Selected introductory topics in derived algebraic geometry

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Plan of the talk

1. Homotopical algebra of dg-modules and cdga's (char 0)
2. Cotangent complex for cdga's
3. Infinitesimal extensions
4. Derived commutative algebra: flat, smooth, étale,....
5. DAG explains classical deformation theory

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DAG and classical deformation theory

Derived deformation theory (:= deformation theory in DAG) fills the 'gaps' in classical deformation theory ($k = \mathbb{C}$ here).

Given a

classical moduli problem:

$F : \mathsf{commalg}_{\mathbb{C}} \longrightarrow \mathrm{Grpds} \; : \; R \mapsto \{ Y \to \mathrm{Spec}\, R \, , \text{ proper \& smooth} \}$

Fixing a $\mathbb{C}$-point $\xi = (f : X \to \mathrm{Spec}\, \mathbb{C}) \in F(\mathbb{C})$, we get a

Formal moduli problem :

$\widehat{F}_{\xi}(A) := \mathit{hofiber}(F(\mathit{augment}) : F(A) \to F(\mathbb{C}); \xi)$, i.e.

$\hat{F}_{\xi} : \mathsf{Artin}_{\mathbb{C}} \longrightarrow \mathrm{Grpds}$

$A \mapsto \{ Y \to \mathrm{Spec}\, A \, , \text{ proper \& smooth } + \text{ iso}\, X \simeq Y \times_A \mathbb{C} \}$

Classical deformation theory:

1. $\widehat{F}_{\xi}(\mathbb{C}[t]/t^{n+1})$ groupoid of infinitesimal n-th order deformations of ξ
2. if $\xi_1 \in \widehat{F}_{\xi}(\mathbb{C}[\varepsilon] = \mathbb{C}[t]/t^2)$, then $\mathrm{Aut}_{\widehat{F}_{\xi}(\mathbb{C}[\varepsilon])}(\xi_1) \simeq H^0(X, T_X)$
3. $\pi_0(\widehat{F}_{\xi}(\mathbb{C}[\varepsilon])) \simeq H^1(X, T_X)$
4. If $\xi_1 \in \widehat{F}_{\xi}(\mathbb{C}[\varepsilon])$, $\exists\, \mathit{obs}(\xi_1) \in H^2(X, T_X)$ which vanishes iff ξ_1 extends to a 2^{nd} order deformation $\xi_2 \in \widehat{F}_{\xi}(\mathbb{C}[t]/t^3)$.

DAG and classical deformation theory

Items 1-3 above are satisfying. Much less about item 4.

Critique of obstructions:

1. what is the deformation theoretic interpretation of the whole $H^2(X, T_X)$?
2. how to determine the subspace of obstructions inside $H^2(X, T_X)$?

These questions

- are important classically: often $H^2(X, T_X) \neq 0$ but $\{\text{obstructions}\} = 0$ (e.g. X smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ of degree ≥ 6);
- have no general answers inside classical deformation theory

Let us see how how derived algebraic geometry answers to both.

DAG and classical deformation theory

Extend the functor $F : \mathsf{commalg}_{\mathbb{C}} \longrightarrow \mathrm{Grpds}$ as follows:

Derived Moduli problem (derived stack):

$\mathbb{R}F : \mathsf{cdga}_{\mathbb{C}} \longrightarrow \mathrm{SSets}$

$A^\bullet \mapsto \mathit{Nerve}(\,(\mathit{dSch}/\mathbb{R}\mathrm{Spec}\, A^\bullet)^{\mathrm{sm,\ proper}}, \mathrm{equivalences}\,)$

then $\mathbb{R}F(R) \simeq F(R)$ for $R \in \mathsf{commalg}_{\mathbb{C}} \hookrightarrow \mathsf{cdga}_{\mathbb{C}}$ (since a derived scheme Y mapping smoothly to $\mathrm{Spec}\, R$ is underived : Exercise)

Derived formal moduli problem (formal derived stack):

$\widehat{\mathbb{R}F}_{\xi} := \mathbb{R}F \times_{\mathrm{Spec}\,\mathbb{C}} \xi : \mathsf{dgArtin}_{\mathbb{C}} \longrightarrow \mathsf{sSets}$

$\widehat{\mathbb{R}F}_{\xi}(A^\bullet) := \mathit{hofiber}(\mathbb{R}F(A^\bullet) \to \mathbb{R}F(\mathbb{C}); \xi)$

where:

$\mathsf{dgArtin}_{\mathbb{C}} := \{A^\bullet \in \mathsf{cdga}_{\mathbb{C}} \,|\, H^0(A^\bullet) \in \mathsf{Artin}_{\mathbb{C}},\ H^i(A^\bullet)\,\text{of finite type over}$ $\mathrm{H}^0(A^\bullet),\ H^i(A^\bullet) = 0, i >> 0\}$

DAG and classical deformation theory

Anwer to Question 1: what is the deformation theoretic interpretation of the whole $H^2(X, T_X)$?

Proposition

There is a canonical isomorphism $\pi_0(\widehat{\mathbb{R}F}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X, T_X)$

I.e. $H^2(X, T_X)$ classifies derived deformations over $\mathbb{R}\mathrm{Spec}(\mathbb{C} \oplus \mathbb{C}[1])$! (derived deformations $:=$ deformations over a derived base).

This also explains why classical deformation could not answer this question: it only deals with deformations over underived bases.

Rmk. This same answer to Question 1 was obtained by M. Manetti, who only worked formally at ξ (i.e. without having the global derived stack $\mathbb{R}F$). Thanks to B. Fantechi for pointing this out.

Derived def-theory explains classical def-theory

Anwer to Question 2: how to determine the subspace of obstructions inside $H^2(X, T_X)$?

Lemma

The following (obvious) diagram is h-cartesian

$$\begin{tikzcd} \mathbb{C}[t]/t^3 \arrow[r] \arrow[d] & \mathbb{C}[\varepsilon] = \mathbb{C}[t]/t^2 \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} \oplus \mathbb{C}[1] \end{tikzcd}$$

Proof: Exercise (or see Porta- V., arXiv:1310.3573).

Observe: both $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}[1]$ and $\mathbb{C}[t]/t^2 \to \mathbb{C} \oplus \mathbb{C}[1]$ are surjective on H^0, with a nilpotent kernel.

DAG and classical deformation theory

Since $\mathbb{R}F$ is a derived Artin stack, it is infinitesimally cohesive (homotopy version of Schlessinger condition), so

$$\begin{tikzcd}
\mathbb{R}F(\mathbb{C}[t]/t^3) = F(\mathbb{C}[t]/t^3) \arrow[r] \arrow[d] & \mathbb{R}F(\mathbb{C}[\varepsilon]) = F(\mathbb{C}[\varepsilon]) \arrow[d] \\
\mathbb{R}F(\mathbb{C}) = F(\mathbb{C}) \arrow[r] & \mathbb{R}F(\mathbb{C} \oplus \mathbb{C}[1])
\end{tikzcd}$$

is h-cartesian; this whole diagram maps to $F(\mathbb{C})$, and the h-fibers at ξ yield

$$\begin{tikzcd}
\widehat{F}_\xi(\mathbb{C}[t]/t^3) \arrow[r] \arrow[d] & \widehat{F}_\xi(\mathbb{C}[\varepsilon]) \arrow[d] \\
\mathrm{pt} \arrow[r] & \widehat{\mathbb{R}F}_\xi(\mathbb{C} \oplus \mathbb{C}[1])
\end{tikzcd}$$

h-cartesian (formal consequence) of pointed simplicial sets.

DAG and classical deformation theory

Hence, get an exact sequence of vector spaces

$$\pi_0(\widehat{F}_\xi(\mathbb{C}[t]/t^3)) \longrightarrow \pi_0(\widehat{F}_\xi(\mathbb{C}[\varepsilon])) \xrightarrow{obs} \pi_0(\widehat{\mathbb{R}F}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X, T_X)$$

Therefore : a 1st order deformation $\xi_1 \in \pi_0(\widehat{F}_\xi(\mathbb{C}[\varepsilon]))$ of ξ, extends to a 2nd order deformation $\xi_2 \in \pi_0(\widehat{F}_\xi(\mathbb{C}[t]/t^3))$ iff the image of ξ_1 vanishes in $H^2(X, T_X)$.
So, Question 2 : how to determine the subspace of obstructions inside $H^2(X, T_X)$?
Answer: The subspace of obstructions is the image of the map *obs* above.

So, in particular, classical obstructions **are** derived deformations ($:=$ deformations over a derived base).

Exercise: extend this argument to all higher orders infinitesimal deformations.