A remark on the Hochschild-Kostant-Rosenberg theorem in characteristic p

Benjamin Antieau* and Gabriele Vezzosi

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Abstract

We prove a Hochschild-Kostant-Rosenberg decomposition theorem for smooth proper schemes X in characteristic p when $\dim X \leqslant p$. The best known previous result of this kind, due to Yekutieli, required $\dim X < p$. Yekutieli's result follows from the observation that the denominators appearing in the classical proof of HKR do not divide p when $\dim X < p$. Our extension to $\dim X = p$ requires a homological fact: the Hochschild homology of a smooth proper scheme is self-dual.

Key Words. HKR theorems, duality, characteristic p.

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1 Introduction

The classical Hochschild-Kostant-Rosenberg theorem of [HKR62] states that if k is a commutative ring and R is a smooth commutative k-algebra, then there is a natural isomorphism $\Omega^*_{R/k} \cong \mathrm{HH}_*(R/k)$ of graded-commutative R-algebras. In fact, *when k is a $\mathbb{Q}$-algebra*, this isomorphism lifts to the level of complexes, giving a natural quasi-isomorphism $\mathrm{HH}(R/k) \simeq \bigoplus_t \Omega^t_{R/k}[t]$. Here, $\mathrm{HH}(R/k)$ denotes Hochschild chains, any one of the natural complexes computing Hochschild homology, and $\bigoplus_t \Omega^t_{R/k}[t]$ is viewed as a complex with zero differential. In particular, we see that the Hochschild chains are naturally formal for smooth affine schemes over characteristic 0 fields.

Swan extends the HKR decomposition in [Swa96, Corollary 2.6] to smooth quasi-projective schemes in characteristic 0 using related work of Gerstenhaber-Schack [GS87].[1] Swan's work implies in particular that there are canonical decompositions

$$\mathrm{HH}_n(X/k) \cong \bigoplus_{t-s=n} \mathrm{H}^s(X, \Omega^t_{X/k})$$

for all n. Using the fact that Hochschild homology for all commutative k-algebras is determined by its values on smooth k-algebras, the HKR decomposition was extended to all commutative k-algebras (still in characteristic 0) in work of Buchweitz and Flenner [BF08], Schuhmacher [Sch04], and Toën and Vezzosi [TV11]. One obtains a natural decomposition

$$\mathrm{HH}(R/k) \simeq \bigoplus_{t \geqslant 0} \mathrm{L}\Lambda^t \left(\mathrm{L}_{R/k}\right)[t], \tag{1}$$

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[1] Often these authors are more concerned with Hochschild cohomology, or Hochschild cochains. The results typically dualize without trouble.

where $\mathrm{L}_{R/k}$ is the cotangent complex and $\mathrm{L}\Lambda^t$ is the derived functor of the tth exterior power. Note that Hochschild homology is not typically formal in the non-smooth case but that the differentials are all supported in the cotangent complex direction and not in the exterior algebra direction. Of course, there is a global version of this decomposition for schemes as well.

Much less is known in characteristic p. The main result to date is due to Yekutieli who proves in [Yek02, Theorem 4.8] that there is a natural HKR quasi-isomorphism in characteristic p for smooth schemes of dimension less than p. To make this precise, we let $\underline{\mathrm{HH}}_X$ denote the complex of quasi-coherent sheaves on $\mathcal{O}_X$ computing Hochschild homology. In other words, if $U \subseteq X$ is an affine open subscheme, then $\underline{\mathrm{HH}}_X(U) \simeq \mathrm{HH}(U/k)$. One model for this complex is $\Delta_X^*(\mathcal{O}_{\Delta_X})$, where $\Delta_X : X \to X \times_k X$ is the diagonal morphism, Δ_X^* is the derived pullback functor,[2] and $\mathcal{O}_{\Delta_X}$ is the structure sheaf of the diagonal inside $X \times_k X$.

Yekutieli proved that if k is a commutative ring and if $X \to \operatorname{Spec} k$ is a smooth morphism of relative dimension d, then

$$\underline{\mathrm{HH}}_X \simeq \bigoplus_{t=0}^{d} \Omega^t_{X/k}[t]$$

as complexes of sheaves on X under the assumption that $d!$ is invertible in k. Again, $\bigoplus \Omega^t_{X/k}[t]$ is viewed as a complex of sheaves with zero differential. This implies that there are natural isomorphisms $\mathrm{HH}_n(X/k) \cong \bigoplus_{s-t=n} \mathrm{H}^s(X, \Omega^t_{X/k})$ for all n under the assumption that $d!$ is invertible in k.

Question 1.1. Is there an HKR theorem for smooth schemes in characteristic p?

There are multiple ways that this question might be answered. The explicit isomorphism constructed in [HKR62] does not extend in general to smooth schemes X in characteristic p (unless $\dim X < p$). However, if R is *any* smooth smooth k-algebra, no matter what the dimension, then there is an HKR-like quasi-isomorphism.[3]

Proposition 1.2. *Let k be a commutative ring and let R be a smooth commutative k-algebra. Then, there is a quasi-isomorphism $\bigoplus_t \Omega^t_{R/k}[t] \simeq \mathrm{HH}(R/k)$.*

Proof. This is a special case of a more general fact: any complex with projective homology is formal. A quasi-isomorphism is obtained by choosing maps from the homology into the complex, which can always be done thanks to projectivity. ∎

The crucial point in the proof above is that we must choose a lift to get our map $\Omega^1_{R/k}[1] \to \mathrm{HH}(R/k)$. Without characteristic or dimension assumptions, it is not known how to make this choice natural in R, which would be necessary to extend the result to schemes.[4]

Given the Hochschild-Kostant-Rosenberg theorem, it is not hard to prove that the Hochschild homology sheaves $\mathcal{HH}_t$ are canonically isomorphic to Ω^t_X for smooth schemes X. In particular, there is a natural local-global spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(X, \Omega^t_X) \Rightarrow \mathrm{HH}_{t-s}(X).$$

We say that the **weak HKR theorem holds for X over k** if this spectral sequence degenerates at the E_2-page. We say that the **strong HKR theorem holds for X over k** if $\underline{\mathrm{HH}}_X$ is formal as a complex of

[2] Here and elsewhere we mean derived functors unless specified otherwise.

[3] Using simplicial commutative rings, one may show that the quasi-isomorphism in the proposition can be chosen to respect multiplicative structures by showing that $\mathrm{HH}(R/k)$ is weakly equivalent to the free simplicial commutative R-algebra on $\Omega^1_{R/k}[1]$. This is a way of saying that $\mathrm{HH}(R/k)$ is formal as a simplicial commutative ring.

[4] The issue is the same as in the gap in the proof of the main theorem of [MM03]. In the proof of Theorem 6.1, they argue that their HKR theorem is true étale locally and hence it is true globally. However, for this argument to work, they must give a globally defined map.

sheaves; i.e., if there is a quasi-isomorphism

$$\bigoplus_t \Omega^t_{X/k}[t] \simeq \underline{\mathrm{HH}}_X.$$

Evidently, the strong HKR theorem for X implies the weak HKR theorem for X.

Summarizing the past work in this language, we know by [HKR62] and Proposition 1.2 that the strong HKR theorem holds for $X \to \operatorname{Spec} k$ smooth affine. By Yekutieli [Yek02, Theorem 4.8], we also know that the strong HKR theorem holds for X over k when $X \to \operatorname{Spec} k$ is smooth of relative dimension d and $d!$ is invertible in k. When k is a field of characteristic p, this condition is the same as asking for $\dim X < p$.

Our main results establish the strong HKR theorem for smooth proper schemes X with $\dim X \leqslant p$ over characteristic p fields.

Theorem 1.3. *Suppose that k is a field of characteristic positive p and that X is a smooth proper k-scheme of dimension at most p. Then, the weak HKR theorem holds for X. Specifically, if $\dim X = p$, then for each n there is a canonical short exact sequence*

$$0 \to \mathrm{H}^{p-n}(X, \Omega^p_X) \to \mathrm{HH}_n(X/k) \to \bigoplus_{t=0}^{p-1} \mathrm{H}^{t-n}(X, \Omega^t) \to 0,$$

which is split (but possibly not canonically split).

The main idea in the proof of this theorem is the use of self-duality for $\mathrm{HH}(X/k)$ in the local-global spectral sequence $\mathrm{H}^s(X, \Omega^t_{X/k}) \Rightarrow \mathrm{HH}_{t-s}(X/k)$. A similar move occurs in [DI87], where the authors use the compatibility of Serre duality with the Cartier isomorphism and a trace argument to boost their result on degeneration of the Hodge-de Rham spectral sequence for smooth X/k with $\dim X < p$ to smooth X/k with $\dim X \leqslant p$. See [DI87, Corollaire 2.3].

Example 1.4. The theorem in particular implies the HKR decomposition for smooth projective surfaces in characteristic 2. This answers the explicit form of Question 1.1 asked by Daniel Pomerleano on `mathoverflow` in [Pom].

The theorem has the following corollary.

Corollary 1.5. *Suppose that k is a field of positive characteristic p and that X is a smooth proper k-scheme of dimension at most p. Then, the strong HKR theorem holds for X.*

We briefly review Hochschild homology in Section 2 and we give the proofs of the theorem and corollary as well as some more examples in Section 3.

Remark 1.6. It is interesting to wonder at the connection between the weak HKR theorem and degeneration of the Hodge-de Rham spectral sequence. Given that there is no liftability hypothesis in Theorem 1.3, it is not at all clear if or when there should be a relation. We can however make the following weak statement. If X is a smooth scheme over a characteristic p field k that lifts to a smooth proper map $\mathcal{X} \to \operatorname{Spec} R$ where R is any discrete valuation ring with characteristic zero fraction field and residue field k, and if the cohomology groups $\mathrm{H}^s(X, \Omega^t_{\mathcal{X}/R})$ are p-torsion-free for all s, t, then the weak HKR theorem holds for X over t. Indeed, the differentials must all be p-torsion by the HKR theorem in characteristic 0, but the groups are p-torsion-free. Hence, the differentials all vanish for the spectral sequence of $\mathcal{X}$ over R. But, the hypotheses also imply that the spectral sequence of X over k is the mod p reduction of that for $\mathcal{X}$ over R.

Example 1.7. By lifting to characteristic 0 and [DK73, Exposé XI, Théorème 1.5], the remark implies that the weak HKR theorem holds for smooth complete intersections in projective space. Despite the sparsity of the local-global spectral sequence in that case, this statement is not entirely trivial. For example, for such a 5-fold X in characteristic 2, the differential $d_2 : \mathrm{H}^2(X, \Omega^3_{X/k}) \to \mathrm{H}^4(X, \Omega^4_X)$ could, for all we would know otherwise, be non-zero. In fact, we can also see that this differential must be zero by using the duality arguments used in the proof of Theorem 1.3 the pullback map from the local-global spectral sequence for $\mathbb{P}^n$ to X.

Finally, let us say a word about multiplicative structures. In [TV11], Toën and Vezzosi prove the strong HKR theorem in characteristic zero and show that the equivalence of (1) is naturally S^1-equivariant and multiplicative. The proof of Corollary 1.5 does in particular induce an equivalence between the sheaf of simplicial commutative rings $\underline{\mathrm{HH}}_X$ and the sheaf of free simplicial commutative rings on $\Omega^1_{X/k}$ and there is a corresponding formality result upon taking global sections. We emphasize again that we do not know how to make this S^1-equivariant or functorial in X, or even if that is possible.

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BA was first asked this question by Yankı Lekili in the fall of 2014 and thanks him for many conversations about this and many other topics. Special thanks also go to Bhargav Bhatt who suggested to BA the idea of bringing duality into play, which had already been considered by GV. The resulting conversations led directly to the present note.

2 Hochschild homology

Let k be a commutative ring and let Cat_k denote the ∞-category of small idempotent complete pretriangulated k-linear dg categories up to derived Morita invariance. The ∞-category Cat_k is equivalent to the Dwyer-Kan localization of the category of small k-linear dg categories (or even small *flat* k-linear dg categories) at the class of k-linear derived Morita equivalences. Objects of Cat_k will be called **k-linear categories** for short.[5]

Let $\mathcal{D}(k)$ denote the derived ∞-category of k. The homotopy category of $\mathcal{D}(k)$ is a triangulated category equivalent to $\mathrm{D}(k)$, the derived category of k. We let $\mathrm{HH}(-/k) : \mathrm{Cat}_k \to \mathcal{D}(k)$ be the Hochschild homology functor as studied in [Kel99].[6] The following proposition is well-known (see [CT12, Example 8.9] or [BGT14, Corollary 6.9]); we give a sketch of the proof for the reader's convenience.

Proposition 2.1. *The functor* $\mathrm{HH}(-/k) : \mathrm{Cat}_k \to \mathcal{D}(k)$ *is symmetric monoidal.*

Proof. We will indicate how the functor from small flat k-linear dg categories to chain complexes over k given by taking the mixed complex as in [Kel99, Section 1.3] is symmetric monoidal. Indeed, given a small flat k-linear dg category $\mathcal{C}$, the mixed complex $C(\mathcal{C})$ of [Kel99] is the total complex of the bicomplex associated to the simplicial chain complex $C_\bullet(\mathcal{C})$ with nth term

$$\bigoplus_{(x_n,\cdots,x_0)} \mathcal{C}(x_n, x_0) \otimes_k \mathcal{C}(x_{n-1}, x_n) \otimes_k \mathcal{C}(x_{n-2}, x_{n-1}) \otimes_k \cdots \otimes_k \mathcal{C}(x_1, x_2) \otimes_k \mathcal{C}(x_0, x_1),$$

[5] Equivalently, Cat_k is the ∞-category of small idempotent complete k-linear stable ∞-categories.

[6] To be precise, $\mathrm{HH}(-/k)$ is the derived functor of Keller's mixed complex construction (obtained by taking flat resolutions of dg categories). Since the mixed complex construction inverts derived Morita equivalences (by [Kel99, Section 1.5]) it descends to a map $\mathrm{Cat}_k \to \mathcal{D}(k)$ of ∞-categories.

where the sum is over all $(n+1)$-tuples of objects of $\mathcal{C}$. The differentials are as usual for the cyclic bar complex. (See [Kel99, Section 1.3].) Given a second small flat k-linear dg category, we see that there is a natural map of simplicial chain complexes $C_\bullet(\mathcal{C}) \otimes_k C_\bullet(\mathcal{D}) \to C_\bullet(\mathcal{C} \otimes_k \mathcal{D})$ obtained using the symmetric monoidal structure on chain complexes. This map is a level-wise quasi-isomorphism. Taking the associated total complexes (and using the shuffle product), we obtain a natural quasi-isomorphism $C(\mathcal{C}) \otimes_k C(\mathcal{D}) \to C(\mathcal{C} \otimes_k \mathcal{D})$ giving a symmetric monoidal structure. Compare with the statement of the Hochschild homological Eilenberg-Zilber theorem (see [Lod92, Theorem 4.2.5]). Since Cat_k is the localization of small k-linear dg categories at the k-linear derived Morita equivalences, and since the mixed complex functor C inverts k-linear derived Morita equivalences by [Kel99, Section 1.5], the proposition follows. □

Let $\mathcal{C}$ be a k-linear category. In this case, $\mathrm{Ind}(\mathcal{C})$ is dualizable in $\mathrm{Mod}_{\mathcal{D}(k)}(\mathrm{Pr}^{\mathrm{L}})$. Let $\mathcal{D}(k) \xrightarrow{\mathrm{coev}_{\mathcal{C}}} \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ be the coevaluation map and let $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathcal{C}}} \mathcal{D}(k)$ be the evaluation map. We say that $\mathcal{C}$ is **smooth** if the coevaluation map preserves compact objects. We say that $\mathcal{C}$ is **proper** if the evaluation map preserves compact objects. Note that $\mathcal{C}$ is smooth and proper if and only if it is dualizable in Cat_k. See [TV07, Definition 2.4] or [AG14, Lemma 3.7]. If $\mathcal{C}$ is dualizable, then the dual is equivalent to the opposite dg category $\mathcal{C}^{\mathrm{op}}$.

Corollary 2.2. *If $\mathcal{C}$ is a smooth and proper k-linear category, then $\mathrm{HH}(\mathcal{C}/k)$ is perfect as a complex over k.*

Proof. In this case, $\mathcal{C}$ is dualizable in Cat_k. Since symmetric monoidal functors preserve dualizable objects, the corollary follows at once from the fact that the dualizable objects of $\mathcal{D}(k)$ are precisely the perfect complexes. □

Remark 2.3. It is important to use Cat_k as opposed to the more familiar ∞-category dgAlg_k of dg k-algebras in the corollary because smooth and proper k-linear categories are typically *not* dualizable in dgAlg_k.

3 Proofs

We give the proofs of Theorem 1.3 and Corollary 1.5 in this section after a couple more preliminaries. The next statement is well-known over fields (see for example [Orl16]). The proof over a general commutative ring k is the same, so we omit it.

Proposition 3.1. *Let k be a commutative ring. If $X \to \operatorname{Spec} k$ is a smooth and proper morphism of schemes, then Perf_X is dualizable as a k-linear category.*

Corollary 3.2. *In the situation above, $\mathrm{HH}(X/k)$ is dualizable as a complex over k with dual the Hochschild homology of the smooth and proper k-linear category $(\mathrm{Perf}_X)^{\mathrm{op}}$.*

Proof. This follows from Corollary 2.2 taking into account that the dualizable objects of $\mathcal{D}(k)$ are precisely the perfect complexes. □

The k-linear categories of the form Perf_X are very special: they are equivalent to their own opposites. Below, if $\mathcal{E}$ and $\mathcal{F}$ are complexes of quasi-coherent sheaves on X, we write $\underline{\mathrm{Map}}_X(\mathcal{E}, \mathcal{F})$ for the mapping spectrum, a spectrum whose homotopy groups $\pi_i \underline{\mathrm{Map}}_X(\mathcal{E}, \mathcal{F})$ are given by $\mathrm{Ext}_X^{-i}(\mathcal{E}, \mathcal{F})$. Similarly, $\mathcal{M}\mathrm{ap}_X(\mathcal{E}, \mathcal{F})$ is the complex of quasi-coherent sheaves on X whose homotopy sheaves are $\pi_i \mathcal{M}\mathrm{ap}_X(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}\mathrm{xt}_X^{-i}(\mathcal{E}, \mathcal{F})$.

Lemma 3.3. *Let X be a k-scheme and let $\mathcal{L}$ be a line bundle on X. Then, there is an equivalence $\mathrm{Perf}_X \simeq \mathrm{Perf}_X^{\mathrm{op}}$ of k-linear categories obtained by $\mathcal{F} \mapsto \mathcal{M}\mathrm{ap}_X(\mathcal{F}, \mathcal{L})$.*

Proof. The claim can be checked Zariski locally on X, hence for affine schemes, where it is obvious. □

Corollary 3.4. *If X is a smooth and proper k-scheme, then the complex of k-modules $\mathrm{HH}(X/k)$ is self-dual. That is, the evaluation map is a non-degenerate pairing $\mathrm{HH}(X/k) \otimes_k \mathrm{HH}(X/k) \to k$.*

Proof. This is an immediate consequence of Corollary 3.2 and Lemma 3.3. ☐

The existence of such a pairing has been observed in the literature before. For example, it appears implicitly in Shklyarov [Shk13, Theorem 1.4] and is studied by Căldăraru and Willerton in [CW10], who call it the Mukai pairing. It also occurs in the preprint [TV17] of Toën and Vezzosi.

Our point of departure is to pair the pairing of Corollary 3.4 with the convergent local-global spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(X, \Omega^t_{X/k}) \Rightarrow \mathrm{HH}_{t-s}(X/k). \tag{2}$$

We will see that this quickly leads to a proof of the main theorem. We need one more lemma, which is also implied by [Yek02, Theorem 4.8].

Lemma 3.5. *Let k be a ring in which p acts nilpotently and let $X \to \operatorname{Spec} k$ be a smooth proper morphism. Let $\tau_{\leqslant p-1}\underline{\mathrm{HH}}_X$ denote the $(p-1)$st truncation of $\underline{\mathrm{HH}}_X$. Then, there is a natural quasi-isomorphism*

$$\tau_{\leqslant p-1}\underline{\mathrm{HH}}_X \simeq \bigoplus_{t=0}^{p-1} \Omega^t_{X/k}[t].$$

Proof. In general, for any smooth scheme, there is a natural map of complexes of sheaves $\underline{\mathrm{HH}}_X \to \Omega^t_{X/k}[t]$ constructed in [Lod92, Section 1.3]. On smooth affine schemes $X = \operatorname{Spec} R$, this map can be described as taking a Hochschild chain $r_0 \otimes r_1 \otimes \cdots \otimes r_t$ to the differential t-form $r_0 dr_1 \cdots dr_t$. This map is *not* the map arising in the Hochschild-Kostant-Rosenberg theorem. Rather, there is an isomorphism $\Omega^t_{X/k} \to \mathcal{HH}_t$ of sheaves. The induced composition $\Omega^t_{X/k} \to \Omega^t_{X/k}$ is multiplication by $t!$ by [Lod92, Proposition 1.3.16]. But, this implies that the induced map $\tau_{\leqslant p-1}\mathrm{HH}_X \to \bigoplus_{t=0}^{p-1} \Omega^t_{X/k}[t]$ is a quasi-isomorphism under the present hypotheses. ☐

We are now ready to give the proof of the main theorem.

Proof of Theorem 1.3. If $\dim X < p$, then the theorem follows from Lemma 3.5 or [Yek02, Theorem 4.8]. So, assume that $\dim X = p$. It follows from Lemma 3.5 that the only possibly non-zero differentials in the local-global spectral sequence (2) are those hitting $\mathrm{E}_r^{s,p} = \mathrm{H}^s(X, \Omega^p_{X/k})$. These groups are only possibly non-zero for $0 \leqslant s \leqslant p$, and they contribute to Hochschild homology in degrees $p, p-1, \ldots, 0$. The differential d_r has bidegree $(r, r-1)$, which lowers the total degree by 1. In particular, the only terms that support a non-zero differential out must in particular have total degree $1, \ldots, p$. By Serre duality, $\mathrm{H}^s(X, \Omega^t_{X/k}) \cong \mathrm{H}^{p-s}(X, \Omega^{p-t}_{X/k})^\vee$, the k-linear dual of $\mathrm{H}^{p-s}(X, \Omega^{p-t}_{X/k})$. In particular, at the E_2-page, the sum of the dimensions of total degree $a \geqslant 1$ is equal to the sum of the dimensions of total degree $-a$. If some term of degree $a \geqslant 1$ supports a differential to $\mathrm{E}_r^{s,p}$, then $\dim_k \mathrm{HH}_a(X/k) < \dim_k \mathrm{HH}_{-a}(X/k)$, which contradicts Corollary 3.4. It follows that there are no non-zero differentials, so that X satisfies the weak HKR theorem over k. The last statement follows from the fact that

$$\Omega^p_{X/k}[p] \to \underline{\mathrm{HH}}(X/k) \to \bigoplus_{t=0}^{p-1} \Omega^t_{X/k}[t]$$

is a cofiber sequence and the lack of differentials in (2) implies we get short exact sequences in global sections, as desired. ☐

Now, we prove Corollary 1.5.

Proof of Corollary 1.5. We can again assume that $\dim X = p$. By Lemma 3.5, it is enough to construct a map $\underline{\mathrm{HH}}_X \to \omega_{X/k}[p]$ such that the composition $\omega_{X/k}[p] \to \underline{\mathrm{HH}}_X \to \omega_{X/k}[p]$ is the identity. In other words, we are interested in the restriction map

$$\underline{\mathrm{Map}}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p]) \to \underline{\mathrm{Map}}_X(\omega_{X/k}[p], \omega_{X/k}[p]).$$

This is the map on global sections of the map of (sheaves of) mapping complexes

$$\mathcal{M}\mathrm{ap}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p]) \to \mathcal{M}\mathrm{ap}_X(\omega_{X/k}[p], \omega_{X/k}[p]),$$

and there is a corresponding map of local-global spectral sequences which is a surjection on the E_2-pages since $\mathcal{HH}_p \cong \omega_{X/k}$. We will be done if we show that the local-global spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(X, \pi_t \mathcal{M}\mathrm{ap}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p])) \Rightarrow \pi_{t-s}\underline{\mathrm{Map}}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p])$$

collapses at the E_2-page. However, $\pi_t \mathcal{M}\mathrm{ap}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p]) \cong \Omega^t_{X/k}$ using the natural isomorphisms

$$\mathcal{T}^{p-t}_{X/k} \otimes \omega_{X/k} \cong \Omega^t_{X/k},$$

where $\mathcal{T}^{p-t}_{X/k}$ denotes the $(p-t)$th exterior power of the tangent bundle of X over k; By Grothendieck duality, $\mathrm{Map}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p]) \simeq \mathrm{HH}(X/k)^\vee$, the k-dual of $\mathrm{HH}(X/k)$. But, by Corollary 3.4, $\mathrm{HH}(X/k)^\vee \simeq \mathrm{HH}(X/k)$. By a dimension count and using Theorem 1.3, we see that the local-global spectral sequence computing $\pi_*\underline{\mathrm{Map}}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[p])$ does indeed collapse. ☐

Given the proof above, we ask the following question.

Question 3.6. Is $\underline{\mathrm{HH}}_X \simeq \mathcal{M}\mathrm{ap}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[d])$ when X is a smooth proper k-scheme of dimension d?

Lemma 3.7. *Let X be a smooth proper d-dimensional scheme over a field k of characteristic $p > 0$. If the weak HKR theorem holds for X, then $\underline{\mathrm{HH}}_X \simeq \mathcal{M}\mathrm{ap}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[d])$.*

Proof. The proof above of Corollary 1.5 applies equally well here to show that there is a map $\underline{\mathrm{HH}}_X \to \omega_{X/k}[d]$ such that the composition $\omega_{X/k}[d] \to \underline{\mathrm{HH}}_X \to \omega_{X/k}[d]$ is the identity. Now, consider the composition $\underline{\mathrm{HH}}_X \otimes_{\mathcal{O}_X} \underline{\mathrm{HH}}_X \to \underline{\mathrm{HH}}_X \to \omega_{X/k}[d]$ induced by the multiplicative structure on $\underline{\mathrm{HH}}_X$. The reader can check that the adjoint map $\underline{\mathrm{HH}}_X \to \mathcal{M}\mathrm{ap}_X(\underline{\mathrm{HH}}_X, \omega_{X/k}[d])$ is an equivalence using that the multiplication on the homotopy sheaves of $\underline{\mathrm{HH}}_X$ agrees with the exterior power multiplication on $\Omega^*_{X/k}$. ☐

We end the paper with a brief connection to algebraic K-theory.

Proposition 3.8. *Let X be a smooth projective 3-fold over a field k of characteristic 2. If the image of the first Chern class map $c_1 : \mathrm{K}_0(X) \to \mathrm{H}^1(X, \Omega^1_X)$ generates $\mathrm{H}^1(X, \Omega^1_X)$ as a k-vector space, then the weak HKR theorem holds for X.*

Proof. It is not hard using that we can make $\underline{\mathrm{HH}}_X$ into an $\mathcal{O}_X$-algebra to see that there are no non-zero differentials leaving the terms $\mathrm{H}^s(X, \mathcal{O}_X)$. The only other possible differential that hits a class of negative degree is $d_2 : \mathrm{H}^1(X, \Omega^1_X) \to \mathrm{H}^3(X, \Omega^2_X)$. This differential vanishes as all of the classes must be permanent thanks to the hypothesis and the trace map $\mathrm{K}_0(X) \to \mathrm{HH}_0(X/k)$. Now, all remaining classes involve terms of total degrees -2, -1, or 0. In any case, if one of these differentials is non-zero, then the self-duality of $\mathrm{HH}(X/k)$ is violated, just as in the proof of Theorem 1.3. ☐

This hypothesis is satisfied for smooth complete intersections in $\mathbb{P}^4$ by Deligne's theorem that these have the same Hodge numbers as their characteristic 0 counterparts [DK73, Exposé XI]. In this case the result also follows as a special case of Example 1.7.

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