LECTURE 1

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Model structure on $\overset{\mathrm{dgmod}_k^{\le 0}}{\mathrm{cdga}_k^{\le 0}}$

{ Kontsevich : "…Good that we don't need model categories anymore." [referring to ∞-categories ofc] }

Modules

k : ring of char 0 (i.e. k is a $\mathbb{Q}$-algebra)

$\mathrm{dgmod}_k^{\le 0}$: $\mathrm{Ch}^{\le 0}(k)$ (cohomological dif.)

$$\cdots \to X^{-2} \xrightarrow{d} X^{-1} \xrightarrow{d} X^0 \to 0$$

Theorem. There is a cofibrantly generated, proper model structure on $\mathrm{dgmod}_k^{\le 0}$ where:

W = q-isos

Fib = surjections in deg < 0

with:

traditional notations

generating trivial cofibrations $\equiv J_{\mathrm{mod}} := \{0 \overset{j_n}{\xrightarrow{\sim}} D^n(k) \mid n > 0\}$

generating cofibrations $\equiv I_{\mathrm{mod}} := \{S^{n-1}(k) \overset{i_n}{\hookrightarrow} D^n(k),\ n > 0\} \cup$
$\cup \{0 \xrightarrow{i_0} \underset{\underset{k[0]}{\shortparallel}}{S^0(k)}\}$

where:

- $S^m(k)$: $\cdots \to 0 \to \overset{\boxed{-m}}{k} \to 0 \to \cdots$: "m-sphere" k-dgmodule $m \ge 0$
- $D^m(k)$: $\cdots \to 0 \to \overset{\boxed{-m}}{k} \xrightarrow{id} \overset{\boxed{-m+1}}{k} \to 0 \to \cdots$: "m-disk" k-dg module $m > 0$

[Rmk. Note that $\mathrm{Hom}_{\mathrm{dgmod}_k^{\le 0}}(D^m(k), C^\cdot) = C^{-m}$, $\forall m > 0$; $\mathrm{Hom}_{\mathrm{dgmod}_k^{\le 0}}(S^m(k), C^\cdot) = Z^{-m}(C^\cdot)$ $\forall m \ge 0$]

(1.2)

<u>Rmks</u>. (1) $\mathrm{Ho}(\mathrm{dgmod}_k^{\leq 0}) \equiv W^{-1}(\mathrm{dgmod}_k) \simeq D^{\leq 0}(k)$

(~~devious~~ tautological). ~~[illegible]~~

~~[illegible]~~

(2) What are generating (trivial) cofibrations good for? [in any cof.ly generated model cat]

- A map is a fibration / trivial fibration iff it has the lifting pty with respect to all generating ~~trivial~~ cofibrations / generating cofibrations.
$\underbrace{\text{generating trivial cofibrations}}_{J}$ / $\underbrace{\text{generating cofibrations}}_{I}$

- (all) cofibrations = retracts of relative I-cell ~~[illegible]~~ maps

(all) trivial cofibrations = retracts of relative J-cell ~~[illegible]~~ maps

what does it mean? In general:

$\mathcal{C}$ category with colimits and $\mathcal{S}$ be a set of maps in $\mathcal{C}$.

(•) an <u>$\mathcal{S}$-cell attachment in $\mathcal{C}$</u> is any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ sitting in a pushout

$$\begin{tikzcd}
\coprod_\alpha X_\alpha \arrow[r] \arrow[d, "\coprod_\alpha s_\alpha"'] & X \arrow[d, "f"] \\
\coprod_\alpha Y_\alpha \arrow[r] & Y \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

$s_\alpha \in \mathcal{S}$

(••) A morphism $g : X \to Y$ in $\mathcal{C}$ is a <u>relative $\mathcal{S}$-cell ~~object~~ map</u> if $\exists$ (possibly transfinite) ordinal λ and a λ-sequence

$X = K_0 \to K_1 \to \dots \to K_\nu \xrightarrow{f_\nu} K_{\nu+1} \to \dots$ $\nu < \lambda$

such that ~~[illegible]~~ $\left\{ \begin{array}{l} \mathrm{colim}_{\nu<\lambda} K_\nu \cong Y;\ f_\nu \text{ is an } \mathcal{S}\text{-cell attachment} \\ \text{the canonical map } X = K_0 \to \mathrm{colim}_{\nu<\lambda} K_\nu = Y \text{ is } f. \end{array} \right.$

(1.3) [in words: "f is ~~the~~ (possibly transfinite) composite of $\mathcal{S}$-cell attachment"]

Rmk. Recall that, in $\mathcal{C}$, $u : A \to B$ is retract of $f : X \to Y$ if the object u is a retract of f in the cat $\mathrm{Arr}(\mathcal{C})$ of maps in $\mathcal{C}$

i.e if $\exists$ maps in $\mathrm{Arr}(\mathcal{C})$

$$u \xrightarrow{i} f \xrightarrow{p} u \quad \text{s.t.} \quad p \circ i = id_u$$

i.e if $\exists$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] \arrow[rr, bend left, "id"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "u"] \\ B \arrow[r] \arrow[rr, bend right, "id_B"'] & Y \arrow[r] & B \end{tikzcd}$$

(p is called ~~the~~ a "retraction of p onto u")

In the case of $\mathrm{dgmod}_k^{\leq 0}$, I_{mod} we have:

Proposition. A map $E \xrightarrow{i} F$ in $\mathrm{dgmod}_k^{\leq 0}$ is a cofibration iff is injective with degreewise projective $\underbrace{\text{cokernels}}_{\text{(in every degree} \leq 0}$ ($\mathrm{coker}(E^n \xrightarrow{i_n} F^n)$ is a k-module and we ask that it is projective as a k-module)

cofibrant dgmodules in $\Leftarrow$ = complexes of projectives i.e cofibrant resolutions are projective resolutions.

Rmk. If k is a field, a cofibration in $\mathrm{dgmod}_k^{\leq 0}$ is then just a map which ~~in degrees ≤ 0~~ is degreewise injective.

Rmk. If I change Fib to Fib': surjective in every degree (≤ 0) Then this + same W, still yields a cofibrantly model structure on $\mathrm{dgmod}_k^{\leq 0}$, whose cofibrations are ~~injective~~ $u : A \to B$ s.t u^i is injective and has projective cokernel $\forall\, i < 0$.

[haven't checked details here, but it sounds ok].

1.4

⌈ **Exercise**. Let X, ~~and~~ $Y \in \mathrm{dgmod}_k^{\leq 0}$ ~~be a map in $\mathrm{dgmod}_k^{\leq 0}$~~.

~~and~~ Let $P(Y^\cdot) := \bigoplus_{n>0} \bigoplus_{y \in Y^{-n}} D^n(k)$.

(1) Construct a "natural" ~~map~~ fibration $P(Y^\cdot) \to Y^\cdot$

(2) Construct, ∀ map $X \xrightarrow{f} Y$ in $\mathrm{dgmod}_k^{\leq 0}$ a factorization of f as

$$X \xrightarrow{u} X \oplus P(Y) \xrightarrow{p} Y$$

s.t $\begin{cases} u \text{ is a trivial cofibration} \\ p \text{ is a fibration.} \end{cases}$ ⌋

1.5

Theorem. $M = dg\,mod_k^{\leq 0}$ is a :

- combinatorial model cat (combinatorial = locally presentable cat + cofibrantly generated model structure ; loc pres := (essentially) $\exists$ set of objects generating all objects by (small colimits)

| Mauro's talk |

- Symmetric monoidal model category:

  (· symm. monoidal cat
  · model cat) compatibly i.e.

→ pushout-product axiom holds i.e $\otimes : M \times M \to M$ is a Quillen bifunctor

or eq.ly, $\forall$ $u : X \to Y$, $u' : X' \to Y'$ cofibrations, the map $X \otimes Y' \coprod_{X \otimes X'} (Y \otimes X') \to Y \otimes Y'$ is a cofibration

$$\begin{tikzcd}
X \otimes X' \arrow[r, "id \otimes u'"] \arrow[d, "u \otimes id"'] & X \otimes Y' \arrow[d] \arrow[rd, bend left, "u \otimes id"] & \\
Y \otimes X' \arrow[r] \arrow[rr, bend right, "id \otimes u'"'] & \square & Y \otimes Y'
\end{tikzcd}$$

[composite is $u \otimes u'$ in both cases]

moreover trivial cof. if either u or u' is a trivial cof. [consequence of the p-p axiom $- \otimes Q$ preserves cofibrations and trivial cof.'s for Q cofibrant]

→ $\forall$ X cofibrant $Q1 \otimes X \xrightarrow{u} 1 \otimes X = X$ $\left(Q1 \overset{\sim}{\twoheadrightarrow} 1 \text{ cof.tres.n} \right)$ is a weak eq.ce [automatic if e is cofibrant like in our case $M = dgmod_k^{\leq 0}$]

- **Freely powered** (as a SMMC)

[see Lurie, HA, 4.5.4.2

Rmks : (i) $\Rightarrow \Sigma_n$ acts freely on $X^{\otimes n}$ (any X)

(ii) [For the experts] [redacted] $\Rightarrow$ as a SMMC satisfies the monoid axiom (Model?) $\Rightarrow$ [redacted] id $\otimes$ trivial cof. is a weak eq.ce

(iii) $dgmod_{\mathbb{F}_p}$ is a [redacted] SMMC satisfying the monoid axiom but not freely powered.]

Theorem (Lurie, HA, Prop. 4.5.4.6) If M is a combinatorial SMMC freely powered,

(1) Then $CommAlg(M)$ has a combinatorial model structure whose fib / W are detected via the forgetful functor $CommAlg(M) \xrightarrow{U} M$

(2) $U : CommAlg(M) \to M$ is right Quillen.

Corollary. $CommAlg(dgmod_k^{\leq 0}) = cdga_k^{\leq 0}$ has combinatorial model structure with W = q-iso, Fib = surjections in deg < 0, [redacted] and

$$\begin{tikzcd} cdga_k^{\leq 0} \arrow[r, shift left, "\overset{Ⓡ}{U}"] & dgmod_k^{\leq 0} \arrow[l, shift left, "\underset{Ⓛ}{Sym_k}"] \end{tikzcd}$$

is a Quillen adjunction.

with

generating trivial cofibrations : $J_{alg} := \mathrm{Sym}_k(J_{mod})$

generating cofibrations : $I_{alg} := \mathrm{Sym}_k(I_{mod})$

(with obvious notations) and note that

$$\mathrm{Sym}\,(S^n(k) \cong k[n]) = \begin{cases} k[x][0], & \text{if } n=0 \\ k \oplus k[n], & n \text{ odd} \\ \text{sq. zero ext.} & \\ k \oplus k[n] \oplus k[2n] \oplus \cdots, & n \text{ even} \end{cases}$$

⌜Exercise. Compute $\mathrm{Sym}_k(D^n(k))$, $n \geq 1$⌟

As in the case of $dgmod_k^{\leq 0}$, we ~~could~~ can characterize cofibrations in $cdga_k^{\leq 0}$; ~~but we won't give explicit formulas here (keyword : relative cell algebra maps / relative Sullivan algebras).~~ ~~at least when~~

$\#$: $cdga_k^{\leq 0} \longrightarrow cga_k^{\leq 0}$ (forget differential)

A map $A \xrightarrow{f} B$ is a semi-free (or quasi-free) extension if

- $\exists$ free graded k-module V and an isomorphism $A^\# \otimes_k \mathrm{Sym}_k V \xrightarrow[\sim]{\varphi} B^\#$ in $A^\#/cga_k^{\leq 0}$

do we need this? (ofc: no condition if k is a field)

(i.e. the diagram

$$\begin{tikzcd} A^\# \arrow[r, "can"] \arrow[dr, "f^\#"'] & A^\# \otimes_k \mathrm{Sym}_k V \arrow[d, "\wr\varphi"] \\ & B^\# \end{tikzcd}$$

commutes (in $cga_k^{\leq 0}$).)

$A \in cdga_k^{\leq 0}$ is semi-free if $\exists$ free graded k-module V and an isomorphism $A^\# \simeq \mathrm{Sym}_k V$ in $cga_k^{\leq 0}$.

Thm :

cofibrations in cdga$_k^{\le 0}$ = retracts of semi-free extensions

cofibrant cdga$_k^{\le 0}$'s = retracts of semi-free cdga$_k^{\le 0}$'s.

see [Berest-Felder-Ramadoss]

1.9

Example of cofibrant resolutions in $\mathrm{cdga}_k^{\leq 0}$ (Koszul)

R : k-algebra (discrete), $r \in R$; consider

$A = R/(r)$. Construction:

$$\mathrm{Kosz}(R;r) := \Big(0 \to \overset{\textcircled{-1}}{R\cdot y} \xrightarrow{d} \overset{\textcircled{0}}{R} \to 0 \Big) \quad \text{is a } \mathrm{cdga}_k^{\leq 0}$$

$d(y) = r$ $\qquad$ $(y^2 = 0)$

(it is $\mathrm{Sym}^{gr}(R[1] \oplus R)$ with differential as indicated) so it is a k semi-free $\mathrm{cdga}_k^{\leq 0}$ $\Rightarrow$ cofibrant

And $\exists$ map of cdga's $\mathrm{Kosz}(R;r) \longrightarrow A$

$$\begin{tikzcd} R \arrow[r, "proj"] & R/(r) \\ Ry \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(obviously a fibration)

Fact. $r \in NZD(R) \Rightarrow \mathrm{Kosz}(R;r) \xrightarrow[qiso]{\sim} A$ (i.e. a cofibrant resolution of A in $\mathrm{cdga}_k^{\leq 0}$)

(if R is a noetherian local ring, then $\Leftarrow$)

More generally: $(r_1, \dots, r_k)$ sequence in R

(i.e. $r_{i+1} \in NZD\left(R/(r_1, \dots, r_i)\right)$, $\forall i$) $\qquad$ NB order of r_i's matters here!!

Then $\exists$ map

$$\mathrm{Kosz}(R;(r_1,\dots,r_k)) := \mathrm{Kosz}(R,r_1) \otimes_k \mathrm{Kosz}(R,r_2) \otimes_k \cdots \otimes \mathrm{Kosz}(R,r_k)$$

$$\downarrow$$

$$A := R/(r_1, \dots r_k)$$

which is a cofibrant resolution if $(r_1, \dots, r_k)$ is a regular seq.ce ($r_i \in NZD\left(R/(r_1, \dots, r_{i-1})\right)$ $\forall i$ $r_1 := 0$) NB order matters here!

(converse ok for R local noetherian).

1) Let k be a field $\mathrm{char}(k) = p > 0$. Prove that cdga_k cannot have a model structure with weak eq $=$ q-iso, fibrations $=$ levelwise surjections.

↦ Hint. (1) Find a map of cdga's $/k$ $f : A \to B$ s.t. $\exists i, \alpha \in H^i(B)$ [i even!] s.t $\alpha^p \notin \mathrm{Im}(H^{pi}(A) \xrightarrow{H^{ip}(f)} H^{pi}(B))$.

Es $A = k^{\mathbb{Z}/p}$, $B = \mathrm{Sym}_k(0 \to \overset{\boxed{-2}}{ky} \to \overset{\boxed{-1}}{k\varepsilon} \to \overset{\boxed{0}}{0})$

show that $d_B(y^2) = 0$ but $y^2 \notin \mathrm{Im}(d^5 : B^{-5} \to B^{-4})$ $\Rightarrow [y^2] \neq 0$ in $H^{-4}(B)$.

(2) Show that any such f cannot be factored as ~~[illegible]~~ fib ∘ triv cofib

[if $\exists$ fact $A \overset{i}{\underset{\sim}{\hookrightarrow}} A' \overset{\pi}{\twoheadrightarrow} B$ then any $[b] \in H^i(B)$, lift to $a \in A'$, $d_{A'}(a'^p) = p\, a'^{p-1} = 0 \Rightarrow [a'^p] \xmapsto{H^{ip}(\pi)} [b]^p$; but $H^*(A) \simeq H^*(A')$] ↤

2) Let k be a field of char 0. Show that the functor $\mathrm{cdga}_k^{\leq} \hookrightarrow \mathrm{dga}_k$ does not induce a fully faithful functor on the homotopy categories.

↦ Hint: (1) Consider $A := k[\overset{\downarrow}{x}, \overset{\downarrow}{y}]$ (deg 0) $\in \mathrm{cdga}_k$

(2) A is cofibrant in cdga_k, hence $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{cdga}_k)}(A, B) \simeq H^0(A) \times H^0(A)$. $\forall B \in \mathrm{cdga}$

(3) A is not cofibrant in dga_k, show that a cofibrant resolution is given by $\mathrm{Free}_{\mathrm{dga}}(x, y, \underset{\substack{\uparrow \\ \deg 1}}{u})$ with $du = xy - yz$. So $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dga})}(A, B) \simeq H^0(B) \times H^0(B) \times H^1(B)$ for $B \in \mathrm{cdga}_k$. ↤

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Mapping spaces in $\mathrm{cdga}_k^{\leq 0}$

Let $A^n_k := k[t_0, \dots, t_n] / (\sum t_i - 1) \in \mathrm{cdg}_k$

$$\Omega^\bullet_n := \Omega^\bullet_{DR, A^n_k} := \overset{0}{k} \to \overset{1}{\Omega^1_{A^n_k/k}} \to \Omega^2_{A^n_k/k} \to \dots \to \Omega^n_{A^n_k/k} \to 0$$

is a cdga (in $\mathrm{cdga}_k^{\geq 0}$).

If $A \in \mathrm{cdga}_k^{\leq 0}$ then

$A \otimes_k \Omega^\bullet_n =$ is a full cdga (in cdga^-_k)

$$\underbrace{A^{-n-1} \otimes \Omega^n_n \oplus \dots \oplus A^{-2} \otimes \Omega^1 \oplus A^{-1}}_{\boxed{-1}} \to \underbrace{\begin{matrix} A^0 \otimes_k K = A^0 \\ \oplus \Omega^1_n \otimes A^{-1} \oplus \dots \\ \oplus \Omega^n \otimes A^{-n}. \end{matrix}}_{\boxed{0}} \xrightarrow{d} \underbrace{A^0 \otimes_k \Omega^1_n \oplus A^{-1} \otimes_k \Omega^2 \oplus \dots \oplus A^{-n+1} \otimes \Omega^n_n}_{\boxed{-1}} \to$$

$\uparrow$ etc $\qquad\qquad \to$ etc

We consider the ~~stupid~~ intelligent truncation $\tau_{\leq 0}(A \otimes_k \Omega^\bullet_n)$

$$\dots \to \bigoplus_{i+j=-1} \Omega^i \otimes_k A^j \to \mathrm{Ker}\, d \to 0$$

This is in $\mathrm{cdga}_k^{\leq 0}$ (memo $d(1) = 0$ $\forall$ $1 \in B^0$, $B \in \mathrm{cdga}$)

Consider $\left\{ \mathrm{Hom}_{\mathrm{cdga}_k^{\leq 0}} (A, \tau_{\leq 0}(B \otimes_k \Omega^\bullet_n)) \right\}_{n \geq 0}$

$A^\bullet_K$ is a ~~co~~simplicial object in $\mathrm{cdg}_k \Rightarrow$

$\Omega^\bullet_{*}$ is a —"— ———— in cdga_k

1.12 hence $\{\tau_{\leq 0}(B \otimes \Omega^{\cdot}_{n})\}_{n}$ is a simplicial object in $\mathrm{cdga}_{k}^{\leq 0}$ and we put

$$\mathrm{Map}_{\mathrm{cdga}_{k}^{\leq 0}}(A,B) := \mathrm{Hom}_{\mathrm{cdga}_{k}^{\leq 0}}(QA, \tau_{\leq 0}(B \otimes_{k} \Omega_{*}^{\cdot}))$$

Where $QA \overset{\sim}{\twoheadrightarrow} A$ is a cofibrant replacement for A in $\mathrm{cdga}_{k}^{\leq 0}$.

Fact. if $\mathrm{Map}(\,,\,)$ denote the mapping space in the model cat $\mathrm{cdga}_{k}^{\leq 0}$ (well defined up to iso in $\mathrm{Ho}(\mathrm{SSets})$), we have an iso

$$\mathrm{Map}(A,B) \simeq \mathrm{Map}_{\mathrm{cdga}_{k}^{\leq 0}}(A,B)$$

in $\mathrm{Ho}(\mathrm{SSets})$.

(i.e. The $\mathrm{Map}_{\mathrm{cdga}_{k}^{\leq 0}}$ above computes the mapping spaces in the model cat $\mathrm{cdga}_{k}^{\leq 0}$)

Exercise. (1) Show that that the obvious adjunction

$$\begin{tikzcd} \mathrm{cdga}_{k}^{\leq 0} \arrow[r, hook, shift right, "i"', "\text{(L)}"' near end] & \mathrm{cdga}_{k} \arrow[l, shift right, "\tau_{\leq 0}\ \text{(R)}"'] \end{tikzcd}$$

is a Quillen adjunction

(model structure on cdga_{k}: fib's = surjections in any degree
W = q-iso)

(2) Suppose to know that $\mathrm{Map}_{\mathrm{cdga}}(C,D) \simeq (\mathrm{Hom}_{\mathrm{cdga}}(C, D \otimes \Omega^{\cdot}_{n}))_{n}$ in $\mathrm{Ho}(\mathrm{SSets})$

~~(3) Suppose known that $\mathrm{Map}_{\mathrm{cdga}}(C,D) \simeq (\mathrm{Hom}_{\mathrm{cdga}}(C, D \otimes \Omega^{\cdot}_{n}))_{n \geq 0}$~~

Show that, $\forall A, B \in \mathrm{cdga}_{k}^{\leq 0}$:

$$\mathrm{Map}_{\mathrm{cdga}_{k}^{\leq 0}}(A,B) \simeq \left(\mathrm{Hom}_{\mathrm{cdga}}(A', B \otimes_{k} \Omega^{\cdot}_{n})\right)_{n \geq 0} \simeq$$

$\simeq \mathrm{Map}_{\mathrm{cdga}}(A,B)$, where $A' \overset{\sim}{\twoheadrightarrow} A$ cof. repl't in $\mathrm{cdga}_{k}^{\leq 0}$

1.13 Rmk.

With $$\underline{\mathrm{Hom}}(A,B) := \left(\mathrm{Hom}_{\mathrm{cdga}_k^{\leq 0}}(A, B\otimes\Omega^\bullet_n)\right)_{n\geq 0}$$
$\in$ ssets

$\mathrm{cdga}_k^{\leq 0}$ is actually a (combinatorial) simplicial model category.

[Model for def. of simpl. model cat. & Hinich, Homological algebra of homotopy algebras for the proof of the relevant facts].

1.74

Derived pushouts in $cdga_k^{\leq 0}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$ in $cdga_k^{\leq 0}$

$\Longrightarrow$ general fact in any model cat. $\exists$ holim of this i.e. the h-pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{h}{\lrcorner}", very near end] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$ in $cdga_k^{\leq 0}$

How to compute D ? $cdga_k^{\leq 0}$ is $\overbrace{\text{(left) proper}}^{\text{pushout of } W \text{ along cof.s}}$ a $W(*)$

let $A \hookrightarrow B' \overset{\sim}{\twoheadrightarrow} B$ (or $A \hookrightarrow C' \overset{\sim}{\twoheadrightarrow} C$) a factorization of $A \to B$ (or $A \to C$) then $D \simeq B' \otimes_A C$ (or $B \otimes_A C'$) iso in $Ho(cdga_k^{\leq 0})$. [Recall that $\text{holim}_I : Ho(M^I) \longrightarrow Ho(M)$].

<u>Examples</u>. →

(*) in ANY model cat the pushout of a W btw cofibrant objects along a cof. is a W.

1.15 An example of derived $\cap$: derived zero locus (affine case)

R : k-alg (discrete) ; P : R-module, projective

$\sigma \in P \Leftrightarrow \underset{= ev_\sigma}{\sigma} : R \leftarrow P^\vee$ (R-dual) of R-modules $\Leftrightarrow \psi_\sigma : \mathrm{Sym}_R P^\vee \to R$ of R-algebras $\underset{\shortparallel}{X}$

σ is a section of the linear scheme $V(P) := \mathrm{Spec}(\mathrm{Sym}_R(P^\vee)) \to \mathrm{Spec} R$

Aim Construct the derived fiber product (in derived schemes$/_k$):

$$\begin{tikzcd} dZero(s) \arrow[r] \arrow[d, "\pi"] & X \arrow[d, "\sigma"] \\ X \arrow[r, "0"'] & V(P) \end{tikzcd}$$

i.e the homotopy pushout

$$\Updownarrow$$

$$\begin{tikzcd} S := \mathrm{Sym}_R(P^\vee) \arrow[r, "0"] \arrow[d, "\psi_\sigma"'] & R \arrow[d] \\ R \arrow[r] \arrow[ur, phantom, "h"] & B \end{tikzcd}$$

in $cdga_k^{\leq 0}$

[NB. S is graded but we will ignore its grading!]

1st step : resolve $S \xrightarrow{0} R$ (i.e find a cofibrant replacement in $S/cdga_k^{\leq 0}$). We do a Koszul resolution construction:

$K(R;P) := S \otimes_R \Lambda_R^{\bullet} P$, it
- is ≤ 0 graded commutative algebra
- has a deg $+1$ differential induced by contraction and the map $h : R \xrightarrow{can} \mathrm{Hom}_R(P,P) \underset{\underset{P \text{ proj}}{\uparrow}}{\simeq} P^\vee \otimes_R P$

if $h(1) = \sum_j \alpha_j \otimes x_j$

$d(\underset{\underset{S}{\pitchfork}}{a} \otimes (\beta_1 \wedge \dots \wedge \beta_{n+1})) = \sum_j a \cdot \alpha_j \otimes \sum_k (-1)^k \beta_k(x_j) \, \beta_1 \wedge \dots \wedge \check{\beta_k} \wedge \dots \wedge \beta_{n+1}$

d is S-linear
- has augmentation over R:

$K(R,P) \to K(R,P)_0 \to S \to R$ (a map of $S/cdga_k^{\leq 0}$)

$\mathrm{Sym}_S(S \otimes_R P[1]) \simeq S \otimes_R \mathrm{Sym}_R^{\bullet}(P[1])$

Fact : the augmentation $K(R,P) \to R$ is a q-iso.

$\Downarrow$ So, since $\mathrm{Sym}_S(S \otimes_R P[1])$ is free as a graded algebra over S we get $K(R,P) \to R$ is a cof resolution in $S/cdga_k^{\leq 0}$

Briefly:

1.18

Model structure on A-dgmod, $A \in cdga_k^{\leq 0}$

$A \in cdga_k^{\leq 0} \rightsquigarrow$ A-dgmod : $<$ Fib, W detected via the forgetful $\mathcal{U}$ to $dgmod_k$ (surj'ive q(iso))

A-dgmod$^{\leq 0}$ $<$ fib, W detected via the forgetful $\mathcal{U}^{\leq 0}$ to $dgmod_k^{\leq 0}$

are (combinatorial SM) MC's.

$Ho(A\text{-dgmod}) \simeq D(A)$
$Ho(A\text{-dgmod}^{\leq 0}) \simeq D^{\leq 0}(A)$ (as triangulated categories)

Moreover:

<u>Theorem</u> (Invariance) Let $f : A \to B$ in $cdga_k^{\leq 0}$.

Then

$$\begin{tikzcd} A\text{-dgmod} \arrow[r, shift left, "f^* \text{Ⓛ}"] & B\text{-dgmod} \arrow[l, shift left, "f_* \text{Ⓡ}"] \end{tikzcd}$$

and

$$\begin{tikzcd} A\text{-dgmod}^{\leq 0} \arrow[r, shift left, "f^* \text{Ⓛ}"] & B\text{-dgmod} \arrow[l, shift left, "f_* \text{Ⓡ}"] \end{tikzcd}$$

are Quillen ~~equivalence~~ adjunctions $\forall f$ and f q-iso $\Rightarrow$ Quillen equivalence

(i.e. Quillen adjunctions inducing an (adjoint) eq.ce on the homotopy categories:

$\mathbb{L}f^* : Ho(\ ) \leftrightarrows Ho(-) f_* \equiv \mathbb{R}f_*$).

⌈ <u>Rmk</u> (1) Obviously $Comm(A\text{-dgmod}^{(\leq 0)}, \otimes) \simeq A/cdga_k^{(\leq 0)}$

(2) In particular $A\text{-dgmod}^{(\leq 0)} \underset{Quillen}{\simeq} A'\text{-dgmod}$

$\forall$ cofibrant resolution A' of A. ⌋

dg Kähler diff's

$A \xrightarrow{f} B$ of cdga's $\quad M \in B\text{-dgmod}_k^{\le 0}$

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$$\underline{\mathrm{Der}}_A^n(B,M) = \{ \partial \in \underline{\mathrm{Hom}}_A^n(B,M) \mid \partial(bb') = \partial(b)b' + (-1)^{n|b|} b\,\partial(b') \}$$

(degree n-derivations)

n-th deg. of hom-complex (=internal hom in A-dgmod$_k^{\le 0}$)

$\left(\text{i.e. } \underline{\mathrm{Der}}_A^n(B,M) = \{ \partial \in \underline{\mathrm{Hom}}_{k\text{-dgmod}}^n(B,M) \mid \text{''} \text{ and } \partial(A)=0 \} \right)$

Induced differential

$$\underline{\mathrm{Der}}_A^n(B,M) \xrightarrow{d} \underline{\mathrm{Der}}_A^{n+1}(B,M)$$

$$\partial \longmapsto d\partial = d_M\partial - (-1)^n \partial d_A .$$

Hence $\underline{\mathrm{Der}}_A^n(B,M)$ is a B-dgmodule via $(b\cdot\partial)(b') := b\cdot\partial(b')$.

Rmk. If $\partial \in \underline{\mathrm{Der}}_A^n(B,M)$ and $\varphi \in \underline{\mathrm{Hom}}_{B\text{-dgmod}}^m(M,N) \Rightarrow \varphi\circ\partial \in \underline{\mathrm{Der}}_A^{n+m}(B,N)$.

Proposition. $\exists$ B-dgmodule $\Omega_{B/A}$ endowed with a A-derivation $\delta : B \to \Omega_{B/A}$ of degree 0 ($\delta \in Z^0(\underline{\mathrm{Der}}_A(B,\Omega_{B/A}))$) such that the pre-composite with δ induce an iso of B-dgmodules

$$\underline{\mathrm{Der}}_A(B,M) \xleftarrow{\delta\circ(-)} \underline{\mathrm{Hom}}_{B\text{-dgmod}}(\Omega_{B/A}, M).$$

(unbounded a priori)

Exercise. Prove the Prop. by exporting the classical non-dg proof to the dg case.

Rmk. (2) Unfortunately the construction $(A \to B) \mapsto \Omega_{B/A}$ is not invariant under q-isos in $A\text{-cdga}_k^{\le 0}$! [see next Exercise]

(no underline) (1) Important, though easy, observation: I might have considered just $\mathrm{Der}_A(B,-) : B\text{-dgmod} \to \mathrm{Sets} : M \mapsto Z^0\underline{\mathrm{Der}}_A(B,M) = \{ \partial : B \to M \text{ of A-dgmod} \mid \partial(bb') = \dots \}$ and then $\mathrm{Der}_A(B,-)$ is representable by $\Omega_{B/A}$!! So $\mathrm{Der}_A(B,M)$ reconstruct $\underline{\mathrm{Der}}(B,M)$!!

↦ Pf. $A \xrightarrow{f} B$; consider

$$\tilde{\Omega}_{B/A} := \bigoplus_{\substack{x \in B \\ \text{homogeneous}}} B\,\delta x$$

- graded by $|\delta x| = |x|$, $|b\,\delta x| = |b| + |x|$
- B-structure $b'.(b\,\delta x) := (b'b)\,\delta x$
- differential : $d(b\,\delta x) := (d_B b)\,\delta x + (-1)^{|b|}\, b\,\delta(d_B x)$
  (so that $d(\delta x) = \delta(dx)$).
- $N_{B/A} :=$ homogeneous sub-B-dgmodule generated by the subset $\{ \delta(x+y) - (\delta x + \delta y),\ \delta(xy) - x\,\delta y - (-1)^{|x||y|}\, y\,\delta x,\ \delta(f(a)) \mid a \in A,\ x, y \in B^{\text{hom}} \}$.

Verify that $d(N) \subseteq N \Rightarrow \tilde{\Omega}_{B/A}/N$ is a B-dgmodule$^{\leq 0}$.

Put $\Omega_{B/A} := \tilde{\Omega}_{B/A}/N$ with universal derivation

$$\begin{tikzcd} B \arrow[r, "\underline{\delta}"] & \Omega_{B/A} \\ b \arrow[r, mapsto] & {[\delta b]} . \end{tikzcd}$$

Verify that $(\Omega_{B/A}, \underline{\delta})$ does the job. ↤

~~[illegible]~~

~~[illegible]~~

<u>Exercise</u>. Give an example of A, B, B' s.t. $B \underset{qiso}{\simeq} B'$ as A-algebra's but such that $\Omega_{B/A} \underset{qiso}{\not\simeq} \Omega_{B'/A}$ (i.e. the Kahler B-dgmodule is not invariant under equivalence)

$\big[$ $A = k[x]$ $\quad B = k \quad B' = 0 \to k[t] \xrightarrow{t} k[t] \to 0$ $\big]$

cotangent complex

Remark. Let $f : A \to B$ in $\mathrm{cdga}_k^{\leq 0}$, $M \in B\text{-dgmod}_k^{\leq 0}$.
We can express $\underline{\mathrm{Der}}_A(B,M)$ in a less "formulaic" way:

$$Z^0\left(\underline{\mathrm{Der}}_A(B,M)\right) \simeq \mathrm{Hom}_{A/\mathrm{cdga}_k^{\leq}/B}(B, B\oplus M) =$$

$$(b,m)\cdot(b',m') := (bb', bm' + b'm)$$

$$\partial \longmapsto (\mathrm{id}_B, \partial)$$

$$= \mathrm{fib}\left(\mathrm{Hom}_{A/\mathrm{cdga}}(B, B\oplus M) \xrightarrow{pr_{B*}} \mathrm{Hom}_{A/\mathrm{cdga}}(B,B) ; \mathrm{id}_B\right)$$

$$\left(pr_B : B\oplus M \to B,\ (b,m) \mapsto b\right)$$

This suggests immediately a:

Definition. The space of $\overbrace{\text{derived } A\text{-derivations from}}^{\mathbb{R}\mathrm{Der}_A(B,M)}$ B to a dg module M is

$$\mathrm{Map}_{A/\mathrm{cdga}/B}(B, B\oplus M) \in \mathrm{Ho}(\mathrm{SSets})$$

or (isomorphic in $\mathrm{Ho}(\mathrm{SSets})$).

$$\mathrm{hofib}\left(\mathrm{Map}_{A/\mathrm{cdga}}(B, B\oplus M) \xrightarrow{pr_{B*}} \mathrm{Map}_{A/\mathrm{cdga}}(B,B) ; \mathrm{id}_B\right) \in \mathrm{Ho}(\mathrm{SSets})$$

Luckily, we have:

Theorem. $\exists\ \mathbb{L}_{B/A} \in \mathrm{Ho}(\mathrm{dgmod}_B)$ and an iso in $\mathrm{Ho}(\mathrm{SSets})$

$$\mathrm{Map}_{B\text{-dgmod}}(\mathbb{L}_{B/A}, M) \simeq \mathbb{R}\mathrm{Der}_A(B,M),$$

natural in M.

1.23 ↦ Proof. $$\begin{tikzcd} Q_A B \arrow[r, "\sim"] & B \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$ a cofibrant replacement of $B \in A/\text{cdga}$'s

$$\mathbb{L}_{B/A} := \Omega_{Q_A B/A} \underset{Q_A P}{\otimes} B \quad \text{works} \dashv$$

[Rmk. (1) Note that $\mathbb{L}_{B/A}$ is well defined up to iso in Ho(B-dgmod) $\simeq$ Ho($Q_A B$-dgmod) by invariance (see ...). So we might have put $\mathbb{L}_{B/A} := \Omega_{Q_A B/A}$.]

$$\begin{tikzcd} B\text{-dgmod}^{\leq 0} \arrow[r, hook, "\textcircled{R}"] & B\text{-dgmod} \arrow[l, bend left, "\tau_{\leq 0}\,\textcircled{L}"] \end{tikzcd}$$
adjoint.

(2) A priori, $\mathbb{L}_{B/A} \in$ B-dgmod (unbounded) but since we allow M only in B-dgmod$^{\leq 0}$ i.e. $B \simeq \tau_{\leq 0} B$ we have that the canonical $\tau_{\leq 0}\mathbb{L}_{B/A}$ is a map $\mathbb{L}_{B/A} \xleftarrow{\sim}$ q.iso. So, we ~~may~~ (and will) assume $\mathbb{L}_{B/A} \in$ Ho(B-dgmod$^{\leq 0}$).]

$\mathbb{L}_{B/A}$ is called the cotangent complex of $A \to B$. [put here $\mathbb{L}$ Ab remark!]

Functoriality

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "u"'] & B' \end{tikzcd}$$ comm. in Ho(cdga$_k^{\leq 0}$)

$\Rightarrow \mathbb{L}_{B/A} \longrightarrow u_* \mathbb{L}_{B'/A'}$ in Ho(B-dgmod) or, equivalently

$$\mathbb{L}_{B/A} \overset{\mathbb{L}}{\otimes}_B B' \simeq \mathbb{L}u^* \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B'/A'}$$

$\boxed{\text{II Ab}}$ and the cotgt complex

$\mathcal{C}$ category (~~w~~ with limits), $\mathcal{C}_{ab}$: abelian group objects in $\mathcal{C}$.

$$\mathcal{C}_{ab} \xrightarrow[\text{forget}]{} \mathcal{C}$$

If there exists a left adjoint $\mathcal{C} \to \mathcal{C}_{ab}$ we call this functor Ab.

Examples : (1) Ab : Ssets $\longrightarrow$ sAb : $X \longmapsto \mathbb{Z}X$ free abelian group on X

(2) $\mathcal{C} =$ ~~R-alg~~ R-calg (R comm ring)

$\Rightarrow \mathcal{C}_{ab} = \{0\}$ (0 is the terminal object of $\mathcal{C}$)

(3) $\underbrace{\mathcal{C} = R/calg/S}_{S \in R/calg} \underset{\text{Quillen}}{\Longrightarrow} \mathcal{C}_{ab} \simeq S\text{-mod}$
$S \oplus M \longleftarrow M$

and Ab : $R/calg/S \longrightarrow S$-mod exists

$B \longmapsto$ ~~B~~ $S \underset{B}{\otimes} \Omega_{B/R}$

(in fact forget : S-mod $\to R/calg/S : M \longmapsto S \oplus M$

and $\mathrm{Hom}_{R/calg/S}(B, S\oplus M) \simeq \mathrm{Der}_R(B, M_{[\varphi]})$

$\simeq \mathrm{Hom}_{B\text{-mod}}(\Omega_{B/R}, M_{[\varphi]}) \underset{\text{adj.}}{\simeq}$

$\simeq \mathrm{Hom}_{S\text{-mod}}(\Omega_{B/R} \underset{B}{\otimes} S, M)$).

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & B \arrow[d, "\varphi"] \\ & S \arrow[d, squiggly] \\ & M \end{tikzcd}$$

Now observe that ~~[illegible]~~ if $A \to B$ in $\mathrm{cdga}_k^{\leq 0}$: (1) we still have $\left(A/\mathrm{cdga}_k^{\leq 0}/B\right)_{ab} \simeq B\text{-dgmod}^{\leq 0}$

(2) The adjoint pair:

$$\begin{tikzcd} B\text{-dgmod}^{\leq 0} \arrow[r, shift right, "\text{forget}"'] & A/\mathrm{cdga}_k^{\leq 0}/B \arrow[l, shift right, "Ab = \Omega_{C/A} \otimes_C B"'] \end{tikzcd}$$

C

$M \mapsto B \oplus M$

Ⓡ

is a Quillen adjunction. Note that $Ab(B) = \Omega_{B/A}$

So: $\exists\ \mathbb{L}Ab : \mathrm{Ho}\left(A/\mathrm{cdga}_k^{\leq 0}/B\right) \to \mathrm{Ho}\left(B\text{-dgmod}^{\leq 0}\right)$

and $\mathbb{L}Ab(B) \simeq \mathbb{L}_{B/A}$.

1.26

~~Fact~~ ⇒ ∃ natural map $\mathbb{L}_{B/A} \to \mathbb{L}_{H^0B/H^0A}$.

Moreover, ∀ $S \to R$ of discrete comm. k-algebras we have

$$\begin{tikzcd} Q_S R \arrow[r, "\sim", "qiso"'] & R \\ S \arrow[u] \arrow[r, equal] & S \arrow[u] \end{tikzcd} \overset{\text{functoriality of } \Omega_{-/S}}{\Longrightarrow} \overset{\mathbb{L}_{R/S}}{\overset{\shortparallel}{\Omega_{Q_S R} \underset{Q_S R}{\otimes} R}} \longrightarrow \Omega^1_{R/S}[0]$$

(concentrated in degree 0)

$$\Rightarrow \quad H^0(\mathbb{L}_{R/S}) \to \Omega^1_{R/S}$$

<u>Fact</u>: This is an iso (Exercise: prove this)

Upshot (of functoriality of $\mathbb{L}$ + <u>Fact</u>):

The canonical map

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{H^0B/H^0A} \quad \text{induces an iso on } H^0.$$

<u>Basic tools for computing $\mathbb{L}$</u>:

1) <u>Transitivity sequence</u>: $C \to A \to B$ in $cdga_k^{\leq 0}$

$$\mathbb{L}_{A/C} \overset{\mathbb{L}}{\underset{A}{\otimes}} B \longrightarrow \mathbb{L}_{B/C} \longrightarrow \mathbb{L}_{B/A}$$ is a cofiber sequence (exact triangle) in $D^{\leq 0}(B)$

(already useful when $C = k$).

2) h-pushouts I: Let
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\lrcorner"] & D \end{tikzcd}$$
$(\simeq B \overset{\mathbb{L}}{\underset{A}{\otimes}} C)$ be a h-pushout in $\mathrm{cdga}_k^{\leq 0}$.

Then ~~[illegible]~~

(2i) The canonical map $\mathbb{L}_{B/A} \overset{\mathbb{L}}{\underset{B}{\otimes}} D \longrightarrow \mathbb{L}_{D/C}$ is a q-iso [co-base change p.ty]

(2ii) The natural map

$$\mathbb{L}_{C/A} \underset{C}{\otimes} D \oplus \mathbb{L}_{B/A} \underset{B}{\otimes} D \longrightarrow \mathbb{L}_{D/A}$$

(coming from the 2 transitivity sequences for
$$A \to C \to D$$
$$A \to B \to D \,)$$
is a q-iso

3) h-pushouts II: Let
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\lrcorner"] & D \end{tikzcd}$$
h-pushout in $\mathrm{cdga}_k^{\leq 0}$

Then the square

$$\begin{tikzcd} \mathbb{L}_A \underset{A}{\otimes} D \arrow[r] \arrow[d] & \mathbb{L}_B \underset{B}{\otimes} D \arrow[d] \\ \mathbb{L}_C \underset{C}{\otimes} D \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathbb{L}_D \end{tikzcd}$$

is h-~~pu~~pushout in $D(D)$

or, equivalently:

$$\mathbb{L}_A \underset{A}{\otimes} D \longrightarrow \mathbb{L}_C \underset{C}{\otimes} D \oplus \mathbb{L}_B \underset{B}{\otimes} D \longrightarrow \mathbb{L}_D$$

is a triangle in $D(D)$.

Exercise. Let $f \neq 0$ in $k[x_1, \dots, x_n]$, and $A = \frac{k[x_1, \dots, x_n]}{(f)}$

Compute $\mathbb{L}_{A|k}$ using the Koszul resolution.

$\mapsto$ $\mathrm{Kosz}(k[x_1, \dots, x_n]; f) \xrightarrow{\sim} A$ is a cofibrant repl't.

$$\begin{tikzcd} \mathrm{Kosz}(k[x_1,\dots,x_n];f) \arrow[d, phantom, "\parallel"] \\ 0 \to k[x_1,\dots,x_n]y \xrightarrow{d} k[x_1,\dots,x_n] \to 0 \end{tikzcd}$$

$$dy = f \qquad \boxed{y^2 = 0}$$

So $\mathbb{L}_{A/k} \simeq \Omega_{\mathrm{Kosz}/k} \otimes_{\mathrm{Kosz}} A$.

Now:

$\Omega_{\mathrm{Kosz}/k} = \left\{ \overset{\text{Exercise:}}{\text{compute}} \text{ this } \overset{\text{directly}}{\text{using}} \text{ the } \overset{\text{general}}{\text{construction}} \text{ given in the proof of } \exists\, \Omega_{B/A} \right\} =$

- $\Omega^\#_{\mathrm{Kosz}/k} = \mathrm{Kosz} \cdot \delta x_1 \oplus \dots \oplus \mathrm{Kosz} \cdot \delta x_n \oplus \mathrm{Kosz} \cdot \delta y$
  $|\delta y| = -1 \quad |\delta x_i| = 0$
- differential:
  $$\begin{cases} d(\delta x_i) = 0 \\ d(\delta y) = \delta(dy) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \delta x_i \end{cases}$$

Therefore: $\mathbb{L}_{A/k} \underset{\uparrow}{\simeq} \quad 0 \to A \cdot \delta y \xrightarrow{d} \bigoplus_{i=1}^n A \delta x_i \to 0$

(recall : A sits in degree 0) where d is A-linear and
$d(\delta y) = \sum \partial_i f \, \delta x_i$ $\dashv$

⌜ **Exercise** Compute $\mathbb{L}_{A/k}$ for $A = \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_r)}$.
$\nearrow$ a regular sequence. ⌟

Example. A smooth discrete k-algebra

$I_i \to A \twoheadrightarrow B_i$ $\quad i = 1,2 \quad$ ($\operatorname{Spec} B_i \hookrightarrow \operatorname{Spec} A$ smooth)

Derived intersection :

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B_1 \arrow[d] & \\
B_2 \arrow[r] & C \arrow[ul, phantom, "\lrcorner"] \arrow[dr, "x"] & \\
 & & k
\end{tikzcd}$$

h-pushout in $\mathrm{cdga}_k^{\leq 0}$

[note that $\operatorname{Spec} H^0(C)$ is the scheme-theoretic $\cap$]

x : point (denoted since k is in deg 0 so
$$\begin{tikzcd}
C \arrow[r, "x :"] \arrow[d] & k \\
H^0(C) \arrow[ur, "x"'] &
\end{tikzcd}$$
)

$$x^*\left(\mathbb{L}_A \overset{L}{\underset{A}{\otimes}} C \to \mathbb{L}_{B_1} \overset{L}{\underset{B_1}{\otimes}} C \oplus \mathbb{L}_{B_2} \overset{L}{\underset{B_2}{\otimes}} C \to \mathbb{L}_C\right) =$$

$$= \mathbb{L}_{A,x} \to \mathbb{L}_{B_1,x} \oplus \mathbb{L}_{B_2,x} \to \mathbb{L}_{C,x}$$

$\uparrow$

| abuse of notations : x is the point in C, B_1, B_2, A |

dualize $/k$ to $\mathbb{T}_{C,x} \to \mathbb{T}_{B_1,x} \oplus \mathbb{T}_{B_2,x} \to \mathbb{T}_{A,x}$

(complexes in degrees ≥ 0) $\Rightarrow$ les in cohomology

$$\underset{\substack{\text{IS} \\ T_{H^0(C),x}}}{H^0(\mathbb{T}_{C,x})} \to \underset{\substack{\text{S1} \\ T_{B_1,x}}}{H^0(\mathbb{T}_{B_1,x})} \oplus \underset{\substack{\text{S1} \\ T_{B_2,x}}}{H^0(\mathbb{T}_{B_2,x})} \to \underset{\substack{\wr \\ |A \text{ smooth}|}}{T_{A,x}} \to H^1(\mathbb{T}_{C,x}) \to H^1(\mathbb{T}_{B_1,x}) \oplus H^1(\mathbb{T}_{B_2,x}) \to \underset{\substack{\uparrow \\ |A \text{ smooth}|}}{0}$$

$T_{H^0(C),x}$ usual tgt space of scheme theoretic intersection

| 0 if B_1, B_2 are smooth, too |

So : $\underset{\substack{\uparrow \\ \text{derived } \cap}}{H^1(\mathbb{T}_{C,x})}$ witnesses non $\pitchfork$ of intersection.

Basic notions of derived commutative algebra

1.30

DAG

$f: A \to B$ in $\mathrm{cdga}_k^{\leq 0}$ is **hfp** if $\forall$ filtered diagram $I \to C$ in $A/\mathrm{cdga}_k^{\leq 0}$, the canonical map

$$\operatorname{hocolim}_I \mathrm{Map}_{A/\mathrm{cdga}_k^{\leq 0}}(B, C_i) \to \mathrm{Map}_{A/\mathrm{cdga}_k^{\leq 0}}(B, \operatorname{hocolim} C_i)$$

is iso in Ho(SSets).

Proposition. $f: A \to B$ is hfp iff

- $H^0(f): H^0A \to H^0B$ is fp (classically)
- $\mathbb{L}_{B/A}$ is perfect as B-dg-module (i.e. dualizable in B-dg-module)

or, eq.ly, $\mathbb{L}_{B/A}$ is hfp in B-dg-modules (i.e. $\forall$ $I \to B\text{-dgmod}^{\leq 0}$, $i \mapsto M_i$ filtered diagram,

$$\operatorname{hocolim}_{i\in I} \mathrm{Map}_{B\text{-dgmod}^{\leq 0}}(\mathbb{L}_{B/A}, M_i) \xrightarrow{\sim} \mathrm{Map}_{B\text{-dgmod}^{\leq 0}}(\mathbb{L}_{B/A}, \operatorname{hocolim}_I M_i))$$

iff: $\exists$ $A = A_0 \to A_1 \to \dots \to A_n = B$ s.t

$$\begin{tikzcd} A_i \arrow[r] & A_{i+1} \\ C \arrow[u] \arrow[r, "u"] & C' \arrow[u] \end{tikzcd}$$

pushout, $u \in I_{\mathrm{alg}}$ (generating cofibrations)

cell attachment

[i.e. B is obtained from A by finitely many cell-attachment)

B is a finite cell A-algebra i.e.:

$f: A \to B$ **hflat** if $\mathbb{L}f^* = - \overset{\mathbb{L}}{\underset{A}{\otimes}} B : \overbrace{\mathrm{Ho}(A\text{-dgmod}_k^{\leq 0})}^{D^{\leq 0}(A)} \to \underbrace{\mathrm{Ho}(B\text{-dgmod}_k^{\leq 0})}_{D^{\leq 0}(B)}$ preserves homotopy pullbacks

AG

$f: R \to S$ of discrete comm k-algebras is **finitely presented** if $\forall$ filtered diagram $I \to C$, $i \mapsto C_i$ in R/calg_k the canonical map

$$\operatorname{colim}_I \mathrm{Hom}_{R/\mathrm{calg}_k}(B, C_i) \to \mathrm{Hom}_{R/\mathrm{calg}_k}(B, \operatorname{colim}_i C_i)$$

is iso

Rmk. So $R \to S$ fp $\not\Rightarrow$ $R \to S$ hfp.

But $R \to S$ fp and lci $\Rightarrow$ $R \to S$ hfp

$f: R \to S$ of discrete comm k-alg. is **flat** if $f^* = - \underset{R}{\otimes} S : R\text{-mod} \to S\text{-mod}$ preserves pullbacks ($\Leftrightarrow$ preserves kernels)

DAG

$f: A \to B$ is formally h-étale if $\mathbb{L}_{B/A} \simeq 0$ in $D^{\leq 0}(B)$

$f: A \to B$ is formally h-smooth if $\mathrm{Hom}_{D^{\leq 0}(B)}(\mathbb{L}_{B/A}, M) = 0$ $\forall\, M \in B\text{-dgmod}^{\leq 0}$ s.t. $H^0(M) = 0$ (i.e. M has, possibly, only cohomology in degrees ≤ -1)

$f: A \to B$ is h-étale if it is formally h-étale and hfp

$f: A \to B$ is h-smooth if it is formally h-smooth and hfp

AG

$f: R \to S$ is formally étale if $\mathbb{L}_{S/R} \simeq 0$ in $D(S)$

$f: R \to S$ is formally smooth if $\mathrm{Hom}_{D(S)}(\tau_{\geq -1}\mathbb{L}_{S/R}, M) = 0$ $\forall\, M \in S\text{-dgmod}_k^{\leq 0}$ s.t. $H^0(M) = 0$

[Rmk. This is not the standard def. of formally smooth but it's equivalent to it.

Exercise. (1) Show that f is formally smooth in the classical def.n iff $\underbrace{\tau_{\geq -1}\mathbb{L}_f}_{\text{"naive cotangent complex" in Stacks proj.}} \underset{D(S)}{\simeq} P[0]$ with P projective S-module

(2) Conclude that our def. of formally smooth is equivalent to the standard one.]

[Rmk. It is crucial to consider $\tau_{\geq -1}\mathbb{L}_f$ in AG in the above def.! $\exists\, R = k \xrightarrow{f} S$ k field s.t. f is formally étale ($\Rightarrow$ formally smooth) but $H^2(\mathbb{L}_f) \neq 0$ (while $H^0 = H^{-1} = 0$, obviously).]

$f: R \to S$ is étale if it is formally étale and fp.

$f: R \to S$ is smooth if it is formally smooth and fp.

DAG

$f: A \to B$ an h-Zariski open immersion if it is h-flat, hfp and the natural map $B \otimes_A^{\mathbb{L}} B \to B$ is a q-iso.

AG

$f: R \to S$ is a Zariski open immersion if it is flat, fp and $S \otimes_R S \xrightarrow{\mu} S$ is an isomorphism.

Rmk (1) All the given notions are stable under composition and co-base change (h-cobase change in DAG)

(2) h-flat $\Leftarrow$ h-smooth $\Leftarrow$ h-étale $\Leftarrow$ h-Zar open imm.

Exercise [formally ^{h}étale ⇒ formally thh étale]

Assume (or prove) the following fact:

$A \xrightarrow{f} B$ formally h-étale and $H^0(f)$ iso

$\Rightarrow f$ q-iso.

Prove that if $f: A \to B$ is formally étale then the canonical map $t: B \to B \overset{\mathbb{L}}{\otimes} B$ is a q-iso.

$$\begin{tikzcd}
B \otimes^{\mathbb{L}}_A B \arrow[r, "b \mapsto b\otimes 1"] \arrow[d, "b \mapsto 1\otimes b"'] & B \arrow[d] \\
B \arrow[r] & B \overset{\mathbb{L}}{\otimes} B \arrow[ul, phantom, "h\lrcorner", very near start]
\end{tikzcd}$$

$B \otimes^{\mathbb{L}}_A B$

$B \otimes^{\mathbb{L}}_A B = C$

[**Hint**: compute $\mathbb{L}_{C/B} \equiv \mathbb{L}_t$ and ...]

Remarks. (1) $f: R \to S$ is formally smooth iff

- Ω_f is a projective S-module
- the sequence $0 \to \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0$ is split exact.

and, analogously, $f: A \to B$ is formally h-smooth iff

[this is the def. in HAG II] →

- $\mathbb{L}_{B/A}$ is h-projective (i.e. a retract of $\oplus B$ in $\mathrm{Ho}(B\text{-dgmod}^{\leq 0}_k)$)
- The canonical map $\mathbb{L}_A \overset{\mathbb{L}}{\otimes}_A B \to \mathbb{L}_B$ has a retraction in $\mathrm{Ho}(B\text{-dgmod}^{\leq 0}_k)$

(2) $f: R \to S$ is étale iff

- f is flat
- f is fp
- S is a flat $S \otimes_R S$-module

Analogously (I'm not 100% sure...), $f: A \to B$ is h-étale iff

- f is flat
- f hfp
- B is a flat $B \otimes^{\mathbb{L}}_A B$-dg module.

These are good, homotopically compatible, notions, but are not very easy to work with.

But we have the following important result.

Def. (1) $A \in \mathrm{cdga}_k^{\leq 0}$, $M \in A\text{-dgmod}_k^{\leq 0}$. M is strong if the natural map $\pi_i(A) \underset{\pi_0 A}{\otimes} \pi_0 M \longrightarrow \pi_i(M)$ is an iso $\forall i \geq 0$.

(2) A map $f : A \to B$ in $\mathrm{cdga}_k^{\leq 0}$ is strong if B is a strong A-dgmodule.

Theorem. A map $f : A \to B$ is h flat/h smooth/h étale/h Zar imm

iff $\begin{cases} \pi_0(f) : \pi_0 A \to \pi_0 B & \text{is flat/smooth/étale/Zar imm} \\ f \text{ is strong.} \end{cases}$

"strongly flat(smooth)/...": =

i.e "h-P = strongly-P".

[Note that while this result is false for hfp i.e hfp is not a strong pty, this difficulty goes away ~~[illegible]~~ when we add formally h-smooth, formally h-étale, full Zar immersion conditions]

Other related facts: $A \in \mathrm{cdga}_k^{\leq 0}$, $P \in A\text{-dgmod}_k^{\leq 0}$

(1) P is h-projective iff P is strong and $\pi_0 P$ is a projective $\pi_0 A$-mod

(2) P is h-flat iff ——"—— flat

(3) M is perfect iff it is hfp and projective

(4) M is perfect iff it is strong and $\pi_0 M$ is a proj module of finite pr. over $\pi_0 A$

(5) For $f : A \to B$ in $\mathrm{cdga}_k^{\leq 0}$, TFAE

- f is smooth (étale)
- f is flat and $\pi_0 f$ is smooth (étale)
- f is formally smooth and $\pi_0 B$ is fp over $\pi_0 A$ (as an algebra).

Exercise.

[redacted] Recall that a map of comm rings $R \to R'$ is smooth iff (1.36, [redacted]

(i) $R \to R'$ is fp

(ii) $R \to R'$ is flat

(iii) $R' \otimes_R R' \longrightarrow R'$ makes R' into a perfect $R' \otimes_R R'$-module

Prove the following derived analog:

$f : A \to B$ in sAlg_k / $\mathrm{cdga}_k^{\leq 0}$ (if $\mathrm{char}(k) = 0$)

is strongly smooth iff:

(i) f is hfp

(ii) f is strongly flat

(iii) B is perfect as a $B \otimes_A^{\mathbb{L}} B$-dg-module

Mapping spaces in $\mathrm{cdga}_k^{\leq 0}$

Let $A_k^n := k[t_0, \dots, t_n] / (\sum t_i - 1) \in \mathrm{cdga}_k$

$$\Omega_n^{\bullet} := \Omega^{\bullet}_{DR, A_k^n} := \overset{0}{k} \to \overset{1}{\Omega^1_{A_k^n/k}} \to \Omega^2_{A_k^n/k} \to \cdots \to \Omega^n_{A_k^n/k} \to 0$$

is a cdga (in $\mathrm{cdga}_k^{\geq 0}$).

If $A \in \mathrm{cdga}_k^{\leq 0}$ then

$A \otimes_k \Omega_n^{\bullet} =$ is a full cdga (in cdga_k^{-})

$$\underbrace{\overset{\uparrow}{\underset{\text{etc}}{A^{-n-1} \otimes \Omega_n^n}} \oplus \cdots \oplus A^{-2} \otimes \Omega^1 \oplus A^{-1}}_{\boxed{-1}} \to \underbrace{\begin{matrix} A^0 \otimes_k k = A^0 \\ \oplus \Omega_n^1 \otimes A^{-1} \oplus \cdots \\ \oplus \Omega^n \otimes A^{-n}. \end{matrix}}_{\boxed{0}} \xrightarrow{d} \underbrace{A^0 \otimes_k \Omega_n^1 \oplus A^{-1} \otimes_k \Omega^2 \oplus \cdots \oplus A^{-n+1} \otimes \Omega_n^n \to}_{\boxed{-1}}$$

$\to$ etc

We consider the intelligent truncation $\tau_{\leq 0}(A \otimes_k \Omega_n^{\bullet})$

$$\cdots \to \bigoplus_{i+j=-1} \Omega^i \otimes_k A^j \to \ker d \to 0$$

This is in $\mathrm{cdga}_k^{\leq 0}$ (memo $d(1) = 0 \ \forall\, 1 \in B^0$, $B \in \mathrm{cdga}$)

Consider $\left\{ \mathrm{Hom}_{\mathrm{cdga}_k^{\leq 0}}(A, \tau_{\leq 0}(B \otimes_k \Omega_n^{\bullet})) \right\}_{n \geq 0}$

$A_k^{\bullet}$ is a simplicial object in $\mathrm{cdga}_k \implies$

$\Omega^{\bullet}_{*}$ is a — " — — " — in cdga_k

Lecture 2

Postnikov tower for $cdga_k^{\leq 0}$

2.1

Def. A PT for $A \in cdga_k^{\leq 0}$ is a sequence of maps (in cdga):

$$\begin{tikzcd}
 & A \arrow[d, "p_n"'] \arrow[drr, "p_2"] \arrow[drrr, "p_1"] \arrow[drrrr, "p_0"] & & & & \\
\cdots \arrow[r] & P_n(A) \arrow[r, "\pi_n"] & \cdots \arrow[r] & P_2(A) \arrow[r, "\pi_2"] & P_1(A) \arrow[r, "\pi_1"] & P_0(A)
\end{tikzcd}$$

such that

(i) $P_n(A)$ is n-truncated i.e. $H^i(P_n(A)) = 0$ $i > n$

(ii) in the "non-zero" degs, $0 \leq i \leq n$, $H^i(p_n)$ is an iso $H^i(A) \xrightarrow{\sim} H^i(P_n(A))$

(iii) $P_n(A)$ is h-universal for maps to n-truncated cdga's i.e $\forall$ n-truncated cdga B

the map $\mathrm{Map}_{cdga_k^{\leq 0}}(P_n(A), B) \xrightarrow{p_n^*} \mathrm{Map}_{cdga_k^{\leq 0}}(A, B)$ is an iso in $\mathrm{Ho}(sSets)$.

Proposition. A PT for $A \in cdga_k^{\leq 0}$ (if $\exists$) is unique up to iso in $\mathrm{Ho}((cdga_k^{\leq 0})^{\mathbb{N}^{op}})$

$(\mathbb{N}^{op} := (\to n \to (n-1) \to \cdots \to 2 \to 1 \to 0))$.

Moreover, the canonical map (induced by p_n's)

$$A \longrightarrow \underset{n \geq 0}{\mathrm{holim}}\, P_n(A)$$

is an iso in $\mathrm{Ho}(cdga_k^{\leq 0})$.

Existence/construction:

Put $P_0(A) := \quad \cdots \to 0 \to A^0/dA^{-1} \to 0 \quad =: A_{\le 0}$

$P_1(A) := \quad \cdots \to 0 \to A^{-1}/dA^{-2} \xrightarrow{d^{-1}} A^0 \to 0 \quad =: A_{\le 1}$

⋮

$P_n(A) := \to \cdots \to 0 \to A^{-n}/d(A^{-n-1}) \xrightarrow{d^{-n}} A^{-n+1} \xrightarrow[d^{-n+1}]{} A^{-n+2} \to \cdots \to A^0 \to 0 \quad =: A_{\le n}$

⋮

with maps:

$P_{n+1}(A) \xrightarrow{\pi_{n+1}} P_n(A)$

$$\begin{tikzcd}
A^{-n-1}/dA^{-n-2} \arrow[r] \arrow[d] & 0 \arrow[d] \\
A^{-n} \arrow[r, "proj"] \arrow[d] & A^{-n}/dA^{-n-1} \arrow[d] \\
A^{-n+1} \arrow[r, "id"] \arrow[d] & A^{-n+1} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

and,

$p_n : A \to P_n(A)$

$$\begin{tikzcd}
A^{-n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
A^{-n} \arrow[r, "proj"] \arrow[d] & A^{-n}/dA^{-n-1} \arrow[d] \\
A^{-n+1} \arrow[r, "id"] \arrow[d] & A^{-n+1} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

2.3

Exercise. (1) Show that $P_n(A) \in$ cdga and the π_n's, p_n's are maps of cdga's

(2) Prove that $\{A_{\leq n}, \pi_n, p_n\}_{n \in \mathbb{N}}$ is a PT for A, and all π_n's and p_n's are fibrations

(3) Show that $A \longmapsto \{A_{\leq n}, p_n, \pi_n\}$ is functorial cdga's $\longrightarrow A/(\text{cdga}^{\mathbb{N}^{op}})$

(4) Define $P_n(A) := 0 \to \cdots \to 0 \to dA^{-n-1} \hookrightarrow A^{-n} \to A^{-n+1} \to$

$\to \cdots \to A^{-1} \to A^0 \to 0$

Build maps π_n, p_n's, and prove (1)-(3) above for this $P_n(A)$'s.

[This "model" of PT for A is sometimes useful for computations]

(5) Prove that if $\{P_n(A), \pi_n, p_n\}$ is a PT for A then

[π_{n+1} is a fib $\Rightarrow$ hofib(0) = Ker π_{n+1} then produce an explicit qiso $H^{-n-1}(A)[n+1] \to$ Ker]

(in $P_{n+1}(A)$-Mod) $\to$ hofib$(\pi_{n+1} : P_{n+1}(A) \to P_n(A); 0) \simeq$

$\simeq H^{-n-1}(A)[n+1]$

[Hint: prove it first for a model of $P_\bullet(A)$]

(6) Prove that for $\mathrm{Ker}(\pi_n) =: I_n$, $I_n \overset{II}{\otimes} I_n \xrightarrow{\mu} I_n$ is 0 in $\mathrm{Ho}(\mathrm{dgmod}_k)$.

We would now like to say:

- each $\pi_n : P_n(A) \to P_{n-1}(A)$ is a "square zero extension by $H^{-n}(A)$" (inspired by Ex 6)
- Relate "square zero extensions" to derived derivations.

2.4

Square-zero extensions

Def. $A \in$ cdga $B_0, B_1 \in A/$cdga ~~[illegible] a map of A/cdga's~~ and $I \subseteq H^{-n}(B_1)$ a sub $H^0(B_1)$-~~[illegible]~~ module

A map in $A/$cdga $\varphi : B_1 \to B_0$ is a square-zero extension by I if

(i) B_1, B_0 are n-truncated

(ii) φ is an $(n-1)$-eq.ce (i.e $H^{-i}(\varphi)$ iso for $0 \leq i \leq n-1$)

(iii) $\forall$ ~~B~~ A-algebra ~~C~~ C n-truncated, the following diagram is h-cartesian

$$\begin{tikzcd}
\mathrm{Map}_{A/\mathrm{cdga}}(B_0, C) \arrow[r] \arrow[d] & \mathrm{Map}_{A/\mathrm{cdga}}(B_1, C) \arrow[d] \\
{[B_0, C]_{0 \text{ on } I}} \arrow[r] & {[B_1, C]}
\end{tikzcd}$$

$\left\{ \text{htpy classes of maps } B_0 \xrightarrow{f} C \text{, s.t. } H^{-n}(f)_{|I} \equiv 0 \right\} =: [B_0, C]_{0 \text{ on } I}$

(iv) ~~Hence~~ $H^{-n}(\varphi) : H^{-n}(B_1) \to H^{-n}(B_0)$ is surjective with kernel I

(v) if $n = 0$, $I^2 = 0$.

Exercise. Prove that if $\varphi : B_1 \to B_0$ is a square-zero ext'n, then $\mathrm{hofib}(\varphi; 0)$ has only $_{\neq 0}$ cohomology in deg $-n$.

2.5

Oss.ne. (1) [split case] Let $n \geq 0$ and $N \in \pi_0(A)$-module (discrete).

Consider $\varphi : A \oplus N[n] \longrightarrow A$ $\quad (a, n) \longmapsto a$.

Then $\varphi_{\leq n} : (A \oplus N[n])_{\leq n} \longrightarrow A_{\leq n}$ is a square-zero extension by $I = N$.

$(H^{-n}(A \oplus N[n]_{\leq n}) \simeq H^{-n}(A \oplus N[n]) \simeq H^{-n}(A) \oplus N)$

(2) $A_{\leq n+1} \xrightarrow{\pi_{n+1}} A_{\leq n}$ is a square-zero extension by $\pi_{n+1}(A)$

[Exercise: prove it].

Properties (0) If $B_1 \to B_0$ sq. zero-ext'n by I, $I \subseteq H^{-n}(B_1)$, I is a B_0-mod ($\pi_0(B_1) \simeq \pi_0(B_0)$ if $n > 0$, obvious if $n = 0$ $(I^2 = 0)$)

(1) If $B_1 \to B_0$ & $B_1 \to B_0'$ are square-zero ext'n, then $\exists$ iso $B_0 \simeq B_0'$ in B_1/cdga

(2) let $B_1 \in A/\mathrm{cdga}$, n-truncated, Given any $n \geq 0$ and any $I \subseteq H^{-n}(B_1)$ $H^0(B_1)$-submodules (with the further pty that $I^2 = 0$ if $n = 0$), there exists a square-zero extension $B_1 \to B_0$ by I and it is ! up to iso in $\mathrm{Ho}(B_1/\mathrm{cdga})$.

To make the link to derivations let us first see how they produce another kind of "extensions":

(2.6) $\boxed{\text{Infinitesimal extensions}}$. (they come from derivations)

As in the $(\,)^2=0$ ext'n case, let $A \in \mathrm{cdga}$, $B_0 \in (A/\mathrm{cdga})^{\mathrm{cof}}$ and $M \in B_0$-dgmodule (connective).

To any derived derivation $[d] \in \pi_0 \mathrm{Map}_{A/\mathrm{cdga}/B_0}(B_0, B_0 \oplus M[1])$, represented by a map $d : B_0 \to B_0 \oplus M[1]$ in $A/\mathrm{cdga}/B_0$, we can associate the infinitesimal extension of B_0 by d as the object $(B_0 \oplus_d M \xrightarrow{\psi_d} B_0) \in \mathrm{Ho}(A/\mathrm{cdga}/B_0)$ defined by the h-pullback square (in)

$$(\bigstar) \qquad \begin{tikzcd} B_0 \oplus_d M \arrow[r] \arrow[d, "\psi_d"'] \arrow[dr, phantom, "\ulcorner h", very near start] & B_0 \arrow[d, "0"] \\ B_0 \arrow[r, "d"'] & B_0 \oplus M[1] \end{tikzcd}$$

Proposition $\mathrm{hofiber}(\psi_d, 0) \simeq M$ (hofiber taken in A-dgmod).

2.7

$\mapsto$ Pf (*) is a h-pushout $\Longrightarrow$

$$\mathrm{hofiber}(\Psi d;0) \simeq \mathrm{hofiber}(B_0 \xrightarrow{0} B_0 \oplus M[1], 0)$$

[note: hofiber = hoker $\neq$ ker since 0 is not a fibration (unless $M=0$)]

observe then that

$$\begin{tikzcd} B_0 \arrow[r, "0"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B_0 \oplus M[1] \arrow[d, "\mathrm{proj}"] \\ 0 \arrow[r] & M[1] \end{tikzcd}$$

is h-cartesian

(it is obviously cartesian, proj is a fibration and every object is fibrant)

therefore $\mathrm{hofib}(B_0 \xrightarrow{0} B_0 \oplus M[1], 0) \simeq$

$\simeq \mathrm{hofib}(0 \xrightarrow{0} M[1], 0) \simeq$

$\Omega M[1] = \Omega \Sigma M \simeq M$.

$$\begin{tikzcd} \Omega N \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner_h", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & N \end{tikzcd}$$

$\dashv$

2.8

Theorem. Any square-zero extension is an infinitesimal extension

↦ Pf. for A, B_0, B_1 discrete (classical case) $[n=0]$. We are in the situation

" $I \hookrightarrow B_1 \overset{\pi}{\twoheadrightarrow} B_0$, $I^2=0$ " $(\Rightarrow I$ is a B_0-module$)$
(iv)

Consider $B_0 \oplus I[1]$ ~~[illegible]~~ i.e.

$$\begin{tikzcd} & \boxed{-1} & & \boxed{0} & \\ 0 \arrow[r] & I \arrow[rr, "0"] & & B_0 \arrow[r] & 0 \end{tikzcd}$$ (cdga)

The 0 derivation is then the map

$B_0 \to B_0 \oplus I[1]$

$$0: \left(\begin{tikzcd} 0 \\ B_0 \arrow[u] \\ I \arrow[u] \\ 0 \arrow[u] \end{tikzcd}\right) \begin{tikzcd} \\ \arrow[l] \\ \arrow[l] \\ \end{tikzcd} \left(\begin{tikzcd} \\ B_0 \arrow[u] \\ 0 \arrow[u] \\ 0 \arrow[u] \end{tikzcd}\right)$$ (0 derivation)

Note that there is an obvious q iso

$$\begin{tikzcd} & 0 & 0 \\ \boxed{0} & B_1 \arrow[u] \arrow[r, "\pi"] & B_0 \arrow[u] \\ \boxed{-1} & I \arrow[u, hook] \arrow[r] & 0 \arrow[u] \\ & 0 \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

and we can define a $\underbrace{\text{new}}_{\text{derived}}$ derivation $\wedge$ by of B_0 w values in $I[1]$

$$d_\pi: \left(\begin{tikzcd} 0 \\ B_1 \arrow[u] \\ I \arrow[u, hook] \\ 0 \arrow[u] \end{tikzcd}\right) \begin{tikzcd} \\ \arrow[r, "\pi"] & \\ \arrow[r, "id"] & \\ \end{tikzcd} \left(\begin{tikzcd} 0 \\ B_0 \arrow[u] \\ I \arrow[u, "0"] \\ 0 \arrow[u] \end{tikzcd}\right)$$ (new derivation d_π)

Note that d_π is a fibration in cdga's.
Since $\mathrm{cdga}_k^{\le 0}$ is a proper model category,
the homotopy pullback $B_0 \oplus_d I$ is
the ordinary pullback of d_π and 0.
But (easy computation) the ordinary pullback
of 0 and d_π is just

$$\Big(0 \to 0 \to \underset{\boxed{0}}{B_1} \to 0\Big) = B_1[0]$$

Hence $B_1 \simeq B_0 \underset{d}{\oplus} I$ in $\mathrm{Ho}(\mathrm{cdga}_k^{\le 0}/B_0)$ ◄|

[
deg 0 : $B_1 \times_{B_0} B_0 = B_1$

deg -1 : $\{\deg -1 \times_{\deg -1} \deg -1\} = 0 \times_I I = 0$

deg ≤ -2 : 0
]

(2.10) The cotangent complex controls the Postnikov tower.

Let $A \in \mathrm{cdga}_k^{\leq 0}$ and

$$A \to \cdots \to P_{n+1}(A) \xrightarrow{\pi_{n+1}} P_n(A) \to \cdots \xrightarrow{\pi_1} P_0(A)$$

be its PT (! in $\mathrm{Ho}(A/\mathrm{cdga}_k^{\leq})$).

We know that $P_{n+1}(A) \xrightarrow{\pi_{n+1}} P_n(A)$ is a square-zero extension by $H^{-n-1}(A)$. By the previous theorem

$$\exists\ d_{\pi_{n+1}} \in \pi_0 \mathrm{Map}_{A/\mathrm{cdga}/P_n(A)}\left(P_n(A), P_n(A) \oplus H^{-n-1}(A)[n+1]\right) =$$

$\{P_n(A) \longrightarrow H^0 P_n(A) \simeq H^0(A) \Rightarrow H^{-n-1}(A)$ is a $P_n(A)$--module$\}$

$$= \pi_0 \underline{\mathrm{Der}}_A\left(P_n(A), H^{-n-1}(A)[n+1]\right) \simeq$$

$$\simeq \left[\mathbb{L}_{P_n(A)/A}, H^{-n-1}(A)[n+1]\right] \simeq \mathrm{Ext}^{n+1}_{P_n(A)}\left(\mathbb{L}_{P_n(A)/A}, H^{-n-1}(A$$

↖ maps in $\mathrm{Ho}(P_n(A)\text{-dgmod}) = D(P_n($

Such that $P_n(A) \oplus_{d_{\pi_n}} H^{-n-1}(A)[n] \simeq$

$\simeq P_{n+1}(A)$ in $\mathrm{Ho}(A/\mathrm{cdga}/P_n(A))$

i.e.

$$\begin{tikzcd}
P_{n+1}(A) \arrow[r] \arrow[d, "\pi_{n+1}"'] \arrow[dr, phantom, "\ulcorner", very near start] & P_n(A) \arrow[d] \\
P_n(A) \arrow[r, "d_{\pi_n}"'] & P_n(A) \oplus H^{-n-1}(A)[n+1]
\end{tikzcd}$$

is homotopy cartesian in $\mathrm{Ho}(A/\mathrm{cdga}_k^{\leq 0})$.

Connectivity of $\mathbb{L}_f$ vs of f, in cdga's

Def. $R \in$ cdga, $M, N \in R$-dgmod (connective), $n \in \mathbb{N}$.

(1) M is n-connective if $H^{-i}(M) = 0$ $0 \leq i < n$

(2) $\varphi : M \to N$ is n-connected if hofib(φ) is n-connective as an R-module.

Theorem. Let $f : A \to B$ in $\mathrm{cdga}_k^{\leq 0}$ and $n \in \mathbb{N}$. Then:

(1) f n-connected $\Rightarrow$ $\mathbb{L}_f$ is $(n+1)$-connective

(2) $\mathbb{L}_f$ $(n+1)$-connective and $H^0 f$ isomorphism $\Rightarrow$ f is n-connected

Corollary (important). A map $f : A \to B$ in cdga is a q-iso iff:

$$\begin{cases} H^0(f) : H^0A \to H^0B \quad \text{iso} \\ \mathbb{L}_f \simeq 0 \quad \text{in } D(B) \end{cases}$$

$\mapsto$ Pf. $\Rightarrow$ ok; $\Leftarrow$: Theorem tells us that f is n-connected $\forall n \geq 0$; by the long exact sequence of H^{-i}'s get $H^{-i}(f)$ iso $\forall i$ $\dashv$

Exercise / Remark. $A \in$ cdga $\Rightarrow H^{-i}(\mathbb{L}_{H^0A/A}) = 0$ $i = 0, 1$

$\mapsto$ $\mathrm{hofib}(p) \to A \xrightarrow{p} H^0(A)$ fiber seq.ce in A-mod $\Rightarrow$

$\cdots \to 0 \to H^{-1}(\mathrm{hofib}) \xrightarrow{\sim} H^{-1}(A) \to 0 \to H^0(\mathrm{hofib}) \to H^0(A) \xrightarrow{\sim} H^0(A) \to 0$ $\Rightarrow$

$H^0(\mathrm{hofib}) = 0$ i.e p is 1-connected $\underset{\text{Thm}}{\Longrightarrow}$ $\mathbb{L}_p$ is 2-connective

i.e $H^{-i}(\mathbb{L}_{H^0A/A}) = 0$ $i = 0, 1$ $\dashv$

Exercise:

2.12

Extension of maps along infinitesimal extensions

(≡ "infinitesimal liftings")

Let $A' \xrightarrow{p} A$ be an infinitesimal extension of cdgas. Hence (by def.), $\exists$ connective module M & derivation $d_p: A \to A \oplus M[1]$ s.t.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "0"] \\ A \arrow[r, "d_p"] & A \oplus M[1] \end{tikzcd} \quad (*)$$

is h-cartesian.

→ Let $f: B \to A$ a map of cdgas. We want to study the problem of

- $\exists$ce of infinitesimal lifting
$$\begin{tikzcd} A' \arrow[r, "p"] & A \\ B \arrow[u, dashed, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$
- classification of —— " ——

Now:
$$\mathrm{Lift}_p(f) = \mathrm{hofiber}\left(\mathrm{Map}(B, A') \xrightarrow{p_*} \mathrm{Map}(B, A); f\right).$$

By (*)

$$\begin{tikzcd} \mathrm{Map}(B, A') \arrow[r] \arrow[d, "p_*"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}(B, A) \arrow[d, "0_*"] \\ \mathrm{Map}(B, A) \arrow[r, "d_{p*}"'] & \mathrm{Map}(B, A \oplus M[1]) \end{tikzcd}$$

2.13

<u>Lemma</u>. Let

$$\begin{tikzcd} & T \arrow[rr] \arrow[dl] \arrow[ddr, "u"] & & X \arrow[d] \arrow[ddl, "f"'] \\ Y \arrow[rrr] \arrow[dr, "g"'] & & & Z \arrow[dll, "h"] \\ & S & & \end{tikzcd}$$

be a comm diagram in Ho(SSets), with the upper square h-cartesian

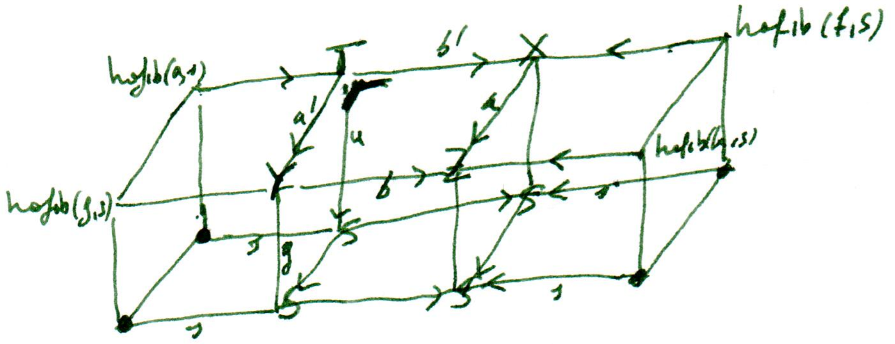
Then, $\forall s \in S$

$$\begin{tikzcd} \mathrm{hofib}(u;s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{hofibr}(f;s) \arrow[d] \\ \mathrm{hofib}(g;s) \arrow[r] & \mathrm{hofib}(h;s) \end{tikzcd}$$

is h-cartesian.

H-Pf of Lemma: The square is also h-cartesian in Ho(SSets$/_S$) and, given $s \in S$, the hofib square is just the pullback of this square by the map $* \xrightarrow[s]{} S$. But pullbacks commute with h-limits ⇒ ok.

A more explicit proof sketch:



$$\begin{tikzcd} \mathrm{hofib}(u,s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d, "a'"] \arrow[dd, bend left=60, "u", "\simeq"'] \\ \mathrm{hofib}(g;s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "g"] \\ * \arrow[r, "s"'] & S \end{tikzcd}$$

⇒ ok the ~~upper~~ top left square is h-cartesian

Same argument for the top right square

⇒ ok! ⊣|

2.14

We apply the lemma to :

$$\begin{tikzcd}
 & \mathrm{Map}(B,A') \arrow[rr] \arrow[dl, "p_*"'] \arrow[dd, "p_*"] & & \mathrm{Map}(B,A) \arrow[dl, "0_*"] \arrow[ddll, bend left, "id"] \\
\mathrm{Map}(B,A) \arrow[rr, "d_{p*}"] \arrow[dr, "id"'] & & \mathrm{Map}(B,A\oplus M[1]) \arrow[dl, "pr_{A*}"] & \\
* \arrow[r, "f"'] & \mathrm{Map}(B,A) & &
\end{tikzcd}$$

and we get a h-cartesian square in Sets

$$\begin{tikzcd}
\mathrm{Lifts}_{\substack{S\\p}}(f) = \mathrm{hofib}(p_*;f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * = \mathrm{hofib}(id;f) \arrow[d, "0_*f = (B \xrightarrow{f} A \xrightarrow{0} A\oplus M[1])"] \\
\mathrm{hofiber}(id;f) = * \arrow[r, "d_{p*}f = (B \xrightarrow{f} A \xrightarrow{d_p} A\oplus M[1])"'] & \mathrm{hofib}(\mathrm{Map}(B,A\oplus M[1]) \xrightarrow{pr_{A*}} \mathrm{Map}(B,A);f)
\end{tikzcd}$$

Observe that $\mathrm{hofib}(\mathrm{Map}(B,A\oplus M[1]) \to \mathrm{Map}(B,A);f) \simeq$

$\mathrm{Map}_{/A}(B,A\oplus M[1])$

Lemma (rework): $\mathrm{Map}_{/A}(B,A\oplus M[1]) \simeq \mathrm{Map}_{/B}(B,B\oplus M[1]_{[f]})$
[Idea (Chronical): The projection $B \xrightarrow{\partial} M[1]$ for $\overset{\psi}{\varphi_\partial}$ determines everything]

Pf.

$$\begin{tikzcd}
B\oplus M[1]_{[f]} \arrow[r] \arrow[d, "pr_B"'] \arrow[dr, phantom, "\ulcorner"] & A\oplus M[1] \arrow[d, "pr_A"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

is h-cartesian in cdga
[why? pr_A is a fibration & the diagram is obviously cartesian ⇒ h-cartesian too]
cdga: right proper (actually proper)

$$\Rightarrow \begin{tikzcd}
\mathrm{Map}(B,B\oplus M[1]_{[f]}) \arrow[r] \arrow[d, "pr_{B*}"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}(B,A\oplus M[1]) \arrow[d, "pr_{A*}"] \\
\mathrm{Map}(B,B) \arrow[r, "f_*"'] & \mathrm{Map}(B,A)
\end{tikzcd}$$

h-cartesian

$$\Rightarrow \underset{\mathrm{Map}_{/B}(B,B\oplus M[1]_{[f]})}{\overset{}{\underset{\text{is}}{\mathrm{hofiber}(pr_{B*};id_B)}}} \simeq \underset{\mathrm{Map}_{/A}(B,A\oplus M[1])}{\underset{\text{S1}}{\mathrm{hofiber}(pr_{A*};f_*(id)=f)}} \quad \dashv$$

But $\mathrm{Map}_{/B}(B, B\oplus M[1]_{[f]}) \simeq \mathrm{Map}_{B\text{-mod}}(\mathbb{L}_B, M[1]_{[f]}) \underset{adj}{\simeq} \mathrm{Map}_{A\text{-mod}}(\mathbb{L}_B \otimes_B A, M[1])$

$\partial \longmapsto \alpha_\partial$

Upshot :

$$\begin{tikzcd} \mathrm{Lifts}_p(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d, "\alpha_0 = 0"] \\ * \arrow[r, "\alpha_{d_p}"'] & \mathrm{Map}_{A\text{-mod}}(\mathbb{L}_B \otimes_B A, M[1]) \end{tikzcd}$$

is h-cartesian.

let's stare at this diagram : it says that

(i) $\mathrm{Lifts}_p(f) = \phi$ unless $[\alpha_{d_p}] = [0] = 0$ in $\mathrm{Hom}_{D(A)}(\mathbb{L}_B \otimes_B A, M[1]) \simeq \mathrm{Ext}^1_A(\mathbb{L}_B \otimes_B A, M)$

(ii) if $[\alpha_{d_p}] \doteq 0$, then $\mathrm{Lifts}_p(f) \underset{\text{noncanonically}}{\simeq} \Omega_0 \mathrm{Map}_{A\text{-mod}}(\mathbb{L}_B \otimes_B A, M[1])$

(: depends on the choice of a path $d_p \sim 0$)

(loops based at 0), i.e.

$$\mathrm{Lifts}_p(f) \simeq \mathrm{Map}_{A\text{-mod}}(\mathbb{L}_B \otimes_A B, \underbrace{\Omega M[1]}_{\Omega\Sigma M} \simeq M)$$

The class $[\alpha_{d_p}] \in \mathrm{Ext}^1_A(\mathbb{L}_B \otimes_B A, M)$ is called (for obvious reasons, by now) the obstruction class $\mathrm{obs}(p;f)$.

If $\mathrm{obs}(p;f) = 0$ then $\pi_0 \mathrm{Lifts}_p(f) \underset{\text{non-canonically}}{\simeq} \mathrm{Ext}^0(\mathbb{L}_B \otimes_B A, M)$.

(a torsor, actually)

under

2.16

Consequence : if we want to study $\exists$.ce of (or construct) maps $B \xrightarrow{f} A$ in cdga we may

- First look if $\exists$ map $\pi_0 B \xrightarrow{f_0} \pi_0 A$

$$\begin{tikzcd} \pi_0 B \arrow[r, "f_0"] \arrow[d, phantom, "\wr"] & \pi_0 A \arrow[d, phantom, "\wr"] \\ B_{\le 0} & A_{\le 0} \end{tikzcd}$$

- Then try to lift $f_{\le 0}$ to

$$\begin{tikzcd} B \arrow[dr, dashed, "f_{\le 1}"] \arrow[ddr, bend right, "f_{\le 0}"'] & & \\ & A_{\le 1} \arrow[r] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & A_{\le 0} \arrow[d, "p"] \\ & A_{\le 0} \arrow[r, "d_{\pi_1}"'] & A_{\le 0} \oplus H^{-1}(A)[2] \end{tikzcd}$$

The lift (by previous exercise) $\exists$ iff:

$$\mathrm{obs}(\pi_1; f_{\le 0}) =: \kappa_1(f_{\le 0}) = 0 \quad \text{in } \mathrm{Ext}^2_{A_{\le 0} = \pi_0 A}\left(\mathbb{L}_B \otimes_B A_{\le 0}, H^{-1}(A)\right)$$

and if this is the case the space of admissible $f_{\le 1}$ is (non canonically) iso (in Ho(SSets)) to $\mathrm{Map}_{A_{\le 0}\text{-mod}}(\mathbb{L}_B \otimes_B A_{\le 0}, H^{-1}(A)[1])$

- if $\kappa_1(f_{\le 0}) = 0$, choose $f_{\le 1}$ in $\pi_0 \mathrm{Map}(\quad) \simeq$

$\underset{\text{n.c.}}{\simeq} \mathrm{Ext}^1_{A_{\le 0}}(\mathbb{L}_B \otimes_B A_{\le 0}, H^{-1}(A))$ and continue...

In other words : the construction of f proceeds by constructing $F_{\le 0} : \pi_0 B \to$ [illegible] and then, by induction, studying the derivations of B into $(\pi_{n+1} A)[n+2]$ (note that once we have $f_{\le 0}$ then any $\pi_i A$ is a B-mod via $B \xrightarrow{\text{can}} \pi_0 B \xrightarrow{f_{\le 0}} \pi_0 A$, $\forall i \ge 1$) and this study is "linear" in that it is governed by linear maps from the cotangent complex of B.

2.17

Exercise (?) Let $A, B \in \mathrm{cdga}$ and $f : A \to B$ a map.

Prove that

$$\pi_i\left(\mathrm{Map}_{\mathrm{cdga}}(A,B); f\right) \simeq \mathrm{Ext}^{-i}_{D(A)}\left(\mathbb{L}_A, B_{[f]}\right)$$

Infinitesimal Deformations

2.18

$$\begin{tikzcd} & C \\ A' \arrow[r, "p"'] & A \arrow[u, "f"'] \end{tikzcd}$$

of $cdga_k^{\leq 0}$

$A' \xrightarrow{p} A$: infinitesimal ext.n by M
($\Leftrightarrow$ $d : A \to A \oplus M[1]$)

Def. Deformation of $(A \to C)$ along p :

$A' \to C'$ s.t. $C' \overset{L}{\underset{A'}{\otimes}} A \simeq C$ in $Ho(A/cdga)$ ~~is a h. pushout~~

Pb: (·) Existence of deformations of a given $A \to C$ along p ?

(··) Classification when they exist?

Solution : (·) $\exists$ $obs(p;f) \in Ext^2_C(\mathbb{L}_{C/A}, C \underset{A}{\otimes} M)$ s.t.

[e.g. Higher Algebra 7.4.2]

$Def(p;f) \neq \emptyset$ iff $obs(p;f) = 0$

(··) if $obs(p;f) = 0$, then $\pi_0 Def(p;f)$ is a torsor ~~under~~ for $Ext^1(\mathbb{L}_{C/A}, C \underset{A}{\otimes} M)$.

~~- Ex $\pi_i(\mathrm{Map}(A,B), f) \simeq \mathrm{Ext}^{-i}_{D(A)}(\mathbb{L}_A, B_{f^*})$ [dunno a proof right now ...]~~

~~**Next:** → connectivity estimates for $\mathbb{L}$ (maybe put it before the Ex on infinitesimal liftings)~~

~~end Cor: f eq.ce iff [illegible] and $\mathbb{L}_f \simeq 0$~~

~~→ slides on "dag explains def.n theory" (precisione! e controllare)~~

— · —

<u>Now</u> move to slides (beamer):

"DAG "explains" classical deformation theory"

— · —

Then: <u>END of notes</u>.