W), where T is a dg-category with cofibrant hom's over A, and W a set of maps in T. Morphisms $(T, W) \longrightarrow (T', W')$ in $dgcat_A^{W,c}$ are dg-functors $T \longrightarrow T'$ sending W to W'.

We fix once for all a factorization

$$\Delta_A^1 \xrightarrow{j} \tilde{I} \xrightarrow{p} \overline{\Delta}_A^1,$$

with j a cofibration and p a trivial fibration. Here Δ^1_A is the A-linearisation of the category Δ^1 that classifies morphisms, and $\overline{\Delta}^1_A$ is the linearisation of the category that classifies isomorphisms. For an object $(T, W) \in dgcat_A^{W,c}$ we define $W^{-1}T$ by the following cocartesian diagram in dg-categories

$$\begin{tikzcd} \coprod_W \Delta^1_A \arrow[r] \arrow[d] & T \arrow[d] \\ \coprod_W \tilde{I} \arrow[r] & W^{-1}T, \end{tikzcd}$$

where $\coprod_W \Delta^1_A \longrightarrow T$ is the canonical dg-functor corresponding to the set W of morphisms in T.

Lemma A.0.6 *The canonical morphism $l : T \longrightarrow W^{-1}T$ defined above is a localization of T along W.*

Proof. According to [To1], the localization of T can be construced as the homotopy push-out of dg-categories

$$\begin{tikzcd} \coprod_W \Delta^1_A \arrow[r] \arrow[d] & T \arrow[d] \\ \coprod_W A \arrow[r] & L_W T. \end{tikzcd}$$

The lemma then follows from the observation that when T has cofibrant hom's, then the push-out diagram defining $W^{-1}T$ is in fact a homotopy push-out diagram. $\square$

The construction $(T, W) \longrightarrow W^{-1}T$ clearly defines a functor

$$dgcat_A^{W,c} \longrightarrow dgcat_A^c$$

from $dgcat_A^{W,c}$ to $dgcat_A^c$, the category of dg-categories with cofibrant hom's. Moreover, this functor comes equipped with a natural symmetric colax monoidal structure. Indeed, $dgcat_A^{W,c}$ is a symmetric monoial category, where the tensor product is given by

$$(T, W) \otimes (T', W') := (T \otimes_A T', W \otimes id \cup id \otimes W').$$

We have a natural map

$$T \otimes_A T' \longrightarrow (W^{-1}T) \otimes_A ((W')^{-1}T'),$$

which by construction has a canonical extension

$$(W \otimes id \cup id \otimes W')^{-1}(T \otimes_A T') \longrightarrow (W^{-1}T) \otimes_A ((W')^{-1}T').$$

The unit in $dgcat_A^{W,c}$ is $(A, \emptyset)$, which provides an canonical isomorphism $(\emptyset)^{-1}A \simeq A$. These data endow the functor $(T, W) \mapsto W^{-1}T$ with a symmetric colax monoidal structure. By composing with the canonical symmetric monoidal ∞-functor $dgcat_A^c \longrightarrow \mathbf{dgCat}_A$, we get a symmetric colax monoidal ∞-functor

$$dgcat_A^{W,c} \longrightarrow \mathbf{dgCat}_A,$$

which sends (T, W) to $W^{-1}T$. By [To3, Ex. 4.3.3], this colax symmetric monoidal ∞-functor is in fact monoidal. We thus have a symmetric monoidal localization ∞-functor

$$W^{-1}(-) : dgcat_A^{W,c} \longrightarrow \mathbf{dgCat}_A.$$

As a result, if T is a (strict) monoid in $dgcat_A^{W,c}$, then $W^{-1}T$ carries a canonical structure of a monoid in $\mathbf{dgCat}_A$. This applies particularly to stict monoidal dg-categories endowed with a compatible notion of equivalences. By MacLane coherence theorem, any such a structure can be turned into a strict monoid in $dgcat_A^{W,c}$, and by localization into a monoid in $\mathbf{dgCat}_A$. In other words, the localization of a monoidal dg-categorie along a set of maps W that is compatible with the monoidal structure, is a monoid in $\mathbf{dgCat}_A$. The same is true for dg-categories which are modules over a given monoidal dg-category.

References

[SGA4-III] Artin, Michael; Alexandre Grothendieck; Jean-Louis Verdier, eds. (1972). *Séminaire de Géométrie Algébrique du Bois Marie - 1963-64 - Théorie des topos et cohomologie étale des schémas - (SGA 4) - vol. 3.* Lecture notes in mathematics 305. Berlin; New York: Springer-Verlag. pp. vi+640. ISBN 978-3-540-06118-2. MR 0354654. doi:10.1007/BFb0070714

[Be-Be] A. A. Beilinson, J. Bernstein, *A proof of Jantzen conjectures*, in Gel'fand, Sergei; Gindikin, Simon, I. M. Gel'fand Seminar, Adv. Soviet Math., 16, Providence, R.I.: American Mathematical Society, 1993, pp. 1-50, ISBN 978-0-8218-4118-1

[Bla] A. Blanc, *Topological K-theory of complex noncommutative spaces.* Compos. Math. **152** (2016), no. 3, 489-555.

[BRTV] A. Blanc, M. Robalo, B. Toën, G. Vezzosi, *Motivic realizations of singularity categories and vanishing cycles*, Journal de l'École Polytechnique, Tome 5 (2018), p. 651-747.

[Bl] S. Bloch, *Cycles on arithmetic schemes and Euler characteristics of curves*, Alg. Geometry, Bowdoin, 1985, 421-450, Proc. Symp. Pure Math. 46, Part 2, A.M.S., Providence, RI (1987).

[SGA7-II] P. Deligne, N. Katz, eds. *Séminaire de Géométrie Algébrique du Bois Marie - 1967-69 - Groupes de monodromie en géométrie algébrique* - (SGA 7) - vol. 2. Lecture notes in mathematics 340. Berlin; New York: Springer-Verlag. pp. x+438 (1973).

[Ga-Lu] D. Gaitsgory, J. Lurie, *Weil's conjecture over function fields*, Preprint available at http://www.math.harvard.edu/∼lurie/papers/tamagawa.pdf

[SGA7-I] A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie - 1967-69 - Groupes de monodromie en géométrie algébrique* - (SGA 7) - vol. 1. Lecture notes in mathematics 288. Berlin; New York: Springer-Verlag. viii+523 (1972).

[Ho] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs, Volume 63, AMS (1999).

[Il1] L. Illusie, *Autour du théorème de monodromie locale*, in "Périodes p-adiques, Séminaire de Bures, 1988", Astérisque 223, 1994, 9-57.

[Il2] L. Illusie, *Around the Thom-Sebastiani theorem, with an appendix by Weizhe Zheng.* Manuscripta Math. **152** (2017), no. 1-2, 61-125.

[ILO] L. Illusie, Y. Laszlo, F. Orgogozo, *Travaux de Gabber sur l'uniformisation locale et la cohomologie etale des schémas quasi-excellents.* Astérisque 363-364, SMF, 2014.

[Ka-Sa] K. Kato and T. Saito, *On the conductor formula of Bloch*, Publ. Math. IHES 100 (2005), 5-151.

[Kap] M. Kapranov, *On DG-modules over the de Rham complex and the vanishing cycles functor.* Algebraic geometry (Chicago, IL, 1989), 57-86, Lecture Notes in Math., **1479**, Springer, Berlin, 1991.

[Lu-COB] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129-280. MR2555928 (2010k:57064)

[Lu-HA] J. Lurie, *Higher Algebra*, http://www.math.harvard.edu/∼lurie/papers/HA.pdf, April 2016 version.

[Or] F. Orgogozo, *Conjecture de Bloch et nombres de Milnor*, Annales de l'Institut Fourier, 53 no. 6, 2003, 1739-1754

[Pr] A. Preygel, *Thom-Sebastiani & Duality for Matrix Factorizations.* Thesis. Preprint arXiv:1101.5834.

[Ro] M. Robalo, *K-Theory and the bridge from Motives to non-commutative Motives*, Advances in Math. Volume 269, 10 January 2015, Pages 399-550.

[Sai] T. Saito, *Characteristic cycles and the conductor of direct image.* Preprint arXiv:1704.04832.

[To1] B. Toën, *DG-categories and derived Morita theory*, Invent. Math. 167 (2007), no. 3, 615-667.

[To2] B. Toën, *Derived Azumaya algebras and generators for twisted derived categories.* Invent. Math. **189** (2012), no. 3, 581-652.

[To3] B. Toën, *Lectures on dg-categories. Topics in algebraic and topological K-theory.* 243-302, Lecture Notes in Math., **2008**, Springer, Berlin, 2011.

[To-Va] B. Toën, M. Vaquié, *Moduli of objects in dg-categories.* Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387-444.

[To-Ve-1] B. Toën, G. Vezzosi, *Caractères de Chern, traces équivariantes et géométrie algébrique dérivée.* Selecta Math. (N.S.) **21** (2015), no. 2, 449-554.

[To-Ve-2] B. Toën, G. Vezzosi, *Géométrie non-commutative, formule des traces et conducteur de Bloch,* Séminaires et Congrès, **31**, Soc. Math. de France (2018), Actes du 1er congrès national de la SMF - C. Lecouvrey, ed., p. 77-107 (2018), p. 1-23

[Vo] V. Voevodsky, $\mathbb{A}^1$*-homotopy theory*, In Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), 579-604 (electronic), 1998.

Bertrand Toën, IMT, CNRS, Université de Toulouse, Bertrand.Toen@math.univ-toulouse.fr
Gabriele Vezzosi, DIMAI, Università di Firenze, gabriele.vezzosi@unifi.it