Quick reminder on sheaves

$\mathcal{E}$ category.

Def: a Grothendieck pre-topology on $\mathcal{E}$ is an assignment to each object $C \in \mathcal{E}$ of a collection $B(C)$ of families of arrows $(f_i : U_i \to C)_{i \in I}$ called covering families such that

(i) every iso to C belongs to $B(C)$. I.e. $(g : D \xrightarrow{\sim} C) \in B(C)$

(ii) pull-back stability: $\forall (f_i : U_i \to C)_{i\in I}^{\in B(C)} \; \forall g : D \to C$

each pull-back $U_i \times_C D$ exists and $(U_i \times_C D \to D)_{i \in I} \in B(D)$.

(iii) Composition stability: $\forall (f_i : U_i \to C)_I^{\in B(C)} \; \forall (g_{ij} : V_{ij} \to U_i)_{j \in J_i} \in B(U_i)$.

$$(V_{ij} \to C)_{i,j} \in B(C).$$

Examples: • X topological space. $\mathcal{E} = \mathcal{O}pen(X)$.

covering families are open covers.

• $Top = \mathcal{E}$. Covering families are ~~open~~ étale cover.

(where étale maps are local homeomorphisms).

• $\mathcal{E} = k\text{-alg}^{op}$. $(A \xrightarrow{f_i} B_i)_{i\in I}$ covering family $\Leftrightarrow$ a) f_i^* étale morphism $\forall i$

b) $\exists J \subset I$ finite subset such that $\coprod_{j \in J} \operatorname{Spec}(B_j) \to \operatorname{Spec}(A)$ surjective.

Def: a sheaf on $\mathcal{E}$ is a presheaf $F : \mathcal{E}^{op} \to Set$ such that

$\forall$ covering family $(f_i : U_i \to C)_{i \in I}$

$$F(C) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_C U_j)$$ exhibits $F(C)$ as a co-equaliser.

Remarks: $\prod_i F(U_i) = Hom(\coprod_i \underline{U_i}, F)$

$\prod_{i,j} F(U_i \times_C U_j) = Hom(\coprod_i \underline{U_i} \times_{\underline{C}} \coprod_j \underline{U_j}, F)$. | let $\underline{U} = \coprod_i \underline{U_i}$

We have an embedding $Sh(\mathcal{E}) \hookrightarrow \mathcal{P}(\mathcal{E})$. $Sh(\mathcal{E})$ is a reflective subcategory of $\mathcal{P}(\mathcal{E})$.

It has a left adjoint: $\mathcal{P}(\mathcal{E}) \to Sh(\mathcal{E})$ (sheafification functor).

∞-categorical version

Def. ~~an ∞-sheaf (or stack)~~ a pretopology on an ∞-category $\mathcal{E}$ is an assignment of a collection $B(c)$ of family of morphisms $(f_i : V_i \to c)_{i \in I}$ such that it induces a pretopology on $Ho(\mathcal{E})$.

Main example: $\mathcal{E} = (\infty\text{-category of cdga}_k^{\leq 0})^{op}$.

$(A \xrightarrow{f_i^*} B_i)_{i \in I}$ covering family $\iff$ a) f_i^* étale morphism for every i.

b) $\exists J \subset I$ finite subset such that

$\coprod_{j \in J} Spec(H^0(B_j)) \to Spec(H^0(A))$ surjective.

Def. An ∞-sheaf on $\mathcal{E}$ is an ∞-functor $F : \mathcal{E}^{op} \to Spaces$ (could be any ∞-category having finite products)

such that $\forall$ covering family $(f_i : V_i \to C)_{i \in I}$

$$F(C) \to \left(\cdots \prod_{i_1, \dots, i_k} F(V_{i_1} \times_C \cdots \times_C V_{i_k}) \cdots \right)$$

exhibits $F(C)$ as the ∞-limit of $\cdots \in \mathcal{P}(\mathcal{E})^{\Delta}$.

Equivalently $Map(\underline{C}, F) \to Map(N(\underline{f}), F)$, where $N(\underline{f})$

$\underline{f} = \coprod_{i \in I} \underline{V_i} \to \underline{C}$.

[∞-hypersheaves (or stacks) = replace $N(\underline{f})$ by hypercovers.]

In other words, we have localized $\mathcal{P}(\mathcal{E})$ at $\underline{C} \to N(\underline{f})$.

Model for ∞-sheaves/stacks

Def: Let M be a model category. A pretopology on M is a choice of covering families such that $Ho(M)$ becomes equipped with a pretopology.

Bousfield localization

Let M be a model category. A left Bousfield localization of M is another model structure on the underlying category of M (we call it M_{loc}) such that

- $cof(M_{loc}) = cof(M)$ (same cofibrations)
- $W(M_{loc}) \supset W(M)$ (more weak equivalences).

$\Rightarrow$ $fib(M_{loc}) \subset fib(M)$ (less fibrations).

$\Rightarrow$ $id : M_{loc} \rightleftarrows M : id$ is a Quillen adjunction.

$\Rightarrow$ at least when M is a combinatorial simplicial model category,

$\infty(M_{loc}) \rightleftarrows \infty(M)$ exhibits $\infty(M_{loc})$ as a reflective ∞-subcategory of $\infty(M)$.

It is completely determined by the collection of local weak equivalences (i.e. $W(M_{loc})$).

Creating left Bousfield localization

Let S be a collection of morphisms in M.

A left ~~Bousfield~~ localization of M w.r.t S is a ~~Bousfield localization of~~ model category N that is universal among ones with a left Quillen functor $M \to N$ that sends S to w.e.

- An object A is S-local if $\forall\, X \to Y \in S$, $Map(Y,A) \to Map(X,A)$ equivalence.
- $f: X \to Y$ S-local equivalence if $\forall$ S-local object A $Map(Y,A) \to Map(X,A)$ equivalence.

A left Bousfield localization is a left localization w.r.t. S that is a Bousfield localization of M w.r.t. S

I.e. it is a model structure $L_S M$ on M such that

$$W(L_S M) = S\text{-local equivalences} \supset W(M).$$

$$\mathrm{Cof}(L_S M) = \mathrm{Cof}(M)$$

Under good assumptions, left Bousfield localization do exist.

Fibrant objects in $L_S M$ are the M-fibrant S-local objects.

Let us now construct a model for ∞-sheaves on a nice model category M equipped with a pre-topology τ. (e.g. $M = (\mathrm{cdga}^{\leq 0})^{op}$).

1) Consider the category of functors $\underset{\underset{\mathrm{cdga}^{\leq 0}}{\shortparallel}}{M^{op}} \longrightarrow sSets$.

Equip it with the projective model structure.

- fibrations / w.e. are objectwise fibrations / w.e.

Does not model ∞-functors $\infty(M^{op}) \longrightarrow$ Spaces !

Namely, does not involve the model structure on M.

2) perform a left Bousfield localization w.r.t.

$$S = \{ \underline{X} \to \underline{Y} \mid X \to Y \text{ w.e. in } M \}.$$

- new fibrant objects are those functor sending w.e. in M to w.e. in $sSets$ and that ~~are objectwise~~ take values in Kan complexes.

3) perform another left Bousfield localization w.r.t.

$$S = \{ \underline{C} \to N(\underline{f}) \mid f = \coprod_{i \in I} V_i \to C \text{ is a cover} \}$$

Remark: any w.e. being a cover, one can skip step 2).

homotopy sheaf / derived stack

$\Rightarrow$ a ~~derived stack~~ is a fibrant object in this left Bousfield localization: I.e.

- F takes values in Kan complexes
- F satisfies ~~stack~~ descent w.r.t. to covers of étale covers.

Examples of a derived stack

- $A \longmapsto A \in \text{dg-}k\text{-mod}^{\leq 0} \Rightarrow \text{sSets}$
- $A \longmapsto \mathbb{L}_A \in \text{dg-}k\text{-mod} = \text{cpxs}$
- $A \longmapsto S^{\cdot}_A(\mathbb{L}_A[-1]) \in \text{graded dg-}k\text{-mod} = \text{graded cpxs.}$
- $A \longmapsto DR(\tilde{A}) = S^{\cdot}_{\tilde{A}}(\Omega^1_{\tilde{A}}[-1])$

$$+ \ \varepsilon: S^k_{\tilde{A}}(\Omega^1_{\tilde{A}}[-1]) \to S^{k+1}_{\tilde{A}}(\Omega^1_{\tilde{A}}[-1])[+1]$$

$\in$ graded mixte cpxs.

- $A \mapsto \mathrm{Map}_{\text{gr-cpx}}\left(k[-n-p](p), S^{\cdot}_A(\mathbb{L}_A[-1])\right) \cong \left(S^p_A(\mathbb{L}_A[-1])[n+p]\right)^{\leq 0}$.

this is called the stack of p-forms of degree n. $A^p(n)$.

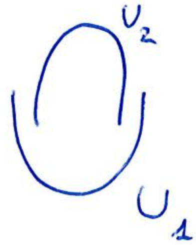
By definition $A^p(X,n) = \mathrm{Map}(X, A^p(n))$.

- $A \longmapsto \mathrm{Map}_{\text{gr-mixt-cpx}}\left(k[-n-p](p), DR(\tilde{A})\right) \cong \left(\prod_{q \geq p} \underbrace{DR(\tilde{A})_{(q)}}_{d_A + \varepsilon_A}[n+p]\right)^{\leq 0}$

Show this as an exercise (actually works for any V ~~$DR(\tilde{A})$~~). not specific to $DR(\tilde{A})$.

More examples

G algebraic group. $A \xmapsto{F} N(G(A))$ does NOT satisfy descent.

E.g.  $= X$ $\qquad \pi_0(F(X)) = *$.

$F(U_1) = *$

$F(U_2) = *$

$\vdots$

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{(g_1,g_2)} & G \\ \downarrow\downarrow & & \downarrow\downarrow \\ U_1 \sqcup U_2 & \longrightarrow & * \end{array}$$

0-simplices of $\mathrm{Map}(\text{nerve of cover}, F)$.
$= G \times G$.

1-simplices $= (g_1, g_2) \xrightarrow{h,k} (k g_1 h, h^{-1} g_2 k^{-1})$.

$\Rightarrow \pi_0(\mathrm{Map}(\text{nerve of cover}, F)) = G/G^{ad}$. exercise.

$\rightsquigarrow$ stackification is BG. $BG(A) = N(\text{gpd of } G\text{-torsors over } \mathrm{Spec}(H^0(A)))$.

$BG = \mathrm{holim}(N(G))$. $\neq$ ~~holim BNG~~

Stackification is nothing but fibrant replacement. ~~given by~~ $\cancel{\coprod id : P(dAff) \to P(dAff)^{\sim,c}}$

More general

Ordinary stacks are derived stacks.

$(M = alg^{op} = Aff$ weak equiv. are iso$)$ $\qquad (M = dAff = (cdga^{\leq 0})^{op})$.

We have a Quillen adjunction $H^0 : cdga^{\leq 0} \rightleftarrows alg : i$

étale covers are sent to each other through it.

$$\Rightarrow \begin{array}{ccc} sP(Aff) & \rightleftarrows & sP(dAff)^{\sim} \\ \downarrow\uparrow & & \downarrow\uparrow \\ sP(Aff)^{\sim,\text{étale}} & \underset{t_0}{\overset{i}{\rightleftarrows}} & sP(dAff)^{\sim,\text{étale}} \end{array}$$

if F is an underived stack then $F \circ H^0$ is a derived stack.

A derived enhancement of an ordinary (underived) stack X is a derived stack $\tilde{X}$ such that $t_0(\tilde{X}) \simeq X$. An obvious choice is $\iota(X)$, but it might not be a smart one.

For instance. Let X, Y be underived stacks.

$$\underline{\mathrm{Map}}(X,Y) : A \longmapsto \mathrm{Map}(X \times^h_X \mathrm{Spec}(A), Y)$$

We have a map $\iota\, \underline{\mathrm{Map}}(X,Y) \longrightarrow \underline{\mathrm{Map}}(\iota(X), \iota(Y))$. Not an equivalence and $t_0\, \underline{\mathrm{Map}}(\iota(X), \iota(Y)) = \underline{\mathrm{Map}}(X,Y)$.

For instance $X = S^2_B$ $Y = BG$.

$$\underline{\mathrm{Map}}_{St}(S^2_B, \underset{P}{BG}) = BG \qquad \underline{\mathrm{Map}}_{dSt}(S^2_B, BG) \simeq [\mathfrak{g}[-1]/G].$$

- de Rham stack: $X_{dR}(A) = X(H^0(A)/_{\text{nil, radical}})$ (1-truncated) $X_{dR} = \iota t_0 X_{dR}$.

Geometric stacks

Inductive definition!

Base step:
1. (-1)-geometric stacks are representable ones. i.e. $\mathrm{Spec}(A)$.
2. $f: F \to G$ is (-1)-representable if $\forall$ A-point of G $F \times^h_G \mathrm{Spec}(A)$ is representable.
3. $f: F \to G$ is (-1)-smooth if it is (-1)-representable and if $\forall$ A-point of G $F \times^h_G \mathrm{Spec}(A)$ is smooth.

Induction:
1. an n-atlas is family $f_i : \mathrm{Spec}(A_i) \to F$ ~~such that~~ of $(n-1)$-smooth morphisms such that $\coprod_i \mathrm{Spec}(A_i) \to F$ is epi.
2. a stack is n-geometric if it admits an n-atlas and the diagonal $F \to F \times F$ is $(n-1)$-representable.
3. $F \to G$ is n-representable if $\forall \mathrm{Spec}(A) \to G$ $F \times^h_G \mathrm{Spec}(A)$ is n-geometric.
4. $F \to G$ is n-smooth if it is n-representable and $\exists$ atlas $(U_i)_{i\in I}$ of $F \times_G \mathrm{Spec}(A)$ such that $U_i \to \mathrm{Spec}(A)$ smooth. $\forall \mathrm{Spec}(A) \to G$

<u>Properties</u> :
- $(n-1)$-representable $\Rightarrow$ n-representable $(n-1)$-geom $\Rightarrow$ n-geom.
- $(n-1)$smooth $\Rightarrow$ n-smooth
- n-representable/n-smooth morphisms are stable by isomorphisms, homotopy pullback and composition.
- n-geometric stacks are stable by homotopy pullback
- G n-geom, $F \to G$ s.t. $F \times^h_G U_i$ n-geom $\Rightarrow$ F n-geom.
  U_i n-atlas of G
- n-representable + m-smooth for $m > n$ $\Rightarrow$ n-smooth.

<u>Slogan</u> : geometric stacks can be obtained from affine ones by successive quotients of smooth groupoids.

Properties of nerves of groupoids

$$X_n \xrightarrow{\sim} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1 \qquad \text{(for nerves of categories).}$$

$$X_2 \longrightarrow X_1 \times^h_{X_0} X_1 \qquad \text{(invertibility of arrows).}$$

<u>Def</u> : a Segal gpd object in derived stacks $X_\bullet$ is a n-smooth Segal gpd if
1) X_0 and X_1 are n-geometric stacks
2) $s = d_0 : X_1 \to X_0$ is n-smooth.

a morphism of Segal gpds $X_\bullet \to Y_\bullet$ is n-smooth if $X_0 \to Y_0$ is n-smooth.

<u>Thm</u> : a) tfae: (1) F is n-geometric (2) $F \simeq |X_\bullet| = \operatorname{hocolim} X_\bullet$ for some $(n-1)$-smooth Segal gpd.

~~Assume F and G are n~~

b) tfae: (1) $F \to G$ n-smooth morphism between n-geom stacks (2) $F \to G$ can be obtained by an $(n-1)$-smooth morphism of $(n-1)$-smooth Segal gpds $X_\bullet \to Y_\bullet$.

Representability theorem

F is an n-geometric stack iff

(a) $t_0(F)$ is an Artin $(n+1)$-stack. I.e. $t_0(F)$ is $\overset{m}{\text{geometric}}$ (for some m) and $F(S)$ is $(n+1)$-truncated $\forall S$ underived scheme.

(b) F has a global cotangent complex

(c) F is infinitesimally coherent. I.e.

$$\begin{tikzcd} A\oplus_d M[-1] \arrow[r] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r] & A\oplus M \end{tikzcd} \quad d\in \mathrm{Der}(A,M) \quad \rightsquigarrow \quad \begin{tikzcd} F(A\oplus_d M[-1]) \arrow[r] \arrow[d] & F(A) \arrow[d] \\ F(A) \arrow[r] & F(A\oplus M) \end{tikzcd}$$

$d: A \to M[-1][1]$

(d) $\forall A$ with Postnikov tower

$$A \to \dots \to A_{\leq k} \to A_{\leq k-1} \to \dots \to H^0(A)$$

$$F(A) \longrightarrow \mathrm{holim}\, F(A_{\leq k}) \text{ is an equivalence.}$$

Meaning of "has a global cotangent complex":

There is a presheaf $dAff^{op} \xrightarrow{QCoh} Cat_\infty$

$$A \longmapsto \infty(A\text{-mod})$$

It is a stack (it satisfies étale descent).

$$QCoh(X) = \underset{\mathrm{Spec} A \to X}{\mathrm{holim}}\ \infty(A\text{-mod}).$$

$\mathrm{Spec}(\mathcal{O}_X \oplus M) = X[M]$

$(M, X) \longmapsto X[M]$

We have
$$\begin{tikzcd} QCoh_\infty^{\infty} \arrow[rr] \arrow[dr] & & Aff_\infty \arrow[dl] \\ & dAff^{op} & \end{tikzcd}$$

$\Rightarrow dSt^{op} \longrightarrow Cat_\infty$ given by a coCartesian fibration
$$\begin{tikzcd} QCoh_\infty \arrow[d] \\ dSt^{op} \end{tikzcd}$$

Similarly:

$dAff^{op} \longrightarrow Cat_\infty$ again a stack $\rightsquigarrow$
$$\begin{tikzcd} dAff_\infty \arrow[d] \\ dAff^{op} \end{tikzcd}$$

$A \longmapsto dAff_{/\mathrm{Spec}(A)}$

We say that X has a global cotangent cpx if $\exists\, \mathbb{L}_X \in QCoh(X)$

Such that $\mathrm{Map}_{AFF}(Y[M], X) \simeq \mathrm{Map}_{QCoh_\infty}(\mathbb{L}_X, M)$

$$\begin{array}{cc} \uparrow & \uparrow \\ (X, \mathbb{L}_X) & (Y, M) \end{array}$$

And moreover $\mathbb{L}_X$ is $(-n)$-connective for some n.

<u>Remark</u>: existence of $\mathbb{L}_X$ as above is equivalent to the following, when $Y = \mathrm{Spec}(A)$.

$$\mathrm{Map}_{AFF}(\mathrm{Spec}(A \oplus M), X) = \mathrm{Map}_{QCoh_\infty}(\mathbb{L}_X, M)$$

taking the fibre at a fixed map $\mathrm{Spec}(A) \xrightarrow{f} X$

$$\mathrm{Map}_{\mathrm{Spec}(A)/}(\mathrm{Spec}(A \oplus M), X) = \mathrm{Map}(f^* \mathbb{L}_X, M).$$

$A \to B$

$$+\ \mathbb{L}_X \in QCoh(X) \Rightarrow \begin{tikzcd} \mathrm{Spec}(A) \arrow[r, "f"] & X \\ \mathrm{Spec}(B) \arrow[u, "u^*"] \arrow[ur, "g"'] & \end{tikzcd} \text{ then } u^* f^* \mathbb{L}_X \simeq f^* \mathbb{L}_X \overset{\mathbb{L}}{\otimes}_A B \simeq g^* \mathbb{L}_X.$$

~~<u>Reminder</u>: a morphism $f: X \to Y$ is a Q-morphism (Q = locally of f.p., Zariski open imm, flat, ...)~~

~~iff it is n-representable for some n~~

~~and $\forall\, \mathrm{Spec}(A) \to Y$, $\exists$ n-atlas $\{U_i\}$ of $X \times_Y \mathrm{Spec}(A)$~~

~~s.t. each $U_i \to \mathrm{Spec}(A)$ is Q.~~

[illegible]