MOTIVIC REALIZATIONS OF SINGULARITY CATEGORIES AND VANISHING CYCLES

ANTHONY BLANC, MARCO ROBALO, BERTRAND TÖEN, AND GABRIELE VEZZOSI

Abstract. In this paper we establish a precise comparison between vanishing cycles and the singularity category of Landau–Ginzburg models over a complete discrete valuation ring. By using noncommutative motives, we first construct a motivic $\ell$-adic realization functor for dg-categories. Our main result, then asserts that, given a Landau–Ginzburg model over a complete discrete valuation ring with potential induced by a uniformizer, the $\ell$-adic realization of its singularity category is given by the inertia-invariant part of vanishing cohomology. We also prove a functorial and $\infty$-categorical lax symmetric monoidal version of Orlov’s comparison theorem between the derived category of singularities and the derived category of matrix factorizations for a Landau–Ginzburg model over a noetherian regular local ring.

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1. Introduction

The main objective of this paper is to use both derived and non-commutative geometry in order to establish a precise and fairly general relation between singularity categories and vanishing cycles. In a first section we work with $S = \text{Spec } A$ with $A$ a commutative noetherian regular local ring (e.g. a discrete valuation ring). We will consider LG-pairs or LG-models over $S$ (LG for Landau-Ginzburg), i.e. pairs $(X/S, f)$ where $X$ is a scheme flat of finite type over $S$, and $f : X \to \mathbb{A}^1_S$ is an arbitrary map. One can associate to an LG-pair $(X/S, f)$ two, a priori different, triangulated categories: the derived category of matrix factorization $\text{MF}(X/S, f)$, and the derived category of singularities $\text{Sing}(X/S, f)$ ([Orl04], [EP15]). A fundamental insight of Orlov ([Orl04]) is that, whenever $X$ is regular, and $f$ is not a zero-divisor, then there is an equivalence\(^1\) of triangulated categories between $\text{MF}(X/S, f)$ and $\text{Sing}(X/S, f)$.

Now, both $\text{MF}(X/S, f)$ and $\text{Sing}(X/S, f)$ can be naturally enhanced to dg-categories over $A$ whose associated homotopy categories are the given triangulated categories of matrix factorizations and of singularities. Moreover, Orlov’s functor can be enhanced (using derived geometry) to a functor of $A$-dg-categories, which is furthermore natural in the pair $(X/S, f)$, and shown to be an equivalence under appropriate hypotheses. This is the content of Theorem 2.35, below, where we also discuss a lax monoidal strengthening.

Once we have the $A$-dg-category $\text{Sing}(X/S, f)$ at our disposal, following [Rob15], we may look at it as an object in the $\infty$-category $\text{SHnc}_S$ of non-commutative motives over $S$. By [Rob15], we have an $\infty$-functor $M_S : \text{SHnc}_S \to \text{SH}_S$, from non-commutative motives to the ($\infty$-categorical version of) Morel and Voevodsky stable category $\text{SH}_S$ of commutative motives over $S$. The functor $M_S$ is the lax monoidal right adjoint to the functor $\text{SH}_S \to \text{SHnc}_S$ canonically induced by the rule $Y \mapsto \text{Perf}(Y)$, where $\text{Perf}(Y)$ denotes the $A$-dg-category of perfect complexes on a smooth $S$-scheme $Y$. By [Rob15, Theorem 1.8], $M_S$ sends the image of the tensor unit $A \in \text{SHnc}_S$ to the object $\text{BU}_S$, representing homotopy algebraic K-theory, i.e. to the commutative motive identified by the fact that $\text{BU}_S(Y)$ is the spectrum of non-connective homotopy invariant algebraic K-theory of $Y$, for any smooth $S$-scheme $Y$. As a consequence, $\text{BU}_S$ is endowed with

\(^1\)Orlov works with pairs $(X/S, f)$ defined over $S$ the spectrum of a field, and where $f$ is actually flat. This is not enough for our purposes, and we refer the reader to Section 2 of this paper for a discussion of this point.
the structure of a commutative algebra in the symmetric monoidal ∞-category SH\textsubscript{S}. Therefore, M\textsubscript{S} actually factors, as a lax monoidal functor, M\textsubscript{S} : SH\textsubscript{nc\textsubscript{S}} → Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}) via the category of BU\textsubscript{S}-modules in SH\textsubscript{S}.

The first main idea in this paper (see Section 3.2) is to modify the functor M\textsubscript{S} in order to obtain different informations, better suited to our goal. Instead of M\textsubscript{S}, we consider a somewhat dual version M\textsubscript{∨S} : SH\textsubscript{nc\textsubscript{S}} → Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}) via the category of BU\textsubscript{S}-modules in SH\textsubscript{S}.

which is, roughly speaking, defined by sending an A-dg-category T to the commutative motive M\textsubscript{∨S}(T) that sends a smooth S-scheme Y to the spectrum KH(Perf(Y)⊗\textsubscript{A} T) of (non-connective) homotopy invariant algebraic K-theory of the A-dg-category Perf(Y)⊗\textsubscript{A} T (see Section 3.2 for details). In particular, for p : X → S, with X quasi-compact and quasi-separated, we get (Proposition 3.9) an equivalence M\textsubscript{∨S}(Perf(X)) ≃ p\textsubscript{*}(BU\textsubscript{X}) in Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}) where BU\textsubscript{X} denotes a relative version homotopy invariant algebraic K-theory, and (Proposition 3.20) an equivalence M\textsubscript{∨S}(Sing(S,0\textsubscript{S})) ≃ BU\textsubscript{S} ⊕ BU\textsubscript{S}[1] in Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}). As consequence of this, the motive M\textsubscript{∨S}(Sing(X,f)) is a module over BU\textsubscript{S} ⊕ BU\textsubscript{S}[1], for any LG-pair (X,f) over S.

The second main idea in this paper (see Section 3.7) is to compose the functor M\textsubscript{∨S} : SH\textsubscript{nc\textsubscript{S}} → Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}) with the ℓ-adic realization functor R\textsubscript{ℓS} : SH\textsubscript{S} → Sh\textsubscript{Q\textsubscript{ℓ}}(S) with values in the ∞-categorical version of Ind-constructible ℓ-adic sheaves on S with Q\textsubscript{ℓ}-coefficients of [Eke90]. Building on results of Cisinski-Deglise and Riou, we prove (see Section 4.2.1) that one can refine R\textsubscript{ℓS} to a functor, still denoted by the same symbol,

\[ R\textsubscript{ℓS} : Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}) → Mod\textsubscript{Q\textsubscript{ℓ}}(β)(Sh\textsubscript{Q\textsubscript{ℓ}}(S)), \]

where β denotes the algebraic Bott element. We then denote by R\textsubscript{ℓS} the composite

\[ R\textsubscript{ℓS} : SH\textsubscript{nc\textsubscript{S}} \xrightarrow{M\textsubscript{∨S}} Mod\textsubscript{BU\textsubscript{S}}(SH\textsubscript{S}) \xrightarrow{R\textsubscript{ℓS}} Mod\textsubscript{Q\textsubscript{ℓ}}(β)(Sh\textsubscript{Q\textsubscript{ℓ}}(S)). \]

We are now in a position to state our main theorem comparing singularity categories and vanishing cycles (see Section 4). Let us take S = Spec A to be a henselian trait with A excellent \textsuperscript{2} and with algebraically closed residue field k, quotient field K, and let us fix a uniformizer π in A, so that A/π = k. We denote by i\textsubscript{σ} : σ := Spec k → S the canonical closed immersion, and by η the generic point of S.

\textsuperscript{2}In practice we will be working with complete discrete valuation rings. In this case excellence conditions is always verified.
Given now a regular scheme $X$, together with a morphism $p : X \to S$ which is proper and flat, we consider the LG-pair $(X/S, \bar{\pi})$, where $\bar{\pi}$ is defined as the composite

$$\bar{\pi} : X \xrightarrow{p} S \xrightarrow{\pi} \mathbb{A}^1_S.$$ 

For a prime $\ell$ different from the characteristic of $k$, we may consider the following two objects inside $\text{Sh}_{\mathbb{Q}_\ell}(\sigma) = \text{Sh}_{\mathbb{Q}_\ell}(k)$:

- the homotopy invariants $H_{\text{ét}}(X_k, \mathcal{V}(\beta)[-1])^hI$, where $H_{\text{ét}}$ denotes $\ell$-adic étale hypercohomology, $\mathcal{V}$ is the complex of vanishing cycles relative to the map $p$, $\mathcal{V}(\beta)[-1] := \mathcal{V}[-1] \otimes \mathbb{Q}_\ell(\beta)$ (see Remark 4.15) and $I = \text{Gal}(K_\text{sep}/K)$ is the inertia group;

- the derived pullback $i^*_\sigma(\mathcal{R}_S^\ell(\text{Sing}(X, \bar{\pi})))$ of the $\ell$-adic realization of the commutative motive given by the image under $M^\vee_S$ of the dg-category of singularities for the LG-pair $(X/S, \bar{\pi})$.

One way to state our main result is then

**Main Theorem.** (see Theorem 4.11) There is a canonical equivalence

$$i^*_\sigma(\mathcal{R}_S^\ell(\text{Sing}(X, \bar{\pi}))) \simeq H_{\text{ét}}(X_k, \mathcal{V}(\beta)[-1])^hI$$

inside $\text{Sh}_{\mathbb{Q}_\ell}(k)$.

What this theorem tells us is that one can recover vanishing cohomology through the dg-category of singularities, i.e. in a purely non-commutative (and derived) geometrical setting. We think of this result as both an evidence and a first step in the application of non-commutative derived geometry to problems in arithmetic geometry, that we expect to be very fruitful. It is crucial, especially for future applications, to remark that our result holds over an arbitrary base henselian trait $S$ (with perfect residue field), so that it holds both in pure and mixed characteristics. In particular, we do not need to work over a base field. The main theorem above is also at the basis of the research announcement [TV16], where a trace formula for dg-categories is established, and then used to propose a strategy of proof of Bloch’s conductor conjecture ([Blo85, KS05]). Full details will appear in [TV17].

**Remark 1.1.** The result of Theorem 4.11 is stated for the $\ell$-adic realization but can also be given a motivic interpretation. Indeed, one can use the formalism of motivic vanishing cycles of [Ayo07c, Ayo07b, Ayo14] to produce a motivic statement that realizes to our
formula. We thank an anonymous referee for his comments and suggestions regarding this motivic presentation. The proof is mutatis mutandis the one presented here for the \( \ell \)-adic realization and we will leave it for further works.

**Related works.** The research conducted in the second part of this work has its origins in Kashiwara’s computation of vanishing cycles in terms of D-modules via the Riemann-Hilbert correspondence [Kas83]. A further deep and pioneering work is undoubtedly Kapranov’s influential paper [Kap91] which starts with the identification of D-modules with modules over the de Rham algebra. This relation between vanishing cycle cohomology and twisted de Rham cohomology as been fully understood by Sabbah and Saito in [SS14, Sab10] establishing proofs for the conjectures of Kontsevich-Soibelman in [KS11]. In parallel, the works of [CaaT13, Dyc11a, Seg13, Shk14] established the first link between twisted de Rham cohomology and the Hochschild cohomology of matrix factorizations and more recently, the situation has been clarified with Efimov’s results [Efi12]. The combination of these results express a link between the theory of matrix factorizations and the formalism of vanishing cycles. The recent works of Lunts and Schnürer’s [LS17, Theorem 1.2] built upon Efimov’s work, combined with those of [IS13], show that this connection between the two theories can be expressed as an equivalence of classes in a certain Grothendieck group of motives. The main result (Theorem 4.11) of this paper might, in some sense, be seen as categorification of Lunts and Schnürer’s results.

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2. **Matrix factorizations and derived categories of singularities**

We let \( A \) be a commutative Noetherian regular local ring and \( S := \text{Spec } A \). For most of the applications in this paper \( A \) will actually be a complete discrete valuation ring.

Let \( \text{Sch}_S \) denote the category of schemes of finite type over \( S \). We introduce the category of Landau-Ginzburg models over \( S \) as the subcategory of \( (\text{Sch}_S) / A_S^1 \) spanned by those pairs

\[
(p : X \to S, f : X \to A_S^1)
\]
where $p$ is a flat morphism. We will denote it by $LG_S$. We will denote by $LG^\text{aff}_S$ its full subcategory spanned by those LG-models where $X$ is affine over $S$.

We also introduce the category of flat Landau-Ginzburg models over $S$ as the full subcategory of $LG_S$ consisting of $(S, 0 : S \to \mathbb{A}^1_S)$ (where $0$ denotes the zero section of the canonical projection $\mathbb{A}^1_S \to S$) together with those pairs

$$(p : X \to S, f : X \to \mathbb{A}^1_S)$$

where both $p$ and $f$ are flat morphisms. We will denote it by $LG^\text{fl}_S$. We will denote by $LG^\text{fl,aff}_S$ its full subcategory spanned by those LG-models where $X$ is affine over $S$.

The category $LG_S$ (resp. $LG^\text{fl}_S$) has a natural symmetric monoidal structure $\boxtimes$ given by the fact that the additive group structure on $\mathbb{A}^1_S$ defines a monoidal structure on the category $(\text{Sch}_S)/\mathbb{A}^1_S$, $\boxtimes : (\text{Sch}_S)/\mathbb{A}^1_S \times (\text{Sch}_S)/\mathbb{A}^1_S \to (\text{Sch}_S)/\mathbb{A}^1_S$ given by

$$(X, f), (Y, g) \mapsto (X, f) \boxtimes (Y, g) := (X \times_S Y, f \boxplus g)$$

where $f \boxplus g := p^*_X(f) + p^*_Y(g)$ for $p_X : X \times_SY \to X$ and $p_Y : X \times_SY \to Y$ the two projections. Notice that as the maps to $S$ are flat, the fiber product $X \times_SY$ is also flat over $S$, and thus it belongs to $LG_S$. Moreover, if $f$ and $g$ are flat, then $f \boxplus g$ is also flat (since flatness is stable by arbitrary base-change, and the sum map $\mathbb{A}^1_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S$ is flat), and if $g = 0 : S \to \mathbb{A}^1_S$, then $f \boxplus g = f$, which is again flat. In particular, the unit for this monoidal structure is $(S, 0)$.

We will denote this monoidal structure on $LG_S$ (resp. $LG^\text{fl}_S$) by $LG_S^\boxtimes$ (resp. $LG^\text{fl}\boxtimes$). Obviously $LG^\text{fl}\boxtimes$ is a full symmetric monoidal subcategory of $LG^\boxtimes_S$. Notice however that $(S, 0)$ is not in $LG^\text{fl}$ so the monoidal structure is non-unital. We use similar notations for the symmetric monoidal subcategories of (flat) affine LG-pairs (recall that $S$ is affine, so that also the affine versions have $(S, 0)$ as the unit).

**Remark 2.1.** Orlov works inside $LG^\text{fl}_S$ in [Orl04], while Efimov and Positselski work in the whole $LG_S$ in [EP15].

In this section we discuss two well known constructions, namely matrix factorizations and categories of singularities. For us, these will be defined as $\infty$-functors with values in dg-categories

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$^3$on functions given by $A[T] \mapsto A[X] \otimes_A A[Y], \quad T \mapsto X \otimes 1 + 1 \otimes Y$
from the category of LG-models to the ∞-category of (small) A-linear idempotent complete dg-categories. Our MF will be in fact a lax symmetric monoidal ∞-functor MF : LG_S^{op} → dgcat_A^{idem}. The first construction we want to describe sends (X, f) to the dg-category MF(X, f) of matrix factorizations of f. The second one sends (X, f) to Sing(X, f), the dg-category of (relative) singularities of the scheme X_0 of zeros of f. We compare these two constructions by means of the so-called Orlov’s equivalence, which for us will be stated as the existence of a natural transformation of ∞-functors.

Sing → MF

which is an equivalence whenever X is a regular scheme.

The results of this section consist mainly in ∞-categorical enhancements of well known results in the world of triangulated categories.

2.1. Review of dg-categories. In this paragraph we fix our notations for the theory of dg-categories, by recalling the main definitions and constructions used in the rest of the paper. Our references for dg-categories will be [Toē11] and [Robdf, Section 6.1.1 and 6.1.2].

As an ∞-category, dgcat_A^{idem} is a Bousfield localization of the ∞-category of (small) A-linear dg-categories with respect to Morita equivalences, namely dg-functors inducing equivalences on the respective derived categories of perfect dg-modules. The ∞-category dgcat_A^{idem} is naturally identified with the full sub-∞-category of dgcat_A consisting of triangulated dg-categories in the sense of Kontsevich. Recall that these are small dg-categories T such that the Yoneda embedding T ↦ T.pe := {Perfect T-dg-modules}, is an equivalence (i.e., any perfect T^{op}-dg-module is quasi-isomorphic to a representable dg-module). With this identification the localization ∞-functor

dgcat_A → dgcat_A^{idem} (2.1.1)

simply sends T to T.pe.

The ∞-category dgcat_A can be obtained as a localization of the 1-category dgcat_A^{strict} of small strict A-dg-categories with respect to Dwyer-Kan equivalences. This localization is enhanced by the existence of a model structure on dgcat_A^{strict} [Tab05].

Moreover, both dgcat_A^{idem} and dgcat_A come canonically equipped with symmetric monoidal structures induced by the tensor product of locally flat dg-categories

4This terminology is not standard, as for us triangulated also includes being idempotent complete.
dgcat_A^{ strict, loc-flat} ⊆ dgcat_A^{ strict} - namely, those strict dg-categories whose enriching hom-complexes are flat in the category of chain complexes. The localization functor
\[
\text{dgcat}_A^{ strict, loc-flat} \rightarrow \text{dgcat}_A
\] (2.1.2)
is monoidal with respect to these monoidal structures. We address the reader to [Toë11] for a complete account of the dg-categories and to [Robdf, Section 6.1.1 and 6.1.2] for an ∞-categorical narrative of these facts 5.

**Notation 2.2.** For a dg-category \( T \), we will denote as \([T]\) its homotopy category.

At several occasions we will need to take dg-quotients: if \( T_0 \rightarrow T \) is a map in \( \text{dgcat}_A^{idem} \), one considers its cofiber as a pushout
\[
\begin{array}{ccc}
T_0 & \longrightarrow & T \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & T \coprod_{T_0} \{0\} := T'
\end{array}
\]
in \( \text{dgcat}_A^{idem} \). By a result of Drinfeld [Dri] the homotopy category of \( T' \) can be canonically identified with the classical Verdier quotient of \( T \) by the image of \( T_0 \). This pushout is equivalent to the idempotent completion of the pushout taken in the \( \text{dgcat}_A \), thus given by \( \hat{T}' \) and its homotopy category can be identified with the idempotent completion of the Verdier quotient of \( T \) by \( T_0 \).

To conclude this review, let us mention that for any ring \( A \), \( \text{dgcat}_A^{idem} \) can also be identified with the ∞-category of small stable idempotent complete \( A \)-linear ∞-categories. The proof of [Coh13] adapts to any characteristic. We address the reader to the discussion [Robdf, Section 6.2] for more helpful comments.

**2.2. Matrix factorizations.** This section deals with the construction of the symmetric lax monoidal ∞-functor
\[
\text{MF} : \text{LG}_S^{aff, op} \rightarrow \text{dgcat}_A^{idem}
\] (2.2.1)

To define this lax monoidal ∞-functor we will first construct an auxiliary strict version and explain its lax structure.

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5 where we use locally cofibrant dg-categories instead of locally flat. The two strategies are equivalent as every locally cofibrant dg-category is locally flat [Toë07, 2.3(3)] and a cofibrant replacement functor is an inverse. See also the discussion in [Robdf, p. 222].
2.2.1. Let \((X, f) \in \text{LG}_{\text{aff}}^\text{S}\) and let us write \(X := \text{Spec } B\), for \(B\) flat of finite type over \(A\). The function \(f\) is thus identified with an element \(f \in B\). We associate to the pair \((B, f)\) a strict \(\mathbb{Z}/2\)-graded \(A\)-dg-category \(\text{MF}(B, f)\) as follows.

**Construction 2.3.** First we construct \(\text{MF}(B, f)\) as an object in the theory of small strict \(\mathbb{Z}/2\)-graded \(B\)-dg-categories, meaning, small strict dg-categories enriched in \(\mathbb{Z}/2\)-graded complexes of \(B\)-modules. Its objects are pairs \((E, \delta)\), consisting of the following data.

1. A \(\mathbb{Z}/2\)-graded \(B\)-module \(E = E_0 \oplus E_1\), with \(E_0\) and \(E_1\) projective and of finite rank over \(B\).
2. A \(B\)-linear endomorphism \(\delta : E \to E\) of odd degree, and satisfying \(\delta^2 = \cdot f\).

In a more explicit manner, objects in \(\text{MF}(X, f)\) can be written as 4-tuples, \((E_0, E_1, \delta_0, \delta_1)\), consisting of \(B\)-modules projective and of finite type \(E_i\), together with \(B\)-linear morphisms

\[
E_0 \overset{\delta_0}{\longrightarrow} E_1 \quad E_1 \overset{\delta_1}{\longrightarrow} E_0
\]

such that \(\delta_0 \circ \delta_1 = \delta_1 \circ \delta_0 = \cdot f\).

For two objects \(E = (E, \delta)\) and \(F = (F, \delta)\), we define a \(\mathbb{Z}/2\)-graded complex of \(B\)-modules of morphisms \(\text{Hom}(E, F)\) in the usual manner. As a \(\mathbb{Z}/2\)-graded \(B\)-module, \(\text{Hom}(E, F)\) simply is the usual decomposition of \(B\)-linear morphisms \(E \to F\) into odd and even parts. The differential is itself given by the usual commutator formula: for \(t \in \text{Hom}(E, F)\) homogenous of odd or even degree, we set

\[
d(t) := [t, \delta] = t \circ \delta - (-1)^{\deg(t)} \delta \circ t
\]

Even though \(\delta\) does not square to zero, we do have \(d^2 = 0\). This defines \(\mathbb{Z}/2\)-graded complexes of \(B\)-modules \(\text{Hom}(E, F)\) and sets \(\text{MF}(B, f)\) as a \(\mathbb{Z}/2\)-graded \(B\)-dg-category.

Composing with the structure map \(\text{Spec } B \to \text{Spec } A = S\), one can now understand \(\text{MF}(B, f)\) as \(\mathbb{Z}/2\)-graded \(A\)-linear dg-category. Notice that as \(A\) is by hypothesis a local ring, and \(B\) is flat over \(A\), being projective of finite rank over \(B\) implies being projective of finite rank over \(A\), as flat and projective become equivalent notions as soon as the modules are finitely generated.

**Construction 2.4.** The assignment \((X, f) \mapsto \text{MF}(X, f)\) acquires a pseudo-functorial structure

\[
\text{LG}_{\text{aff}}^\text{S} \to \text{dgcat}_{\text{strict}}^{\mathbb{Z}/2, A}
\]

as any morphism of \(A\)-algebras \(q : B \to B'\), with \(q(f) = f'\), defines by base change from \(B\) to \(B'\) a \(\mathbb{Z}/2\)-graded \(A\)-linear dg-functor

\[
B' \otimes_B - : \text{MF}(B, f) \to \text{MF}(B', f')
\]
Notation 2.5. Throughout this work we will always allow ourselves to freely interchange the notions of $\mathbb{Z}/2$-graded complexes and 2-periodic $\mathbb{Z}$-graded complexes via an equivalence of strict categories

$$\text{dgMod}^{\mathbb{Z}/2}_A \xrightarrow{\theta} A[u, u^{-1}] - \text{dgMod}$$

(2.2.3)

where $A[u, u^{-1}]$ is the free strictly commutative differential graded algebra over $A$ with an invertible generator $u$ sitting in cohomological degree 2. The functor $\theta$ sends a $\mathbb{Z}/2$-graded complex $(E_0, E_1, \delta_0, \delta_1)$ to the $A[u, u^{-1}]$-dg-module $[E_0 \to E_1 \to E_0 \to E_1 \to E_0 \to ...]$

where $u$ acts via the identity. The inverse equivalence to $\theta$ sends an $A[u, u^{-1}]$-dg-module $F$ to the 2-periodic complex with $F_0$ in degree 0 and $F_1$ in degree 1, together with the differential $F_0 \to F_1$ of $F$ and the new differential $F_1 \to F_2 \simeq F_0$ using the action of $u^{-1}$. Moreover, $\theta$ is symmetric monoidal: the tensor product of 2-periodic complex identifies with the tensor product over $A[u, u^{-1}]$. In particular, this induces an equivalence between the theory of $\mathbb{Z}/2$-graded $A$-dg-categories and that of $A[u, u^{-1}]$-dg-categories :

$$\text{dgcat}^{\text{strict}, \otimes}_{\mathbb{Z}/2,A} \xrightarrow{\theta} \text{dgcat}^{\text{strict}, \otimes}_{A[u,u^{-1}]}$$

(2.2.4)

Applying $\theta$ to the enriching 2-periodic hom-complexes on the l.h.s of (2.2.3), the equivalence (2.2.3) becomes an equivalence of strict $A[u, u^{-1}]$-dg-categories.

This discussion descends to a monoidal equivalence of between the homotopy theories of dg-categories, as explained in [Dyc11b, Section 5.1]. Moreover, the results of [Toë07] remains valid.

For our purposes we will work with the version of $\text{MF}$ obtained by the composition of (2.2.2) with (2.2.4).

2.2.2. We will now give a strict version of the symmetric lax monoidal structure on $\text{MF}$:

Construction 2.6. Let $(X, f)$ and $(Y, g)$ be two objects in $\text{LG}_S^{\text{aff}}$, with $X = \text{Spec } B$ and $Y = \text{Spec } C$ (so $f \in B$ and $g \in C$). We consider the pair $(D, h)$, where $D = B \otimes_A C$ and $h = f \otimes 1 + 1 \otimes g \in D$. We have a natural $A$-linear dg-functor

$$\boxtimes : \text{MF}(B, f) \otimes_A \text{MF}(C, g) \to \text{MF}(D, h),$$

(2.2.5)

obtained by the external tensor product as follows. For two objects $E = (E, \delta) \in \text{MF}(B, f)$ and $F = (F, \partial) \in \text{MF}(C, g)$, we define a projective $D$-module of finite type

$$E \boxtimes F := E \otimes_A F,$$
with the usual induced $\mathbb{Z}/2$-gradation: the even part of $E \boxtimes F$ is $(E_0 \otimes_A F_0) \oplus (E_1 \otimes_A F_1)$, and its odd part is $(E_1 \otimes_A F_0) \oplus (E_0 \otimes_A F_1)$. The odd endomorphism $\delta : E \boxtimes F \to E \boxtimes F$ is given by the usual formula on homogeneous generators

$$\delta(x \otimes y) := \delta(x) \otimes y + (-1)^{\deg x} x \otimes \partial(y).$$  \hfill (2.2.6)

All together, this defines an object $E \boxtimes F \in \text{MF}(D,h)$, and with a bit more work a morphism in $\text{dgcat}^{\text{strict}}_{\mathbb{Z}/2,A}$ giving shape to (2.2.5). These are clearly symmetric and associative. Finally, the construction $\text{MF}$ is also lax unital with unity given by the natural $\mathbb{Z}/2$-graded $A$-linear dg-functor

$$A \to \text{MF}(A,0),$$ \hfill (2.2.7)

sending the unique point of $A$ to $(A[0],0)$ where $A[0]$ is $A$ considered as a $\mathbb{Z}/2$-graded $A$-module pure of even degree (and $A$ is considered as an $\mathbb{Z}/2$-graded $A$-linear dg-category with a unique object with $A$ as endomorphism algebra). This finishes the description of the lax symmetric structure on

$$\text{MF} : \text{LG}_{S}^{\text{aff,op}} \to \text{dgcat}_{\mathbb{Z}/2,A}^{\text{strict,} \otimes} \simeq \text{dgcat}_{A[u,u^{-1}]}^{\text{strict,} \otimes}$$ \hfill (2.2.8)

We must now explain how to make this a lax structure in the homotopy theory of dg-categories. Following the discussion in the Section 2.1, it will be enough to show that $\text{MF}$ has values in locally-flat dg-categories. But this is indeed the case as by definition the objects of $\text{MF}(B,f)$ are pairs of finitely generated projective $B$-modules, therefore flat over $B$, which being $B$ flat over $A$, are all flat over $A$. Therefore, the enriching hom-complexes are flat over $A$. Following this discussion, the lax symmetric monoidal functor (2.2.2) factors as

$$\text{LG}^{\text{aff,op,} \boxtimes} \to \text{dgCat}_{\mathbb{Z}/2,A}^{\text{strict,loc-} \boxtimes}$$ \hfill (2.2.9)

and by composition with (2.2.4) and the restriction along $A \to A[u,u^{-1}]$ we obtain a lax symmetric monoidal functor

$$\text{LG}^{\text{aff,op,} \boxtimes} \to \text{dgCat}_{\mathbb{Z}/2,A}^{\text{strict,loc-} \boxtimes} \overset{(2.2.4)}{\longrightarrow} \text{dgCat}_{A[u,u^{-1}]}^{\text{strict,loc-} \boxtimes} \overset{\text{rest.}}{\longrightarrow} \text{dgCat}_{A}^{\text{strict,loc-} \boxtimes}$$ \hfill (2.2.10)

Finally, we compose this with the monoidal localization $\infty$-functor (2.1.2) followed by (2.1.1), to obtain a new lax monoidal $\infty$-functor

$$\text{MF} : \text{LG}^{\text{aff,op,} \boxtimes} \to \text{dgcat}_{A}^{\text{idem,} \boxtimes}$$ \hfill (2.2.11)
Remark 2.7. The restriction of scalars along $A \to A[u, u^{-1}]$ forgets the 2-periodic structure. However, it is a consequence of the construction that we can recover this 2-periodic structure from the lax monoidal structure (2.2.11).

Indeed, the lax monoidal structure endows $\text{MF}(S, 0)$ with a structure of object in $\text{CAlg}(\text{dgcat}^{\text{idem}}_A)$. At the same time, as $\text{MF}(S, 0)$ admits a compact generator given by $A$ in degree 0, it follows that in $\text{dgcat}^{\text{idem}}_A$, $\text{MF}(S, 0)$ is equivalent to perfect complexes over the dg-algebra $\theta(\text{End}_{\text{MF}(S, 0)}(A))$. But an explicit computation shows that this is a strict-dg-algebra given by $A[u, u^{-1}]$ with $u$ a generator in cohomological degree 2. In addition to this, the symmetric monoidal structure on $\text{MF}(S, 0)$, which by construction of the lax structure in fact exists in the strict theory of strict dg-categories, produces a structure of strict commutative differential graded algebra on $A[u, u^{-1}]$ which corresponds to the standard one. It follows that in $\text{CAlg}(\text{dgcat}^{\text{idem}}_A)$, we have a monoidal equivalence

\[
\text{MF}(S, 0)^{\otimes} \simeq \text{Perf}(A[u, u^{-1}])^{\otimes} \quad (2.2.12)
\]

where the r.h.s is equipped with the relative tensor product of $A[u, u^{-1}]$. It follows from the lax structure and the monoidal equivalence (2.2.12) that $\text{MF}$ extends to a lax monoidal functor

\[
\text{MF} : \text{LG}^{\text{aff, op}} \to \text{ModPerf}(A[u, u^{-1}])^{\otimes} \quad (2.2.13)
\]

We observe that this is precisely the 2-periodic structure of (2.2.10) before restricting scalars along $A \to A[u, u^{-1}]$. Indeed, this follows from the commutativity of the square of lax monoidal $\infty$-functors

\[
\begin{array}{cccc}
\text{N(dgCat}_{A[u, u^{-1}]^{\text{strict, loc-flat/A, } \otimes})} & \text{N(dgCat}_{A}^{\text{strict, loc-flat/A, } \otimes}) \\
\downarrow & \downarrow \\
\text{dgcat}_{A[u, u^{-1}]}^{\text{idem, } \otimes} & \text{dgcat}_{A}^{\text{idem, } \otimes}
\end{array}
\quad (2.2.14)
\]

and the monadic equivalence

\[
\text{dgcat}_{A[u, u^{-1}]}^{\text{idem, } \otimes} \simeq \text{ModPerf}(A[u, u^{-1}])^{\otimes} \quad (\text{dgcat}_{A}^{\text{idem}})^{\otimes}
\]

The construction of $\text{MF}$ as a lax monoidal functor from affine LG-pairs to $A$-dg-categories, can be extended to all LG-pairs: one way is to interpret $\text{MF}$ as a functor $\text{LG}^{\text{aff}} \to \text{dgcat}_{A}^{\text{idem, op}}$ and take its monoidal left Kan extension Kan $\text{MF}$ to presheaves of spaces $\mathcal{P}(\text{LG}^{\text{aff}})$ [Lur16, 4.8.1.10]. Now $\text{LG}_S$ embeds fully faithfully by Yoneda inside $\mathcal{P}(\text{LG}^{\text{aff}})$ in a monoidal way with respect to the Day product. The restriction to this full
subcategory defines an \( \infty \)-functor
\[
\text{Kan MF} : \text{LG}\,^\text{op} \rightarrow \text{dgcat}_{A}^{\text{idem}}
\]  
(2.2.15)
of matrix factorizations over \( S \), naturally equipped with a lax symmetric monoidal enhancement
\[
\text{Kan MF}^{[\otimes]} : \text{LG}_{S}^\text{op}[\otimes] \rightarrow \text{dgcat}_{A}^{\text{idem}, \otimes}.
\]  
(2.2.16)
Alternatively, there is a definition of matrix factorizations on non-affine LG-pairs \( (X, f) \) under the assumption that \( X \) has enough vector bundles. Indeed, one should work with matrix factorizations \( (E_0, E_1, \delta_0, \delta_1) \) where \( E_0 \) and \( E_1 \) are vector bundles on \( X \). See [Orl04, Orl12, LP11]. Under this assumption, this second definition agrees with the Kan extension. See [LP11, 2.11], [Efi12, Section 5] or [BW12, Section 3].

2.3. \textbf{dg-Categories of singularities.}

2.3.1. Let \( (X, f) \) be an LG-model. In this section we will study an invariant that captures the singularities of \( X_0 \subset X \), the closed subscheme of zeros of \( f \). As we will not impose any condition on \( f \), for instance \( f \) can be a zero divisor, we have to allow \( X_0 \) to be eventually a derived scheme. More precisely, we consider the derived fiber product
\[
\begin{array}{ccc}
S & \xrightarrow{i} & X \\
\downarrow & & \downarrow f \\
A_1^S & \xrightarrow{\scriptstyle 0} & X_0
\end{array}
\]  
(2.3.1)
where the canonical map \( i \) is an lci closed immersion, as it is the pullback of the lci closed immersion \( S \rightarrow A_1^S \).

\textbf{Remark 2.8.} Note that if \( (X, f) \in \text{LG}_{S}^{\text{fl}} \) (i.e. \( f \) is flat), then \( X_0 \) is just the scheme theoretic zero locus of \( f \). In particular it coincides with the \( X_0 \) consider by Orlov in [Orl04]. Therefore, in this case, the derived categories of singularities considered in this paper (see Def. 2.13 and Remark 2.15 below) is (an \( \infty \)- or dg-categorical version of) the derived category of singularities of [Orl04].

For an LG-model \( (X, f) \), with associated derived scheme \( X_0 \) of zeros of \( f \), we consider \( \text{Qcoh}(X_0) \) the \( A \)-linear dg-category of quasi-coherent complexes on \( X_0 \) (see [Toë14, Section 3.1] for a survey). We consider the following full sub-dg-categories of \( \text{Qcoh}(X_0) \):

- \( \text{Perf}(X_0) \): perfect objects over \( X_0 \), meaning, objects \( E \in \text{Qcoh}(X_0) \) such that locally it belong to the thick sub-category of \( \text{Qcoh}(X_0) \) generated by the structure
sheaf of $X_0$. These are exactly the $\otimes$-dualizable objects in $\text{Qcoh}(X_0)$ and agree with the compact objects as soon as $X_0$ is quasi-compact and quasi-separated. See [BZFN10, 3.6].

- $\text{Coh}^b(X_0)$: cohomologically bounded objects $E \in \text{Qcoh}(X_0)$ whose cohomology $H^*(E)$ is a coherent $H^0(O_{X_0})$-module.
- $\text{Coh}^{-}(X_0)$: cohomologically bounded above objects $E \in \text{Qcoh}(X_0)$ whose cohomology $H^*(E)$ is a coherent $H^0(O_{X_0})$-module.
- $\text{Coh}^b(X_0)_{\text{Perf}(X)}$: cohomologically bounded objects $E \in \text{Qcoh}(X_0)$ whose cohomology $H^*(E)$ is a coherent $H^0(O_{X_0})$-module and such that the direct image of $E$ under $i_*$ is perfect over $X$;

where we always have inclusions

$$\text{Perf}(X_0) \subseteq \text{Coh}^b(X_0) \subset \text{Coh}^{-}(X_0) \subset \text{Qcoh}(X_0)$$

$$\text{Coh}^b(X_0)_{\text{Perf}(X)} \subseteq \text{Coh}^b(X_0)$$

and the fact the map $X_0 \to X$ is a lci closed immersion (of derived schemes), thus preserving perfect complexes, gives us another inclusion

$$\text{Perf}(X_0) \subseteq \text{Coh}^b(X_0)_{\text{Perf}(X)}$$

\textbf{Remark 2.9.} The construction $(X, f) \mapsto X_0$ can be presented as an $\infty$-functor. We leave this as an easy exercise to the reader. Moreover, for any map of LG-pairs $u : (X, f) \to (Y, g)$ there is a well-defined pullback functor

$$\text{Coh}^b(Y_0)_{\text{Perf}(Y)} \to \text{Coh}^b(X_0)_{\text{Perf}(X)} \quad (2.3.2)$$

Notice that the pullback map $\text{Coh}^b(Y_0) \to \text{Coh}^b(X_0)$ is not necessarily defined as one would need the map $X_0 \to Y_0$ to be of finite Tor-amplitude. What is true in general is that $\text{Coh}^{-}(Y_0) \to \text{Coh}^{-}(X_0)$ is defined\footnote{These are also known as pseudo-perfect complexes ([MR071]) or almost perfect complexes ([Lur16]).} and this is enough to show that the restriction (2.3.2) is always well-defined. Indeed the proper base change formula applied to the

\footnote{This is [MR071, Exp. I - 2.16.1] for underived $X_0$, i.e. when $f$ is flat. For a general derived $X_0$ one can use the characterization of almost perfect complexes as almost compact objects given in [Lur16, 7.2.4.10, 7.2.4.17] plus the fact that for a map of connective $E_\infty$-rings, base change preserves the condition of being cohomologically bounded above as the forgetful functor is both left and right $t$-exact. Alternatively, and more immediately, use [Lur16, 7.2.4.11-(4) and (5)].}
cartesian diagram

$$\begin{array}{ccc}
X_0 & \overset{i}{\rightarrow} & X \\
\downarrow_{u_0} & & \downarrow_{u} \\
Y_0 & \rightarrow & Y
\end{array} \tag{2.3.3}$$

together with the fact the pullback of perfect complexes is always perfect, tells us that if $E \in \text{Coh}^b(Y_0)_{\text{Perf}(Y)}$ then the direct image $i_* (u_0^*(E))$ in $X$ is perfect and therefore of finite Tor-amplitude. Moreover, as $X$ is an underived scheme, finite Tor-amplitude implies finite cohomological amplitude. As the map $i : X_0 \rightarrow X$ is a closed immersion, $i_* : \text{Qcoh}(X_0) \rightarrow \text{Qcoh}(X)$ is t-exact and therefore conservative as the induced functor $i_*^\triangledown : \text{Qcoh}(X_0)^\triangledown \rightarrow \text{Qcoh}(X)^\triangledown$ is the classical pushforward functor on the truncations $t(X_0) \rightarrow t(X)$ and therefore conservative. It follows that $u_0^*(E)$ is also of finite cohomological amplitude.

As we know that $u_0^*(E)$ is also in $\text{Coh}^-(X_0)$ we conclude the proof. The construction $(X, f) \mapsto \text{Coh}^b(X_0)_{\text{Perf}(X)}$ can easily be written as part of an $\infty$-functor.

For future reference, we include here the following result:

**Proposition 2.10.** For any LG-model $(X, F)$, the inclusion

$$\text{Coh}^b(X_0)_{\text{Perf}(X)} \rightarrow \text{Coh}^-(X_0)_{\text{Perf}(X)} \tag{2.3.4}$$

is an equivalence. Therefore, the category $\text{Coh}^b(X_0)_{\text{Perf}(X)}$ fits in a pullback square of idempotent complete $A$-linear dg-categories

$$\begin{array}{ccc}
\text{Coh}^-(X_0) & \overset{i_*}{\rightarrow} & \text{Coh}^-(X) \\
\downarrow & & \downarrow \\
\text{Coh}^b(X_0)_{\text{Perf}(X)} & \rightarrow & \text{Perf}(X)
\end{array} \tag{2.3.5}$$

and the assignment $(X, f) \mapsto \text{Coh}^b(X_0)_{\text{Perf}(X)}$ has descent with respect to $h$-Cech covers.

**Proof.** The fact that $i_* : \text{Qcoh}(X_0) \rightarrow \text{Qcoh}(X)$ in the Remark 2.9 is t-exact and conservative implies that the diagram

$$\begin{array}{ccc}
\text{Coh}^-(X_0) & \overset{i_*}{\rightarrow} & \text{Coh}^-(X) \\
\downarrow & & \downarrow \\
\text{Coh}^b(X_0) & \rightarrow & \text{Coh}^b(X)
\end{array} \tag{2.3.6}$$

is cartesian. Combining this with the fact that (2.3.5) is cartesian, allows us to conclude the equivalence (2.3.4). The descent statement now follows from (2.3.4) and the fact

---

*The diagram is cartesian by definition of morphism of LG-pairs.*
that by [HLP14, Thm 4.12] both almost perfect complexes and perfect complexes satisfy $h$-descent for Cech covers. □

Remark 2.11. Let us remark that the Ind-completion $\text{Ind}(\text{Coh}^b(X_0)_{\text{Perf}(X)})$ embeds fully faithfully inside the presentable $\infty$-category $\text{IndCoh}(X_0)_{\text{Qcoh}(X)}$ obtained via the pullback of presentable $A$-linear dg-categories

\[
\begin{array}{ccc}
\text{IndCoh}(X_0) & \xrightarrow{i_*} & \text{IndCoh}(X) \\
\theta \downarrow & & \phi \downarrow \\
\text{IndCoh}(X_0)_{\text{Qcoh}(X)} & \xrightarrow{} & \text{Qcoh}(X)
\end{array}
\]  

and the inclusion $\text{Ind}(\text{Coh}^b(X_0)_{\text{Perf}(X)}) \subseteq \text{IndCoh}(X_0)_{\text{Qcoh}(X)}$ is closed under filtered colimits. Let us remark first that the inclusion $\text{Perf}(X) \subseteq \text{Coh}^{-}(X)$ is fully-faithful, so is the inclusion $\phi$ after Ind-completion $\text{Qcoh}(X) \subseteq \text{IndCoh}(X)$ and therefore, so is the map $\theta$ by definition of pullbacks in $\text{Pr}^L$ [Lur09, 5.5.3.13] and the definition of mapping spaces in a pullback. Moreover, by the description of colimits in a pullback [Lur09, 5.4.5.5], $\theta$ preserves filtered colimits because the same is true for $\phi$.

The natural inclusions of bounded coherent inside Ind-coherent and perfect inside all quasi-coherent, give us a canonical fully faithful embedding

\[
\text{Coh}^b(X_0)_{\text{Perf}(X)} \subseteq \text{IndCoh}(X_0)_{\text{Qcoh}(X)}
\]  

We will now check that the image of this embedding lives in the full subcategory of the r.h.s spanned by compact objects. Indeed, let $U := (F, E, \alpha : \phi(E) \simeq i_*(F))$ be an object of $\text{IndCoh}(X_0)_{\text{Qcoh}(X)}$ living in the l.h.s. and consider a filtered system $U_j = (F_j, E_j, \alpha_j : \phi(E_j) \simeq i_*(F_j))$ in $\text{IndCoh}(X_0)_{\text{Qcoh}(X)}$. We want to show that the natural map

\[
\text{colim}_j \text{Map}_{\text{IndCoh}(X_0)_{\text{Qcoh}(X)}}(U, U_j) \to \text{Map}_{\text{IndCoh}(X_0)_{\text{Qcoh}(X)}}(U, \text{colim}_j U_j)
\]  

is an equivalence. Now, as $\theta$ is fully faithful and preserves filtered colimits, the r.h.s is equivalent to the mapping space $\text{Map}_{\text{IndCoh}(X_0)}(F, \text{colim}_j F_j)$. By the same arguments, the l.h.s of equation (2.3.9) is equivalent to $\text{colim}_j \text{Map}_{\text{IndCoh}(X_0)}(F, F_j)$. Finally, as $F$ is assumed to be in $\text{Coh}^b(X_0)$, we conclude that (2.3.9) is indeed an equivalence. This proofs the claim that (2.3.8) indeed factors through the compact objects of the r.h.s. so that after passing to Ind-completion we obtain a fully faithful map that preserves filtered colimits.

We start with an absolute version of the definition of the derived category of singularities:
Definition 2.12. Let $Z$ be a derived scheme of finite type over $S$. The (absolute) derived category of singularities of $Z$ is the dg-quotient $\text{Sing}(Z) := \text{Coh}^b(Z)/\text{Perf}(Z)$ taken in $\text{dgcat}_{idem}^A$.

We will now consider the derived category of singularities of an LG-pair $(X,f)$. If $(X,f)$ is an LG-model, we first observe that the canonical closed immersion $i : X_0 \to X$ is lci (of derived schemes) regardless the fact that $f$ is flat, since $i$ is the derived base change of the lci closed immersion $0_S : \text{Spec } S \to \text{Spec } A_S^1$ (note that when $f$ is flat, then the derived fiber $X_0$ of $f$ at 0 coincides with the classical scheme-theoretic fiber, and this assertion boils down to the lci property being preserved under flat base-change). Thus, there are well-defined push-forward $i_* : \text{Perf}(X_0) \to \text{Perf}(X)$ and $i_* : \text{Coh}^b(X_0) \to \text{Coh}^b(X)$, and therefore a well-defined induced functor $\text{Sing}(X_0) \to \text{Sing}(X)$. Moreover, $i$ being of finite Tor dimension, the pullback $i^*$ preserves both bounded coherent and perfect complexes, and induces a well-defined map in the quotient $i^* : \text{Sing}(X) \to \text{Sing}(X_0)$ with $i^*$ left adjoint to $i_*$. In this paper we will use the following definition:

Definition 2.13. The dg-category of singularities of the pair $(X,f)$ is the homotopy fiber in $\text{dgcat}_{idem}^A$

$$\text{Sing}(X,f) := \text{Ker}(i_* : \text{Sing}(X_0) \to \text{Sing}(X)).$$

Proposition 2.14. For any $(X,f) \in \text{LG}_S$ the canonical functor

$$\text{Coh}^b(X_0)_{\text{Perf}(X_0)} / \text{Perf}(X_0) \simeq \text{Sing}(X,f)$$

is an equivalence. Here the dg-quotient on the lhs is taken in $\text{dgcat}_{idem}^A$.

Proof. We start with the observation that as $X$ is assumed to be of finite type over $S$, it is quasi-compact and quasi-separated and in particular, $\text{Perf}(X)$ admits a compact generator [BvdB03]. It is then a consequence of [Rob15, Prop. 1.18] that the exact sequence of idempotent complete dg-categories

$$\begin{array}{ccc}
\text{Perf}(X) & \longrightarrow & \text{Coh}^b(X) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \text{Sing}(X)
\end{array} \quad (2.3.10)$$
is also a pullback in $\text{dgcat}^{\text{idem}}_A$. This cartesian diagram together with the cartesian diagram (2.3.5) fit together in a commutative cube

\[
\begin{array}{ccc}
\text{Sing}(X_0) & \rightarrow & \text{Sing}(X) \\
\downarrow & & \downarrow \\
\text{Sing}(X, f) & \rightarrow & 0 \\
\text{Coh}^b(X_0) & \rightarrow & \text{Coh}^b(X) \\
\downarrow & & \downarrow \\
\text{Coh}^b(X_0)_{\text{Perf}(X)} & \rightarrow & \text{Perf}(X)
\end{array}
\]

where the right, bottom and upper faces are cartesian. In particular, it follows that the face on the left is cartesian and again by [Rob15, Prop. 1.18] applied to $\text{Perf}(X_0) \rightarrow \text{Coh}^b(X_0) \rightarrow \text{Sing}(X_0)$, combined with the fact the face on the left is now known to be cartesian, gives us two cartesian squares

\[
\begin{array}{ccc}
\text{Perf}(X_0) & \leftarrow & \text{Coh}^b(X_0)_{\text{Perf}(X)} \\
\downarrow & & \downarrow \\
* & \rightarrow & \text{Sing}(X, f) \\
\downarrow & & \downarrow \\
\text{Sing}(X_0) & \rightarrow & \text{Sing}(X_0)
\end{array}
\]

where the vertical right arrow is essentially surjective (being the pullback of $\text{Coh}^b(X_0) \rightarrow \text{Sing}(X_0)$ which is essentially surjective). This shows that the canonical map induced by the universal property of the quotient

\[
\text{Coh}^b(X_0)_{\text{Perf}(X)} / \text{Perf}(X_0) \rightarrow \text{Sing}(X, f)
\]

is essentially surjective. It remains to check it is fully faithful. For that purpose we use the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Coh}^b(X_0)_{\text{Perf}(X)} / \text{Perf}(X_0) & \rightarrow & \text{Sing}(X_0) \\
\downarrow & & \downarrow \\
\text{Sing}(X, f) & \rightarrow & \text{Sing}(X_0)
\end{array}
\]

and show that both maps to $\text{Sing}(X_0)$ are fully faithful, thus deducing the fully faithfulness of (2.3.13). The fact that the diagonal arrow is fully faithful follows from the definition of $\text{Sing}(X_0)$ as a fiber of $i_*$. Indeed, this fiber is computed in $\text{dgcat}^{\text{idem}}_A$ but, as the inclusion $\text{dgcat}^{\text{idem}}_A \subseteq \text{dgcat}_A$ commutes with limits (with left adjoint the idempotent completion), we conclude from the formula of the mapping spaces in the fiber product in $\text{dgcat}_A$
and using the fact that 0 is a terminal object, that the diagonal arrow is fully faithful.

It remains to show that the quotient map \( \operatorname{Coh}^b(X_0)_{\operatorname{Perf}(X)} / \operatorname{Perf}(X_0) \to \operatorname{Sing}(X_0) = \operatorname{Coh}^b(X_0) / \operatorname{Perf}(X_0) \) is fully-faithful. This is true as the inclusion \( \operatorname{Coh}^b(X_0)_{\operatorname{Perf}(X)} \to \operatorname{Coh}^b(X_0) \) is fully faithful and the map induced in the quotient corresponds to a quotient by a common subcategory \( \operatorname{Perf}(X_0) \). Notice that as \( \operatorname{Perf}(X_0) \) has a compact generator this corresponds to killing a common object. The claim now follows from the following general observation: if \( C \subseteq D \) is a fully faithful inclusion of idempotent complete \( A \)-dg-categories and \( K \) is an object in \( C \), then the quotient map \( C / K \to D / K \) is fully faithful. This follows from the universal property of the dg-localizations: notice that by Yoneda, a functor \( A \to B \) is fully faithful if and only if for any third category \( \mathcal{H} \) the composition \( \operatorname{Fun}(A, \mathcal{H}) \to \operatorname{Fun}(B, \mathcal{H}) \) is fully faithful. Now, since \( C / K \simeq C[\{ K \to 0 \}^{-1}] \) and \( D / K \simeq D[\{ K \to 0 \}^{-1}] \) and since by the nature of the situation the inclusion \( C \to D \) preserves weak-equivalences, the universal property of localizations gives us fully faithful compositions \( \operatorname{Fun}(C[\{ K \to 0 \}^{-1}], \mathcal{H}) \to \operatorname{Fun}(D[\{ K \to 0 \}^{-1}], \mathcal{H}) \). Conclusion: both \( \operatorname{Coh}^b(X_0)_{\operatorname{Perf}(X)} / \operatorname{Perf}(X_0) \) and \( \operatorname{Sing}(X, f) \) identify with the same full sub-category \( \operatorname{Sing}(X_0) \).

The description of \( \operatorname{Sing}(X, f) \) given in Prop. 2.14 will be recurrent in this paper.

**Remark 2.15.** Following [EP15], one could also define the relative derived category of singularities with respect to \( X_0 \to X \), \( \operatorname{Sing}(X_0 / X) \), as the dg-quotient of \( \operatorname{Sing}(X_0) \) by the image of \( i^* \) taken in \( \operatorname{dgcat}^\text{idem}_A \). This differs from our Definition 2.13 (as explained in [BW12, Remark 6.9]). Nevertheless, one can understand both choices of definition as variations of the situation when \( X \) is regular, where both agree with \( \operatorname{Sing}(X_0) \). Our choice has the advantage of being always equivalent to matrix factorizations of projective modules (as it is proven by [EP15, Proof of Theorem 2.7, p.47] and we shall revisit it in Section 2.4), contrary to the one of [EP15] where one needs to use coherent matrix factorizations.

**Remark 2.16.** All \( \operatorname{Coh}^b(X_0) \), \( \operatorname{Perf}(X_0) \) and \( \operatorname{Coh}^b(X_0)_{\operatorname{Perf}(X)} \) are idempotent complete \( A \)-dg-categories. This is well-known for \( \operatorname{Coh}^b(X_0) \) and \( \operatorname{Perf}(X_0) \). For \( \operatorname{Coh}^b(X_0)_{\operatorname{Perf}(X)} \) this follows because both \( \operatorname{Coh}^b(X_0) \) and \( \operatorname{Perf}(X) \) are idempotent complete and the pushforward along \( i : X_0 \to X \) is an exact functor thus preserving all retracts that exist.
In what follows we will first construct $\text{Sing}$ as an $\infty$-functor defined on affine LG-pairs. Our strategy will be to build a strict model for $\text{Coh}^b(X_0)_{\text{Perf}(X)}$ (see below) and construct the functorialities in this strict setting, transferring them later to the homotopical setting via the localization functor of Section 2.1.

**Remark 2.17.** The reader should be aware that the construction of $\text{Sing}$ as an $\infty$-functor can be done using only $\infty$-categorical methods, without any rectification step, as suggested in the Remark 2.9. Note however that the comparison with the construction of matrix factorizations requires some steps with strict dg-categories, as our initial definition of $\text{MF}$ (Construction 2.3) was indeed given in this setting.

We start with a quick review of a strict model for the derived intersection $X_0$. Let $(X = \text{Spec } B, f) \in LG^{\text{aff}}_S$, corresponding to $f \in B$ for $B$ a flat and finitely presented $A$-algebra. We consider $K(B, f)$, the Koszul algebra associated to the element $f \in B$. It is the commutative $B$-dg-algebra whose underlying complex is $B \rightarrow B/f \rightarrow 0$, with the standard multiplicative structure. We have maps $B \rightarrow K(B, f) \rightarrow B/(f)$. When $f$ is not a zero divisor, these maps make $K(B, f)$ into a cofibrant model for $B/(f)$ as a commutative $B$-dg-algebra (i.e. the diagram above is a factorization of $B \rightarrow B/(f)$ as a cofibration followed by a trivial fibration). More generally, even if $f$ is a zero divisor, $K(B, f)$ is always a cofibrant commutative $B$-dg-algebra which is an algebraic model for the derived scheme $X_0$ of zeros of $f$.

**Example 2.18.** Let $B = A$ and $f = 0$. Then $S_0 := S \times_{A_1} A_1$ is the derived self-intersection of zero inside $A_1$. This is explicitly given by the commutative differential graded algebra $K(A, 0) = A[\epsilon]$ with $\epsilon$ a generator in cohomological degree $-1$ with $\epsilon^2 = 0$, with underlying complex

$$
\begin{align*}
0 & \longrightarrow A.\epsilon \quad 0 \\
\epsilon & \longrightarrow A \\
0 & \longrightarrow 0
\end{align*}
$$

(2.3.16)

**Remark 2.19.** This explicit model for the derived intersection gives us explicit models for $\text{Perf}$, $\text{Coh}^b$, and $\text{Qcoh}$. For instance, there is a canonical equivalence of $A$-dg-categories between the dg-category $\text{Qcoh}(X_0)$ of quasi-coherent complexes on $X_0$, and the $A$-dg-category of cofibrant $K(B, f)$-dg-modules, which we will denote as $K(B, f)$. The full subcategory $\text{Coh}^b(X_0) \subset \text{Qcoh}(X_0)$ (resp. $\text{Perf}(X_0)$) identifies with the full dg-subcategory of $K(B, f)$ spanned by those complexes which are of bounded cohomological amplitude and with coherent cohomology (resp. the full sub dg-category of $K(B, f)$ spanned by cofibrant dg-modules which are homotopically finitely presented). A priori,
the functor $i_*$ can be described as

$$i_*: \hat{K}_B(B,f) \xrightarrow{Q_B \circ \text{Forget}} \hat{B}$$

where $Q_B$ is a cofibrant replacement functor in $B$-dg-modules and Forget is the restriction of scalars along $B \to K(B,f)$. But as $K(B,f)$ is already cofibrant over $B$, any cofibrant $K(B,f)$-dg-module will also be cofibrant over $B$. This $Q_B$ is not necessary.

This strict description of the derived zero locus leads us to the following useful result, which is essentially the observation that for the computation performed in the proof of [Pre11, Prop. 3.1.4] to work we don’t need $A$ to be a field of characteristic zero. In fact, it works whenever $A$ is regular:

**Proposition 2.20.** We have an equivalence in $\text{dgcat}^{\text{idem}}_A$

$$\text{Coh}^b(S \times_{A_S} h_S)_{\text{Perf}(A)} \simeq \text{Perf}(A[u]) \quad (2.3.17)$$

where $u$ has degree 2.

**Proof.** Since $A$ is regular, we have $\text{Coh}^b(S \times_{A_S} h_S)_{\text{Perf}(S)} \simeq \text{Coh}^b(S \times_{A_S} h_S)$, where $S \times_{A_S} h_S$ is the derived zero locus of the zero-section $0 : S \to A_S$. Now, this derived zero-locus is the spectrum of the simplicial commutative ring $\text{Sym}^*_A(A[1])$, whose normalization is the commutative differential graded ring $K(A,0)$ of Example 2.18. Therefore $\text{Coh}^b(S \times_{A_S} h_S)$ is equivalent to $\text{Coh}^b(K(A,0))$, i.e. to dg-modules over $K(A,0)$ which are coherent on the truncation $H^0(K(A,0)) = A$. It is easy to verify that $\text{Coh}^b(K(A,0))$ is generated by the $A$-dg-module $A$, via the homotopy cofiber-sequence

$$A \xrightarrow{0} A \xrightarrow{0} K(A,0) \quad (2.3.18)$$

so that $\text{Ind}(\text{Coh}^b(K(A,0)))$ is equivalent to dg-modules over $\mathbb{R}\text{Hom}_{K(A,0)}(A,A)$ and $\text{Coh}^b(K(A,0))$ to perfect dg-modules over $\mathbb{R}\text{Hom}_{K(A,0)}(A,A)$. Now, we remark the existence of an infinite resolution

$$\cdots \xrightarrow{\text{id}} A \xrightarrow{0} A \xrightarrow{\text{id}} A \xrightarrow{0} A \xrightarrow{\text{id}} A \quad (2.3.19)$$

of $A$ as a $K(A,0)$-dg-module. This can be obtained as an homotopy colimit in $\text{Qcoh}(K(A,0))$ induced by the multiplication by $\epsilon$ as follows: let $K(A,0)\{1\}$ denote the cofiber of
$\epsilon : K(A, 0)[1] \to K(A, 0)$ and by induction, we construct $K(A, 0)\{n + 1\}$ by the cofiber

$$K(A, 0)[2n - 1] \longrightarrow K(A, 0)\{n\} \quad (2.3.20)$$

and obtain the infinite resolution (2.3.19) as the homotopy colimit in in $\text{Qcoh}(K(A, 0))$

$$\text{colim} \ (K(A, 0) \longrightarrow K(A, 0)\{1\} \longrightarrow K(A, 0)\{2\} \longrightarrow \cdots) \simeq A \quad (2.3.21)$$

Using this resolution we can directly compute

$$\mathbb{R}\text{Hom}_{K(A, 0)}(A, A) \simeq A[u] \quad (2.3.22)$$

with $\deg(u) = 2$. Let us briefly describe this computation. It is clear that as $A$-modules, we get an isomorphism of complexes

$$\mathbb{R}\text{Hom}_A(\cdots \longrightarrow \frac{A}{i} \longrightarrow \frac{A}{2i} \longrightarrow \frac{A}{i} \longrightarrow A \longrightarrow 0 \longrightarrow \cdots ) \simeq (A \longrightarrow \frac{A}{2} \longrightarrow \frac{A}{2i} \longrightarrow \frac{A}{i} \longrightarrow A \longrightarrow 0 \longrightarrow \cdots) \quad (2.3.23)$$

where each degree $\frac{A}{i}$ on the r.h.s is a disguise of $\text{Hom}_A(\frac{A}{i}, A)$. The extra demand for a $K(A, 0)$-linear compatibility forces every map $f$ to verify the relation $f(\epsilon.(-)) = \epsilon f$ with $\epsilon$ corresponding to the unity of $A$ in degree $-1$ in $K(A, 0)$. As the action of $\epsilon$ is zero on the trivial $K(A, 0)$-module $A$ concentrated in degree $0$, the $K(A, 0)$-linear structure gives $f(\epsilon.-) = 0$ imposing that for odd $i$ only the zero map in $\text{Hom}_A(\frac{A}{i}, A)$ is allowed. This shows (2.3.22) and concludes the proof of the equivalence (2.3.17). Under this equivalence $u$ corresponds to $1 \in A \simeq \text{Ext}_{K(A, 0)}^2(A, A)$.

For completeness we describe the computation of $\mathbb{R}\text{Hom}_{K(A, 0)}(A, E)$ for $E \in \text{Coh}^{b}(K(A, 0))$. One shows that level $n$ of $\mathbb{R}\text{Hom}_{K(A, 0)}(A, E)$ is the level $n$ of the complex $E \otimes_A A[u]$. However, the differential on $\mathbb{R}\text{Hom}_{K(A, 0)}(A, E)$ is not the naive tensor product differential. Indeed, using the same infinite resolution of $A$ as a $K(A, 0)$-module and from the relation $f(\epsilon.(-)) = \epsilon f$ one obtains that the elements of odd degree are determined by the antecedent element even degree under multiplication by $\epsilon$. Therefore, the level $n$ of $\mathbb{R}\text{Hom}_{K(A, 0)}(A, E)$ is the direct sum $\bigoplus_{i \geq 0} E_{n-2i}$ and the differential $\bigoplus_{i \geq 0} E_{n-2i} \to \bigoplus_{i \geq 0} E_{n+1-2i}$ is given by $d + \epsilon.(-)$ where $d$ is the native differential of $E$. It is clear now that the resulting level $n$ of $\mathbb{R}\text{Hom}_{K(A, 0)}(A, E)$ identifies with the resulting level $n$ of the naive tensor product $E \otimes_A A[u]$ but the differential is twisted by the action of $\epsilon$. To encode the result of this computation we will write

$$\mathbb{R}\text{Hom}_{K(A, 0)}(A, (E, d)) \simeq (E \otimes_A A[u], d + \epsilon) \quad (2.3.24)$$

for future use.
Remark 2.21. Notice that the equivalence (2.3.21) is valid only in $\text{Qcoh}(K(A, 0))$ and not in $\text{IndCoh}(K(A, 0))$.

We now discuss a strict model for $\text{Coh}^b(\text{X}_0)_{\text{Perf}(X)}$, for $X = \text{Spec} B$. We consider the full sub $\text{dg}$-category $\text{Coh}^s(B, f)$ of the strict $\text{dg}$-category of (all) $K(B, f)$-$\text{dg}$-modules, spanned by those whose image along the restriction of scalars along the structure map $B \to K(B, f)$

$$K(B, f) - \text{dgMod}_A \to B - \text{dgMod}$$

are strictly perfect as complexes of $B$-modules (i.e. strictly bounded and degreewise projective $B$-modules of finite type). Notice that as $X = \text{Spec}(B)$ is an affine scheme, the sub $\text{dg}$-category $\text{Perf}(X) \subseteq \hat{B}$ is equivalent to its full sub-$\text{dg}$-category spanned by strict perfect complexes (see [TT90, 2.4.1]). Note also that we do not make the assumption that objects in $\text{Coh}^s(B, f)$ are cofibrant as $K(B, f)$-$\text{dg}$-modules, so there is no fully faithful embedding from $\text{Coh}^s(B, f)$ to the $\text{dg}$-category $K(B, f)$-$\text{dgMod}_A^{\text{cof}} = \text{Qcoh}(\text{X}_0)$ of cofibrant $K(B, f)$-$\text{dg}$-modules.

Remark 2.22. More explicitly, an object in $\text{Coh}^s(B, f)$ is the data of a strictly bounded complex $E$ of projective $B$-modules of finite type, together with a morphism of graded modules $h : E \to E[1]$ of degree 1, satisfying $[d, h] = dh + hd = f$. In fact, given a $B$-$\text{dg}$-module $E$, the datum of a $K(B, f)$-$\text{dg}$-module structure on $E$, restricting to the given $B$-$\text{dg}$-module structure via the canonical map $B \to K(B, f)$, amounts to a pair $(m_0, m_1)$ of morphisms $m_\alpha : E \to E[-\alpha]$ of graded $B$-modules, where $m_0$ is forced to be the identity by the fact that the $B$-$\text{dg}$-module structure is assigned, and $h := m_1$ is subject to the condition $dh + hd = f$. Note also that, as a strict $A$-$\text{dg}$-category, $\text{Coh}^s(B, f)$ is locally flat. This follows because by assumption $B$ is flat over $A$ and $A$ is a regular local ring.

The $\text{dg}$-category $\text{Coh}^s(B, f)$ is a strict model for the $\text{dg}$-category $\text{Coh}^b(\text{X}_0)_{\text{Perf}(X)}$, as stated by the following lemma.

Lemma 2.23. Let $\text{Coh}^{s, \text{acy}}(B, f) \subset \text{Coh}^s(B, f)$ be the full sub-$\text{dg}$-category consisting of $K(B, f)$-$\text{dg}$-modules which are acyclic as complexes of $B$-modules. Then, the cofibrant replacement $\text{dg}$-functor induces an equivalence of $\text{dg}$-categories

$$\text{Coh}^s(B, f)[q, \text{iso}^{-1}]_{\text{dg}} \simeq \text{Coh}^s(B, f)/\text{Coh}^{s, \text{acy}}(B, f) \simeq \text{Coh}^b(\text{X}_0)_{\text{Perf}(X)}$$

(2.3.25)

In particular, we have a natural equivalence of $\text{dg}$-categories

$$\text{Coh}^s(B, f)/\text{Perf}^s(B, f) \simeq \text{Coh}^b(\text{X}_0)_{\text{Perf}(X)}/\text{Perf}(\text{X}_0) = \text{Sing}(X, f),$$
where $\text{Perf}^a(B, f)$ is by definition the full sub-dg-category of $\text{Coh}^a(B, f)$ consisting of objects which are perfect as $K(B, f)$-dg-modules.

Proof. The category of $K(B, f)$-dg-modules admits a combinatorial model structure inherited by the one from complexes of $B$-modules. Therefore, it admits a functorial cofibrant replacement

$$Q : K(B, f) - \text{dgMod}_A \to K(B, f) - \text{dgMod}^\text{cof}_A$$

which is not a priori a dg-functor. In our case we are interested in applying this to the inclusion $\text{Coh}^a(B, f) \subseteq K(B, f) - \text{dgMod}_A$ and it happens that for objects $E \in \text{Coh}^a(B, f)$ we can model $Q$ by a dg-functor as follows: as $E$ is strictly perfect over $B$, in particular $E$ is cofibrant over $B$. Therefore, by definition $E \otimes_B K(B, f)$ is a cofibrant $K(B, f)$-dg-module. So are the powers $E \otimes_B B^n \otimes_B K(B, f)$. This gives us a resolution of $E \simeq E \otimes_{K(B, f)} K(B, f)$ by a simplicial diagram. Extracting its totalization we obtain a cofibrant resolution of $E$ in a functorial way. This way we get a strict cofibrant-replacement dg-functor

$$Q : \text{Coh}^a(B, f) \to \hat{K}(B, f)$$  \hspace{1cm} (2.3.26)

which by definition, sends weak-equivalences to equivalences. By the universal property of the dg-localization we have a factorization in $\text{dgcat}_A$

$$Q : \text{Coh}^a(B, f)[q, \text{iso}^{-1}]_{\text{dg}} \to \hat{K}(B, f)$$  \hspace{1cm} (2.3.27)

Notice also that by the universal property of the quotient, this dg-localization is equivalent in $\text{dgcat}_A$ to $\text{Coh}^a(B, f)/\text{Coh}^{a, \text{acy}}(B, f)$ and the map (2.3.27) is the one induced by the fact that $Q$ sends the full subcategory $\text{Coh}^{a, \text{acy}}(B, f)$ to zero.

We show that the dg-functor (2.3.26) is fully faithful with essential image given by $\text{Coh}^b(K(B, f))_{\text{Perf}(B)}$. More precisely, we show that:

1. The functor (2.3.27) factors through the full-subcategory $\text{Coh}^b(K(B, f))_{\text{Perf}(B)}$, where it is essentially surjective;

2. (2.3.27) is fully-faithful.

Let us start with (1). Of course, as an object $E \in \text{Coh}^a(B, f)$ is strictly bounded, any cofibrant replacement will remain cohomologically bounded. The cohomology groups of $E$ carry a natural structure of $\pi_0(K(B, f)) = B/f$-module. Moreover, being $E$ levelwise made of projective $B$-modules of finite type, these same cohomology groups are coherent when seen as $B$-modules via composition with the surjective map $B \to B/f$ and therefore are coherent as $\pi_0(K(B, f)) = B/f$-modules. Therefore, its cofibrant replacement $Q(E)$ is
in $\text{Coh}^b(K(B, f))$. Indeed, notice that by definition of $\text{Coh}^a(B, f)$ the image of $E$ under composition with $B \to K(B, f)$, which we will denote as $\text{Forget}(E)$, is a strict perfect complex and therefore, is perfect. As the forgetful functor along $B \to K(B, f)$ preserves all weak-equivalences of dg-modules, $\text{Forget}(Q(E))$ is weak-equivalent to $\text{Forget}(E)$. Finally, by definition of $i_* := Q_B \circ \text{Forget}$ (see the Remark 2.19 for notations) we find that $i_*(E)$ is quasi-isomorphic to $\text{Forget}(E)$ and therefore is perfect.

To show that (2.3.27) is essentially surjective on $\text{Coh}^b(K(B, f))_{\text{Perf}(B)}$ we notice first that as $X$ is affine, the inclusion of strictly perfect complexes over $B$, $\text{Perf}^a(B)$, inside $\text{Perf}(B)$ is an equivalence. In this case so is the inclusion $\text{Coh}^b(K(B, f))_{\text{Perf}^*(B)} \subseteq \text{Coh}^b(K(B, f))_{\text{Perf}(B)}$. Suppose $M \in \text{Coh}^{b}(K(B, f))_{\text{Perf}^*(B)}$ is in cohomological degree 0, a $B/f$-module of finite type. In this case, take any simplicial resolution of $M$ by free $K(B, f)$-dg-modules $E \to M$. This might be unbounded because $M$ itself is not strictly perfect over $K(B, f)$. The restriction of scalars of $E$ to $B$ is cofibrant and is degreewise projective over $B$ as $K(B, f)$ itself is strictly perfect over $B$ and $M$ is by hypothesis strictly perfect over $B$. One can now truncate the resolution $\tau_{\leq b+1}E$ for $b$ the tor-amplitude of $M$ over $B$. This new resolution is now strictly bounded as $K(B, f)$-dg-module and remains quasi-isomorphic to $M$.

Let us now show (2). As $\text{Coh}^a(B, f)$ has a canonical triangulated structure (coming from the strict dg-enrichment) to show that the map (2.3.27) is fully faithful it is enough to show that it is fully faithful on the homotopy categories because of the triangulated nature of the dg-localization. In this case it is enough so show that for any $E \in \text{Coh}^a(B, f)$ and for any quasi-isomorphism $P \to E$ with $P$ a $K(B, f)$-dg-module, it is possible to find an object $P' \in \text{Coh}^a(B, f)$ and a second quasi-isomorphism $P' \to P \to E$. But this follows using free resolutions like in (1) above. $\Box$

**Construction 2.24.** The construction $(B, f) \mapsto \text{Coh}^a(B, f)$ is functorial in the pair $(B, f)$: if $B \to B'$ is a morphism sending $f \in B$ to $f' \in B'$, the base change along $K(B, f) \to K(B', f')$ is induced by the base change $B' \otimes_B -$ given by an $A$-linear dg-functor

$$B' \otimes_B - : K(B, f) - \text{dgMod}_A \longrightarrow K(B', f') - \text{dgMod}_A$$

This restricts to an $A$-dg-functor

$$B' \otimes_B - : \text{Coh}^a(B, f) \longrightarrow \text{Coh}^a(B', f')$$

(2.3.28)

---

9As $A$ is Noetherian and $B$ is of finite type over $A$, it is of finite presentation as an $A$-algebra. Then it is also Noetherian and therefore coherent modules are the same as finitely generated modules.
Indeed, the base change of a strictly bounded complex remains strictly bounded and if $E$ is a $K(B, f)$-dg-module whose levels $E_i$ are projective $B$-modules of finite type, then the base change $E_i \otimes_B B'$ are $B'$-modules of finite type and again projective. Together with the Remark 2.22, we get this way a pseudo-functor

\[ \text{Coh}^s : \text{LG}_{S, \text{aff}, \text{op}} \rightarrow \text{dgCat}_{A}^{\text{strict, loc-flat}} \] (2.3.29)

which sends $(\text{Spec } B, f)$ to $\text{Coh}^s(B, f)$.

One can now use (2.3.29) combined with the Lemma 2.23 to exhibit the assignment $(X, f) \mapsto \text{Coh}_b(X_0)_{\text{Perf}(X)}$ as an $\infty$-functor

\[ \text{Coh}^b_{\text{Perf}} : \text{LG}_{S, \text{aff}, \text{op}} \rightarrow \text{dgCat}_{A}^{\text{idem}} \] (2.3.30)

For this purpose we remark that the base change maps (2.3.28) preserve quasi-isomorphisms. Indeed, if $E \rightarrow F$ is a quasi-isomorphism between objects in $\text{Coh}^s(B, f)$ then $E \rightarrow F$ is a quasi-isomorphism between the underlying strictly perfect $B$-dg-modules. As strictly perfect $B$-complexes are cofibrant as $B$-dg-modules, and every $B$-dg-module is fibrant, $E \rightarrow F$ is an homotopy equivalence so that the base change $E \otimes_B B' \rightarrow F \otimes_B B'$ remains an homotopy equivalence and therefore a quasi-isomorphism (alternatively, use Ken Brown’s Lemma [Hov99, 1.1.12]). In this case the functoriality (2.3.29) can be refined

\[ \text{Coh}^s : \text{LG}_{S, \text{aff}, \text{op}} \rightarrow \text{PairsdgCat}_{A}^{\text{strict, loc-flat}} \] (2.3.31)

where $\text{PairsdgCat}_{A}^{\text{strict}}$ is the 1-category whose objects are pairs $(T, W)$ with $T$ a strict small $A$-dg-category and $W$ a class of morphisms in $T$. This encodes the fact that weak-equivalences are stable under base change and sends a pair $(B, f)$ to the pair $(\text{Coh}^s(B, f), W_{q, \text{iso}})$ with $W_{q, \text{iso}}$ the class of quasi-isomorphisms. In the 1-category $\text{PairsdgCat}_{A}^{\text{strict, loc-flat}}$ we have a natural notion of weak-equivalence, namely, those maps of pairs $(T, W) \rightarrow (T', W')$ whose underlying strict dg-functor $T \rightarrow T'$ is a Dwyer-Kan equivalence of dg-categories. This produces a map between the $\infty$-categorical localizations

\[ \text{loc}_{\text{dg}} : N(\text{PairsdgCat}_{A}^{\text{strict, loc-flat}})[W_{DK}^{-1}] \rightarrow \text{dgCat}_{A} \simeq N(\text{dgCat}_{A}^{\text{strict, loc-flat}})[W_{DK}^{-1}] \] (2.3.32)

sending a pair $(T, W)$ to its dg-localization $T[W_{-1}]_{dg}$ in $\text{dgCat}_{A}$. To give a concrete description this $\infty$-functor, we remark the existence of another 1-functor

\[ \text{dgCat}_{A}^{\text{strict, loc-flat}} \rightarrow \text{PairsdgCat}_{A}^{\text{strict, loc-flat}} \]

sending a strict small $A$-dg-category $T$ to the pair $(T, W_T)$ where $W_T$ is the class of equivalences in $T$. By definition, this functor sends weak-equivalences of dg-categories to weak-equivalences of pairs therefore induces a functor between their $\infty$-localizations

\[ \text{dgCat}_{A} \simeq N(\text{dgCat}_{A}^{\text{strict, loc-flat}})[W_{DK}^{-1}] \otimes N(\text{PairsdgCat}_{A}^{\text{strict, loc-flat}})[W_{DK}^{-1}] \]

\[ \rightarrow \text{dgCat}_{A} \simeq N(\text{dgCat}_{A}^{\text{strict, loc-flat}})[W_{DK}^{-1}] \otimes N(\text{PairsdgCat}_{A}^{\text{strict, loc-flat}})[W_{DK}^{-1}] \]
We now claim that this functor admits a left adjoint, which will be our model of (2.3.32). By a dual version of [Lur09, Lemma 5.2.4.9] it suffices to check that for every pair \((T, S)\), the left fibration
\[
\text{dgcat}_A \times \text{N(PairsdgCat}^{\text{strict,loc-flat}}_A[W_{DK}^{-1}] \text{N(PairsdgCat}^{\text{strict,loc-flat}}_A[W_{DK}^{-1}]_C[T, S]) \rightarrow \text{dgcat}_A
\]
is co-representable. But this follows because of the existence of dg-localizations - see [Toë07, Corollary 8.7] \(^{10}\).

**Construction 2.25.** We construct a lax symmetric monoidal structure on (2.3.30). For this purpose with start by discussing a lax monoidal structure on the functor (2.3.29). Given two \(\text{LG-pairs}\) \((X := \text{Spec}(B), f)\) and \((Y := \text{Spec}(C), g)\) one must specify a functor
\[
\text{Coh}^s(B, f) \otimes_A \text{Coh}^s(C, g) \rightarrow \text{Coh}^s(B \otimes_A C, f \otimes 1 + 1 \otimes g)
\] (2.3.33)
verifying the conditions of lax symmetric structure. To construct (2.3.33) let us start by introducing some notation. For an \(\text{LG-pair}\) \((X, f)\) we denote by \(Z^h(f)\) the derived zero locus of \(f\) so that in the affine case, with \(X = \text{Spec}(B)\), we have \(Z^h(f) = \text{Spec}(K(B, f))\). By construction, given two affine \(\text{LG-pairs}\) as above, one obtains a commutative diagram
\[
\begin{array}{ccc}
Z^h(f) \times_S Z^h(g) & \rightarrow & X \times_S Y \\
\downarrow & & \downarrow (f,g) \\
S & \rightarrow & A^1_S \\
\downarrow & & \downarrow \\
0 & \rightarrow & A^1_S \\
\end{array}
\] (2.3.34)
where each face is cartesian and all the horizontal maps are lci closed immersions (as a consequence of the same property for the zero section \(S \hookrightarrow A^1_S\)). Moreover, we remark that the arrows (1) and (2) in the diagram can be given strict models
\[
\begin{array}{ccc}
B \otimes_A C & \rightarrow & K(B \otimes_A C, f \otimes 1 + 1 \otimes g) \\
(1) & \rightarrow & (1) \circ (2) \\
& & (2)
\end{array}
\] (2.3.35)
where (1) is completely determined by an element \(\alpha\) of degree \(-1\) in \(K(B, f) \otimes_A K(C, g)\) satisfying
\[
d(\alpha) = f \otimes 1 + 1 \otimes g \quad \alpha^2 = 0.
\]
\(^{10}\)see also the higher categorical comments in [Robdf, Section 6.1].
We set
\[\alpha := h \otimes 1 + 1 \otimes k\] (2.3.36)
where \(h\) and \(k\) are the canonical element in \(K(B, f)\) and \(K(C, g)\) respectively, of degree \(-1\) with
\[d(h) = f \quad d(k) = g \quad h^2 = k^2 = 0.\] (2.3.37)

To define the lax symmetric structure (2.3.33) one is reduced to explain that the composition
\[\text{Coh}^s(B, f) \otimes_A \text{Coh}^s(C, g) \xrightarrow{k} \text{Coh}^s(K(B, f) \otimes_A K(C, g)) \xrightarrow{(1)_s} \text{Coh}^s(B \otimes_A C, f \otimes 1 + 1 \otimes g)\] (2.3.38)
is well-defined. \((1)_s\) is given by the forgetful functor and as such it is well-defined the level of the categories \(\text{Coh}^s\): indeed, if \(E\) is strictly bounded \(K(B, f) \otimes_A K(C, g)\)-dg-module whose image under the forgetful functor \((1) \circ (2)\) is strictly perfect over \(B \otimes_A C\), then by commutativity of the diagram (2.3.35), \((1)_s(E)\) is in \(\text{Coh}^s(B \otimes_A C, f \oplus g)\).

It remains to provide an argument for the box product \(\boxtimes\): is defined by sending a pair \((E, F)\) to \(\pi_f^s(E) \otimes \pi_g^s(F)\) with \(\pi_f\) and \(\pi_g\) the projections of \(Z^h(f) \times_S Z^h(g)\) in each coordinate. Using the projection formulas and base change for affines\(^{11}\), the underlying \(A\)-module of \(E \boxtimes F\) is just the \(A\)-tensor product \(E \otimes_A F\). One must show that if \(E\) (resp. \(F\)) is strictly perfect over \(B\) (resp. \(C\)) then \(E \boxtimes F\) is strictly perfect over \(B \otimes_A C\). The fact that \(E \boxtimes F\) is strictly bounded follows immediately from the definition of the strict tensor product of complexes and the fact both \(E\) and \(F\) are strictly bounded. The fact that level of the complex \(E \boxtimes F\) is projective over \(B \otimes_A C\) follows because each level \(E^i\) (resp. \(F^k\)) is by assumption projective over \(B\) (resp. over \(C\)) so that each graded piece of the tensor product \(E^i \otimes_A F^k\) is projective over \(B \otimes_A C\): \(E^i\) (resp. \(F^k\)) being projective over \(B\) (resp. \(C\)) gives us a retract via a map of \(B\)-modules (resp. \(C\)-modules) of an inclusion of \(B\)-modules (resp. \(C\)-modules) \(E^i \subseteq B^{\oplus l}\) for some \(l\) (resp. \(F^k \subseteq C^{\oplus s}\)). Via base change we obtain the graded piece \(E^i \otimes_A F^k\) as a retract of \((B \otimes_A C)^{\oplus l+s}\) via a map of \(B \otimes_A C\)-modules, for some \(l, s\). This proves the claim. To conclude, we define the lax unit via the map
\[A \rightarrow \text{Coh}^s(K(A, 0)),\] (2.3.39)
sending the unique point to \(A\) itself (with its trivial structure of \(K(A, 0)\)-dg-module).

The construction (2.3.33) is clearly symmetric and associative and this concludes the construction of a lax symmetric monoidal enhancement of (2.3.29)
\[\text{Coh}^s,\mathbb{I} : \text{LG}_s^{\text{aff,op},\mathbb{I}} \rightarrow \text{dgCat}_{A,\text{strict,loc−flat,}\otimes}^\text{strict,loc−flat,}\otimes\] (2.3.40)

One now proceeds as in the Construction 2.24 to obtain a lax symmetric monoidal

\(^{11}\) Notice that by definition of \(LG\)-pairs, both \(B\) and \(C\) are flat over \(A\). In particular, the derived tensor product is the usual one.
structure on (2.3.30): one remarks that the category of pairs $\text{Pairs}_{\text{dgCat}}^{\text{strict}}$ introduced in the Construction 2.24 comes naturally equipped with a tensor structure: if $(T, W)$ and $(T', W')$ are two pairs, the pair $(T \otimes T', W \otimes W')$ is defined by $(T \otimes T', W \otimes W')$. It is clear that the lax structure of (2.3.40) can be lifted

$$\text{Coh}^\otimes : \text{LG}_{\text{aff, op}} \rightarrow \text{Pairs}_{\text{dgCat}}^{\text{strict, loc-flat}}$$

(2.3.41)

It follows from the definition of locally-flat dg-categories that the tensor structure in $\text{Pairs}_{\text{dgCat}}^{\text{strict, loc-flat}}$ is compatible with weak-equivalences in each variable so that the localization functor along Dwyer-Kan equivalences of pairs is a monoidal $\infty$-functor

$$\text{N}(\text{Pairs}_{\text{dgCat}}^{\text{strict, loc-flat}}) \rightarrow \text{N}(\text{Pairs}_{\text{dgCat}}^{\text{strict, loc-flat}})[W_{\text{DK}}^{-1}]$$

(2.3.42)

It remains to check that (2.3.32) is strongly monoidal. This follows from [Lur16, 7.3.2.12] as the required hypothesis follow from the definition of the tensor structure on pairs, together with the fact that for any two pairs $(T, S), (T', S')$ the canonical morphism

$$(T \otimes T')[S \otimes S'^{-1}]_{\text{dg}} \rightarrow T[S^{-1}]_{\text{dg}} \otimes T'[S'^{-1}]_{\text{dg}}$$

is an equivalence in $\text{dgcat}_A$ (this is an immediate consequence of the universal property of dg-localizations combined with the existence of internal-homs in $\text{dgcat}_A$).

Finally, the composition of the lax monoidal $\infty$-functors (2.3.41), (2.3.42), (2.3.32) and idempotent completion, combined with the result of Lemma 2.23, achieve the construction of the lax monoidal structure on (2.3.30).

**Remark 2.26.** The lax symmetric monoidal structure of Construction 2.25 produces a symmetric monoidal structure on $\text{Coh}^b(K(S, 0))$, which we shall denote as $\text{Coh}^b(K(S, 0))^{\otimes}$. Its monoidal unit is the $K(A, 0)$-dg-module $A$ in degree 0 with zero $\epsilon$-action. Via the identification of $K(S, 0)$ as a strict model for the derived tensor product $S \times_{\mathbb{A}^1} S$, this symmetric monoidal structure corresponds to the convolution product induced by the additive group structure on $\mathbb{A}^1$. This symmetric monoidal structure has a geometric origin: in fact $S \times_{\mathbb{A}^1} S$ is a derived group scheme with operation induced by the additive group structure on the affine line. By unfolding the definition, given $E, F \in \text{Coh}^b(K(S, 0))$, $E \boxtimes F$ is given by the underlying tensor $E \otimes_A F$ equipped with an action of $K(S, 0)$ via the map $K(S, 0) \rightarrow K(S, 0) \otimes_A K(S, 0)$ of (1) in (2.3.35). In the case when $A$ is a field of characteristic zero this recovers the monoidal structure described in [Pre11, Construction 3.1.2].

Moreover, given an LG-pair $(X, f)$, the action of $\text{Coh}^b(K(S, 0))^{\otimes}$ on $\text{Coh}^b(X_0)_{\text{Perf}(X)}$ also has a geometric interpretation: indeed, the derived fiber product $X_0$ carries a canonical
action of the derived group scheme $S \times \mathcal{A}$. This is obtained via the cartesian cube

$\begin{align*}
X_0 \times_S (S \times \mathcal{A}) &\xrightarrow{v = \text{action}} X_0 \\
pr \downarrow &\quad \quad pr(S,0) \downarrow \\
pr(S,0) &\xrightarrow{i} X \\
pr \downarrow &\quad \quad \downarrow i \\
X_0 &\xrightarrow{i} X \\
S \times \mathcal{A} &\xrightarrow{f} S \\
S &\xrightarrow{\nu} \mathcal{A}
\end{align*}$

(2.3.43)

Let us describe this action more precisely. In the affine case this is given by the formula (2.3.38). In geometric terms this is explained by the derived fiber product in the diagram (2.3.43) whose top face is the self-intersection square

$\begin{align*}
X_0 \times_S (S \times \mathcal{A}) &\xrightarrow{v} X_0 \\
pr \downarrow &\quad \quad pr(S,0) \downarrow \\
pr(S,0) &\xrightarrow{i} X \\
pr \downarrow &\quad \quad \downarrow i \\
X_0 &\xrightarrow{i} X \\
S \times \mathcal{A} &\xrightarrow{f} S \\
S &\xrightarrow{\nu} \mathcal{A}
\end{align*}$

(2.3.44)

and the action of $F \in \text{Coh}^b(K(A,0))$ on $M \in \text{Coh}^b(X_0)_{\text{Perf}(X)}$ is given by $F \boxslash M := v_* (pr^*(M) \otimes pr^*_{(S,0)}(F))$. In particular, by derived base change, we have

$$K(A,0) \boxslash M \simeq i^* i_* (M)$$

(2.3.45)

Moreover, the action of $A$ (as a $K(A,0)$-module in degree 0 with a trivial action of $\epsilon$) is given by

$$A \boxslash M := v_* (pr^*(M) \otimes pr^*_{(S,0)}(A)) \simeq M$$

(2.3.46)

To show this last formula we remark that $A$ as a trivial $K(A,0)$-module is given by $t_*(A)$ where $t : S = \text{Spec}(A) \to \text{Spec}(K(A,0) = S \times \mathcal{A}$ is the inclusion of the classical truncation. By derived base-change we get that

$$A \boxslash M \simeq v_* (pr^*(M) \otimes pr^*_{(S,0)}(t_*(A))) \simeq v_* (pr^*(M) \otimes (Id_{X_0} \times t)_* O_S)$$

which by the projection formula, is equivalent to

$$v_* \circ (Id_{X_0} \times t)_* ((Id_{X_0} \times t)^* pr^*(M)) \simeq M$$

(2.3.47)

We conclude that $A$, as a trivial $K(A,0)$-module, acts via the identity map on $\text{Coh}^b(X_0)_{\text{Perf}(X)}$.

The following result was proved in [Pre11, Prop. 3.1.4] in the case where $A$ is a field of characteristic zero. We remark that it is still valid whenever $A$ is regular:
Lemma 2.27. The equivalence (2.3.17) of the Prop. 2.20 has a symmetric monoidal enhancement
\[
\text{Coh}^b(K(A,0)) \cong \text{Perf}(A[u])^{\otimes A[u]} \tag{2.3.48}
\]
where on the r.h.s we have the standard tensor product over \(A[u]\) induced by the fact \(A[u]\) is naturally a commutative algebra-object in \(\text{Mod}_F(Sp)^\otimes\).

Proof. The proof relies in the formula (2.3.24). We show that the arguments given in [Pre11, Prop. 3.1.4] work for a general regular ring \(A\) using our infinite resolution (2.3.19) instead of the Koszul-Tate resolution used in loc.cit. We start by showing that the strict dg-functor
\[
\mathcal{E} : K(A, 0) - \text{dgMod}^{\text{strict}}_A \rightarrow A[u] - \text{dgMod}^{\text{strict}}_A \tag{2.3.49}
\]
sending
\[
(E, d_E) \mapsto \mathcal{E}(E, d_E) := (E \otimes_A A[u], d_E + \epsilon) \tag{2.3.50}
\]
is symmetric monoidal with respect to the convolution \(\oplus\) of the Remark 2.26 on the l.h.s and the usual tensor product over \(A[u]\) on the r.h.s. This follows essential from the definition of \(\oplus\): given two pairs \((E, d_E)\) and \((F, d_F)\) in \(K(A, 0) - \text{dgMod}_A\) their box product is given by the pair that consists of the usual tensor product over \(A, E \otimes_A F\) equipped with the action of \(K(A,0)\) given by the formula (2.3.37) which in this case is explicitly given by \(\epsilon \otimes Id + Id \otimes \epsilon\). In this case the natural lax structure is an equivalence as
\[
\mathcal{E}((E, d_E) \oplus (F, d_F)) = \mathcal{E}(E \otimes_A F, d_E \otimes Id + Id \otimes d_F) \simeq
\]
\[
((E \otimes_A F) \otimes_A A[u]), (d_E \otimes Id + Id \otimes d_F) \simeq (E \otimes_A A[u], d_E + \epsilon) \otimes_A (F \otimes_A A[u], d_F + \epsilon)
\]
\[
\simeq \mathcal{E}(E, d_E) \otimes_A (F, d_F)
\]
As a second step we notice that \(\mathcal{E}\) sends \(\text{Coh}^*(K(A,0)) \subseteq K(A,0) - \text{dgMod}_A^{\text{strict}}\) to the full subcategory of \(A[u] - \text{dgMod}_A^{\text{strict}}\) spanned by cofibrant objects for the projective structure model structure. Indeed, given \(E \in \text{Coh}^*(K(A,0))\), the action of \(\epsilon\) is zero, and in this case, \(\mathcal{E}(E)\) is isomorphic to \((E \otimes_A A[u], d_E)\) which is a free \(A[u]-\text{dg-module}\) and therefore, by definition of the projective model structure, cofibrant.

Now we remark that the restriction \(\mathcal{E} : \text{Coh}^*(K(A,0)) \rightarrow A[u] - \text{dgMod}_A^{\text{cofibrant}}\) sends weak-equivalences to weak-equivalences. Indeed, this follows because by definition each \(E \in \text{Coh}^*(K(A,0))\) is strictly bounded levelwise made of projective and therefore flat, \(A\)-modules so that if \(E \rightarrow E'\) is a quasi-iso, then so it is \(E \otimes_A A[u] \rightarrow E' \otimes_A A[u]\). As seen in the construction of the strict lax structure on \(\text{Coh}^*(-)\), \(\text{Coh}^*(K(A,0))\) is stable under the \(\oplus\) tensor product on \(K(A, 0) - \text{dgMod}_A^{\text{strict}}\) and in this case, by the universal property
of the dg-localization combined with the equivalence (2.3.25) we obtain a symmetric monoidal dg-functor in $\mathbf{dgcat}^\text{big}_A$

$$E : \text{Coh}^b(K(A, 0))^\oplus \to \text{Qcoh}(A[u])^{\otimes A[u]}$$

(2.3.51)

As seen in the proof of Pro. 2.20, every $E \in \text{Coh}^b(K(A, 0))$ can be obtained from the $K(A, 0)$-dg-module $A$ under finite shifts and cones so that $E(E)$ will be a perfect $A[u]$-module. Therefore (2.3.51) factors through $\text{Perf}(A[u])^{\otimes A[u]}$ and this factorization in $\mathbf{dgcat}^\text{idem}_A$ recovers the equivalence (2.3.17). □

**Remark 2.28.** It is a consequence of the symmetric monoidal equivalence (2.3.48) in small idempotent complete dg-categories that the equivalence (2.3.22) identifies the algebra-structure of composition of endomorphisms with the standard multiplication on $A[u]$.

**Remark 2.29.** As in the Remark 2.7, using the equivalence (2.3.48) we recover the lax symmetric monoidal structure on the $\infty$-functor (2.3.30)

$$\text{LG}_{\text{aff}, \text{op}} \xrightarrow{\text{Coh}^b(-)_{\text{Perf}(-)}} \text{Mod}_{\text{Perf}(A[u])}(\mathbf{dgcat}^\text{idem}_A)^{\otimes}$$

(2.3.52)

restricted to affine LG-pair $(X, f)$ and the induced action of the small stable idempotent complete symmetric monoidal $(\infty, 1)$-category $\text{Perf}(A[u])^{\otimes A[u]}$ on the $\infty$-category $\text{Coh}^b(X_0)_{\text{Perf}(X)}$.

**Proposition 2.30.** Under the symmetric monoidal equivalence (2.3.48), the full subcategory $\text{Perf}(S \times^h A^1_S) \subseteq \text{Coh}^b(S \times^h A^1_S)$ is identified with the full subcategory $\text{Perf}(A[u])^{u^{-}\text{Torsion}}$ of u-torsion modules, i.e., those perfect dg-modules $M$ over $A[u]$ such that there exists an $N \geq 0$ such that the multiplication by $u^N$, $M[-2n] \to M$ is null-homotopic. Moreover

$$\text{Sing}(S, 0) \simeq \text{Perf}(A[u, u^{-1}])$$

(2.3.53)

**Proof.** Let us start with the first claim. The argument is similar to the one of [Pre11, Lemma 3.1.9]. Using the formula (2.3.24) one obtains

$$\mathbb{R}\text{Hom}_{K(A, 0)}(A, K(A, 0)) \xrightarrow{\sim} (A \xrightarrow{c = \text{id}} A \xrightarrow{c = \text{id}} A \xrightarrow{c = \text{id}} \cdots)$$

(2.3.54)

and observe that the r.h.s is quasi-isomorphic to $A[1]$. In this case, the equivalence (2.3.17) maps the full subcategory $\text{Perf}(S \times^h A^1_S) \subseteq \text{Coh}^b(S \times^h A^1_S)$, by definition, generated by $K(A, 0)$ under finite colimits and retracts, to the full subcategory of $\text{Perf}(A[u])$ generated
by the object $A[1]$. This is equivalent to the (stable and idempotent complete) subcategory generated by $A$ as a trivial $A[u]$-module. We remark that as an $A[u]$-dg-module, $A$ fits in a cofiber-fiber sequence

$$A[u][-2] \xrightarrow{u} A[u] \quad \text{(2.3.55)}$$

which can be obtained using the explicit model for the cone of the multiplication by $u$ (given by the identity on each level).

We use this to conclude that the thick subcategory generated by $A$ in $\text{Perf}(A[u])$ is exactly the full subcategory spanned by the $u$-torsion dg-modules. Indeed, $A$ is by construction $u$-torsion, as $u$ acts null-homotopically. Moreover, the full subcategory of $u$-torsion modules is by its nature a thick stable and idempotent complete subcategory of $\text{Perf}(A[u])$. It remains to show that every $M \in \text{Perf}(A[u])^{u-\text{Torsion}}$ can be obtained as a retract of a homotopy finite cellular object built from $A$ under shifts and cones. For this purpose we use the cofiber/fiber-sequence (2.3.55): given $M \in \text{Perf}(A[u])^{u-\text{Torsion}}$, using the relative tensor product over $A[u]$ (which as explained in the Remark 2.28 carries its standard structure of $E_\infty$-algebra) we obtain a cofiber-fiber sequence

$$M[-2n] \simeq M \otimes_{A[u]} A[u][-2n] \xrightarrow{u^n \sim 0} M \simeq M \otimes_{A[u]} A[u] \quad \text{(2.3.56)}$$

The assumption that $M$ is perfect over $A[u]$ means by definition that it is obtained under finite shifts and cones of $A[u]$. In particular, $M \otimes_{A[u]} (\bigoplus_{0 \leq i \leq n-1} A[-2i])$ is then obtained as a finite cell-object from $A$ and has $M$ as a direct factor.

Let us now address the second claim. First notice that since $A$ is regular, we have

$$\text{Sing}(S,0) \simeq \text{Sing}(S \times_{k[S]} \mathbb{A}^1_k, S).$$

The strategy to show the second claim is to show that the quotient map $\text{Coh}^b(S,0) \to \text{Sing}(S,0)$ identifies under (2.3.48) with the symmetric monoidal base change map

$$- \otimes^{\mathbb{L}}_{A[u]} A[u,u^{-1}] : \text{Perf}(A[u]) \to \text{Perf}(A[u,u^{-1}]). \quad \text{(2.3.57)}$$

Thanks to the half of the proposition already proved, to establish this identification one is reduced to present the base change $- \otimes^{\mathbb{L}}_{A[u]} A[u,u^{-1}]$ as a Verdier quotient with respect to the thick subcategory of $u$-torsion dg-modules. For this purpose we remark that at
the level of the presentable ∞-categories of modules, the base-change in Pr^L

\[ - \otimes_{A[u]}^L A[u, u^{-1}] : \text{Mod}_{A[u]}(\text{Sp}) \to \text{Mod}_{A[u,u^{-1}]}(\text{Sp}) \]

admits the restriction of scalars along the map \( A[u] \to A[u, u^{-1}] \) as a fully faithful right adjoint, whose image is the full subcategory of \( \text{Mod}_{A[u]}(\text{Sp}) \) spanned by those dg-modules where the multiplication by \( u \) is invertible. In other words, such objects become the local objects for the presentation of \( \text{Mod}_{A[u,u^{-1}]}(\text{Sp}) \) as a Bousfield localization of \( \text{Mod}_{A[u]}(\text{Sp}) \).

As a consequence of this fact, \( - \otimes_{A[u]}^L A[u, u^{-1}] \) has an alternative description in terms of a colimit in (the big category of) \( A[u] \)-modules in spectra given by multiplication by \( u \):

\[ M \otimes_{A[u]}^L A[u, u^{-1}] \simeq \text{colim}_n (\cdots \to M \to M[2] \to M[4] \to \cdots) \]

This follows from the combination of the universal properties of base change, of colimits in \( A[u] \)-modules and fully faithfulness along restriction of scalars. The formula remains valid for perfect complexes because base-change preserves perfect complexes (the colimit being always taken in spectra). In this case, if an \( A[u] \)-module \( M \) is \( u \)-torsion the colimit by multiplication by \( u \) is by cofinality equivalent to the colimit of the zero diagram so that \( M \otimes_{A[u]} A[u, u^{-1}] \simeq 0 \). Conversely, if \( M \otimes_{A[u]} A[u, u^{-1}] \simeq 0 \) and \( M \in \text{Perf}(A[u]) \), then \( M \) is compact and we have

\[ * \simeq \text{Map}_{A[u]}(M, \text{colim}_n M[2n]) \simeq \text{colim}_n \text{Map}_{A[u]}(M, M[2n]) \]

in \( \text{Mod}_{A[u]}(\text{Sp}) \), so that there is an \( n \) such that the power \( u^n \) is the zero map. \( \square \)

The following proposition achieves the main goal of this section of exhibiting a lax symmetric structure on the ∞-functor \( \text{Sing} \) (2.3.15). In particular this extends [Pre11, Prop. 3.4.3] to a base ring which we only require to be regular, instead of a field of characteristic zero.

**Proposition 2.31.** There is a natural equivalence between the composition

\[
\begin{array}{c}
\text{LGaff,op} \\
\xrightarrow{\text{Coh}^b(-) \otimes_{A[u]} (-)} \\
\xrightarrow{\text{Mod}\text{Perf}_{A[u]}(\text{dgc}^\text{lem})} \\
\xrightarrow{(-) \otimes_{A[u,u^{-1}]} A[u, u^{-1}]^{-1}} \\
\xrightarrow{\text{Forget}} \\
\text{dgc}^\text{lem}
\end{array}
\]

and the ∞-functor \( \text{Sing} \). By transfer under this equivalence, the functor \( \text{Sing} \) acquires a lax symmetric monoidal enhancement. In particular, \( \text{Sing}(S, 0) \) acquires a symmetric monoidal structure equivalent to the natural one on 2-periodic complexes.

**Proof.** The proof is similar to [Pre11, 3.4.1]. As \( \text{Coh}^b(K(A, 0)^[\mathbb{H}] \simeq \text{Perf}(A[u]) \otimes_{A[u]} A[u,u^{-1}] \) is a rigid symmetric monoidal idempotent-complete dg-category and

\( \text{Perf}(A[u])^{u^-\text{Torsion}} \subseteq \text{Perf}(A[u]) \to \text{Perf}(A[u, u^{-1}]) \)
is an exact sequence of $\text{Perf}(A[u])$-linear dg-categories, one can use exactly the same arguments as in [Pre11, Lemma 3.4.2] to deduce that for any LG-pair $(X, f)$ the base-change sequence

$$\text{Coh}^b(X_0)_{\text{Perf}(X)} \otimes_{\text{Perf}(A[u])} \text{Perf}(A[u])^{u-\text{Torsion}} \subseteq \text{Coh}^b(X_0)_{\text{Perf}(X)} \to \text{Coh}^b(X_0)_{\text{Perf}(X)} \otimes_{\text{Perf}(A[u])} \text{Perf}(A[u, u^{-1}])$$

remains a cofiber-fiber sequence in $\text{Mod}_{\text{Perf}(A[u])}(\text{dgcat}^\text{idem}_A)$. It follows from the definition of the tensor product in $\text{dgcat}^\text{idem}_A$ that the localization functor $\text{Coh}^b(X_0)_{\text{Perf}(X)} \to \text{Coh}^b(X_0)_{\text{Perf}(X)} \otimes_{\text{Perf}(A[u])} \text{Perf}(A[u, u^{-1}])$ can be described as in the proof of the Prop. 2.31 by the $\infty$-functor sending $M \in \text{Coh}^b(X_0)_{\text{Perf}(X)}$ to the colimit in $\text{Ind}(\text{Coh}^b(X_0)_{\text{Perf}(X)})$.

Moreover, by definition, $\text{Coh}^b(X_0)_{\text{Perf}(X)} \otimes_{\text{Perf}(A[u])} \text{Perf}(A[u])^{u-\text{Torsion}}$ identifies with the full subcategory of $\text{Coh}^b(X_0)_{\text{Perf}(X)}$ spanned by those objects $M$ such that there exists $N \geq 0$ such that $u^N : M \to M[2n]$ is null-homotopic. In this case we are reduced to show that $M$ is in $\text{Perf}(X_0)$ if and only there exists $N \geq 0$ such that $u^N \sim 0$. For this we follow the steps of [Pre11, 3.4.1 (i) ⇔ (ii)]. We use the description of the action of $\text{Coh}^b(K(A, 0))$ on $\text{Coh}^b(X_0)_{\text{Perf}(X)}$ given in the remark 2.26, namely the formulas (2.3.45), (2.3.46). Combining these formulas with our resolution (2.3.21) for $A$ as a $K(A, 0)$-module, we get a diagram of natural transformations

$$\cdots i^* i_s[n] \to i^* i_s[n-1] \to \cdots \to i^* i_s[1] \to i^* i_s$$

induced by the multiplication by $\epsilon$. As the resolution works only in $\text{Qcoh}(K(A, 0))$ (Remark 2.21), the formula (2.3.21) yields a canonical equivalence of $\infty$-functors

$$\text{colim}_n (i^* i_s \to i^* i_s[1] \to i^* i_s[2] \to \cdots) \simeq \text{Id}_{\text{Qcoh}(X_0)}$$

(2.3.59)

where the colimit is taken in $\text{Qcoh}(X_0)$ (by definition of $\text{Coh}^b_{\text{Perf}}$, $i^* i_s$ has values in the quasi-coherent category). We now remark that $M$ is in $\text{Perf}(X_0)$ if and only if there exists an $N \geq 0$ such that $M$ is an homotopy retract of some $i^* i_s\{N\}(M)$. Indeed, suppose that $M$ is perfect. Then it is a compact object in $\text{Qcoh}(X_0)$ and the identity map $M \to M \simeq \text{colim}_n i^* i_s\{n\}(M)$ factors through some finite stage $i^* i_s\{n\}(M)$. Conversely, suppose that $M \in \text{Coh}^b(X_0)_{\text{Perf}(X)}$ is an homotopy retract of some $i^* i_s\{n\}(M)$. Then because of the definition of $\text{Coh}^b(X_0)_{\text{Perf}(X)}$, for any $M$, $i^* i_s\{n\}(M)$ is always a perfect complex so that the finite colimit $i^* i_s\{n\}(M)$ is also perfect. Being a retract, $M$ will also be perfect.

It remains to identity modules obtained as homotopy retracts of some $i^* i_s\{N\}(M)$ exactly with those modules where the action of $u$ is torsion. Given $T \in \text{Perf}(A[u])$.
and $M \in \text{Coh}^b(X_0)_{\text{Perf}(X)}$ let us introduce the notation $T \otimes_{A[u]} M$ for the action of $T$ on $M$ via the monoidal equivalence (2.3.48). By definition of (2.3.48), we get that $M \simeq A[u] \otimes_{A[u]} M \simeq A \otimes M$ and that $A[1] \otimes_{A[u]} M \simeq K(A,0) \otimes M \simeq i^*i_* M$. In particular, as the action by construction commutes with colimits in each variable, the cofiber sequence (2.3.55) produces a cofiber sequence

\[
M[-2] \simeq A[u][-2] \otimes_{A[u]} M \xrightarrow{u} M \simeq A[u] \otimes_{A[u]} M
\]

So that

\[
\text{cofib } u \simeq i^*i_*(M)[-1]
\]

Using this one can easily construct a cofiber-fiber sequence

\[
i^*i_! M[1] \longrightarrow 0 \quad \text{cofib}(u^2)[3]
\]

By induction, one shows that

\[
\text{cofib } u[-1] \longrightarrow 0
\]

\[
\text{cofib } u^N[-2] \longrightarrow \text{cofib } u^{N+1}
\]

are cofiber-diagrams and more generally, using the diagrams (2.3.20) we get

\[
\text{cofib}(u^N)[2N-1] \simeq i^*i_! \{N-1\}(M)
\]

In particular, if the action is torsion, we have $u^N \sim 0$ for some $N \geq 0$, and $\text{cofib}(u^N) \simeq M[-2N+1] \oplus M$ and $\text{Cofib}(u^N)[2N-1] \simeq M \oplus M[2N-1]$. In this case we get $M \oplus M[2N-1] \simeq i^*i_! \{N-1\}(M)$ so that $M$ is a retract of the finite colimit. Conversely, if $M$ is a retract of the cofiber of $u^N$ then this $u^N$ is null-homotopic as the retract gives a splitting of the cofiber sequence $M[-2N] \rightarrow M \rightarrow \text{cof } u^N$. □

**Remark 2.32.** Notice that in the previous proof it is never used that $(X,f)$ is an affine LG-pair. In fact, the proof of the previous proposition can be used to conclude that the functor $(X,f) \mapsto \text{Sing}(X,f)$ is lax monoidal on non-affine LG-pairs, independently of the strict model for bounded coherent on $X_0$ perfect on $X$ of 2.23 in the affine case. Indeed, the lax monoidal structure on $(X,f) \mapsto \text{Sing}(X,f)$ can be obtained using the fact that $\text{Qcoh}$ is a lax monoidal $\infty$-functor (obtained from the lax monoidal structure on the construction $A \mapsto \text{Mod}_A(\text{Sp})$ - see [Lur16]), combined with the cartesian property.
of the diagram (2.3.5), the Proposition 2.30, the Lemma 2.27, the lax monoidality of inverting $u$ and the proof of the proposition 2.31.

**Remark 2.33.** The fact that $\text{Sing}(S,0)$ is *monoidal* equivalent to 2-periodic complexes has been proved in the case where $A$ is a field of characteristic zero, see for instance [Pre11, Prop. 3.1.9] and [AG15, Section 5.1]. This is an instance of Koszul duality for modules.

**Remark 2.34.** One should also remark that as for MF, under some hypothesis, the functor $\text{Sing}$ on non-affine LG-pairs matches the result of the Kan extension of its restriction to affine LG-pairs. This follows from a combination of Cech descent for $\text{Coh}_{\text{Perf}}$ of the Prop. 2.10, together with the fact that for schemes of Noetherian schemes of finite Krull dimension the Zariski topos is hypercomplete [Lur17, 3.7.7.3]. Knowing Zariski descent for $\text{Coh}_{\text{Perf}}$ it suffices, after the Prop. 2.31 and the Remark 2.32, to remark that the base-change $-\otimes_{\text{Perf}(A[u])}\text{Perf}(A[u, u^{-1}])$ in idempotent complete $A$-dg-categories, preserves finite limits because both $\text{Perf}(A[u])$ and $\text{Perf}(A[u, u^{-1}])$ have single compact generators, and the localization $A[u, u^{-1}]$ can be obtained as a filtered colimit under multiplication by $u$ in $\text{Sp}$, and filtered colimits preserve finite limits.

### 2.4. Comparison

In this § we prove the following

**Theorem 2.35.** There is a lax symmetric natural transformation of $\infty$-functors

$$\text{Orl}^{-1} : \text{Sing} \to \text{MF} : \text{LG}_S^{\text{op}} \to \text{dgcat}_A^\text{idem}$$

which is an equivalence when restricted to the sub-category of LG-models $(X,f)$ where $f$ is a non-zero divisor on $X$ (i.e. the induced morphism $\mathcal{O}_X \to \mathcal{O}_X$ is a monomorphism), $X/S$ is separated, and $X$ has the resolution property (i.e. every coherent $\mathcal{O}_X$-Module is a quotient of a vector bundle).

Theorem 2.35 provides a $\infty$-functorial, dg-categorical lax symmetric monoidal version of the so-called Orlov’s comparison theorem, comparing matrix factorizations and categories of singularities (see [EP15, Theorem 2.7] and [Orl12, Thm 3.5]). We will see below (Remark 2.40) that the lax symmetric monoidal natural transformation $\text{Orl}^{-1}_\otimes$, identifies the *symmetric monoidal* structure of $\text{Sing}(S,0)$ given in the Prop. 2.31 with the one in $\text{MF}(S,0)$ of the Remark 2.7. Moreover, for any LG-model $(X,f)$ over $S$ with $f$ a non-zero divisor $X/S$ separated and $X$ having the resolution property (e.g. $X$ regular), the two dg-categories $\text{MF}(X,f)$ and $\text{Sing}(X,f)$ are then equivalent as $A[u, u^{-1}]$-linear idempotent complete dg-categories.
Remark 2.36. (derived vs classical zero locus) Note however, that our natural transformation $\text{Orl}^{-1}$ is defined also for non-flat LG-models $(X, f)$, and this was made possible by considering the derived zero locus of $f$ instead of the classical scheme-theoretic zero locus in the definition of the functor $\text{Sing}$ (while $\text{MF}$ is defined using only non-derived ingredients). Note that for flat LG-models $(X, f)$, $f$ is indeed a non-zero divisor on $X$. If we restrict the functor $\text{Sing}$ to LG-models $(X, f)$ where $f$ is a non-zero divisor on $X$, i.e. $f_U$ is a non-zero divisor for all Zariski open affine subschemes $U \subseteq X$ (this case is of particular interest for us, see Remark 4.1), then the derived fiber $X_0$ coincides with the classical scheme-theoretic fiber $X_0^{\text{cl}}$ (i.e. the truncation of $X_0$), and one does not need to use derived algebraic geometry at all in the definition of $\text{Sing}$ (Definition 2.13). Note, however, that if these conditions are not met, there is no way to avoid taking the derived fiber $X_0$. In fact, the push-forward along the closed immersion $X_0^{\text{cl}} \rightarrow X$ does not necessarily preserve perfect complexes, so that a purely classical analogue $\text{Sing}^{\text{cl}}$ of our definition of $\text{Sing}$ is simply impossible. And this, regardless, the fact that $X$ may or may not enjoy the resolution property. Moreover, even when the pushforward along $X_0^{\text{cl}} \rightarrow X$ does preserve perfect complexes, so that both our definition of $\text{Sing}$ and its purely classical analogue $\text{Sing}^{\text{cl}}$ make sense, then they might differ. As an important example, one could take $(X, f) := (S, 0)$: here $X_0^{\text{cl}} = X$, so that $\text{Sing}^{\text{cl}}(S, 0)$ is defined and is trivial, while $\text{Sing}(S, 0)$ is equivalent to the dg-category of 2-periodic complexes $\text{Perf}(A[u, u^{-1}])$ (Proposition 2.31), and is therefore equivalent to $\text{MF}(S, 0)$, as an object in $\text{CAlg}(\text{dgc\text{at}}^{\text{idem}}_A)$. In particular there is no hope for $\text{MF}$ to be equivalent to $\text{Sing}^{\text{cl}}$, when $f$ is allowed to be a zero-divisor.

Remark 2.37. The considerations of the previous remarks lead us to believe that the $\infty$-functor $\text{Orl}^{-1}$ of Theorem 2.35 is an equivalence even without restricting to flat or non-zero divisors LG-pairs: we think this generalization of Theorem 2.35 is important, and will be discussed elsewhere. Granting this fact, we can make a few more observations. First, note that flat LG-pairs $(X, f)$ where $X/S$ is separated and $X$ is regular belong to the subcategory for which $\text{Orl}^{-1}$ is an equivalence. But unfortunately, the property of being regular is not preserved under base-change, so that these regular flat LG-pairs do not form a monoidal subcategory of the category of flat LG-pairs (recall from Section 2 that $(X, f) \boxtimes (Y, g) := (X \times_S Y, f \boxtimes g)$). However, if we denote by $\text{LG}_S^{\text{fl-qproj}}$ the subcategory of flat LG-pairs $(X, f)$ over $S$ where $X/S$ is quasi-projective (hence separated), then $\text{LG}_S^{\text{fl-qproj}}$ is a symmetric monoidal subcategory of $\text{LG}_S^{\text{fl}}$, and $\text{Orl}^{-1}$ will remain an equivalence when restricted to $\text{LG}_S^{\text{fl-qproj}}$ (granting the validity of Theorem 2.35 without the flatness hypothesis) since any quasi-projective scheme over an affine scheme has the resolution property, see e.g. [TT90, 2.1]).
We now address the proof of the Theorem 2.35. By Kan extension and descent, it is enough to perform the construction of \( \text{Orl} : \text{Sing} \to \text{MF} \) for affine LG-models. For \((\text{Spec } B, f)\) an affine LG-model, we first define a strict \( A \)-linear dg-functor

\[
\psi : \text{Coh}^a(B, f) \longrightarrow \text{MF}(B, f)
\]
as follows.

**Construction 2.38.** Recall the description of objects of \( \text{Coh}^a(B, f) \) from Remark 2.22: they are pairs \((E, h)\) consisting of a strictly bounded complex \( E \) of projective \( B \)-modules of finite type, together with a morphism of graded modules \( h : E \to E \) of degree \(-1\), satisfying the equation \([d, h] = dh + hd = f\). Given such a pair \((E, h)\), we define \( \psi(E) \) to be the \( \mathbb{Z}/2 \)-graded \( B \)-module associated to \( E \), that is

\[
\psi(E)_0 = \bigoplus_n E_{2n} \quad \psi(E)_1 = \bigoplus_n E_{2n+1}.
\]

We endow \( \psi(E) \) with the odd endomorphism

\[
\delta := h + d : \psi(E) \longrightarrow \psi(E).
\]
We clearly have \( \delta^2 = f \), so this defines an object \((\psi(E), \delta)\) in \( \text{MF}(B, f) \). Indeed, as \( E \) is strictly bounded and each \( E_i \) is a projective \( B \)-module, each sum above is finite and remains projective over \( B \). This defines an \( A \)-linear dg-functor

\[
\psi_{(B, f)} : \text{Coh}^a(B, f) \longrightarrow \text{MF}(B, f).
\]
The \( \psi_{(B, f)} \) are part of a natural transformation between the pseudo-functors

\[
\psi : (2.3.40) \longrightarrow (2.2.8)
\]
This is clear from the pseudo-functorial structure on \((2.3.41)\) described in the beginning of the Construction 2.24 and the pseudo-functorial behavior of \((2.2.10)\) described in the Construction 2.4. Moreover, \( \psi \) has a lax symmetric monoidal enhancement \( \psi^\otimes \) with respect to the lax monoidal enhancements \((2.3.40)\) and \((2.2.10)\). Indeed, given affine LG-pairs \((B, f)\) and \((C, g)\) the commutativity of the diagram in \( \text{dgCat}^\text{strict,loc−flat}_A \)

\[
\begin{array}{c}
\text{Coh}^a(B, f) \otimes_A \text{Coh}^a(C, g) \\
\downarrow^{(2.3.33)} \quad \downarrow^{(2.2.5)}
\end{array}
\]

\[
\begin{array}{c}
\text{MF}(B, f) \otimes_A \text{MF}(C, g) \\
\psi_{(B, f)} \otimes \psi_{(C, g)}
\end{array}
\]

comes from the explicit descriptions of each composition: if \( E \in \text{Coh}^a(B, f) \) and \( F \in \text{Coh}^a(C, g) \), the composition \( \psi_{(B \otimes_A C, f \otimes 1 + 1 \otimes g)} \circ (2.3.33)(E, F) \) gives a 2-periodic complex

\[
\bigoplus_{(\alpha, \beta) : \alpha + \beta \text{ even}} E_{\alpha} \otimes_A F_{\beta} \longrightarrow \bigoplus_{(\alpha, \beta) : \alpha + \beta \text{ odd}} E_{\alpha} \otimes_A F_{\beta}
\]
If \( h \) (resp. \( k \)) denotes the element of degree \(-1\) in \( K(B, f) \) (resp. \( K(C, g) \)) explained in the Remark 2.22, then the formula (2.3.36) and the formula for the differential of the tensor product of complexes combined, describe the differential \( \delta \) of (2.4.4) (defined by (2.4.2)) as

\[
\delta_E \otimes \text{Id}_F + \text{Id}_E \otimes \delta_F \quad (2.4.5)
\]

By re-indexing (2.4.4)

\[
(\oplus_{\alpha \text{even}, \beta \text{even}} E_{\alpha} \otimes_A F_{\beta}) \oplus (\oplus_{\alpha \text{odd}, \beta \text{odd}} E_{\alpha} \otimes_A F_{\beta}) \rightrightarrows (\oplus_{\alpha \text{even}, \beta \text{odd}} E_{\alpha} \otimes_A F_{\beta}) \oplus (\oplus_{\alpha \text{odd}, \beta \text{even}} E_{\alpha} \otimes_A F_{\beta})
\]

we recover the composition \((2.2.5) \circ \psi_{(B, f)} \otimes \psi_{(C, g)}(E, F)\), as the definition of the product differential (2.2.6) also gives (2.4.5). To conclude we have to check that \( \psi \) is compatible with the lax units, meaning, that it makes the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{(2.3.39)} & \text{Coh}^s(A, 0) \\
\downarrow(2.2.7) & & \downarrow \psi_{(A, 0)} \\
& \xrightarrow{\psi(A, 0)} & \text{MF}(A, 0)
\end{array}
\]

commute. This is immediate from the definitions.

**Lemma 2.39.** The dg-functor defined above

\[
\psi_{(B, f)} : \text{Coh}^s(B, f) \longrightarrow \text{MF}(B, f)
\]

sends quasi-isomorphisms to equivalences.

**Proof.** We will prove the equivalent statement that \( \psi \) sends the full sub-dg-category \( \text{Coh}^{s,\text{acy}}(B, f) \) of acyclic complexes, to zero. Again, recall from the Remark 2.22 the description of objects in \( \text{Coh}^s(K(B, f)) \) as pairs \((E, h)\). Such a pair \((E, h)\) sits in \( \text{Coh}^{s,\text{acy}}(K(B, f)) \) if and only if there exists a degree \(-1\) endomorphism \( k \) of \( E \), with \( kd + dk = id \). The endomorphism \( k \) defines an odd degree endomorphism of \( \psi(E) \) as a \( \mathbb{Z}/2 \)-graded \( B \)-module, so an element \( \psi(k) \) of degree \(-1\) in the complex of endomorphism \( \text{End}_{\text{MF}(B, f)}(\psi(E)) \) (i.e. \( \psi(k) \in \text{End}_{\text{MF}(B, f)}(\psi(E))_{-1} \)). By construction this element is a homotopy between 0 and \( id + hk + kh \). The endomorphism \( u = hk + kh \) is of degree \(-2\) and thus, because \( E \) is bounded, we have \( u^n = 0 \) for some integer \( n \). We see in particular that the identity of \( \psi(E) \) becomes a nilpotent endomorphism in the homotopy category \([\text{MF}(B, f)]\). This implies that \( \psi(E) \approx 0 \) in \([\text{MF}(B, f)]\) as stated. \( \square \)

A consequence of the Lemma 2.39 is that \( \psi^\otimes \) has an enhancement as a lax symmetric monoidal natural transformation between

\[
\begin{array}{ccc}
\mathbf{LG}_{S}^{\text{aff, op,}[]} & \xrightarrow{(2.3.41)} & \mathbf{Pairsdgc}_{A}^{\text{strict, loc-flat,} \otimes} \\
\downarrow \Downarrow & & \downarrow \\
\mathbf{MF}^\otimes & &
\end{array}
\]

(2.4.6)
where MF is seen as an object in \( \text{Pairs}_{\text{dgCat}}^{\text{idem,loc-flat}} \) taking the equivalences as the distinguish class of morphisms. Finally, composing with the (symmetric monoidal) functors (2.3.42) and (2.3.32) we obtain a lax symmetric monoidal transformation

\[
\begin{array}{c}
\text{LG}_{S}^{\text{op,aff}} \xrightarrow{\text{Coh(\text{\text{-}B})_{\text{Perf}(\text{\text{-})}}}^{\text{\text{-}B}} \downarrow \text{MF}^{\text{idem,\text{\text{-}B}}} \\
\text{MF}^{\text{idem,\text{\text{-}B}}}
\end{array}
\] (2.4.7)

**Remark 2.40.** As part of the lax symmetric monoidal enhancement (2.4.7), we obtain a symmetric monoidal \( \infty \)-functor

\[
\psi^{\otimes}_{(A,0)} : \text{Coh}^{b}(K(A,0))_{\text{idem}} \to \text{MF}(A, 0)_{\text{idem}}
\] (2.4.8)

It is now immediate to check using the definitions that under the symmetric monoidal equivalences (2.3.48) and (2.2.12) , \( \psi^{\otimes}_{(A,0)} \) identifies with the symmetric monoidal base change functor (2.3.57).

It is a consequence of the Remark 2.40 that the lax natural transformation (2.4.7) has in fact values in \( \text{Mod}_{\text{Perf}(A[u])}(\text{dgcat}_{A}^{\text{idem}})_{\otimes} \). Moreover, as the lax symmetric monoidal \( \infty \)-functor \( \text{MF}^{\text{idem}} \) has values in \( \text{Mod}_{\text{Perf}(A[u,v^{-1}])(\text{dgcat}_{A}^{\text{idem}})_{\otimes}} \) (because of the equivalence (2.2.12)), by base-change, (2.4.7) is in fact equivalent to the data of a lax symmetric monoidal transformation

\[
\begin{array}{c}
\text{Coh}_{\text{Perf}(\text{\text{-}B})}^{\text{\text{-}B}} \otimes_{A[u]} A[u, u^{-1}] \longrightarrow \text{MF}^{\text{idem}}
\end{array}
\] (2.4.9)

Finally, composing with the equivalence of the Prop. 2.31 we obtain a lax symmetric monoidal transformation

\[
\psi^{\otimes} : \text{Sing}^{\otimes} \to \text{MF}^{\text{idem}}
\] (2.4.10)

**Proof of Theorem 2.35.** We set \( \text{Orl}^{-1,\otimes} := \psi^{\otimes} : \text{Sing}^{\otimes} \longrightarrow \text{MF}^{\text{idem}} \). The explicit description of \( \psi \) given in the Construction 2.38 is all we need to conclude. Indeed, as observed in [EP15, p. 47], for each fixed \((B, f)\), the induced triangulated functor

\[
[\text{Orl}^{-1}] : [\text{Sing}(B, f)] \longrightarrow [\text{MF}(B, f)]
\]

(there denoted as \( \Delta \)) is an inverse to the functor \( \Sigma \) described in [EP15, Theorem 2.7] (which is an analogue of Orlov’s “Cok” functor in [Orl12, Thm 3.5]), and thus is an equivalence (by [EP15, Theorem 2.7]) on those LG-pairs \((X, f)\) where \( f \) is flat (so that the derived fiber \( X_{0} \) coincides with the scheme theoretic fiber considered in [EP15, Theorem 2.7]), \( X/S \) is separated (hence \( X \) is), and \( X \) has the resolution property, so that the
standing hypotheses of [EP15]) are met. See also [BW12, Theorem 6.8.]. As all the categories involved are stable, this implies the equivalence of the dg-enrichments. □

3. Motivic Realizations of dg-categories

In this section we explain how to associate to every dg-category $T$ a motivic $BU$-module, where $BU$ is the motivic ring-object representing algebraic $K$-theory. At first we describe some general features of this motivic incarnation of $T$ and then we will study several of its realizations. If $R$ is any realization of motives (e.g. $\ell$-adic, étale, Hodge, de Rham, etc), the realization $R(T)$ will carry a structure of $R(BU)$-module.

3.1. Motives, $BU$-modules and noncommutative motives. Let again $S = \text{Spec} \, A$ be any affine base scheme. By [Rob15] we have a symmetric monoidal $\infty$-category $\text{SH}^S_\otimes$, which is an $\infty$-categorical version of Morel-Voevodsky’s stable homotopy category of schemes over $S$ [MV99]. We let $\text{Sm}_S$ be the category of smooth and affine schemes over $S$. It is a symmetric monoidal category for the cartesian product. By definition, $\text{SH}^S_\otimes$ is a presentable stable symmetric monoidal $\infty$-category together with a symmetric monoidal $\infty$-functor $\text{Sm}_S \times S \to \text{SH}^S_\otimes$ universal with respect to the following properties (see [Rob15, Cor. 1.2]):

1. The image of an elementary Nisnevich square in $\text{Sm}_S$ is a pushout square in $\text{SH}^S_\otimes$.
2. (Homotopy invariance) The natural projection $\mathbb{A}^1_S \to S$ is sent to an equivalence.
3. (Stability) Let $S \to \mathbb{P}^1_S$ be the point at infinity and consider its image in $\text{SH}^S_\otimes$. The cofiber of this map in $\text{SH}^S_\otimes$, denoted as $(\mathbb{P}^1_S, \infty)$, is $\otimes$-invertible.\[13\]

In the following, we will denote by $1_S \in \text{SH}_S$ the unit of the tensor structure in $\text{SH}^S_\otimes$. We will also be using the standard notation $1_S(1) := (\mathbb{P}^1_S, \infty)[-2] = \Omega \mathbb{G}_{m,S}$, and $(-)(d) := (-) \otimes 1_S(1)^{\otimes d}$ for the the motivic Tate $d$-twist, $d \in \mathbb{Z}$, where, as usual, we denote by $[1]$ the shift given by smashing with the topological circle $S^1$.

If we denote by $\text{CAlg}(\text{SH}_S)$ the $\infty$-category of commutative algebras in $\text{SH}_S$, then there exists an object in $\text{CAlg}(\text{SH}_S)$ representing homotopy invariant algebraic $K$-theory of Weibel [Wei89, Cis13]. As a functor, it sends $Y \in \text{Sm}_S$ to $KH(Y)$, the (non-connective) spectrum of homotopy invariant algebraic $K$-theory of $Y$. This motive is usually denoted as $KH_S$ but we will denote it here as $BU_S$, inspired by the topological analogy. The relation between the motive $BU_S$ and the theory of non-commutative motives was studied in [Rob15]. Let us briefly recall this relation. First of all, to every $Y \in \text{Sm}_S$ we can assign a dg-category over $S$, $\text{Perf}(Y)$, of perfect complexes on $Y$. This dg-category is of

\[13\] As an object in $\text{SH}_S$, this is equivalent to the tensor product of the topological circle $S^1$ and the algebraic circle $\mathbb{G}_{m,S}$. 
finite type in the sense of [TV07]. This assignment can be organized into a symmetric monoidal $\infty$-functor

$$\text{Perf} : \text{Sm}_S \times S \to \text{dgcat}_{idem, ft, op}^\otimes$$

where $\text{dgcat}_{idem, ft, op}^\otimes$ denotes the monoidal full $\infty$-subcategory of $\text{dgcat}_{op}^\otimes$ consisting of $S$-dg-categories of finite type. One can mimic the construction of motives starting from the theory of dg-categories. More precisely, one constructs a presentable stable symmetric monoidal $\infty$-category $\text{SH}_{nc}^\otimes$ together with a symmetric monoidal functor

$$\iota : \text{dgcat}_{idem, ft, op}^\otimes \to \text{SH}_{nc}^\otimes$$

satisfying a universal property analogous to (1), (2), (3) above for $\text{SH}$, namely, every Nisnevich square of dg-categories (see [Rob15, Section 33.1]) is sent to a pushout diagram, the pullback along the canonical projection $\text{Perf}(S) \to \text{Perf}(A_S^1)$ is sent to an equivalence and the image of the cofiber $(P^1_S, \infty)$ is $\otimes$-invertible. From the universal property of $\text{SH}_{S}^\otimes$, one then obtains a symmetric monoidal $\infty$-functor

$$\text{RPerf} : \text{SH}_{S}^\otimes \to \text{SH}_{nc}^\otimes$$

informally defined by sending a motive $Y$ to the motive of its dg-category $\text{Perf}(Y)$. For formal reasons, this admits a lax monoidal right adjoint $M_{S}^\otimes$. By [Rob15, Theorem 1.8] this adjoint sends the image of the tensor unit in $\text{SH}_{nc}^\otimes$ to the object $BU_S$, thus endowing it with a structure of commutative algebra in the $\infty$-category $\text{SH}_{S}$ (the construction of this algebra structure has also been obtained using other methods in [GS09, Section 5.2]). Formal reasons then imply that $M_{S}^\otimes$ factors as a lax monoidal functor via the theory of $BU$-modules

$$\text{SH}_{nc}^\otimes \to \text{Mod}_{BU}(\text{SH}_{S})^\otimes$$

which we will again denote as $M_{S}^\otimes$.

**Remark 3.1.** (Algebraic Bott periodicity) The object $BU_S$ reflects the projective bundle theorem in algebraic K-theory in the form of a periodicity given by the Bott isomorphism

$$BU_S \xrightarrow{\sim} R\text{Hom}_{\text{SH}_S}((\mathbb{P}^1_S, \infty), BU_S) \simeq BU_S(-1)[-2] \quad (3.1.1)$$

One can find this as a consequence of the fact that the non-commutative motive of $(\mathbb{P}^1_S, \infty)$ is a tensor unit (see [Rob15, Lemma 3.25]):

$$BU_S \simeq M_S(R\text{Hom}_{\text{SH}_{nc}S}(1^ne_S^1, 1^ne_S^1)) \simeq M_S(R\text{Hom}_{\text{SH}_{nc}S}(R\text{Perf}_S(\mathbb{P}^1_S, \infty), 1^ne_S^1)) \xrightarrow{\sim} R\text{Hom}_{\text{SH}_S}((\mathbb{P}^1_S, \infty), M_S(1^ne_S^1)) \simeq BU_S(-1)[-2]$$
Notice that as all the functors used here are lax monoidal and the tensor unit in non-commutative motives has a unique structure of commutative algebra object, the map in the equivalence (3.1.1) is in fact $B_U S$-linear. Therefore, it is completely determined by a map in $SH_S$

$$u : 1_S(1)[2] \to B_U S$$

which, unwinding the argument in the proof of [Rob15, Lemma 3.25] one sees, corresponds to the element $\beta^{-1} := [O(1)] - [0]$ in $K_0(\mathbb{P}^1_S)$.

Moreover, as $(\mathbb{P}^1_S, \infty) \simeq 1_S(1)[2]$ is $\otimes$-invertible, we can tensor (3.1.1) on both sides by $1_S(1)[2]$ and obtain

$$B_U S(1)[2] \sim \to B_U S$$

The map (3.1.3) corresponds to the composition

$$B_U S(1)[2] \xrightarrow{Id \otimes \beta^{-1}} B_U S \otimes B_U S \to B_U S$$

where the last map is the multiplication map of the commutative algebra structure on $B_U S$. The element $\beta^{-1}$ is invertible with inverse corresponding to a map

$$u := \beta : 1_S(-1)[-2] \to B_U S$$

3.2. The realization of dg-categories as $BU$-modules. The construction $M_S$ gives us a way to assign a motive to a dg-category of finite type via the composition with the universal map $dgcat_S^{idem, ft, op, \otimes} \to SHnc_S^\otimes$. We will use it to produce a more interesting assignment. By construction, $SHnc_S^\otimes$ is a stable presentable symmetric monoidal $\infty$-category. As such, it admits internal-hom objects $RHom_{SHnc}$ and in particular, there exists an $\infty$-functor

$$RHom_{SHnc}(-, 1_S^{nc}) : SHnc^{op, \otimes} \to SHnc$$

where $1_S^{nc}$ is the tensor unit. Of most importance to us is the fact that this functor can be endowed with a lax monoidal structure [Lur16, 5.2.2.25, 5.2.5.10, 5.2.5.27 ]

$$RHom_{SHnc}(-, 1_S^{nc}) : SHnc^{op, \otimes} \to SHnc_S^\otimes$$

The composition

$$dgcat_S^{idem, ft, \otimes} \xrightarrow{i} SHnc_S^{op, \otimes} \xrightarrow{RHom_{SHnc}(-, 1_S^{nc})} SHnc_S^\otimes$$

is lax monoidal. We now recall that dg-categories of finite type generate all the Morita theory of dg-categories under filtered colimits i.e. $Ind(dgcat_S^{idem, ft})^{\otimes} \simeq dgcat_S^{idem, \otimes}$.

\footnote{See [TV07] and the $\infty$-categorical narrative in [Robdf, 6.1.27]}
As $\text{SHnc}_S$ is presentable, we obtain via the universal property of the convolution product in Ind-objects [Lur16, 4.8.1.10], an induced lax symmetric monoidal functor

$$\text{dgcat}_S^{\text{idem}, \otimes} \xrightarrow{\mu_S^\otimes} \text{SHnc}_S^\otimes$$

(3.2.1)

Informally, if $T$ is an $S$-dg-category of finite type, and we write again $T$ for its image in $\text{SHnc}_S$, the object $\mu_S^\otimes(T) = R\text{Hom}_{\text{SHnc}}(T, 1_S^{\text{nc}})$ can be described\footnote{It is helpful to remind the reader that $\text{SHnc}_S$ can be constructed as a localization of the of presheaves in $\text{dgcat}_S^{\text{mor}, \text{op}, \mathfrak{R}}$ with values in the $\infty$-category of spectra $\text{Sp}$, by forcing Nisnevich descent and $\text{Perf}(\mathbb{A}_{S}^1)$-invariance - see [Rob15, Section 33]} as the $\infty$-functor sending a dg-category of finite type $T'$ over $S$ to the mapping spectrum

$$\text{Map}_{\text{SHnc}_S}(T', R\text{Hom}_{\text{SHnc}}(T, 1_S^{\text{nc}})) \simeq \text{Map}_{\text{SHnc}_S}(T' \otimes_S T, 1_S^{\text{nc}})$$

which following [Rob15, Theorem 1.8 (ii) and Cor 4.8] is the spectrum of homotopy invariant K-theory $KH(T' \otimes_S T)$

More generally, for $T \in \text{dgcat}_S^{\text{idem}}$ we can write $T$ as a filtered colimit of dg-categories of finite type and as $KH$ commutes with filtered colimits (this is well known but see [Bla13, Prop. 2.8]), and the same holds for tensor product of dg-categories, we conclude that $T$ is sent to the object in $\text{SHnc}_S$, defined by the $\infty$-functor $KH(- \otimes_S T)$.

We will denote by $M^\vee_S$ the composition of the lax monoidal functors

$$\text{dgcat}_S^{\text{idem}, \otimes} \xrightarrow{\mu_S^\otimes} \text{SHnc}_S^\otimes \xrightarrow{M^\otimes_S} \text{Mod}_{\text{B}_{\mathbb{U}}S}(\text{SH}_S)^\otimes$$

**Remark 3.2.** Notice that as $M_S$ commutes with filtered colimits, this composition is also the functor obtained by the monoidal universal property of the Ind-completion. To see that $M_S$ commutes with filtered colimits it is enough to test on compact generators [Robdf, Prop. 5.3.3 and 6.4.24] and use the fact $R\text{Perf}$ preserve compact generators.

By the definition of $M_S$ as a right adjoint to $R\text{Perf}$ and following the previous discussion, the motive $M^\vee_S(T)$ in $\text{SH}_S$ represents the $\infty$-functor sending a smooth scheme $X$ over $S$ to the spectrum $KH(\text{Perf}(X) \otimes_S T)$.

**Corollary 3.3.** $M^\vee_S$ sends exact sequences of dg-categories to cofiber-fiber sequences in the stable $\infty$-category of $\text{B}_{\mathbb{U}}S$-modules.
Proof. Indeed, this follows because cofiber sequences of dg-categories are stable under tensor products and because homotopy K-theory sends exact sequences of dg-categories (see the discussion in [Rob15, Section 1.5.4]) to cofiber-fiber sequences in spectra (see the details in [Rob15, Prop. 417 and Prop. 3.19]). □

Remark 3.4. In [CT11, Tab08, CT12] Cisinski and Tabuada introduced an alternative category of non-commutative motives $M_{\text{Loc}}^\otimes$ which is dual to the one used here. Indeed, there is a duality blocking a direct comparison between $M_{\text{Tab}}^\otimes$ and $SH^\otimes$. The category $SHnc^\otimes$ was designed to avoid this obstruction (see [Rob15, Appendix A] for the comparison between the two approaches). It is exactly this duality that we encode in our construction of motivic realizations of dg-categories via the functor $\mu^\otimes$. As in [Rob15, Appendix A], let $M_{\text{Tab}}^{\text{Nis}}$ be the Nisnevich version of the construction of Tabuada-Cisinski. Then, by definition of $M_{\text{Tab}}^{\text{Nis}}$, the functor $\mu^\otimes$ of the diagram (3.2.1) factors in a unique way as a lax monoidal functor

$$
\begin{array}{ccc}
dgcat^\text{idem.ft,}\otimes_S & \xrightarrow{\mu^\otimes} & SHnc^\otimes_S \\
\downarrow & & \downarrow \\
M_{\text{Tab}}^{\text{Nis,}\otimes} & \xrightarrow{\Leftarrow} & \end{array}
$$

This exhibits $M_{\text{Tab}}^{\text{Nis,}\otimes}$ as a universal motivic realization of dg-categories (see next section for more on realizations).

3.3. The six operations in $BU$-modules and realizations. Before continuing towards our main goals we will need to discuss some functorial aspects. In the last two section we were working with motives over a fixed base $S$. It is possible to work in a relative setting. Let $\text{Sch}/S$ denote a category of $S$-schemes satisfying the properties [CD12, 2.0] 16. For every $X \in \text{Sch}/S$ we can construct a stable presentable symmetric monoidal $\infty$-category $SH^\otimes_X$ encoding the motivic homotopy theory of Morel-Voevodsky over $X$. Moreover, we can make the assignment $X \mapsto SH^\otimes_X$ functorial in $X$, given by an $\infty$-functor

$$
SH^\otimes : \text{Sch}^\text{op}/S \to C\text{Alg}(\text{Pr}^L_{\text{Stab}})
$$

where $\text{Pr}^L_{\text{Stab}}$ denotes the $\infty$-category of stable presentable $\infty$-categories with colimit preserving functors, and $C\text{Alg}(\text{Pr}^L_{\text{Stab}})$ is the $\infty$-category of commutative monoids in $(\text{Pr}^L_{\text{Stab}})^\otimes$, i.e. $C\text{Alg}(\text{Pr}^L_{\text{Stab}})$ is the $\infty$-category of stable presentable symmetric monoidal $\infty$-categories (see [Lur16, 4.8] recalled in [Robdf, 3.8.1]). This is done in [Robdf, Section 9.1]. The $\infty$-functor $SH^\otimes_X$ comes together with a more complex system of functorialities encoding the six operations of Grothendieck (see Appendix A).

16In the case of interest to this paper, $S$ will be an henselian trait and $\text{Sch}/S$ will be all Noetherian $S$-schemes of finite dimension. See [CD12, 2.0, footnote 35]
We will be interested in several motivic realizations. For us, a motivic realization consists of an $\infty$-functor
\[ D^\otimes : Sch^{op}_{/S} \to CAlg(Pr^L_{Stb}) \]
enriched with a system of six operations, plus the data of a monoidal natural transformation
\[ SH^\otimes \to D^\otimes \]
and a system of compatibilities between the systems of six operations on $SH^\otimes_X$ and $D^\otimes$ (see Prop. A.4). Of major importance to us are the étale and the $\ell$-adic realizations which we will explore later in this section.

**Remark 3.5.** We recall also that in [Robdf, Chapter 9] it is shown that the theory of non-commutative motives also admits relative versions encoded by an $\infty$-functor
\[ SHnc^\otimes : Sch^{op}_{/S} \to CAlg(Pr^L_{Stb}) \]
together with a natural transformation
\[ R_{Perf} : SH^\otimes \to SHnc^\otimes \]
However, it is not known if the necessary localization and proper base change conditions to endow $SHnc^\otimes$ with a system of six operations hold.

Another important example is that of $BU$-modules. For each $X \in Sch_{/S}$ there exists a commutative algebra object $BU_X \in CAlg(SH_X)$ representing a relative version of homotopy invariant algebraic $K$-theory. This commutative algebra structure can also be obtained from a relative version of the results in [Rob15] which the reader can consult in [Robdf, Chapter 9]. For this, it is crucial that these relative versions $BU_-$ are compatible under pullbacks (see [Cis13, 3.8]). This allows us to construct an $\infty$-functor
\[ \text{Mod}_{BU}(SH)^\otimes : Sch^{op}_{/S} \to CAlg(Pr^L_{Stb}) : (X/S) \mapsto \text{Mod}_{BU_X}(SH_X) \]
together with a natural transformation
\[ - \otimes BU : SH^\otimes \to \text{Mod}_{BU}(SH)^\otimes \]
which for each $X \in Sch_{/S}$ admits a conservative right adjoint $\text{Mod}_{BU_X}(SH_X) \to SH_X$ that forgets the module structure. As explained in [CD12, 13.3.3] (see also the discussion in [Robdf, pg 260, 9.4.38, 9.4.39]), the conservativity of the forgetful functor and the fact it commutes with the functorialities $(-)_*$, $(-)^*$ and $(\cdot)_!$, and verifies the projections formulas, are enough to deduce the conditions endowing $\text{Mod}_{BU}(SH)^\otimes$ with a system of six operations (see Prop. A.2), to make the natural transformation $- \otimes BU$ compatible with the operations in the sense of Prop.A.4, and to make the forgetful functor $\text{Mod}_{BU} \to SH$ compatible with all the operations (meaning that the natural transformations at the end of A.4 are natural isomorphisms; see [CD12, Section 7.2]).
Remark 3.6. Notice that the algebraic Bott isomorphism of the Remark 3.1 forces the functorialities \((-)^\sharp\) and \((-)^!\) to be the same for smooth maps (see equation (A.0.4) in Appendix A).

To conclude this preliminary section, we must also remark that if \(R^\otimes : SH^\otimes \to D^\otimes\) is a motivic realization, being monoidal, it preserves algebra-objects and thus sends \(BU\) to an algebra object \(R(BU)\). Therefore, it produces a new realization

\[
R^\otimes_{\text{mod}} : \text{Mod}_{BU}(SH)^\otimes \to \text{Mod}_{R(BU)}(D)^\otimes
\]

Throughout the next sections we will analyse several realizations of dg-categories, all obtained by pre-composition with \(M^\otimes_{\text{S}}\)

\[
dgcat^\otimes_{\text{S}} \xrightarrow{M^\otimes_{\text{S}}} \text{Mod}_{BU_S}(SH_S)^\otimes \xrightarrow{R^\otimes_{\text{mod}}_S} \text{Mod}_{BU}(D_S)^\otimes
\]

Remark 3.7. By 3.3, every realization of dg-categories sends exact sequences to exact sequences.

Example 3.8. When the base ring is \(\mathbb{C}\), we have a Betti realization \(SH^\otimes_{\mathbb{C}} \to Sp^\otimes\). In [Bla15, Section 4.6] it is shown that the realization of \(BU_{\mathbb{C}}\) is the spectrum representing topological \(K\)-theory \(BU_{\text{top}}\). The composite realization

\[
dgcat^\otimes_{\text{S}} \rightarrow \text{Mod}_{BU_S}(SH_S) \rightarrow \text{Mod}_{BU_{\text{top}}}(Sp)
\]

recovers what in [Bla15] is called the topological \(K\)-theory of dg-categories.

3.4. The \(BU\)-motives of \(\text{Perf}, \text{Coh}^b\) and \(\text{Sing}\).

3.4.1. Let

\[
\begin{array}{ccc}
  X & \xrightarrow{p} & \text{S} \\
  \downarrow & & \downarrow \\
  \text{S}
\end{array}
\]

be any \(S\)-scheme with \(X\) quasi-compact and quasi-separated. Then \(\text{Perf}(X)\) is an object in \(\text{dgcat}^\otimes_{\text{S}}\) and through the construction explained in Section 3.2 it produces a \(BU_S\)-module \(M^\otimes_{\text{S}}(\text{Perf}(X))\). At the same time, \(BU_X\) is an object in \(SH_X\) and, as in the previous section, we can consider its direct image \(p_* (BU_X) \in \text{Mod}_{BU_S}(SH_S)\).

Proposition 3.9. Assume \(p : X \to S\) as in (3.4.1). Then the two objects \(M^\otimes_{\text{S}}(\text{Perf}(X))\) and \(p_* (BU_X)\) are canonically equivalent as \(BU_S\)-modules.
Proof. The first ingredient is the fact that by [Rob15, Theorem 1.8] and its extension to general basis in [Robdf, Cor. 9.3.4], we have canonical equivalences

\[ p_*(BU_X) \simeq p_*(M_X(1_{NC_X})) \]

By formal adjunction reasons, \( M \) and \((-)_* \) are compatible, so that \( p_*(M_X(1_{NC_X})) \) is canonically equivalent to \( M_S(p_*(1_{NC_X})) \), where now \( p_* \) denotes the direct image functoriality in \( SH_{nc} \). By the analysis in Section 3.2, we are reduced to showing that \( p_*(1_{NC_X}) \) is equivalent to the object in \( SH_{ncS} \) given by \( \mu_S(Perf(X)) \) (where \( \mu_S \) is defined in diagram (3.2.1)). Unwinding the adjunctions, the first corresponds to the \( \infty \)-functor sending an \( S \)-dg-category of finite type \( T \) to the homotopy K-theory spectrum \( KH_X(p^*(T)) \). The second corresponds to the \( \infty \)-functor sending a dg-category of finite type \( T \) to the spectrum \( KH_S(T \otimes_S Perf(X)) \). In the case where \( X = \text{Spec } B \) is an affine scheme over \( S = \text{Spec } A \), the equivalence between the two follows from the same arguments as in [Robdf, Prop. 10.1.4].

We deduce the general case from the affine case using the Zariski descent property for \( X \mapsto SH_{ncX} \) of [Robdf, 9.21]. If \( \{\phi_i : U_i \hookrightarrow X\}_{i \in I} \) is a Zariski cover of \( X \) by affine open immersions then we have from the Zariski descent for homotopy K-theory that

\[ \text{colim}_i (\phi_i)_2 \circ (\phi_i)^*1_{NC_X} \simeq 1_{NC_X} \]

in \( SH_{ncX} \). Equivalently, passing to right adjoints, we also have

\[ 1_{NC_X} \simeq \text{lim}_i (\phi_i)_* \circ (\phi_i)^*1_{NC_X} \]

Since \( p_* \) is a right adjoint, this gives

\[ p_*(1_{NC_X}) \simeq \text{lim}_i (p \circ \phi_i)_*1_{NC_{U_i}} \]

which by the result in the affine case is equivalent to

\[ p_*(1_{NC_X}) \simeq \text{lim}_i \mu(Perf(U_i)) \]

Finally, by the Zariski descent for \( KH \) of dg-categories (in fact \( KH \) has non-commutative Nisnevich descent [Rob15, 4.16, 4.18]), and the fact that Zariski squares of dg-categories with compact generators - as it is the case of \( Perf(X) \) for \( X \) quasi-compact and quasi-separated by [BvdB03, 3.1.7] - are stable under tensor products (see the proof of [Rob15, Prop 3.19, Rem 3.20]), \( \text{lim}_i \mu(Perf(U_i)) \) is canonically equivalent to \( \mu(Perf(X)) \). \( \square \)

Remark 3.10. As a consequence of the six operations for \( BU \)-modules, if \( p \) is a smooth map, then \( p_! BU_X \) is also equivalent to \( BU_X^Y := R\text{Hom}_{SH_S}(p_*1_X, BU_S) \) (by projection formula).

\(^{17}\)Notice that the arguments in [Robdf, Prop. 10.1.4] are written for \( p \) a closed immersion but in fact work in the general case.
Remark 3.11. For $X$ as in Proposition 3.9, $\text{Perf}(X)$ carries a symmetric monoidal structure given by the tensor product of perfect complexes. This can be understood as a commutative algebra object $\text{Perf}(X)^{\otimes} \in \text{CAlg}(\text{dgcat}_S^{\text{bim}})$. As $\mathcal{M}_S^\otimes(\text{Perf}(X))$ is lax monoidal, $\mathcal{M}_S^\otimes(\text{Perf}(X))$ is an object in $\text{CAlg}(\text{Mod}_{BU}(\text{SH}_S))$.

3.4.2. We now extend the context of Subsection 3.4.1 by considering

$$U := X - Z \xrightarrow{j} X \xrightarrow{i} Z$$ (3.4.2)

with $X$ a regular scheme, quasi-compact quasi-separated, $i$ a closed immersion and $j$ its open complementary. It follows also from Prop. 3.9 that $(p \circ j)_! B_U$ is equivalent to $\mathcal{M}_S^\otimes(\text{Perf}(U))$ where $\text{Perf}(U)$ is seen as an $S$-dg-category via the composition $p \circ j$. In the same way we have that $(p \circ i)_! B_U \simeq \mathcal{M}_S^\otimes(\text{Perf}(Z))$. Moreover, pullback along $j$ produces a map of $B_U$-modules $\mathcal{M}_S^\otimes(j^* : \mathcal{M}_S^\otimes(\text{Perf}(X)) \to \mathcal{M}_S^\otimes(\text{Perf}(U))$ which via the equivalence of Prop. 3.9 is identified with the map induced by the unit of the adjunction

$$p_!(BU_X) \to p_! j_! j^* BU_X$$ (3.4.3)

This morphism fits into an exact sequence in $\text{SH}_S$

$$p_! i^! BU_X \to p_! BU_X \to p_! j_! j^* BU_X$$ (3.4.4)

given by the localization property of $\text{SH}$ [MV99] (see A.2 and A.3 in Appendix A).

Proposition 3.12. Assume the conditions and notations as in context (3.4.2). Then $p_! i^! BU_X$ is equivalent as a $B_U$-module to $\mathcal{M}_S^\otimes(\text{Coh}^b(Z))$, where the dg-category $\text{Coh}^b(Z)$ of bounded coherent complexes on $Z$ is regarded as a $S$-dg-category via the composition $p \circ i$.

Proof. As $X$ is assumed to be regular, and $U$ is open, $U$ is also regular and we have $\text{Perf}(X) = \text{Coh}^b(X)$ and $\text{Perf}(U) = \text{Coh}^b(U)$. In this context $\mathcal{M}_S^\otimes(\text{Coh}^b(X))$ is given by the $\infty$-functor sending a smooth $S$-scheme $Y$ to the spectrum $KH(\text{Perf}(Y) \otimes_S \text{Coh}^b(X))$ and $\mathcal{M}_S^\otimes(\text{Coh}^b(U))(Y)$ is given by $KH(\text{Perf}(Y) \otimes_S \text{Coh}^b(U))$. Let now $X$ be an $S$-scheme of finite type and let $Y$ be smooth over $S$. We have an equivalence of $S$-dg-categories

$$\text{Coh}^b(X) \otimes_S \text{Perf}(Y) \simeq \text{Coh}^b(X \times_S Y)$$ (3.4.5)

This is [Pre11, Prop. B.4.1] together with the fact that as $Y$ is smooth over $S$ and $S$ is assumed to be regular, $Y$ is regular and therefore $\text{Perf}(Y) = \text{Coh}^b(Y)$. The equivalence
(3.4.5) holds for $X$ and also for both $U$ and $Z$. With this identification, $\mathcal{M}_{S}(\text{Coh}^{b}(X))$ can now be identified with the $\infty$-functor $Y \mapsto \text{KH}(\text{Coh}^{b}(X \times S Y))$ which is equivalent to the $G$-theory spectrum of $X \times S Y$ by $\mathbb{A}^{1}$-invariance of $G$-theory. As the same holds for $U$, the map 3.4.3 can be identified with the $G$-theory pullback along $j$

$$G(X \times S -) \to G(U \times S -)$$

whose fiber is well-known from Quillen’s localization theorem for $G$-theory [Qui73, §7 Prop 3.2] to be the homotopy invariant $K$-theory of the dg-category of bounded coherent sheaves in $Z$, $\text{Coh}^{b}(Z \times S Y)$ which again by the lemma is equivalent to $\text{Perf}(Y) \otimes_{S} \text{Coh}^{b}(Z)$ from which the propositions follows. □

**Remark 3.13.** Combining the result of the Prop. 3.12 with the discussion in [CD12, Section 13.4.1] one finds that $\mathcal{M}_{S}(\text{Coh}^{b}(Z))$ can also be described as $K$-theory with support in $Z$.

Finally, following 3.3, the exact sequence of $S$-dg-categories

$$\text{Perf}(Z) \to \text{Coh}^{b}(Z) \to \text{Sing}(Z)$$

creates a cofiber-fiber sequence of $B\text{U}_{S}$-modules

$$\mathcal{M}_{S}(\text{Perf}(Z)) \to \mathcal{M}_{S}(\text{Coh}^{b}(Z)) \to \mathcal{M}_{S}(\text{Sing}(Z))$$

which, thanks to Proposition 3.9 and Proposition 3.12 (applied to $p \circ i$), can now be identified with the cofiber-fiber sequence

$$p_{*}i_{*}B\text{U}_{Z} \to p_{*}i_{*}i_{!}B\text{U}_{X} \to \mathcal{M}_{S}(\text{Sing}(Z)) \quad (3.4.6)$$

**Remark 3.14.** Following Remark 3.11, $\text{Perf}(Z)$ defines an object $\text{Perf}(Z)^{\otimes} \in \text{CAlg}(\text{dgcat}_{S}^{\text{idem}})$, and thus $\mathcal{M}_{S}(\text{Perf}(Z))$ as an object in $\text{CAlg}(\text{Mod}_{B\text{U}_{S}(\text{SH}_{S}))}$. Now, since the tensor product of coherent by perfect is coherent, the inclusion $\text{Perf}(Z) \subseteq \text{Coh}^{b}(Z)$ make $\text{Coh}^{b}(Z)$ an object in $\text{Mod}_{\text{Perf}(Z)^{\otimes}}(\text{dgcat}_{S}^{\text{idem}, \otimes})$. In this case $\mathcal{M}_{S}(\text{Coh}^{b}(Z))$ defines an object in $\text{Mod}_{\mathcal{M}_{S}(\text{Perf}(Z))}(\text{Mod}_{B\text{U}_{S}(\text{SH}_{S}))}$. Moreover, as the inclusion $\text{Perf}(Z) \subseteq \text{Coh}^{b}(Z)$ is a map of $\text{Perf}(Z)$-modules, the induced map

$$u : p_{*}i_{*}B\text{U}_{Z} \simeq \mathcal{M}_{S}(\text{Perf}(Z)) \to \mathcal{M}_{S}(\text{Coh}^{b}(Z)) \simeq p_{*}i_{*}i_{!}B\text{U}_{X} \quad (3.4.7)$$

is defined in $\text{Mod}_{\mathcal{M}_{S}(\text{Perf}(Z))}(\text{Mod}_{B\text{U}_{S}(\text{SH}_{S}))}$. By adjunction, this is the same as a map $1_{S} \to \mathcal{M}_{S}(\text{Coh}^{b}(Z))$ in $\text{SH}_{S}$ which corresponds to an element $u$ in the Grothendieck group $\text{KH}_{0}(\text{Coh}^{b}(Z))$. We will discuss this element $u$ in the Remark 3.17.

---

18 Notice that this works without any hypothesis on the regularity of Z
Remark 3.15. In particular, when $Z$ is itself regular we recover the purity isomorphism in algebraic K-theory

$$p_*i_*BU_Z \simeq p_*i_*j!BU_Z$$

And, more generally, the motive $M^\vee_S(S\text{ing}(Z))$ measures the obstruction to purity.

Combining (3.4.4) and (3.4.6) we get two cofiber-fiber sequences

$$
\begin{array}{cccc}
p_*i_*BU_Z & \xrightarrow{u} & p_*i_*i^!BU_X & \rightarrow M^\vee_S(S\text{ing}(Z)) \\
p_*BU_X & \downarrow & & \downarrow \\
p_*j_*j^*BU_X & & &
\end{array}
$$

(3.4.8)

3.4.3. We now consider $X$ as in subsection 3.4.2, together with a function $f : X \rightarrow \mathbb{A}_S^1$. In this case we get derived fiber products over $S$

$$
\begin{array}{cccc}
U & \xleftarrow{j} & X & \xrightarrow{i} X_0 \\
\downarrow & & \downarrow & \downarrow \\
Gm,S & \xleftarrow{i_0} & \mathbb{A}_S^1 & \xrightarrow{i_0} S
\end{array}
$$

(3.4.9)

where $i_0$ is the zero section, map $i$ is an lci closed immersion and $j$ is its open complementary. The classical truncation of this diagram brings us to the context of subsection 3.4.2 with $Z := t(X_0)$ the classical underived zero locus of $f$.

The following result tells us that the $BU$-motive $M^\vee_S(Coh^b(-))$ is invariant under derived thickenings:

**Proposition 3.16.** Let $\tilde{Z}$ be a derived scheme over $S$ with classical underlying scheme $Z$ and canonical closed immersion $Z \hookrightarrow \tilde{Z}$. Then the push forward along the inclusion

$$M^\vee_S(Coh^b(Z)) \rightarrow M^\vee_S(Coh^b(\tilde{Z}))$$

is an equivalence of $BU_S$-modules.

**Proof.** Analyzing the definitions, we are reduced to show that for any smooth scheme $Y \rightarrow S$, the induced map of K-theory spectra

$$KH(\text{Perf}(Y) \otimes_S Coh^b(Z)) \rightarrow KH(\text{Perf}(Y) \otimes_S Coh^b(\tilde{Z}))$$
is an equivalence. But again, thanks to [Pre11, Prop. B.4.1] we have the formula (3.4.5) so that it is enough to show that the map

$$\text{KH}(\text{Coh}^b(Y \times S Z)) \to \text{KH}(\text{Coh}^b(Y \times S \tilde{Z}))$$

is an equivalence. Here we mean the derived fiber product, which as $Y$ is flat over $S$, equals the usual fiber product. But now this equivalence follows from the theorem of the heart: for any derived scheme $V$ with truncation $t(V)$, $\text{Coh}^b(t(V))$ and $\text{Coh}^b(V)$ both carry $t$-structures with the same heart [Lur11, 2.3.20], so that by [Bar12], their K-theory spectra are equivalent via pushforward.

□

In the setting of the diagram (3.4.9), for the derived scheme $X_0$, the exact sequence of $S$-dg-categories

$$\text{Perf}(X_0) \to \text{Coh}^b(X_0) \to \text{Sing}(X_0)$$

creates, by Cor. 3.3, a cofiber-fiber sequence of $BU_S$-modules

$$M^\vee_S(\text{Perf}(X_0)) \to M^\vee_S(\text{Coh}^b(X_0)) \to M^\vee_S(\text{Sing}(X_0)) \quad (3.4.10)$$

where, thanks to Prop. 3.16 and Prop. 3.12, the middle term is canonically identified with $p_* i_* i! BU_X$ via the commutative diagram

$$p_* i_* i! BU_X \xrightarrow{\sim} p_* BU_X \quad (3.4.11)$$

Using the fact that the closed immersion $X_0 \to X$ is lci, we know that the push-forward $i_*$ preserves perfect complexes [Toe12], and thus provides a map $M^\vee_S(\text{Perf}(X_0)) \to M^\vee_S(\text{Perf}(X)) \simeq p_* BU_X$ that we will still denote as $i_*$. By projection formula this is a map of $p_* BU_X$-modules, and moreover it fits into a commutative 2-simplex

$$M^\vee_S(\text{Perf}(X_0)) \xrightarrow{i_*} p_* i_* i! BU_X \xrightarrow{\sim} p_* BU_X \quad (3.4.12)$$

**Remark 3.17.** In the context of the diagram (3.4.9), $i_*$ preserves perfect complexes and it follows from Remark 3.13 and the same arguments used in the discussion of [CD12,
Section 13.4.1] that the element $u$ of the Remark 3.14 corresponds to the element $i_\ast(\mathcal{O}_Z)$ in $K$-theory with support $K_Z(X)$.

One can now put together the exact sequence (3.4.10), the localization sequence for the $j^\ast$-pullback (3.4.4) and the localization sequence for the $i^\ast$-pullback 

$$p_\ast j_\sharp \mathbb{B}U \to p_\ast \mathbb{B}U_X \to p_\ast i_\ast \mathbb{B}U_Z$$

(3.4.13)

to get a commutative diagram (recall that $Z = t(X_0)$)

$$
\begin{array}{ccc}
\mathcal{M}_S^\psi(\operatorname{Perf}(X_0)) & \longrightarrow & p_\ast i_\ast i^\ast \mathbb{B}U_X \\
\downarrow & & \downarrow \text{\textit{i-\textit{pullback}}} \\
p_\ast j_\sharp \mathbb{B}U & \longrightarrow & p_\ast \mathbb{B}U_X & \longrightarrow & p_\ast i_\ast \mathbb{B}U_Z \\
\downarrow h=i & & \downarrow j^\ast \text{-pullback} & & \\
p_\ast j_\sharp j^\ast \mathbb{B}U_X & & \\
\end{array}
$$

In the next lemma, we will denote by $i^\ast : p_\ast \mathbb{B}U_X \simeq \mathcal{M}_S^\psi(\operatorname{Perf}(X)) \to \mathcal{M}_S^\psi(\operatorname{Perf}(X_0))$ the morphism of $\mathbb{B}U_S$-modules $\mathcal{M}_S^\psi(i^\ast : \operatorname{Perf}(X) \to \operatorname{Perf}(X_0))$.

**Lemma 3.18.** The composition $i^\ast i_\ast : \mathcal{M}_S^\psi(\operatorname{Perf}(X_0)) \to \mathcal{M}_S^\psi(\operatorname{Perf}(X_0))$ is null-homotopic in $\mathbb{B}U_S$-modules. Therefore, the octahedral property ([Lur16, Thm 1.1.2.15 (TR4)]) for the stable $\infty$-category of $\mathbb{B}U_S$-modules gives us a cofiber-fiber sequence of $\mathbb{B}U_S$-modules

$$\mathcal{M}_S^\psi(\operatorname{Sing}(X_0)) \to \mathcal{M}_S^\psi(\operatorname{Perf}(X_0))[1] \oplus p_\ast i_\ast \mathbb{B}U_Z \to \operatorname{cofib}(h : p_\ast j_\sharp \mathbb{B}U \to p_\ast j_\ast \mathbb{B}U_U)$$

(3.4.15)

**Proof.** Recall that in the context of the diagram (3.4.9) we have the cartesian cube (2.3.43), and that, by Remark 2.26, we have 

$$i^\ast i_\ast \simeq v_\ast (pr)^\ast \simeq K(A,0) \boxplus -$$

and 

$$\operatorname{Id}_{X_0} \simeq A \boxplus -$$

Finally, using the cofiber-sequence (2.3.18) we get a cofiber sequence of dg-functors

$$
\begin{array}{ccc}
\operatorname{Id}_{X_0} & \longrightarrow & \operatorname{Id}_{X_0} \\
0 & \longrightarrow & i^\ast i_\ast \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0 \\
\end{array}
$$

(3.4.16)

which shows that $i^\ast i_\ast$ is zero in $K$-theory of $X_0$. Finally, to conclude that the induced map $i^\ast i_\ast$ in $\mathbb{B}U_S$-modules is zero, it is enough now to base change the diagram (2.3.43) along any $Y$ smooth affine over $S$ and apply the same arguments. \qed
Remark 3.19. In the case where \( Z = t_0(X_0) = X_0 \) (as will be the case considered later in section 4.2), the conclusion of the lemma 3.18 tells us that the class \( u \in K_Z(X) \) of the Remark 3.17 factors as a map of \( p_*i_*B_UZ \)-modules

\[
p_*i_*B_UZ \xrightarrow{\theta} \text{cofib}(p_*j_*BU_X \to p_*j_*BU_U)[-1] \xrightarrow{p_*i_*} BU_X
\]

where the last map is the boundary map of the localization sequence. Using Bott periodicity (Remark 3.1), the map \( \theta \) can also be interpreted as a map of \( p_*BU_X \)-modules

\[
p_*i_*B_UZ(-1)[-2] \simeq p_*i_*B_UZ \xrightarrow{\theta} \text{cofib}(p_*j_*BU_X \to p_*j_*BU_U)[-1]
\]

This is the same as a map of \( p_*BU_X \)-modules

\[
p_*i_*B_UZ \xrightarrow{\theta} \text{cofib}(p_*j_*BU_X \to p_*j_*BU_U)(1)[1]
\]

We will see in Remark 4.6 that \( \theta \) is a K-theoretic version of the cycle class defined in [Del77, Cycle §2.1] associated to the closed pair \( (Z, X) \).

3.5. The \( BU \)-motives of 2-periodic complexes. As a first application of the cofiber-fiber sequence (3.4.15) we compute the motivic \( BU \)-module of the \( dg \)-category of 2-periodic complexes.

Proposition 3.20. There is a canonical equivalence of \( BU_S \)-modules

\[
M^0_S(S, 0) \simeq BU_S \oplus BU_S[1]
\]

As explained in Prop. 2.31, \( Sing(S, 0) \) is equivalent to \( Perf(A[u, u^{-1}]) \) and obtained as the cofiber sequence in \( dgecat_{idem}^A \)

\[
Perf(A[e]) \subseteq \text{Coh}\,^b(A[e]) \to Sing(S, 0)
\]

In this case, as \( U \) is empty and \( S \) is regular (so that \( Sing(S, 0) \simeq Sing(S_0) \) where \( S_0 \) is the derived pullback of \( 0_S : S \to \mathbb{A}^1_S \) along itself), the cofiber-fiber sequence of (3.4.15) gives an equivalence

\[
M^0_S(S, 0) \simeq M^0_S(S_0) \simeq M^0_S(Perf(A[e]))[1] \oplus BU_S
\]

and we are left to show that \( M^0_S(Perf(A[e])) \) is equivalent to \( BU_S \). But this follows from Prop. 3.9, and the following remark applied to the graded algebra \( A[e] \).
Remark 3.21. Let $R = \bigoplus_{i \leq 0} R_i$ be a graded algebra over $A$ concentrated in non-positive degrees. Then the canonical inclusion and projection
\[ q : R_0 \to R \quad \text{pr} : R \to R_0 \]
seen as maps of $A$-dg-categories with single objects, are $A_A^1$-homotopy inverse. Indeed, the composition $R_0 \to R \to R_0$ is the identity. For the other composition notice that by definition the grading in $R$ is the data of a map of $A$-modules
\[ R \to R \otimes_A A[t] \]
sending an element $r \in R$ of degree $i \leq 0$ to $r \otimes t^{-i}$. This map provides the required $A_A^1$-homotopy between the composition $q \circ \text{pr}$ and the identity via the respective evaluations at 0 and 1.

3.6. Rational Coefficients and Tate-2-periodicity. $\text{SH}^\otimes$ carries a canonical action of the $\infty$-category of spectra $\text{Sp}^\otimes$ seen as a constant system of monoidal categories indexed by $\text{Sch}_{/S}$. More precisely, since for any $X$, $\text{SH}^\otimes_X$ is a symmetric monoidal $\infty$-category and $\text{SH}_X$ is stable and presentable, there exists a unique (up to a contractible space of choices) natural transformation $a : \text{Sp}^\otimes \to \text{SH}^\otimes$ of $\infty$-functors $\text{Sch}_{/S}^{\text{op}} \to \text{CAlg}(\text{Pr}_{\text{Stb}}^L)$: this follows from the universal property of the smash product symmetric monoidal structure on spectra [Lur16, Cor. 4.8.2.19]. This provides for each $\Lambda \in \text{CAlg}(\text{Sp})$, a family of commutative algebra objects $\Lambda_X \in \text{CAlg}(\text{SH}_X)$ indexed by $X \in \text{Sch}_{/S}$ and stable under pullbacks. A perhaps more concrete, though less structured way of understanding the natural transformation $a : \text{Sp}^\otimes \to \text{SH}^\otimes$ is as follows. If $\Lambda \in \text{CAlg}(\text{Sp})$ and $X \in \text{Sch}_{/S}$, then we can identify $\Lambda$ with the constant object $\Lambda^s_X$ in the stable homotopy category $\text{SH}^s_X$ of schemes where only the simplicial suspension $\Sigma^1_s$ has been inverted (this is because we already have bonding maps $\Sigma^1_s \wedge \Lambda_n \to \Lambda_{n+1}$, as $\Lambda \in \text{Sp}$). Now, $\text{SH}_X$ can be constructed by further inverting the Tate suspension in $\text{SH}^s_X$, and we define $a(X)(\Lambda) = \Lambda_X$ as the image of $\Lambda^s_X$ via the canonical functor $\text{SH}^s_X \to \text{SH}_X$. Note that, in particular, via the natural transformation $a$, $\text{SH}$ is tensored over $\text{Sp}$ (see [Lur16, Rmk. 4.8.2.20]), and by the same discussion as for $B\underline{U}$-modules above, we thus have a system of categories $\text{Mod}_\Lambda(\text{SH})^\otimes$ together with a realization map
\[ \otimes \Lambda : \text{SH}^\otimes \to \text{Mod}_\Lambda(\text{SH})^\otimes. \]

Remark 3.22. For each $X \in \text{Sch}_{/S}$, the category $\text{Mod}_{\Lambda_X}(\text{SH}_X)$ can be identified with the tensor product in $\text{Pr}_{\text{Stb}}^L$
\[ \text{SH}_X \otimes_{\text{Sp}} \text{Mod}_\Lambda(\text{Sp}) \]
This is follows from [Lur16, Thm. 4.8.4.6 and Section 4.5.1].
Remark 3.23. The natural transformation of (3.6.1) is the universal \(\Lambda\)-linear realization. Indeed, recall that, by definition, \(\Lambda\)-linear stable \(\infty\)-categories are objects in \(\text{Mod}_{\text{Mod}_A(Sp)^\otimes(\text{Pr}_L^{\text{Stb}})}\). In particular, if \(R : \text{SH}^\otimes \to D^\otimes\) is a realization where \(D^\otimes\) takes values in \(\Lambda\)-linear categories, the universal property of base change [Lur16, 4.5.3.1] tells us that \(R\) factors in a unique way by a \(\Lambda\)-linear realization \(R : \text{SH}^\otimes \otimes_{\text{Sp}} \text{Mod}_A(Sp)^\otimes \to D^\otimes\).

Let \(\Lambda = HQ\) be the Eilenberg-Maclane spectrum representing rational singular cohomology. It has the structure of algebra-object in \(\text{CAlg}(Sp)\) given by the cup product in cohomology. This is an idempotent ring-object, in the sense that the multiplication map \(HQ \otimes HQ \simeq HQ\) is an equivalence (i.e. the localization at \(HQ\) is smashing). Therefore the universal \(Q\)-linear realization

\[- \otimes Q := - \otimes HQ : \text{SH}^\otimes \to \text{Mod}_{HQ}(\text{SH}^\otimes)\]

identifies \(\text{Mod}_{HQ}(\text{SH})\) with the full subcategory \(\text{SH}_Q\) of \(\text{SH}\) spanned by non-torsion objects.

Moreover, whenever \(S\) is of finite Krull dimension, this localization is strongly compatible with all the six operations in the sense that the natural transformations (A.0.8) are natural isomorphisms. This follows from the arguments in the proof of [Ayo14, A.14].

Following the discussion in Section 3.3, rationalization carries over to \(BU\)-modules

\[- \otimes Q : \text{Mod}_{BU}(\text{SH}^\otimes) \to \text{Mod}_{BU Q}(\text{SH}_Q^\otimes) \simeq \text{Mod}_{BU Q}(\text{SH}^\otimes)\]  
(3.6.2)

where \(BU_Q := BU_S \otimes HQ\) and the last equivalence follows from [Lur16, 3.4.1.9].

Thanks to the strong compatibility with the six operations, all the constructions performed in the previous sections at the level of \(BU_S\)-modules can now be repeated after rationalization, without changing the results. This follows because the realization at the level of modules is determined by the underlying realization via the forgetful functors.

The main reason why we are interested in passing to rational coefficients is the following result:

**Proposition 3.24.** Let \(X\) be any scheme of finite Krull-dimension. Then the morphism \(u : 1_X(1)[2] \to BU_X \otimes Q\) of (3.1.2) induces an equivalence of commutative algebra objects

\[\text{MB}_X(\beta) := \text{Free}(\text{MB}_X(1)[2])[u^{-1}] \xrightarrow{\sim} BU_X Q := BU_X \otimes Q\]  
(3.6.3)

\[\text{This is equivalent to the rational sphere spectrum.}\]
where $M_B X$ is the commutative algebra-object representing Beilinson’s motivic cohomology of [CD12, Def. 14.1.2]\textsuperscript{20}. In particular, if $X$ is a geometrically unibranch excellent scheme, then we can replace Beilinson’s motivic cohomology by the spectrum $MQ_X$ representing rational motivic cohomology, via the equivalence of commutative algebra objects

$$M_B \rightarrow MQ_X$$

induced by the Chern character.

Proof. This is already proven in [CD12, 14.2.17, 16.1.7] using the results of Riou in [Rio10] on the $\gamma$-filtration. The only remaining issue is to construct this as an equivalence of $E_\infty$-algebras in motives. But this follows just by describing the image of $u$ and showing it is invertible. But again this is verified at the classical level.

The fact that the map (3.6.4) corresponds to the Chern character is explained in [Rio10, Def 6.2.3.9 and Rem 6.2.3.10].

Remark 3.25. In practice the hypothesis that the schemes are excellent and geometrically unibranch will be verified in the cases that interests us, namely, when $S$ is a complete Henselian trait and $X$ is a regular scheme of finite type over $S$. This follows because complete local rings are excellent [Gro65, Scholie 7.8.3 page 214] and $S$ being a discrete valuation ring it is regular, so it is normal (in fact in dimension 1 the two are equivalent) and therefore by a direct checking of the definitions, geometrically unibranch. See [CD12, Thm 8.3.30]. Moreover, following [Gro65, Prop. 7.8.6 page 217] if $X \rightarrow S$ is a scheme of finite type over $S$ with $S$ excellent then $X$ is excellent. Again if $X$ is regular, it is normal [Ser65, Cor 3 , IV-39] and therefore it is geometrically unibranch.

Following from [CD12, 14.1.6] one obtains for any $X$ a canonical isomorphism

$$BU_{X,Q} \otimes M_B X \simeq BU_{X,Q}$$

and the realization (3.6.2) is equivalent to

$$\text{Mod}_{BU}(SH) \rightarrow \text{Mod}_{BUQ}(\text{Mod}_{MB}(SH))$$

which is therefore strongly compatible with all the six operations.

\textsuperscript{20}The structure of commutative algebra object of $M_B$ in the symmetric monoidal $\infty$-category $SH$ follows from the equivalences [CD12, (5.3.35.2)], the definition [CD12, 15.2.1] and [CD12, 14.2.9]. The combination of these results characterizes $M_B$-modules as a monoidal reflexive localization of $SH_Q$, so that as explained in [CD12, 14.2.2], $M_B$ is the image of the monoidal unit in $SH_Q$ under a monoidal localization functor and a lax monoidal inclusion, so, it acquires a natural structure of commutative algebra object.
3.7. \(\ell\)-adic realization. In this section we discuss the \(\ell\)-adic realization of \(\text{BU}\)-modules and dg-categories.

**Assumption 3.26.** Throughout this section we assume that \(S\) is an excellent scheme of dimension less or equal than one and we work with \(\text{Sch}/S\) the category of schemes of finite type over \(S\). We fix \(\ell\) a prime invertible in \(S\).

We describe below, working over schemes under the assumption 3.26, the construction of a monoidal realization functor

\[
\mathbf{R}^\ell : \text{Mod}_{\text{BU}}(\text{SH}) \otimes \rightarrow \text{Sh}_{\mathbb{Q}_\ell}(-) \otimes
\]

where for each \(X \in \text{Sch}/S\), \(\text{Sh}_{\mathbb{Q}_\ell}(X)\) denotes the symmetric monoidal \(\infty\)-category of \(\text{Ind-constructible} \ \mathbb{Q}_\ell\)-adic sheaves on \(X\). Let us first explain what is the definition of \(\text{Sh}_{\mathbb{Q}_\ell}(X)\) and the reason for the assumption 3.26 We will need to consider for each \(n \geq 0\), \(\text{Sh}^\ell(X_{\text{ét}}, \mathbb{Z}/\ell^n)\) the \(\infty\)-category of étale sheaves with \(\mathbb{Z}/\ell^n\)-coefficients. We will denote by \(\text{Sh}^\ell(X_{\text{ét}}, \mathbb{Z}/\ell^n)\) the full subcategory of étale sheaves with \(\mathbb{Z}/\ell^n\)-coefficients spanned by constructible sheaves as in [LG14, 4.2.5] or in [CD16, page 598]. We have the following crucial result:

**Proposition 3.27.** Let \(S\) be a base scheme satisfying the Assumption 3.26. Then for any scheme \(X\) of finite type over \(S\), the étale topos of \(X\) is of finite cohomological dimension and the étale cohomological dimension of its points are uniformly bounded. In this case \(\text{Sh}^\ell(X_{\text{ét}}, \mathbb{Z}/\ell^n)\) is compactly generated and its compact objects are exactly the constructible sheaves.

**Proof.** The first statement follows as in [CD16, Prop. 1.1.5] as explained in [CD16, Remark 1.1.6]. The second statement now follows exactly as in [LG14, 4.2.2] replacing the use of [LG14, Lemma 4.1.13] by the first statement. \(\square\)

In this case we consider the limit of \(\infty\)-categories

\[
\text{Sh}^\ell(X) := \lim_{n \geq 0} \text{Sh}^\ell(X_{\text{ét}}, \mathbb{Z}/\ell^n)
\]

and it follows that because of Prop. 3.27 and the same arguments as in [LG14, 4.3.17] that \(\text{Sh}^\ell(X)\) is an \(\infty\)-categorical enhancement of the derived category of constructible \(\ell\)-adic sheaves on \(X\), in the sense that its homotopy category is equivalent (as a triangulated category) to the so-called constructible derived \(\ell\)-adic category \(D_c(X, \mathbb{Z}_\ell)\) of [Eke90] and [BBD82].

As a second step, we define the \(\infty\)-category of \(\ell\)-adic sheaves on \(X\) as the Ind-completion of \(\text{Sh}^\ell(X)\) ([LG14, 4.3.26])

\[
\text{Sh}_\ell(X) := \text{Ind}(\text{Sh}^\ell(X))
\]
Therefore one should think of $\mathcal{SH}(X)$ as the derived $\infty$-category of Ind-constructible $\ell$-adic sheaves on $X$. Finally, we define $\mathcal{SH}_{\mathbb{Q}_\ell}(X) := \mathcal{SH}(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Note that $\mathcal{SH}_{\mathbb{Q}_\ell}(X)$ can also be identified as the full subcategory of $\mathcal{SH}(X)$ spanned by those objects $F$ such that the natural morphism $F \mapsto F[\ell^{-1}]$ is an equivalence. Also note that the limit (3.7.3) can be taken inside the theory of symmetric monoidal small idempotent-complete stable and $\mathbb{Z}_\ell$-linear $\infty$-categories. Therefore, by working inside $\text{CAlg(Pr}_L\text{Mod}_{\mathbb{Z}_\ell}(\text{Pr}^L))$, and taking first Ind-completion, and then applying $- \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ (or, equivalently, inverting $\ell$), we get a symmetric monoidal small idempotent-complete stable and $\mathbb{Q}_\ell$-linear structure on $\mathcal{SH}_{\mathbb{Q}_\ell}(X)$. We will denote this monoidal structure by $\mathcal{SH}_{\mathbb{Q}_\ell}(X) \otimes$.

Remark 3.28. The discussion in [LG14, Section 4] is written for quasi-projective schemes over a field as it requires the étale topos of $X$ to be of finite cohomological dimension [LG14, Lemma 4.1.13]. In our case this follows from the Assumption 3.26 and Prop. 3.27.

Remark 3.29. The construction of an $\infty$-functor $X \mapsto \mathcal{SH}_{\mathbb{Q}_\ell}(X)^\otimes$ can be obtained using the arguments of [Robdf, Chapter 9].

The monoidal realization (3.7.1) could be obtained using the universal property of $\text{SH}$ proved in [Rob15]. However, we will need to show that it is strongly compatible with all the six operations and with the classical notion of Tate twists. For this purpose, we describe an alternative construction of (3.7.1) using results of [CD16]. Once (3.7.1) is available and strongly compatible with all operations, we can then pass to $\text{BU}_{\mathbb{Q}}$-modules and deduce that the composition

$$
\text{Mod}_{\text{BU}(\text{SH})} \xrightarrow{(3.6.5)} \text{Mod}_{\text{BU}_{\mathbb{Q}}(\text{Mod}_{\text{MB}(\text{SH})})} \xrightarrow{(3.7.1)} \text{Mod}_{R(\text{BU}_{\mathbb{Q}})(\text{Sh}_{\mathbb{Q}_\ell}(-))}
$$

is again compatible with all the six operations and twists. **Construction of (3.7.1).**

Let us now review the construction of (3.7.1). This can be done as in [CD16] using the theory of $h$-motives. Recall from [Voe96, Def 3.1.2] that the $h$-topology on Noetherian schemes is the topology whose covers are the universal topological epimorphisms. It is the minimal topology generated by open coverings and proper surjective maps (see for the case of excellent schemes [Voe96, Def 3.1.2]). In [CD16] the authors constructed for any noetherian scheme $X$ and any ring $R$ a theory of $h$-motives, $\text{DM}_h(X,R)$. See [CD16, Section 5.1]. The constructions in loc.cit can be formulated in the language of higher categories, using the arguments and steps of [Rob15] and an $\infty$-functor

$$
\text{DM}_h(-,R)^\otimes : \text{Sch}^{op}_{/S} \rightarrow \text{CAlg(Pr}^L)
$$

can be provided as in [Robdf, Chapter 9]. Following [CD16, 5.6.2] and Prop. A.2 this $\infty$-functor satisfies all the formalism of the six operations over Noetherian schemes of finite Krull dimension. We now recall how to relate $h$-motives both to the r.h.s and l.h.s
of (3.7.1). Let \( R \) be the localization of \( \mathbb{Z} \) at the prime \( \ell \). To understand the r.h.s of (3.7.1) we use a form of rigidity theorem given by [CD16, Thm 5.5.3 and Thm 4.5.2]: for any Noetherian scheme \( X \), \( \ell \) invertible in \( \mathcal{O}_X \) and for each \( n \geq 0 \), we have a monoidal equivalence
\[
DM_h(X, R/\ell^n) \simeq Sh(X_{\et}, R/\ell^n)
\]
with the last being the standard infinite-category of \( \ell^n \)-torsion étale sheaves on \( X \). This equivalence is compatible with all the six operations over Noetherian schemes of finite Krull dimension. Following [CD16, Thm 6.3.11] for all Noetherian schemes of finite dimension, (3.7.5) restricts to an equivalence
\[
DM_{h,lc}(X, R/\ell^n) \simeq Sh_c((X)_{\et}, R/\ell^n)
\]
where on the l.h.s we have the full subcategory of locally constructible objects of [CD16, Def. 6.3.1]. Thanks to the uniformization results of Gabber (see [CD16, 6.3.15] for the l.h.s and [IL014, Exposé 0 Thm 1] for the r.h.s) the constructions \( X \mapsto DM_{h,lc}(X, R/\ell^n) \) are stable under the six operations when restricted to quasi-excellent noetherian schemes of finite dimension.

**Remark 3.30.** Via (3.7.5) motivic Tate twists are sent to the usual \( \ell \)-adic twists given by the roots of unity. This is a consequence of the Kummer exact sequence as explained in [CD16, Section 3.2].

As a result, the equivalences (3.7.6) assemble to an equivalence of infinite-functors, strongly compatible with all the six operations and twists
\[
DM_{h,lc}(-, R/\ell^n) \simeq Sh^c((-)_{\et}, R/\ell^n)
\]
whenever \( \ell \) is invertible in \( S \)

**Remark 3.31.** Under the assumption 3.26 and because of Prop. 3.27 and [CD16, Prop 6.3.10], the notion of locally constructible objects in h-motives coincides with the notion of constructible of [CD16, 5.1.3] which also coincides with the notion of compact object [CD16, Thm 5.2.4].

As a conclusion to this discussion, (3.7.7) provides an equivalence
\[
\lim_{n \geq 0} DM_{h,lc}(-, R/\ell^n) \simeq \lim_{n \geq 0} Sh^c((-)_{\et}, R/\ell^n)
\]
Now, the l.h.s of (3.7.1) is related to the theory of h-motives via the combination of [CD16, Thm 5.2.2] and [CD12, 14.2.9]: when \( R = \mathbb{Q} \) we have an equivalence of infinite-functors defined on Noetherian schemes of finite Krull dimension
\[
DM_h( -, \mathbb{Q}) \simeq Mod_{MB}(SH)
\]
strongly compatible with the six operations. We recall that for each scheme $X$ the $\infty$-category $\mathrm{SH}_X$ is compactly generated and so is $\mathrm{Mod}_{\mathbf{MB}_X}(\mathrm{SH}_X)$. See [Robdf, Section 4.4. and Prop.3.8.3]. Thanks to [CD16, Prop. 6.3.3 and Thm 5.2.4], the equivalence (3.7.9) identifies the compact objects of $\mathrm{Mod}_{\mathbf{MB}_X}(\mathrm{SH}_X)$ with the subcategory of locally constructible objects $\mathrm{DM}_{h,lc}(-, \mathbb{Q})$ as defined in [CD16, 5.1.3]. By [CD16, 6.2.14] it is stable under all the six operations. Having these characterizations of both the r.h.s and l.h.s of (3.7.1), in order to achieve the construction of the natural transformation (3.7.1), we need to exhibit a natural transformation of $\infty$-functors with values in small stable idempotent complete $R[\ell^{-1}] \simeq \mathbb{Q}$-linear $\infty$-categories

$$
\mathrm{DM}_{h,lc}(-, \mathbb{Q}) \simeq \mathrm{DM}_{h,lc}(-, R) \otimes R[\ell^{-1}] \to (\lim_{n \geq 0} \mathrm{DM}_{h,lc}(-, R/\ell^n)) \otimes R[\ell^{-1}]
$$

For that purpose we use the results of [CD16] that explain the $\ell$-adic realization functor (3.7.1) as an $\ell$-adic completion of $h$-motives. The system of base changes along the maps of rings $R \to R/\ell^n$ produces natural transformations $\mathrm{DM}_h(-, R) \to \mathrm{DM}_h(-, R/\ell^n)$ and by the standard procedure one can construct the data of a cone over the diagram indexed by $n \geq 0$ and obtain a natural transformation between the $\infty$-functors with values in presentable stable $R$-linear $\infty$-categories

$$
\mathrm{DM}_h(-, R) \to \lim_{n \geq 0} \mathrm{DM}_h(-, R/\ell^n)
$$

(3.7.10)

It follows from the same arguments as in [LG14, 4.3.9] that this homotopy limit identifies with the construction $\mathrm{DM}_h(-, \hat{R}_\ell)$ of [CD16, Def 7.2.1] and from [CD16, Thm 7.2.11] that it commutes with all the six operations over Noetherian schemes of finite Krull dimension. Moreover, by [CD16, 7.2.16], it restricts to a natural transformation between locally constructible objects

$$
\mathrm{DM}_{h,lc}(-, R) \to \lim_{n \geq 0} \mathrm{DM}_{h,lc}(-, R/\ell^n)
$$

(3.7.11)

again compatible with all operations. Following the discussion in [CD16, Section 7.2.18, Prop. 7.2.19, Thm 7.2.21], using the Prop. 3.27, one can mimic the arguments of [LG14, 4.3.17] to deduce that the homotopy category of the r.h.s recovers the classical derived category of constructible $\ell$-adic sheaves of [BBD82] and [Eke90]. Finally, the realization $R^\ell$ of (3.7.1) is defined via the composition
\[
\text{Mod}_{\text{MB}}(\text{SH}) \xrightarrow{(3.7.9)} \text{Ind}(\text{DM}_{h,c}(-, R) \otimes_R \mathbb{Q}) \xrightarrow{(3.7.11)} \text{Ind}(\lim_{n \geq 0} \text{DM}_{h,c}(-, R/\ell^n) \otimes_R \mathbb{Q}) \xrightarrow{(3.7.8)} \text{Sh}_{\mathbb{Q}_\ell}(-)
\]
and by the preceding discussion it is strongly compatible with all the six operations and Tate twists.

**Remark 3.32.** As the \(\ell\)-adic monoidal realization functor \(R^\ell : \text{Mod}_{\text{MB}}(\text{SH})^\otimes \to \text{Sh}_{\mathbb{Q}_\ell}(-)^\otimes\) is monoidal, for any scheme \(X\) we have \(R^\ell(\text{MB}_X) \simeq \mathbb{Q}_{\ell,X}\), the monoidal unit of \(\mathbb{Q}_{\ell}\)-adic sheaves over \(X\). As it is a left adjoint and commutes with Tate twists, Prop. 3.24 implies that that
\[
R^\ell(\text{BU}_X, \mathbb{Q}_{\ell}) \simeq \text{Free}(\mathbb{Q}_{\ell,X}(1)[2])[u^{-1}] \simeq \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}_{\ell,X}(n)[2n] =: \mathbb{Q}_{\ell,X}(\beta)
\]
By Remark 3.30, \(\mathbb{Q}_{\ell,X}(i)\) are the usual \(\ell\)-adic Tate twists given by the roots of unity.

In particular, the extension (3.7.4) of \(R^\ell\) to \(\text{BU}\)-modules takes values in Tate-twisted 2-periodic objects inside \(\text{Sh}_{\mathbb{Q}_\ell}(X)\), i.e. objects \(E\) together with an equivalence \(E \simeq E(1)[2]\).

**Notation 3.33.** Throughout the rest of this paper we will write
\[
R^\ell : \text{Mod}_{\text{BU}}(\text{SH}) \to \text{Mod}_{\mathbb{Q}_{\ell}(\beta)}(\text{Sh}_{\mathbb{Q}_\ell}(-))
\]
to denote the natural transformation obtained via the composition (3.7.4). As already observed, it is strongly compatible with all six operations and Tate twists.

**Remark 3.34.** Note that if \(p : X \to S\) is a smooth finite type morphism of schemes, and we denote by \([X] := p_!(1_X) \in \text{SH}_S\) its motive over \(S\), then
\[
\text{Map}_{\text{Sh}_{\mathbb{Q}_\ell}(S)}(R^\ell([X] \otimes \text{BU}_S), \mathbb{Q}_{\ell,S}) \simeq H^\bullet_{\ell}(X, \mathbb{Q}_{\ell}) \otimes \mathbb{Q}_{\ell}(\beta),
\]
where \(H^\bullet_{\ell}(X, \mathbb{Q}_{\ell}) \simeq \text{Map}_{\text{Sh}_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}, \mathbb{Q}_{\ell,X})\) denotes the \(\ell\)-adic cohomology of \(X\). In other words, the \(\mathbb{Q}_{\ell,S}\)-dual of \(R^\ell([X] \otimes \text{BU}_S)\) is a Tate 2-periodized version of the \(\ell\)-adic cohomology of \(X\). Note that, instead, if we denote by \(R^\ell : \text{Mod}_{\text{MB}}(\text{SH})^\otimes \to \text{Sh}_{\mathbb{Q}_\ell}(-)^\otimes\) the realization functor (3.7.1), we have
\[
\text{Map}_{\text{Sh}_{\mathbb{Q}_\ell}(S)}(R^\ell([X] \otimes \text{MB}_S), \mathbb{Q}_{\ell,S}) \simeq H^\bullet_{\ell}(X, \mathbb{Q}_{\ell}).
\]
In the same situation, we have
\[
R^\ell(\mathcal{M}(\text{Perf}(X))) \simeq p_!(\mathbb{Q}_{\ell,X}(\beta)).
\]
This follows from Prop. 3.9, projection formula, and the fact that \( p_\ast \) commutes with \( \ell \)-adic realization. In other words, the \( \ell \)-adic realization of rationalized \( M^\wedge_S(\text{Perf}(X)) \) is equivalent to the Tate 2-periodized version of the \( \ell \)-adic cohomology of \( X \) relative to \( S \).

4. Vanishing Cycles and Singularity Categories

In this section we prove our main Theorem 4.11 which establishes the link between the results of the previous section and the theory of vanishing cycles. Namely, it says that the \( M^\wedge_S \) motive of the dg-category of singularities is a model for the \( \ell \)-adic cohomology of the 2-periodized sheaf of vanishing cycles. Before coming to the precise statement, we will first recall the theory of nearby and vanishing cycles in the motivic setting.

4.1. Nearby and Vanishing Cycles. In this section we recall the formalism of nearby and vanishing cycles in the \( \ell \)-adic setting as presented in [MR072]. More recently, a motivic formulation was developed by Ayoub in [Ayo07b]. There are several technical steps required to express the formalism of [Ayo07b] in a higher categorical setting. We provide these details in Appendix A of this paper.

Throughout this section we fix a diagram of schemes

\[
\eta \xrightarrow{j_\eta} \pi \xleftarrow{i_\sigma} \sigma
\]

with \( S \) an excellent henselian trait, namely, the spectrum of a an excellent henselian discrete valuation ring \( A \), with uniformizer \( \pi \), generic point \( \eta = \text{Spec}(K) \) and closed point \( \sigma = \text{Spec}(A/m) \) with \( m = (\pi) \) the maximal ideal and \( k := A/m \) is a perfect field of exponential characteristic \( p \geq 0 \). The pair \( (\eta, \sigma) \) forms a closed-open complement pair (and the maps are, respectively, an open and a closed immersion). In practice we will take \( S \) to be the spectrum of a complete discrete valuation ring. Roughly speaking, the scheme \( S \) plays the role of a formal disk, \( \sigma \) of the center of the disk and \( \eta \) of the punctured disk (this is quite precise in the equicharacteristic zero case). We will say that a henselian trait is strictly local if \( k \) is algebraically closed.

Remark 4.1. The choice of a uniformizer \( \pi \in A \) defines a map \( \pi : S \to A^1_S \). In this case we have two cartesian diagrams

\[
\begin{array}{c}
\eta \xrightarrow{j_\eta} \pi \xleftarrow{i_\sigma} \\
\downarrow \quad \downarrow \pi \\
\mathbb{G}_{m,S} \xleftarrow{j_0} A^1_S \xrightarrow{i_0} S
\end{array}
\]

which allow us to reduce ourselves to working over an affine line, even in mixed characteristic. Notice that both diagrams are in fact derived fiber products. Indeed, for the left
This diagram is immediate because the inclusion of $G_{m,S}$ is an open immersion, while for the diagram on the right we find that the derived tensor product is given by the spectrum of the commutative differential graded algebra

$$0 \longrightarrow A \xrightarrow{\pi} A \longrightarrow 0$$

which is, in fact, quasi-isomorphic to $A/\pi$, since $\pi$ is a non-zero divisor.

In what follows, we fix:

- a separable closure $\overline{k}$ of $k$ (inside a fixed algebraic closure), and denote by $\overline{S} := S(\overline{\sigma})$ the strict localization of $S$ at the corresponding geometric point $\overline{\sigma} = \text{Spec}(k)$ (which is localized at $\sigma$): in other words $\overline{S}$ the spectrum of the strict henselization of $A$ along $k \hookrightarrow \overline{k}$. Note that $\overline{S}$ is now a strictly henselian trait, with closed point $\overline{\sigma}$ and fraction field $K_{\text{unr}}$ a maximal unramified extension of $\overline{K}$ (see [Ser62, Chapitre II, §2 Prop 3 and Ex.4]). We set $\eta^\text{unr} = \text{Spec}(K_{\text{unr}})$ and denote as $j_{\eta^\text{unr}} : \eta^\text{unr} \rightarrow \overline{S}$ the corresponding open immersion.
- a separable closure $\overline{K}$ of $K_{\text{unr}}$ (inside a fixed algebraic closure), and put $\overline{\eta} := \text{Spec} \overline{K}$ and $j_{\overline{\eta}} : \overline{\eta} \rightarrow \overline{S}$.
- $\eta^t = \text{Spec}(K^t)$ a maximal tamely ramified extension of $K$ inside $\overline{K}$.

All this information fits in a commutative diagram

$$\xymatrix{ & \bar{\eta} \ar[d]_{u_{\eta}} \ar[r]^{j_{\bar{\eta}}} & \eta^t \ar[d]_{u_{\eta}} & S_{\bar{\sigma}} \ar[d]_{\eta} \ar[r]^{i_{\bar{\sigma}}} & S_{\sigma} \ar[d]_{u_{\sigma}} \ar[l]_{j_{\eta}} & \bar{\sigma} \ar[l]_{\bar{i}_{\sigma}} \ar[d]_{i_{\bar{\sigma}}} \ar[l]_{u_{\sigma}} \\ \bar{\eta} \ar[r]_{j_{\bar{\eta}}} & \bar{S} & S \ar[l]_{j_{\eta}} & S_{\sigma} \ar[l]_{i_{\bar{\sigma}}} \ar[r]_{u_{\sigma}} & \bar{\sigma} }$$

which the reader should keep in mind throughout this section.

We also recall the existence of an exact sequences of groups (see [Ser62, Chap. III §5 Thm 2, Thm 3, Cor 1])

$$1 \rightarrow I := \text{Gal}(\bar{\eta}/\eta^\text{unr}) \rightarrow \text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{\sigma}/\sigma) \rightarrow 1$$

where $I$ is inertia group, which fits in a exact sequence (see [Ser62])

$$1 \rightarrow 1$$
where $I_t$ is the tame inertia group, isomorphic to $\lim_{(n,p)=1} \mu_n$ where $\mu_n$ is the group of $n$th-roots of unit in $K^{unr}$.

Consider a map $p : X \to S$. We recall the definition of the nearby and vanishing cycles in the $\ell$-adic setting. Consider the commutative diagram:

$$
\begin{array}{ccc}
X_{\bar{j}} & \xrightarrow{\bar{j}} & X_{\bar{\sigma}} \\
\downarrow^{v_{\bar{\eta}}} & & \downarrow^{v_{\bar{\sigma}}} \\
X_{\eta} & \xrightarrow{p_{\bar{\eta}}} & X_{\sigma} \\
\downarrow^{p_{\bar{\eta}}} & & \downarrow^{p_{\bar{\sigma}}} \\
\eta & \xrightarrow{\bar{\eta}} & \bar{S} \\
\downarrow^{\bar{\eta}} & & \downarrow^{\bar{\sigma}} \\
\eta_{\bar{\eta}} & \xrightarrow{\bar{\eta}} & \bar{S}_{\bar{\sigma}} \\
\end{array}
$$

(4.1.7)

where the rightmost base square, the two front-faces, the two back-faces, and the central transverse square are all cartesian (the maps $v_{\bar{\eta}}$ and $v_{\eta}$ are then uniquely induced).

By definition, the object of nearby cycles associated to $p$ and $E \in \text{Sh}_{Q_\ell}(X)$ is given by $\Psi_p(E) := \bar{i}^* \bar{j}_* E_{\bar{\eta}}$ (where $E_{\bar{\eta}} := v_{\bar{\eta}}^* j^* E \simeq \bar{j}^* v^* E$) in the $\infty$-category $\text{Sh}_{Q_\ell}(X_{\bar{\sigma}})^{\text{Gal}(\bar{\eta}/\bar{\eta})}$, of objects in $\text{Sh}_{Q_\ell}(X_{\bar{\sigma}})$ equipped with an equivariant structure with respect to the continuous action of $\text{Gal}(\bar{\eta}/\bar{\eta})$ on $X_{\bar{\sigma}}$ via the map (4.1.5). When the map $p$ is uniquely determined by our context, we will often write $\Psi(E)$ for $\Psi_p(E)$.

**Remark 4.2.** We will not give here the details for a precise definition of the continuous Galois-equivariant $\infty$-category $\text{Sh}_{Q_\ell}(X_{\bar{\sigma}})^{\text{Gal}(\bar{\eta}/\bar{\eta})}$ of the previous paragraph. A precise construction can be obtained using the $\infty$-categorical analogue of the Deligne topos described in [MR073, Exp. XIII]. The ingredient here is the fact that the étale topos of $S$ can be described as a lax limit (in the sense of $\infty$-categories) of the diagram given by the specialization map from the étale topos of the generic point and the étale topos of the closed point.

**Remark 4.3.** Notice that by definition, the inertia group $I$ acts trivially on $X_{\bar{\sigma}}$. In this case, every object of $\text{Sh}_{Q_\ell}(X_{\bar{\sigma}})^{\text{Gal}(\bar{\eta}/\bar{\eta})}$ produces an object in the $\infty$-category of objects in $\text{Sh}_{Q_\ell}(X_{\bar{\sigma}})^{\text{Gal}(\bar{\sigma}/\bar{\sigma})}$ equipped with a continuous action of $I$. 
Set $E_{\bar{\sigma}} := v_{\sigma}^* i^* E$. There is a canonical adjunction morphism

$$sp : E_{\bar{\sigma}} \to \Psi_p(E)$$

that is compatible with the action of $\text{Gal}(\bar{\eta}/\eta)$. On the source, this action comes via (4.1.5). In other words, (4.1.8) is a morphism in the $\infty$-category $\text{Sh}_{\mathbb{Q}_\ell}(X_{\bar{\sigma}})_{\text{Gal}(\bar{\eta}/\eta)}$ of $\ell$-adic sheaves on $X_{\bar{\sigma}}$ endowed with a $\text{Gal}(\bar{\eta}/\eta)$-action.

Now, the object of vanishing cycles $\mathcal{V}_p(E)$ is defined as the cofiber of (4.1.8), i.e.

$$E_{\bar{\sigma}} \to \Psi_p(E) \to \mathcal{V}_p(E)$$

is a cofiber sequence in the $\infty$-category $\text{Sh}_{\mathbb{Q}_\ell}(X_{\bar{\sigma}})_{\text{Gal}(\bar{\eta}/\eta)}$. Again, when the map $p$ is uniquely determined by our context, we will often write $\mathcal{V}(E)$ for $\mathcal{V}_p(E)$.22

4.2. Comparison between Vanishing Cycles and the singularity category. Throughout this section we fix an excellent henselian trait $S = \text{Spec} A$ with uniformizer $\pi$. We also fix $p : X \to S$ a proper flat scheme of finite type over $S$ with $X$ regular. We consider the LG-pair given by $X$ with the function $f$ defined as the composite

$$f := (X \xrightarrow{p} S \xrightarrow{\pi} \mathbb{A}^1_S)$$

where $\pi$ is our fixed uniformizer. We will simply denote this LG-pair by $(X, \pi)$. Our main result establishes a comparison between the motivic realization of the corresponding singularity dg-category $\text{Sing}(X, \pi)$ and the vanishing cycles for the map $p$. We will consider the following commutative diagram with pullback squares

$$\begin{array}{ccc}
X_{\eta} & \xrightarrow{j} & X & \xleftarrow{i} & X_{\sigma} = X_0 \\
p_{\eta} & \downarrow{j_{\eta}} & p & \downarrow{\pi} & p_{\sigma} \\
\eta & \xrightarrow{j_{\eta}} & S & \xleftarrow{i_{\sigma}} & \mathbb{A}^1_S \\
\mathbb{G}_m, S & \xrightarrow{j_0} & \mathbb{A}^1_S & \xleftarrow{i_0} & S \\
q & \downarrow{i_0} & \downarrow{\pi} & \downarrow{\pi} & \downarrow{\pi} \\
S & \xrightarrow{j_0} & S & \xleftarrow{i_0} & S \\
\end{array}$$

which by Remark 4.1 and under the hypothesis that $p$ is flat, are also derived fiber products.

---

22 A more standard notation for $\mathcal{V}_p(E)$ is $\Phi_p(E)$. 

4.2.1. We start this section, by exploring the motive of the derived category of singularities of the pair $(X, \pi)$. Recall from the discussion on motivic realizations of dg-categories, that diagram (3.4.14) and Lemma 3.18 provide a cofiber-fiber sequence of $B_{US}$-modules

$$M_S^X(Sing(X, \pi)) \to M_S^X(Perf(X_0))[1] \oplus p_*i_*j^*B_{US}X \to \text{cofib}(p_*j_2B_{UX_0} \to p_*j_*B_{UX_0}) \tag{4.2.3}$$

where we have $M_S^X(Perf(X_0)) \simeq p_*i_*B_{US}X_0$, since in this case the canonical inclusion $t(X_0) \to X_0$ is an equivalence and we can apply the discussion for schemes of section 3.4..

As we are now working over a local ring $S = \text{Spec}A$, the localization property for SH (see Prop. A.3) and A.2) gives us a cofiber-fiber sequence of $B_{US}$-modules

$$(j_\eta)_*(i_\sigma)^*(M_S^X(Sing(X, \pi))) \to M_S^X(Sing(X, \pi)) \to (i_\sigma)_* \circ (i_\sigma)^*(M_S^X(Sing(X, \pi))) \tag{4.2.4}$$

**Proposition 4.4.** Assume the conditions in the beginning of Section 4.2. Then the first term in the cofiber-sequence (4.2.4) is a zero object. In this case the last map in (4.2.4) is an equivalence of $B_{US}$-modules and the motivic $B_{US}$-module of $M_S^X(Sing(X, \pi))$ is completely determined by its restriction to the residue field. More precisely, it is determined by a cofiber-fiber sequence of $B_{UK}$-modules

$$(i_\sigma)^*(M_S^X(Sing(X, \pi))) \to (p_\sigma)_*(B_{UX_0}[1] \oplus B_{UX_0}) \to (p_\sigma)_*i^*j_*B_{UX_0} \tag{4.2.5}$$

**Proof.** To show that the first term is zero, as $j_\eta$ is fully faithful\(^{23}\), it is enough to apply $j^*$ to the first row of the the diagram (3.4.14) and check it is sent to zero. Indeed, under the hypothesis that $p$ is proper, proper base change (see Prop. A.3 and A.2) gives us a natural equivalence $(j_\eta)_*p_* \simeq (p_\eta)_*j^*$. But again, the localization property tells us $j^* \circ i_* \simeq 0$, so that the first two terms in the first row of (3.4.14) become zero. So does the cofiber $(j_\eta)_*(M_S^X(Sing(X_0)))$.

To describe $(i_\sigma)^*(M_S^X(Sing(X_0)))$ we apply $(i_\sigma)^*$ to the whole diagram (3.4.14). Again because of proper base change, we have a natural equivalence $(i_\sigma)^*p_* \simeq (p_\sigma)_*i^*$. Moreover, as the counit is an equivalence $(i_\sigma)^* \circ (i_\sigma)_* \simeq Id \tag{4.2.6}$ and because $i_\sigma p_*j_2B_{UX_0} \simeq \text{Id}$

\(^{23}\)Follows from smooth base change for $j$ (see Prop. A.3 and A.2).

\(^{24}\)See [Robdf, Rmk 9.4.19]
(p_\sigma)_*i_*j_!BUX_\eta \text{ with } i_*j_! \text{ being always zero, we recover}
\begin{align*}
(p_\sigma)_*BUX_0 \rightarrow (p_\sigma)_*i_*BUX \rightarrow i^*(M^\vee_S(Sing(X_0)))
\end{align*}
(4.2.6)

Finally, we study the image of $M^\vee_S(Sing(X, \pi))$ under the $\mathbb{Q}_{\ell}$-adic realization functor (3.7.14). By construction $R^\ell(M^\vee_S(Sing(X, \pi)))$ carries the structure of Tate-twisted-2-periodic object in $\text{Sh}_{\mathbb{Q}_{\ell}}(S)$. Moreover, given the fact that $p$ is proper, proper base change, the strong compatibility between the six operations and Prop. 4.4 together, imply that the $\mathbb{Q}_{\ell}$-adic sheaf $R^\ell_S(M^\vee_S(Sing(X, \pi)))$ is again determined by its restriction to the residue field via a cofiber-fiber sequence of $R^\ell_k(BU_k)$-modules
\begin{align*}
(i_\sigma)^*(j_0)_*(p_\eta)_*BUX_\eta \rightarrow (p_\sigma)_*i_*j_*BUX_\eta \rightarrow (p_\sigma)_*BUX_0[1] \oplus (p_\sigma)_*BUX_0
\end{align*}
(4.2.7)

\begin{remark}
Notice that, since $R^\ell_{X_0}(BUX_0) \simeq Q_{\ell,X_0}(\beta)$, one has:
\begin{align*}
(p_\sigma)_*(R^\ell_k(BUX_0)) \simeq (p_\sigma)_*p^\ast_{\ell}(Q_{\ell,\sigma}(\beta)) \simeq (p_\sigma)_*(Q_{\ell,X_0}) \otimes Q_{\ell,\sigma}(\beta) \simeq ((p_\sigma)_*(Q_{\ell,X_0}))(\beta)
\end{align*}
where the second equivalence follows from the projection formula (as $p$ is proper) and the last equivalence from the definitions and because the tensor product commutes with colimits separately in each variable.
\end{remark}

\begin{remark}
One can expand the discussion about the fundamental class constructed in the Remark 3.19 in the $\ell$-adic setting. Following the discussion in [Del77, Cycle §2.1] we have a fundamental cycle class $\theta \in H^1(\eta, Q_{\ell}(1))$ whose image under the boundary map of the localization sequence associated to the closed-open complement $(\sigma, S, \eta)$
\begin{align*}
H^1(\eta, Q_{\ell}(1)) \rightarrow H^2_\sigma(S, Q_{\ell}(1))
\end{align*}
(4.2.8)
refines the first Chern class of the conormal bundle of $\sigma$ in $S$ in $H^2(S, \mathbb{Q}_\ell(1))$. As a cohomology class, it can be interpreted as morphism in $\text{Sh}_{\mathbb{Q}_\ell}(\eta)$

$$\theta : \mathbb{Q}_{\ell,\eta} \rightarrow \mathbb{Q}_{\ell,\eta}(1)[1] \quad (4.2.9)$$

See also [ILO14, Exp. XVI Def. 2.3.1, Prop. 2.3.4] and [Fuj00, §1, Prop.1.1.5]. By pulling back to $X_\eta$, we obtain a map in $\text{Sh}_\ell(X_\eta)$

$$\theta : \mathbb{Q}_{\ell,X_\eta} \rightarrow \mathbb{Q}_{\ell,X_\eta}(1)[1]$$

which we can then tensor to get a map of $R^\ell(\mathbb{B}U_{X_\eta})$-modules

$$\theta : R^\ell(\mathbb{B}U_{X_\eta}) \rightarrow R^\ell(\mathbb{B}U_{X_\eta})(1)[1] \quad (4.2.10)$$

By adjunction, (4.2.10) corresponds to a map

$$\theta : \mathbb{Q}_{\ell,S} \rightarrow (j_\eta)_*(p_\eta)_*R^\ell(\mathbb{B}U_{X_\eta})(1)[1] \quad (4.2.11)$$

whose image under $i^*_\sigma$ provides a class

$$\theta : \mathbb{Q}_{\ell,\sigma} \rightarrow i^*_\sigma(j_\eta)_*(p_\eta)_*R^\ell(\mathbb{B}U_{X_\eta})(1)[1] \quad (4.2.12)$$

We establish an identification between (4.2.12) and the $\ell$-adic realization of the class (3.4.19) given in the Remark 3.19. Indeed, using proper base change for $p$ and the fact $i^*_\sigma j_\sigma = 0$, the class $i^*_\sigma(R^\ell(3.4.19))$ corresponds to a map of $(p_\sigma)_*R^\ell(\mathbb{B}U_Z)$-modules

$$(p_\sigma)_*R^\ell(\mathbb{B}U_Z) \rightarrow i^*_\sigma(j_\eta)_*(p_\eta)_*R^\ell(\mathbb{B}U_{X_\eta})(1)[1] \quad (4.2.13)$$

which by adjunction is equivalent to a map in $\text{Sh}_{\mathbb{Q}_\ell}(\eta)$

$$\mathbb{Q}_{\ell,\sigma} \rightarrow i^*_\sigma(j_\eta)_*(p_\eta)_*R^\ell(\mathbb{B}U_{X_\eta})(1)[1] \quad (4.2.14)$$

The identification between (4.2.14) and (4.2.12) follows from the fact that both have their origin in the localization sequence for $(s, S, \eta)$ inducing (4.2.8), combined with the use of the purity isomorphism\textsuperscript{25} for $(\sigma, S)$.

\textsuperscript{25}See (4.2.22) below in the proof of Prop. 4.9
4.2.2. Let $\mathcal{V}_p(\beta) := \mathcal{V}_p(E)$ where $E := R^1_X(BU_X) \simeq \mathbb{Q}_l(\beta)_X$. The formalism of vanishing cycles associated to the diagram (4.2.2) produces a cofiber-fiber sequence in the equivariant derived $\infty$-category $\text{Sh}_{\mathcal{Q}_l}(\bar{\sigma}^{\text{Gal}(\bar{\eta}/\eta)})$

$$(p_\sigma)_*\mathcal{V}_p(\beta)[-1] \to (p_\sigma)_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}) \to (p_\sigma)_*i^*j_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}) \quad (4.2.15)$$

By the exact sequence (4.1.5), and using the Remark 4.3, taking homotopy fixed points under the action of the inertia group $I \subseteq \text{Gal}(\bar{\eta}/\eta)$ we obtain a cofiber-fiber sequence in $\text{Sh}_{\mathcal{Q}_l}(\bar{\sigma})^{\text{Gal}(\bar{\sigma}/\sigma)}$:

$$((p_\sigma)_*\mathcal{V}_p(\beta)[-1])^{\text{hl}} \to ((p_\sigma)_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}))^{\text{hl}} \to ((p_\sigma)_*i^*j_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}))^{\text{hl}} \quad (4.2.16)$$

**Assumption 4.7.** From now on, for simplicity, we assume that $S$ is strictly local, so that $\bar{\sigma} = \sigma$.

We start by describing the last term of (4.2.16):

**Proposition 4.8.** The canonical map

$$(p_\sigma)_*i^*j_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}) \to ((p_\sigma)_*i^*j_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}))^{\text{hl}}$$

is an equivalence.

**Proof.** The result follows by adjunction and Galois descent. Let $\nu_\eta : X_{\bar{\eta}} \to X_{\eta}$ be the canonical map. Then by Galois theory we know that étale sheaves on $X_{\eta}$ are equivalent to étale sheaves on $X_\eta$ endowed with a continuous action of the Galois group $\text{Gal}(\bar{\eta}/\eta)$. The equivalence is given by $\nu_\eta^*$ in one direction, and by $(\nu_\eta)^{\text{hGal}(\bar{\eta}/\eta)}$, i.e. by taking fixed points of global section, in the other direction. Then we have $\text{Id}_{X_{\eta}} = ((\nu_\eta)_*\nu_\eta^*)^{\text{hGal}(\bar{\eta}/\eta)}$.

As in the strictly local case we have $\bar{j} = j \circ \nu_\eta$ we find that

$$(i^*\bar{j}_*(\nu_\eta^*E_\eta))^{\text{hGal}(\bar{\eta}/\eta)} \simeq i^*(\bar{j}_*(\nu_\eta^*E_\eta))^{\text{hGal}(\bar{\eta}/\eta)} \simeq i^*(j_*(\nu_\eta^*E_\eta))^{\text{hGal}(\bar{\eta}/\eta)}$$

$$\simeq i^*j_*(\nu_\eta^*E_\eta)^{\text{hGal}(\bar{\eta}/\eta)} \simeq i^*j_*E_\eta$$

for any $E_\eta \in \text{Sh}_{\mathcal{Q}_l}(X_{\eta})$. See [MR073, Exp XIII pag 7].

$\square$

Let us now discuss the last term of (4.2.16) using absolute purity for $\ell$-adic sheaves:

**Proposition 4.9.** There is a canonical equivalence

$$((p_\sigma)_*\mathcal{V}_p(\beta)[-1])^{\text{hl}} \simeq (p_\sigma)_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}}) \oplus (p_\sigma)_*R^\ell_{X_{\bar{\eta}}}(BU_{X_{\bar{\eta}}})(-1)[-1]$$
Proof. By construction of vanishing cycles, the action of $I$ on $(p_\sigma)_*R^\ell_X(BU_\sigma)$ is the trivial action so that by a projection formula for the projection $BI \to \sigma$, we have

$$(p_\sigma)_*R^\ell_X(BU_\sigma)^{hl} \simeq (p_\sigma)_*R^\ell_X(BU_\sigma) \otimes (Q_{\ell,\sigma})^{hl}$$

The claim then follows from the computation

$$(Q_{\ell,\sigma})^{hl} \simeq Q_{\ell,\sigma} \oplus Q_{\ell,\sigma}(-1)[-1] \quad (4.2.17)$$

in $\text{Sh}_{Q_{\ell}(\bar{\sigma})}^{\text{Gal}(\bar{\sigma}/\sigma)}$ (which identifies trivially, i.e. with no need of Galois descent, with $\text{Sh}_{Q_{\ell}(\sigma)}$ because we are working with $S$ strictly local). Note that in the l.h.s. of (4.2.17), the inertia $I$ acts trivially on $Q_{\ell,\sigma}$. In order to perform this computation, we proceed in three steps:

Step 1) Let us first remark that given the diagram

$$\begin{array}{ccc}
G_{m,S} & \xrightarrow{j_0} & \mathbb{A}^1_S \\
q & \downarrow & \pi \\
S & \xleftarrow{\text{Id}} & S
\end{array} \quad (4.2.18)
$$

we have

$$(i_0)^*(j_0)_*1_{G_{m,S}} \simeq q_*1_{G_{m,S}} \simeq 1_S \oplus 1_S(-1)[-1] \quad (4.2.19)$$

Here these operations hold for $\text{SH}_S$ but also for any realization compatible with the six operations (like the $\ell$-adic one with $Q_{\ell}$ coefficients).

To show this, we remark first that we have $q_*1_{G_{m,S}} = 1_S \oplus 1_S(1)[1]$. Indeed, by definition of the Tate motive we obtain $1_S(1)[1]$ as the cofiber of the map in motives over $S$ of $1 : S \to G_{m,S}$. As this map has a splitting given by the projection $q : G_{m,S} \to S$ we deduce the formula. By adjunction we obtain $q_*1_{G_{m,S}} = 1_S \oplus 1_S(-1)[-1]$

To show the formula (4.2.19) it is now enough to show that $(i_0)^*(j_0)_*q^* \simeq q_*q^*$. For that, we notice that the projection $p_\sigma p^* \to \text{Id}$ admits a section given by the $1 : S \to G_{m,S}$ and under the adjunctions, this becomes a splitting of the unit $\text{Id} \to p_*p^*$. Now we recall that $q = \pi \circ j_0$ so that this is equivalent to show that

$$(i_0)^*(j_0)_*j_0^*\pi^* \simeq \pi_* (j_0)_*j_0^*\pi^*$$

From the localization property we have an exact sequence

$$\pi_* (i_0)_*i_0^*\pi^* \xrightarrow{\pi_* \text{Id} \pi^*} \pi_* (j_0)_*j_0^*\pi^*$$
Because of $\mathbb{A}^1$-invariance we have $\pi_*Id \pi^* \simeq Id$. Moreover, as $\pi_*(i_0)_* = Id_S$, the last sequence is equivalent to

$$
\begin{array}{c}
\pi_*(i_0)_* i_0^! \pi^* \\ \downarrow \simeq \\
\downarrow \\
i_0^! \pi^*
\end{array}
\to
\begin{array}{c}
\pi_* Id \pi^* \\ \downarrow \simeq \\
\downarrow \\
Id_S
\end{array}
\to
\begin{array}{c}
\pi_*(j_0)_* j_0^* \pi^* \\ \downarrow \simeq \\
\downarrow \\
j_0^* \pi^*
\end{array}
$$

where the horizontal maps on the right split because of the discussion above. Finally, because $(i_0)_*$ is fully faithful and because $i_0^* \pi^* = Id_S$ the sequence is also equivalent to

$$
\begin{array}{c}
\pi_*(i_0)_* i_0^! \pi^* \\ \downarrow \simeq \\
i_0^! \pi^*
\end{array}
\to
\begin{array}{c}
\pi_* Id \pi^* \\ \downarrow \simeq \\
Id_S
\end{array}
\to
\begin{array}{c}
\pi_*(j_0)_* j_0^* \pi^* \\ \downarrow \simeq \\
j_0^* \pi^*
\end{array}
$$

and from the localization sequence

$$
i_0^1 (i_0)_* i_0^! \pi^* \\ \downarrow \\
i_0^* \pi^*
\to
\begin{array}{c}
i_0^1 (i_0)_* i_0^! \pi^* \\ \downarrow \\
\downarrow \\
i_0^1 (j_0)_* j_0^* \pi^* \end{array}
$$

we deduce the result. Moreover, because of the splitting we find

$$i_0^1 1_{\mathbb{A}^1} S \simeq 1_S \bigoplus 1_S(-1)[-2]$$

Step 2) Now we specialize back to the $\mathbb{Q}_\ell$-realization: Absolute purity for $\ell$-adic sheaves, namely, the result of [Ayo14, 7.4] using the fact that the hypothesis [Ayo14, 7.3] holds (as proved in [ILO14, XVI 3.5.1]), says that the transfer map

$$t^* i_0^1 \to i_\sigma^* \pi^*$$

associated to the pullback diagram

$$
\begin{array}{ccc}
s & \xrightarrow{i_\sigma} & S \\
\downarrow t & & \downarrow \pi \\
S & \xrightarrow{i_0} & \mathbb{A}^1_S
\end{array}
$$

(4.2.20)

with $\pi$ the uniformizer, is an equivalence. Therefore we obtain equivalences
so that the last terms on the right are also equivalent and therefore, by the computation in the previous section, equivalent to $\mathbb{Q}_{\ell,\sigma} \oplus \mathbb{Q}_{\ell,\sigma}(-1)[-1]$. Therefore we find an exact sequence

$$\xymatrix{ i_\sigma^! t^* \mathbb{Q}_{\ell,S} \ar[r] \ar[d] & i_\sigma^! t^* \mathbb{Q}_{\ell,S} \ar[r] \ar[d] & i_\sigma^! j_! j^* \mathbb{Q}_{\ell,S} \ar[d] \ar[r] & \mathbb{Q}_{\ell,\sigma} \oplus \mathbb{Q}_{\ell,\sigma}(-1)[-1] \ar[d] \ar[r] & 0 \ar[r] & 0 \ar[r] & t^! (i_0)_0^* \mathbb{Q}_{\ell,\mathbb{G}_m S} }$$

(4.2.21)

where the last map has a splitting. In this case we find the absolute purity isomorphism for $(\sigma, S, \eta)$

$$i_\sigma^! \mathbb{Q}_{\ell,S} \simeq \mathbb{Q}_{\ell,\sigma}(-1)[-2]$$

(4.2.22)

and deduce that

$$i_\sigma^*(j_\eta)_* \mathbb{Q}_{\ell,\bar{\eta}} \simeq \mathbb{Q}_{\ell,\sigma} \oplus \mathbb{Q}_{\ell,\sigma}(-1)[-1]$$

Step 3) As $S$ is trivially smooth over itself via the identity map, the specialization map defining vanishing cycles relative to $S$

$$\mathbb{Q}_{\ell,\sigma} \to i_{\sigma}^!(j_\bar{\eta})_* \mathbb{Q}_{\ell,\bar{\eta}}$$

(4.2.23)

is an equivalence in $\text{Sh}_{\mathbb{Q}_{\ell}}(\sigma)^I$ where in this case as we work under the assumption that $S$ is strictly local, we have $I = \text{Gal}(\bar{\eta}, \eta)$. For tame nearby cycles this follows by the explicit computation of the r.h.s of [Ayo14, Formula (102)] using the fact we are working with torsion coefficients. For the total vanishing cycles functor this follows by passing to the colimit over all wild extensions.\(^\text{26}\)

The specialization map being equivariant, implies that after passing to $I$-invariants we still get an equivalence

$$\mathbb{Q}_{\ell,\sigma}^I \simeq (i_{\sigma}^*(j_\bar{\eta})_* \mathbb{Q}_{\ell,\bar{\eta}})^I$$

(4.2.24)

\(^{26}\text{One can also give a simpler argument for the equivalence (4.2.23) in terms of \'{e}tale cohomology groups. Indeed, one can give an explicit description of the \'{e}tale sheaves $(j_\eta)_* \mathbb{Z}/n\mathbb{Z}$ by noticing that its cohomology groups are the \'{e}tale sheafification of the presheaf of abelian groups sending an \'{e}tale map $V \to S$ to $H^i(V \times_S \bar{\eta}, \mathbb{Z}/n\mathbb{Z})$. As $\bar{\eta}$ is separably closed, $V \times_S \bar{\eta}$ is a disjoint union of copies of $\bar{\eta}$ so that its \'{e}tale cohomology groups vanish for $i \geq 1$. For $i = 0$ we get $\mathbb{Z}/n\mathbb{Z}$ as the set of (underived) global sections of its associated constant sheaf.}\)
Finally, by the same argument preceding the proof of this proposition when describing the last term of (4.2.16), we find

\[(i^*_\sigma (j_\eta)_* Q_{\ell, \eta})_h I \simeq i^*_\sigma (j_\eta)_* Q_{\ell, \eta})\]

which, by the Step 2) above, is equivalent to \(Q_{\ell, \sigma} \oplus Q_{\ell, \sigma}(-1)[-1]\).

This concludes the proof of the proposition. □

**Remark 4.10.** As taking invariants is a lax monoidal procedure (being right adjoint to the trivial representation functor which is monoidal), we deduce that \((Q_{\ell, \sigma})^{hl} \simeq Q_{\ell, \sigma} \oplus Q_{\ell, \sigma}(-1)[-1]\) carries a canonical structure of commutative algebra object in \(Sh_{Q_{\ell}(\sigma)}\)

\[(Q_{\ell, \sigma})^{hl} \otimes (Q_{\ell, \sigma})^{hl} \to (Q_{\ell, \sigma})^{hl}\]

As (4.2.24) is an equivalence of algebras, the formula (4.2.25) tells us that the algebra structure on \((Q_{\ell, \sigma})^{hl}\) is obtained by transferring the canonical algebra structure on \(Q_{\ell, \eta}\) via the lax monoidal functor \(i^*_\sigma (j_\eta)_*\). Following [RZ82, 1.2] we have a cycle class

\[\theta' : Q_{\ell, \eta} \to Q_{\ell, \eta}(1)[1]\]

such as the one of (4.2.9). We refer to loc. cit. for the definition of this new class. One remarks that multiplication by elements in Tate degree \((1, 1)\), defined by the composition

\[Q_{\ell, \sigma}(-1)[-1] \otimes (Q_{\ell, \sigma})^{hl} \xrightarrow{incl \otimes Id} (Q_{\ell, \sigma})^{hl} \otimes (Q_{\ell, \sigma})^{hl} \xrightarrow{(4.2.26)} (Q_{\ell, \sigma})^{hl}\]

(4.2.28)

can also be presented as a map

\[(Q_{\ell, \sigma})^{hl} \to (Q_{\ell, \sigma})^{hl}(1)[1]\]

(4.2.29)

The arguments of [RZ82, 1.2] (see also [Ill94, §3.6]) show that (4.2.29) corresponds under (4.2.24) and (4.2.25), to the image of the cycle class \(\theta'\) by the functor \(i^*_\sigma (j_\eta)_*\). Finally, and central to our work, the comparison of [ILO14, Exp XVI Lemmas 3.4.6, 3.4.7 and 3.4.8] establish the identification of the class \(\theta'\) with the cycle class \(\theta\) of (4.2.24) of the Remark (4.6).

It follows from this discussion that for every object \(\Psi \in Sh_{Q_{\ell}(\sigma)}\), the object \(\Psi^{hl}\) carries a canonical structure of \((Q_{\ell, \sigma})^{hl}\)-module

\[(Q_{\ell, \sigma})^{hl} \otimes \Psi^{hl} \to \Psi^{hl}\]

and by the arguments above, is acted by the class \(\theta\)

\[\Psi^{hl} \xrightarrow{\theta} \Psi^{hl}(1)[1]\]
It is a key idea of [RZ82, 1.4] that $\Psi$ with its $I$-action can be completely recovered from $\Psi^{hI}$ together with the information of the action of $\theta$. This is also the main ingredient used in [Ayo07c, Ayo14] to construct motivic versions of nearby and vanishing cycles.

Finally, as a consequence of Prop. 4.9, (4.2.16) becomes

$$
\left((p_\sigma)_*V_p(\beta)[-1]\right)^{hI} \longrightarrow (p_\sigma)_*R^\ell_X(\mathbb{B}U_{X_\sigma}) \oplus (p_\sigma)_*R^\ell_{X_\sigma}(\mathbb{B}U_{X_\sigma})(-1)[-1]
\quad (4.2.30)
$$

Now we can prove our main theorem:

**Theorem 4.11.** Let $p : X \to S$ with $X$ regular, $p$ a proper flat morphism of finite type over a strictly local excellent henselian trait $S = \text{Spec } A$. There is a canonical equivalence of $\mathbb{Q}_\ell$-adic sheaves over $\sigma$

$$
(i_\sigma)^*R^\ell_S(M_S^\vee(\text{Sing}(X, \pi \circ p))) \simeq ((p_\sigma)_*V_p(\beta)[-1])^{hI}
\quad (4.2.31)
$$

Moreover, thanks to the Proposition 4.4, we also have an equivalence of $\mathbb{Q}_\ell$-adic sheaves over $S$

$$
R^\ell_S(M_S^\vee(\text{Sing}(X, \pi \circ p))) \simeq (i_\sigma)_*((p_\sigma)_*V_p(\beta)[-1])^{hI}
\quad (4.2.32)
$$

**Remark 4.12.** Note that if $p : X \to S$ is a proper morphism, and $X$ regular, $p$ then $\text{Sing}(X, \pi \circ p) \simeq \text{Sing}(X_0)$, where $X_0$ is the derived zero locus of $\pi \circ p$. If, moreover, $p$ is flat, then $X_0 \simeq t(X_0)$ (i.e. the derived zero locus coincides with the scheme theoretic zero-locus). Therefore, the equivalences in Theorem 4.11, can be equivalently re-written as

$$
(i_\sigma)^*R^\ell_S(M_S^\vee(\text{Sing}(X_0))) \simeq ((p_\sigma)_*V_p(\beta)[-1])^{hI},
\quad (4.2.33)
$$

and

$$
R^\ell_S(M_S^\vee(\text{Sing}(X_0))) \simeq (i_\sigma)_*((p_\sigma)_*V_p(\beta)[-1])^{hI}.
\quad (4.2.34)
$$

**Remark 4.13.** Notice that, since the action of $I$ on $X_k$ is trivial and taking homotopy invariants can be represented as a limit, and both $(i_\sigma)_*$ and $(p_\sigma)_*$ are right adjoints, they commute with taking $hI$-invariants. In particular, we have

$$
\left((p_\sigma)_*V_p(\beta)[-1]\right)^{hI} \simeq (p_\sigma)_*\left((V_p(\beta)[-1])^{hI}\right)
$$

This can also be deduced from the monadic argument producing the equivalence $\text{Sh}_{\mathbb{Q}_\ell}(\sigma)^t \simeq \text{Mod}_{\mathbb{Q}_\ell,\sigma}[I](\text{Sh}_{\mathbb{Q}_\ell}(\sigma))$ with $\mathbb{Q}_\ell,\sigma[I]$ the internal group-ring of $I$. 

and

\[(i_\sigma)_*(p_\sigma)_*\mathcal{V}_p(\beta)[-1])^{hl} \simeq (i_\sigma)_*(p_\sigma)_*(\mathcal{V}_p(\beta)[-1])^{hl} \].

**Proof of Theorem 4.11.** The idea of the proof is to show that the two cofiber-fiber sequences (4.2.7) and (4.2.30) are the same.

Let us write \(\phi_\ast : \text{Sh}_\mathbb{Q}_\ell(I_\sigma) \to \text{Sh}_\mathbb{Q}_\ell(I_\sigma)\) for the functor computing inertia (homotopy) invariants. It is right adjoint to the functor \(\phi^\ast\) assigning the trivial representation. Let

\[sp : (p_\sigma)_*R^\ell B\mathbb{U}_{X_0} \to ((p_\sigma)_*i^*j_*R^\ell X_n(B\mathbb{U}_{X_n}))\]  

be the specialization map whose cofiber defines vanishing cycles. This is by construction a map in \(\text{Sh}_\mathbb{Q}_\ell(I_\sigma)\) where the l.h.s is endowed with the trivial action (so that in fact the source of the map (4.2.35) should read as \(\phi^\ast((p_\sigma)_*R^\ell B\mathbb{U}_{X_0})\).) and following the computation in the Prop. 4.8 and base change, we have

\[\phi_\ast(((p_\sigma)_*i^*j_*R^\ell X_n(B\mathbb{U}_{X_n})) \simeq ((p_\sigma)_*i^*j_*R^\ell X_n(B\mathbb{U}_{X_n})) \simeq (i_\sigma)^*(j_\eta)_*(\eta)_*R^\ell B\mathbb{U}_{X_\eta}\] (4.2.36)

Notice that as all the adjunctions are defined at the level \(B\mathbb{U}\)-modules, the map (4.2.35) is also a map of \((p_\sigma)_*R^\ell B\mathbb{U}_{X_0}\)-modules. In particular, after passing to invariants, the map

\[\phi_\ast \phi^\ast((p_\sigma)_*R^\ell B\mathbb{U}_{X_0}) \xrightarrow{\phi^\ast(sp)} (i_\sigma)^*(j_\eta)_*(\eta)_*R^\ell B\mathbb{U}_{X_\eta}\] (4.2.37)

is \(\phi_\ast \phi^\ast((p_\sigma)_*R^\ell B\mathbb{U}_{X_0}) \simeq (p_\sigma)_*R^\ell B\mathbb{U}_{X_0} \otimes \mathbb{Q}^{hl}_\ell\)-linear, with the action on the r.h.s given by the arguments of the Remark 4.10.

Let us now write \(\lambda : (p_\sigma)_*R^\ell B\mathbb{U}_{X_0} \to \phi_\ast \phi^\ast((p_\sigma)_*R^\ell B\mathbb{U}_{X_0})\) for the unit of the adjunction \((\phi_\ast, \phi^\ast)\). Notice that as the adjunctions are lax monoidal, this is a map of algebra-objects. Using (4.2.36), we obtain a commutative triangle

\[\begin{array}{ccc}
(p_\sigma)_*R^\ell B\mathbb{U}_{X_0} & \xrightarrow{\lambda} & \phi_\ast \phi^\ast((p_\sigma)_*R^\ell B\mathbb{U}_{X_0}) \\
\downarrow \rho & & \downarrow \phi^\ast(sp) \\
(i_\sigma)^*(j_\eta)_*(\eta)_*R^\ell B\mathbb{U}_{X_\eta} & & \\
\end{array}\] (4.2.38)

where \(\rho\) is the map appearing in the cofiber-sequence given by the localization property for \((X_\eta, X_\sigma, X)\). The octahedral property gives us a cofiber-fiber sequence.
We remark that:

- As in the diagram (4.2.38), it follows from the localization sequence of $(X_\eta, X_\sigma, X)$ that

\[ \text{Fib } \rho \simeq (p_\sigma)_* i^! R^\ell (B \mathcal{U}_X) \]

- The fact that $\phi_*$ preserves cofiber-fiber sequences, gives us an equivalence

\[ \text{Fib } \phi_*(sp) \simeq ((p_\sigma)_* V_*(\beta)[-1])^{hI} \]

- The computation of the Prop. (4.9) shows that

\[ \phi_* \phi^*((p_\sigma)_* R^\ell B \mathcal{U}_{X_0}) \simeq (p_\sigma)_* R^\ell_{X_\sigma} (B \mathcal{U}_{X_\sigma}) \oplus (p_\sigma)_* R^\ell_{X_\sigma} (B \mathcal{U}_{X_\sigma})(-1)[-1] \]

and as the action is trivial on $(p_\sigma)_* R^\ell B \mathcal{U}_{X_0}$, the map $\lambda$ corresponds to the inclusion on the first component and the canonical map

\[ \text{Fib } \lambda \to (p_\sigma)_* R^\ell B \mathcal{U}_{X_0} \]  

(4.2.40)

is the zero map. It follows that

\[ \text{Fib } \lambda \simeq (p_\sigma)_* R^\ell_{X_\sigma} (B \mathcal{U}_{X_\sigma})(-1)[-2] \]

Combining these remarks, the cofiber-fiber sequence (4.2.39) reads as

\[ \begin{array}{c}
(p_\sigma)_* R^\ell_{X_\sigma} (B \mathcal{U}_{X_\sigma})(-1)[-2] \\
\downarrow \quad \downarrow \quad \downarrow \quad (p_\sigma)_* i^! R^\ell (B \mathcal{U}_X)
\end{array} \]

(4.2.41)

and using the commutativity of the diagram

\[ \begin{array}{c}
\text{Fib } \lambda \quad \mu \\
\downarrow \\
(p_\sigma)_* R^\ell B \mathcal{U}_{X_0}
\end{array} \]  

(4.2.40)$\Rightarrow$0

we remark that $\mu$ factors as
\[ \text{Fib} \lambda \xrightarrow{i} (i_\sigma)^*(j_\eta)_*(p_\eta)_* R^\ell \mathcal{B} \mathcal{U} X_\eta \rightarrow \text{Fib} \rho \]  
\tag{4.2.42}

By adjunction, $\tilde{\theta}$ can be interpreted as a map

\[ \overline{(p_\sigma)_* R^\ell X_\sigma} \xrightarrow{\tilde{\theta}} (i_\sigma)^*(j_\eta)_*(p_\eta)_* R^\ell \mathcal{B} \mathcal{U} X_\eta \]  
\tag{4.2.43}

To conclude it will be enough to identify $\tilde{\theta}$ with the map (4.2.13) of the Remark 4.6. The fact that the two classes match is a consequence of the fact that both are constructed from the cycle class (4.2.9) in the same way via $(i_\sigma)^*(j_\eta)_*(p_\eta)_* R^\ell \mathcal{B} \mathcal{U} X_\eta[1]$.

Corollary 4.14. Under the hypotheses and notations of Theorem 4.11, we have an equivalence of étale $\ell$-adic hyper-cohomologies

\[ \mathbb{H}_{Q_\ell}(S, R^f_\Sigma M^\vee_S(S, \pi \circ p)) \simeq \mathbb{H}_{Q_\ell}(X_k, V_p(\beta)[-1])^{\text{hl}} \]

in the $\infty$-category of $\mathbb{Q}_\ell$-dg-modules.  

Proof. The statement follows by applying the hypercohomology functor $\mathbb{H}_{Q_\ell}(S, -)$ to the equivalence (4.2.32), and using in the r.h.s. that

\[ \mathbb{H}_{Q_\ell}(X_k, -) \simeq \mathbb{H}_\text{ét}(S, (i_\sigma)_*(p_\sigma)_*(-)). \]

Also observe that, since taking homotopy invariants can be represented as a limit, it commutes with taking hyper-cohomology, so that we have

\[ \mathbb{H}_{Q_\ell}(X_k, V_p(\beta)[-1])^{\text{hl}} \simeq \mathbb{H}_{Q_\ell}(X_k, (V_p(\beta)[-1])^{\text{hl}}). \]

Remark 4.15. Notice also that the sheaf of vanishing cycles $V_p(\beta)$ is canonically equivalent to the tensor product $V_p \otimes \mathbb{Q}_\ell(\beta)$ where $V_p$ are vanishing cycles of the structure sheaf $\mathbb{Q}_\ell X$. Indeed, given $E$ a $\mathbb{Q}_\ell$-adic sheaf on $X$, we claim first to have a canonical equivalence between $V_p(E(1))$ and $V_p(E) \otimes \mathbb{Q}_\ell(1)$. To check this we can look at the cofiber sequence defining vanishing cycles

\[ v_\ast^* i_\ast^*(E(1)) \rightarrow \overline{i}^* \overline{j}^* v^*(E(1)) \rightarrow V_p(E(1)) \]  
\tag{4.2.44}

and notice that pullbacks commute with Tate-twists (by definition) and that one has canonical equivalences

\[ 28 \text{Note that in the literature the r.h.s of is often denoted as } \mathbb{R} \Gamma(I, \mathbb{H}_{Q_\ell}(X_k, V_p(\beta)[-1])). \]
\[ j_* j^*(F(1)) \simeq (j_* j^*) F(1) \]  
\[ (4.2.45) \]

for any \( F \) over \( \bar{X} \). This equivalence can be deduced by looking at the mapping spaces from a third \( \ell \)-adic sheaf \( G \) to both sides of (4.2.45) and use the adjunction \((j^*, j_*)\) together with the fact that \( j^* \) is monoidal and that the Tate twist is an invertible object stable under base change. The equivalence \( i_*(E(1)) \simeq i_*(E)(1) \) follows by the same argument. In this case the cofiber sequence (4.2.44) is equivalent to

\[ (v_\sigma^* i_*(E))(1) \rightarrow (\bar{i}_* j^* \bar{j}^* v^* E)(1) \rightarrow \mathcal{V}_\pi(E)(1) \]
\[ (4.2.46) \]

Finally to deduce the equivalence \( \mathcal{V}_p(\beta) \simeq \mathcal{V}_p \otimes Q_\ell(\beta) \) one uses the equivalence \( Q_\ell(\beta) \simeq \bigoplus_{i \in \mathbb{Z}} Q_\ell(i)[2i] \) together with the fact that both \( \ast\)-pullbacks and \( \ast\)-pushforwards preserve arbitrary colimits (see the discussion in [Robdf, Example 9.4.6] for pushfowards, which is the only non obvious verification to be made).

**Remark 4.16.** We believe that the equivalence of Theorem 4.11 (and, correspondingly, of Corollary 4.14) is compatible with the canonical action of \( R_\ell(S_{\text{Sing}(S, 0)}) \simeq R_\ell(BU_S) \oplus R_\ell(BU_S)[1] \) on the l.h.s, and the action of \( Q_\ell(\beta) \oplus Q_\ell(\beta)(-1)[-1] \) on the r.h.s. More precisely, we expect the equivalence (4.2.32) of Theorem 4.11 to be natural in \( (p : X \rightarrow S) \) (where both \( S \) and a uniformizer \( \pi \) in \( S \) are fixed), and the natural transformation \( \mu \) be lax symmetric monoidal with respect to the structure of Proposition 2.31 on the l.h.s., and the so-called additive convolution monoidal structure appearing in Thom-Sebastiani theorem for vanishing cycles ([Ill17, Thm. 4.5]) on the r.h.s. We will address this refined statement in a forthcoming paper.

**Corollary 4.17.** In the situation of Theorem 4.11, let us suppose that \( S \) is an excellent henselian trait (so that its residue field is not necessarily separably closed). Let us fix a separable closure \( \bar{k} \) of \( k \), and let \( \bar{S} = \text{Spec } A^{\text{sh}} \) be the corresponding strict henselization of \( S \). Then, in the notations of diagram (4.1.7), we have an equivalence in \( \text{Sh}_{Q_\ell}(S)^{\text{Gal(\sigma)}} \)

\[ R^s_{\bar{S}}((M^\vee_S(S_{\text{Sing}(X, p)}))) \simeq u^*(i_\sigma)_*(p_\sigma)_*\mathcal{V}_p(\beta)[-1])^{\text{hl}}. \]
\[ (4.2.47) \]

**Proof.** We borrow our notations from diagram (4.1.7). First of all, observe that since \( S \) is an excellent henselian trait, the étale topos of \( \ell \)-torsion sheaves of any \( S \)-scheme of finite type is of finite cohomological dimension (Remark 3.28), therefore we still have an \( \ell \)-adic realization functor

\[ R^nc_{\ell,S} := R^s_{\bar{S}} \circ M^\vee_S : \text{dgcat}_{S}^{\text{idem}} \rightarrow \text{Sh}_{Q_\ell}(S). \]
Analogously, we will write

$$R_{\ell,S}^{nc} := R_{S}^{f} \circ M_{S}^{\neq} : \text{dgcat}_{S}^{\text{idem}} \to \text{Sh}_{\ell}(\mathcal{S}).$$

Notice that $m_{A}A^{sh} = m_{A}A^{h}$, so that any uniformizer $\pi$ of $A$ gives a uniformizer of $A^{sh}$ (i.e. its image via $A \to A^{sh}$ is a uniformizer in $A^{sh}$). Since $u : \bar{S} \to S$ is formally étale and the ramification locus of $u : \bar{S} \to S$ is disjoint from the singularity locus of $p : X \to S$, in the base change $\bar{X}/\bar{S}$ of $X/S$ along $u$, $\bar{X}$ is still regular (since $X$ is). Since $u_{*} : \text{Sh}_{\ell}(\bar{S}) \to \text{Sh}_{\ell}(S)$ is lax monoidal, we have a commutative diagram

$$\begin{array}{ccc}
\text{dgcat}_{S}^{\text{idem}} & \xrightarrow{R_{\ell,S}^{nc}} & \text{Mod}_{R_{\ell,S}^{nc}(A^{sh})}(\text{Sh}_{\ell}(\bar{S})) \\
\downarrow u_{*} & & \downarrow u_{*} \\
\text{dgcat}_{S}^{\text{idem}} & \xrightarrow{R_{\ell,S}^{nc}} & \text{Mod}_{R_{\ell,S}^{nc}(A)}(\text{Sh}_{\ell}(S))
\end{array}$$

As already observed for $R_{\ell,S}^{nc}(A^{sh}) \simeq \mathbb{Q}_{\ell,S}(\beta)$, we have $R_{\ell,S}^{nc}(A) \simeq \mathbb{Q}_{\ell,S}(\beta)$. Since $\bar{X}$ is regular, we have $u_{*}(\text{Sing}(\bar{X}, \pi \circ \bar{p})) \simeq \text{Sing}(X_{\sigma})$, and recall that, by definition of vanishing cycles for $p$ and the fact that $u^{*}(\mathbb{Q}_{\ell,S}(\beta)) \simeq \mathbb{Q}_{\ell,S}(\beta)$, we have $\mathcal{V}_{p}(\mathbb{Q}_{\ell,S}(\beta)) \simeq \mathcal{V}_{p}(\mathbb{Q}_{\ell,S}(\beta))$ inside $\text{Sh}_{\ell}(X_{\sigma})_{\text{Gal}(\eta/\sigma)}$. Now, the above commutative diagram of $\ell$-adic realizations combined with Theorem 4.11, yields an equivalence in $\text{Sh}_{\ell}(S)^{\text{Gal}(\eta/\sigma)}$

$$R_{\ell,S}^{nc}(\text{Sing}(X_{\sigma})) \simeq u_{*}(i_{\sigma})_{*}(p_{\sigma})_{*}(\mathcal{V}_{p}(\beta)[-1])^{\text{hl}}). \quad (4.2.48)$$

Remark 4.18. The equivalence of the Theorem 4.11 also provides a Chern character map from the K-theory of matrix factorizations to $\ell$-adic 2-periodic inertia invariant vanishing cohomology. Indeed, using the functoriality of the $\ell$-adic realization (3.7.14) one obtains a map

$$\begin{array}{ccc}
\text{KH}(\text{MF}(X, \pi \circ p)) & \simeq & \text{Map}_{\text{Mod}_{\ell,h}}(\text{BU}_{S}, M_{S}^{\neq}(\text{MF}(X, \pi \circ p))) \\
\text{H}_{d}(X_{h}, \mathcal{V}_{p}(\beta)[-1])^{\text{hl}} & \simeq & \text{Map}_{\text{Mod}_{\ell,h}}(\text{BU}_{\sigma}, i_{\sigma}^{*}M_{S}^{\neq}(\text{MF}(X, \pi \circ p))) \\
\downarrow (3.7.14)+(4.2.31) & & \\
\text{H}_{d}(X_{h}, \mathcal{V}_{p}(\beta)[-1])^{\text{hl}} & \simeq & \text{Map}_{\text{Mod}_{\ell,h}}(\mathcal{Q}_{\ell,S}(\beta), ((p_{\sigma})_{*}\mathcal{V}_{p}(\beta)[-1])^{\text{hl}})
\end{array}$$

Remark 4.19. Notice that for the conclusion of the Theorem 4.11 to hold, the hypothesis that $X$ is regular is crucial. Indeed, if $X$ is not regular, the relative derived category of singularities $\text{Sing}(X, \pi)$ is not equivalent to the absolute one $\text{Sing}(X_{0})$ as explained in
the Remark 2.15. In the non-regular case we would need to provide a proof for all the statements in Section 3.4 replacing $\text{Coh}^b(X_0)$ by $\text{Coh}^b(X_0)_{\text{Perf}(X)}$.

**APPENDIX A. THE FORMALISM OF SIX OPERATIONS IN THE MOTIVIC SETTING**

The $\infty$-functor $\text{SH}^\otimes$ carries a system of extra functorialities known as the six operations. This means:

(1) For every smooth morphisms $f : X \to Y$ of base schemes, the assignment $f^* : \text{SH}_Y \to \text{SH}_X$ has a left adjoint $f_!$ which is a map of $\text{SH}_Y$-modules with respect to the natural map induced from the unit of the adjunctions $(f_!, f^*)$

$$f_!(- \otimes f^*(-)) \to f_!(-) \otimes -$$

where $\text{SH}_X$ is seen as a $\text{SH}_Y$-module via the monoidal functoriality $f^*$. Moreover, $(-)_!$ should verify smooth base-change (see the Appendix A).

(2) The existence of a second functoriality for the assignment $X \mapsto \text{SH}_X$ encoded by an $\infty$-functor

$$\text{SH}_! : \text{Sch}^{\text{sep, ft}} / S \to \text{Pr}_\text{Stb}$$

defined in $\text{Sch}^{\text{sep, ft}} / S$ - the subcategory of $\text{Sch} / S$ spanned by separated morphisms of finite type.

(3) The existence of natural transformations $(-)_* \to (-)_!$ and $(-)_! \to (-)_!$ defined in $\text{Sch}^{\text{sep, ft}} / S$ which are isomorphisms, respectively, for proper maps and open immersions. 29;

(4) (Projection Formula) The functoriality $\text{SH}_!$ has a module structure over the functoriality $\text{SH}^\otimes$. More precisely, for any map $f : X \to Y$ separated of finite type, we ask for $f_! : \text{SH}_X \to \text{SH}_Y$ to be a map of $\text{SH}_Y$-modules as in (1). Here $\text{SH}_X$ is seen as a $\text{SH}_Y$-module via the monoidal functoriality $f^*$.

(5) For any cartesian square of schemes

$$\begin{array}{ccc}
Y' & \xrightarrow{p'} & X' \\
\downarrow^{f'} & & \downarrow^f \\
Y & \xrightarrow{p} & X
\end{array}$$

with $f$ separated of finite type, we ask for natural equivalences of $\infty$-functors

$$p^* \circ f_! \simeq (f')_! \circ (p')^*$$

(A.0.2)

and

$$f^! \circ p_* \simeq (p')_* \circ (f')^!$$

(A.0.3)

29Here $(-)_!$ denotes the right adjoints of the functoriality $\text{SH}^\otimes$
(6) For any smooth morphism of relative dimension $d$, $f : Y \to X$ the adjunctions $(f_!, f^!)$ and $(f_\# , f^*)$ are related by a natural equivalence

$$f_\# \simeq f_!( - \otimes (P_1^Y, \infty)^{\otimes d})$$

(A.0.4)

Thanks to the main results of [Ayo07a, CD12] (see Prop. A.2), all these operations and coherences can be constructed from the initial $\infty$-functor $\text{SH}^\otimes$. In the setting of higher categories this can be done using the theory of correspondences developed in [GR, Part V.1]. Alternatively, one can also use results of [LZ12a, LZ12b] as explained in [Robdf, Section 9.4]. In this appendix we give a brief survey of the construction based on the techniques of [GR, Part V.1] but we do not look at the necessary Beck-Chevalley conditions. These have now been carefully treated in [Kha16], in a more general setting where base schemes are allowed to be derived.

Fix a base Noetherian scheme $S$ and consider $\text{Sch}_{/S}$ in the sense of [CD12, 2.0]. The important key point is Nagata’s compactification for separated morphisms of finite type between $S$-schemes. Suppose we are given an $\infty$-functor $F : \text{Sch}_{/S}^{op} \to \text{CAlg}(\text{Pr}_{\text{Stb}})$. From this it is possible to extract a new functor $\text{Arr}(\text{Sch}_{/S})^{op} \to \text{Mod}(\text{Pr}_{\text{Stb}})$ where $\text{Mod}(\text{Pr}_{\text{Stb}})$ is the $\infty$-category of pairs $(C, M)$ with $C$ a symmetric monoidal stable presentable $\infty$-category and $M$ a stable presentable category endowed with a structure of $C$-module. The new functor sends $f : Y \to X$ to the pair $(F(X), F(Y))$ with $F(Y)$ seen as a module via $F(f)$. We will also denote by $F$. We redirect the reader to [Robdf, Section 9.4.1.2] for a precise description of this assignment.

The $\infty$-category $\text{Mod}(\text{Pr}_{\text{Stb}})$ is a non-full subcategory of $\text{Mod}(\text{Cat}_{\text{big}}^{\infty})$ which is the maximal $(\infty, 1)$-category of the $(\infty, 2)$-category $\text{Mod}(\text{Cat}_{\text{big}}^{\infty})^{2-\text{cat}}$ where we also include natural transformations of functors. In reality, the initial data we are interested in, is the new $\infty$-functor

$$F : \text{Arr}(\text{Sch}_{/S})^{op} \to \text{Mod}(\text{Cat}_{\text{big}}^{\infty})^{2-\text{cat}}$$

and the six operations will express the coherences between $F$ and the following five distinct classes of maps in $\text{Sch}_{/S}$:

- $\text{spft} :=$ separated morphisms of finite type
- $\text{all} :=$ all morphisms
- $\text{isom} :=$ isomorphisms
- $\text{proper} :=$ proper morphisms
- $\text{smooth} :=$ smooth morphisms
- $\text{open} :=$ open morphisms

These classes verify some standard stability assumptions - see [GR, Part V.1 Sect 1.1.1]. We will use the same notations for the classes of maps in $\text{Arr}(\text{Sch}_{/S})$ given by natural
transformations where the maps belong to the respective classes. We now explain the conditions and their consequences. The reader should consult [GR, Part V.1] for the notations. The first two conditions are:

1. $\mathbb{F}$ satisfies the right Beck-Chevalley condition with respect to the inclusion $\text{vert} := \text{smooth} \subseteq \text{horiz} := \text{all}$ [GR, Def. 3.1.5 Part V.1]. By the universal property of correspondences [GR, Theorem 3.2.2-b) Part V.1] $\mathbb{F}$ extends in a unique way to an $(\infty, 2)$-functor

$$\mathbb{F}^{\text{smooth,all}} : \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth,all}} \to \text{Mod}(\text{Cat}_{\infty}^{\text{big}})^{2-\text{cat}}$$

whose restriction along the inclusion

$$((\text{Sch}/S)_{\text{horiz}})^{\text{op}} \subseteq \text{Arr}(\text{Sch}/S)^{\text{op}} \subseteq \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth,all}}$$

recovers $\mathbb{F}$.

Using the fact that $(\text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth,all}})^{\text{1-op}} = \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth}}$,

passing to the the 1-opposite we obtain a new $(\infty, 2)$-functor

$$((\mathbb{F})^{\text{smooth,all}})^{\text{1-op}} : \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth,all}} \to (\text{Mod}(\text{Cat}_{\infty}^{\text{big}})^{2-\text{cat}})^{\text{1-op}}$$

whose restriction along

$$((\text{Sch}/S))^{\text{all}} \subseteq \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth,all}}$$

recovers $\mathbb{F}^{\text{op}}$.

2. $\mathbb{F}^{\text{op}} : \text{Arr}(\text{Sch}/S) \to (\text{Mod}(\text{Cat}_{\infty}^{\text{big}})^{2-\text{cat}})^{\text{1-op}}$ satisfies the left Beck-Chevalley with respect to the inclusion $\text{horiz} := \text{proper} \subseteq \text{vert} := \text{all}$ [GR, Def. 3.1.2]. By the universal property of correspondences [GR, Theorem 3.2.2-a) Part V.1] $\mathbb{F}^{\text{op}}$ extends in a unique way to an $(\infty, 2)$-functor

$$((\mathbb{F})^{\text{proper}})^{\text{all,proper}} : \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{proper}} \to (\text{Mod}(\text{Cat}_{\infty}^{\text{big}})^{2-\text{cat}})^{\text{1-op}}$$

whose restriction along the inclusion

$$((\text{Sch}/S)_{\text{vert}}) = \text{Arr}(\text{Sch}/S) \subseteq \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{all,proper}}$$

recovers $\mathbb{F}^{\text{op}}$.

We remark that these Beck-Chevalley conditions need to be verified at the level of modules.

We consider now the restriction $((\mathbb{F})^{\text{smooth,all}})^{\text{1-op}}$ along the inclusion

$$\text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{isom}} \subseteq \text{Corr}(\text{Arr}(\text{Sch}/S))^{\text{smooth,all,open}}$$
and observe that we have built a commutative diagram of $(\infty, 2)$-functors

\[
\begin{array}{ccc}
\text{Corr}(\text{Arr}(\text{Sch}/S))_{\text{proper}} & \to & \text{Arr}(\text{Sch}/S) \\
\downarrow & & \downarrow \\
\text{Corr}(\text{Arr}(\text{Sch}/S))_{\text{isom}} & \to & (\text{Mod}(\text{Cat}_{\infty})^\text{big}_{2\text{-cat}})^{1\text{-op}}
\end{array}
\]

The formalism of six operations is constructed by gluing these two functors, merging open immersions and proper maps. More precisely, it follows from Nagata’s compactification that any morphism in spft can be written as a morphism in open composed with a morphism in proper. In this sense what we would like is to produce a new $(\infty, 2)$-functor completing the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Corr}(\text{Arr}(\text{Sch}/S))_{\text{proper}} & \to & \text{Arr}(\text{Sch}/S) \\
\downarrow & & \downarrow \\
\text{Corr}(\text{Arr}(\text{Sch}/S))_{\text{isom}} & \to & (\text{Mod}(\text{Cat}_{\infty})^\text{big}_{2\text{-cat}})^{1\text{-op}}
\end{array}
\]

This is solved by the theorem [GR, Thm 5.2.4 Part V.1] which gives necessary and sufficient conditions for the existence and uniqueness of the dotted map. These are the following:

1. (3) [GR, 5.1.2 Part V.1]: the class open ∩ proper consists of embeddings of connected components and therefore are monomorphisms.
2. (4) [GR, 5.2.2 Part V.1]: For any map $f : X \to Y$ separated of finite type, the space $\text{Fact}(f)$ of factorizations of $f$ as an open immersion followed by a proper map, is contractible. This is a consequence of Nagata’s compactification as explained in [GR, Prop 2.1.6 Part II.2].
3. (5) [GR, 5.2.2 Part V.1] It is the well-known support property of Deligne.

**Definition A.1.** We say that an $(\infty, 1)$-functor $F : \text{Sch}_{/S}^{op} \to \text{CAlg}(\text{Pr}_{\text{Stb}}^\text{op})$ has the six operations if it verifies the conditions (1) to (5) above. We denote by $F_{\text{proper}}^{\text{all,spft}}$ its unique extension

\[
\text{Corr}(\text{Arr}(\text{Sch}/S))_{\text{spft,all}}^{\text{proper}} \to \text{Mod}(\text{Cat}_{\infty})^\text{big}_{2\text{-cat}}
\]

Let $F$ verify the six operations. We will use the following notations:
The following result of Ayoub and Cisinski-Deglise gives sufficient conditions for a given $F$ to have the six operations:

**Proposition A.2** (Ayoub and Cisinski-Deglise). Let $F : \text{Sch}_{/S}^{\text{op}} \to \text{CAlg}(\text{Pr}^L_{\text{Stb}})$ be an $\infty$-functor satisfying the following conditions:

a) $F$ satisfies (1);

b) for each proper map $f : X \to Y$, $F_*(f)$ has a right adjoint;

c) (Localization) For every closed immersion $i : Z \hookrightarrow X$ of base schemes with open complementary $U := X - Z \hookrightarrow X$ the commutative diagram

$$
\begin{array}{ccc}
F(Z) & \xrightarrow{F_*(i)} & F(X) \\
\downarrow & & \downarrow F(f) \\
0 & \longrightarrow & F(U)
\end{array}
$$

is a pullback in $\text{Pr}^L_{\text{Stb}}$

d) (Homotopy Invariance) For any base scheme $X$, the map $F(\pi) : F(X) \to F(\mathbb{A}^1_X)$ is fully faithful. Here $\pi : \mathbb{A}^1_X \to X$ is the canonical projection.

e) (Stability) For any base scheme $X$, the composition $F^\natural(\pi) \circ F_*(s) : F(X) \to F(X)$ maps the tensor unit to a $\otimes$-invertible object (the Tate motive).

Then $F$ satisfies all the conditions (1)-(5).

**Proposition A.3.** The $\infty$-functor $\text{SH}^\otimes$ verifies all the conditions of the Prop A.2.

**Proof.** The localization property was proved by Morel-Voevodsky in [MV99, Thm 2.21 pag. 114]. The other conditions follow from the results of Cisinski-Deglise in [CD12] and Ayoub [Ayo07a, Ayo07b]. The fact it satisfies the necessary Beck-Chevalley conditions in the correct $\infty$-sense is proved in [Kha16]. See also the survey in [Robdf, Sections 9.3 and 9.4.1] for an overview and more precise references.  

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30 $s : X \to \mathbb{A}^1_X$ being the zero section.
To conclude this Appendix we discuss the compatibility of the six operations under natural transformations. We have the following result due to Ayoub (for the projective case) and Cisinski-Deglise (for the generalization to proper morphisms)

**Proposition A.4.** (See [Ayo10, Theorem 3.4] and [CD12]) Let $\phi : F \rightarrow G$ be a natural transformation of $\infty$-functors $\text{Sch}_{/S}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{Stb}})$ such that

1. both $F$ and $G$ satisfy the hypothesis of the Proposition A.2.
2. if $f : X \rightarrow Y$ is a smooth map in $\text{Sch}_{/S}$, then the diagram

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\phi_X} & G(X) \\
\downarrow F(f) & & \downarrow G(f) \\
F(Y) & \xrightarrow{\phi_Y} & G(Y)
\end{array}
$$

is left-adjointable

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\phi_X} & G(X) \\
\downarrow F_{!}(f) & & \downarrow G_{!}(f) \\
F(Y) & \xrightarrow{\phi_Y} & G(Y)
\end{array}
$$

is left-adjointable

Then, the natural transformation induced from the adjunctions

$$
G_{!} \circ \phi \rightarrow \phi \circ F_{!}
$$

is an equivalence. Moreover, the natural transformations

$$
\phi \circ F_{*} \rightarrow G_{*} \circ \phi \quad \phi \circ F^{!} \rightarrow G^{!} \circ \phi
$$

are given by equivalences whenever $f$ is proper, respectively, smooth.

**References**


Max Planck Institute for Mathematics, Bonn, Germany
E-mail address: anthony.blanc@ihes.fr

Institut de Mathématiques de Jussieu - Paris Rive Gauche, UPMC, France
E-mail address: marco.robalo@imj-prg.fr

Institut de Mathématiques de Toulouse, Université Paul Sabatier, France
E-mail address: bertrand.toen@math.univ-toulouse.fr

Dipartimento di Matematica ed Informatica, Firenze, Italy
E-mail address: gabriele.vezzosi@unifi.it