Lecture 4: Shifted Poisson geometry

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Outline

- based on joint works with D.Calaque, B.Toën, G.Vezzosi, M.Vaquié
- shifted Poisson geometry
- non-degenerate Poisson $\Longleftrightarrow$ symplectic

Shifted Poisson structures

Definition:

(a) An ***n*-shifted Poisson structure on X** is a morphism of graded dg Lie algebras $\pi : \mathbb{C}[-1](2) \to \mathsf{Pol}(X, n+1)[n+1]$.

(b) π is **non-degenerate** if the associated element in cohomology π_0 induces a quasi-isomorphism $\pi_0^\flat : \mathbb{L}_X \widetilde{\to} \mathbb{T}_X[-n]$.

$\pi : \mathbb{C}(2) \to \mathsf{Pol}(X, n+1)[n+2]$ gives rise to an element $\pi_0 \in \mathbb{H}^{-n}(X, \Phi_n^{(2)}(\mathbb{T}_X))$, where

$$\Phi_n^{(2)}(\mathbb{T}_X) := \begin{cases} \mathrm{Sym}^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if } n \text{ is odd} \\ \wedge^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if } n \text{ is even.} \end{cases}$$

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Notation:

- $\mathrm{Poiss}(X, n) = \mathrm{Map}_{\mathrm{dgLie}^{\mathrm{gr}}_{\mathbb{C}}}(\mathbb{C}[-1](2), \mathrm{Pol}(X, n+1)[n+1])$ will denote the space of n-shifted Poisson structures.
- $\mathrm{Poiss}(X, n)^{\mathrm{nd}} \subset \mathrm{Poiss}(X, n)$ will denote the space of non-degenerate n-shifted Poisson structures

The equivalence theorem

To quantize all the interesting shifted symplectic structures on moduli spaces we need two comparison results. The first allows us to pass from symplectic to Poisson structures:

Theorem: [CPTVV] Let X be a derived Artin stack locally of finite presentation. Then there exists a natural map of spaces

$$\sigma : \mathsf{Poiss}(X, n)^{\mathsf{nd}} \to \mathsf{Sympl}(X, n)$$

which is a weak homotopy equivalence.

Remark: A version of this theorem for Deligne-Mumford derived stacks was recently proven by J. Pridham by a different method.

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algebraic space

sheaf of $\mathbf{cdga}^{\leqslant 0}$

$\mathbb{P}_{n+1}$-structures

Definition: A $\mathbb{P}_{n+1}$-structure on a derived scheme X is a pair $(\mathcal{O}'_X, \alpha)$, where

- $\mathcal{O}'_X$ is a sheaf of strict $\mathbb{P}_{n+1}$-algebras on t_0X;
- $\alpha : \mathcal{O}'_X \to \mathcal{O}_X$ is a quasi-isomorphism of sheaves of **cdga**$^{\leqslant 0}$.

Goal: Define a map of spaces

$$\begin{array}{ccc} \mathbb{P}_{n+1}(X)^{\mathrm{nd}} & \longrightarrow & \mathrm{Sympl}(X, n) \\ & & \cap \\ & & A^{2,cl}(X, n) \end{array}$$

Construction at the level of points (i)

Let $(\mathcal{O}'_X, \alpha) \in \mathbb{P}_{n+1}(X)$, and let $\mathbb{L}'_X$ and $\mathbb{T}'_X$ be the tangent and cotangent complexes of $\mathcal{O}'_X$. **Comments**

We have a sheaf of graded $\mathrm{dgLie}_{\mathbb{C}}$ algebras

$$\mathrm{Pol}'(X, n)[n+1] = \left(\left(\mathrm{Sym}\, \mathbb{T}'_X[-n-1]\right)[n+1], [\bullet, \bullet], d\right)$$

on the Zariski site of X. By Melani's theorem the strict $\mathbb{P}_{n+1}$-structure on X is encoded in a $\mathrm{dgLie}^{\mathrm{gr}}_{\mathbb{C}}$-map

$$\pi : \mathbb{C}[-1](2) \to \mathrm{Pol}'(X, n)[n+1].$$

Note: π is an actual map of $\mathrm{dgLie}^{\mathrm{gr}}_{\mathbb{C}}$, not just a map in the homotopy category.

Construction at the level of points (ii)

Contraction with π gives a map $\mathbb{L}'_X[-1] \longrightarrow \mathbb{T}'_X[-1-n]$. Passing to Sym we get a map

$$\mathrm{Sym}(\mathbb{L}'_X[-1])[n+1] \xrightarrow{(\dagger)} \mathrm{Sym}(\mathbb{T}'_X[-1-n])[n+1],$$

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Note:

- The mixed structure on the left is d_{DR} while the mixed structure on the right is $[\pi, \bullet]$.
- The compatibility of $(\dagger)$ with the mixed structures follows from the strictness of π.
- The fact that $(\dagger)$ is an equivalence follows from the non-degeneracy of π.

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fails for stacks

Construction at the level of points (iii)

Inverting (†) we get a map of mixed graded complexes

$$\begin{tikzcd} \mathbb{C}[-1](2) \arrow[r, "\pi"] \arrow[dr, "(\dagger)^{-1}\circ\pi"'] & \mathrm{Sym}(\mathbb{T}'_X[-1-n])[n+1] \arrow[d, "(\dagger)^{-1}"] \\ & \mathrm{Sym}(\mathbb{L}'_X[-1])[n+1] \end{tikzcd}$$

which can be viewed as a map

$$\boxed{\mathbb{C}(2) \to \mathrm{Sym}(\mathbb{L}'_X[-1])[n+2]}$$

Construction at the level of points (iv)

By the Dold-Kan correspondence we have

$$A^{2,cl}_{\mathcal{O}'}(X,n) = \left|\left(\mathrm{Sym}^{\geqslant 2}_{\mathcal{O}'}\mathbb{L}'_X[-1]\right)[n+2]\right|,$$

which in turn can be identified with $A^{2,cl}_{\mathcal{O}}(X,n)$ via α.

Conclusion: $\alpha((\dagger)^{-1}\circ\pi[1])$ is a closed non-degenerate n-shifted 2-form on X. This gives a map

$$\sigma : \mathbb{P}_{n+1}(X)^{\mathrm{nd}} \to \mathrm{Sympl}(X,n)$$

at the level of points.

Next:

- Extend σ to a map of spaces (=ssets).
- Show that σ is functorial for étale maps in X'
- Prove that σ is an equivalence.

Map of spaces (i)

Slogan: The construction at the level of points already gives a map of spaces.

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Note: • We can use either C or C^{op} since $\mathrm{Nerve}(C) \cong \mathrm{Nerve}(C^{\mathrm{op}})$.

• For any simplicial set M we can talk about locally constant sheaves on M of $\mathbf{cdga}^{\leqslant 0}$, $\mathbf{cdga}_{\mathrm{gr}}^{\leqslant 0}$, $\varepsilon - \mathbf{cdga}_{\mathrm{gr}}^{\leqslant 0}$, $\mathbb{P}_{n+1}$-algebras, etc.

Map of spaces (ii)

Claim: [CPTVV] There is a universal sheaf $\mathcal{A} \to \mathbb{P}_{n+1}(X)$ of n-shifted Poisson $\mathbf{cdga}_{\mathbb{C}}^{\leqslant 0}$.

Explanation:

- $\mathbb{P}_{n+1}(X) = \text{Nerve}\,(\text{category of pairs } (\mathcal{O}'_X, \alpha))$.
- The locally constant sheaf $\mathcal{A}$ is given by the functor

$$\mathcal{A}: \quad \begin{pmatrix} \text{category} \\ \text{of pairs} \\ (\mathcal{O}'_X, \alpha) \end{pmatrix} \longrightarrow \begin{pmatrix} \text{category of} \\ \text{sheaves of } n\text{-} \\ \text{shifted Poisson} \\ \mathbf{cdga}^{\leqslant 0} \text{ on } X \end{pmatrix}, \quad (\mathcal{O}'_X, \alpha) \mapsto \mathcal{O}'_X.$$

Map of spaces (iii)

Consider now the category

$$C := \begin{pmatrix}\text{category}\\ \text{of pairs}\\ (\mathcal{O}'_X, \alpha)\end{pmatrix} \times \begin{pmatrix}\text{category}\\ \text{of étale}\\ \text{opens in}\\ X\end{pmatrix}^{\mathrm{op}}$$

The sheaf $\mathcal{A}$ gives rise to a functor

$$\begin{aligned} C &\longrightarrow \mathbb{P}_{n+1}(\mathbb{C}) \\ ((\mathcal{O}'_X, \alpha), U) &\longrightarrow R\Gamma(X, \mathcal{O}'_X) \end{aligned}$$

which sends all morphisms in C to étale maps of n-shifted Poisson $\mathbf{cdga}^{\leqslant 0}$.

Map of spaces (iv)

The pointwise construction applied to a diagram of non-degenerate n-shifted Poisson $\mathbf{cdga}^{\leqslant 0}$ with étale maps yields a diagram of n-shifted symplectic forms. This gives the desired map of spaces

$$\sigma : \mathbb{P}^{\mathrm{nd}}_{n+1}(X) \to \mathrm{Sympl}(X, n)$$

The comparison theorem now follows from the following

Theorem: [CPTVV] σ induces an equivalence

$$\mathbb{P}^{\mathrm{nd}}_{n+1}(-) \to \mathrm{Sympl}(-, n)$$

of stacks on $(\mathrm{derSp})_{\text{ét}}$.

The equivalence theorem (i)

$\mathbf{n = 0}$: a $\mathbb{P}_1$ is an ordinary Poisson structure, and X is underived and smooth. In this case the map is simply the usual inversion of Poisson structures.

$\mathbf{n < 0}$: To show that the map of stacks $\mathbb{P}_{n+1}^{\mathrm{nd}}(-) \to \mathrm{Sympl}(-, n)$ is an equivalence, we must show that it is fully faithful and essentially surjective.

The equivalence theorem (ii)

Essential surjectivity: Can be checked locally since these are stacks.

Use the Darboux lemma of Brav-Bussi-Joyce: up to a quasi-isomorphism a pair $(A \in \mathbf{cdga}^{\leqslant 0}, \omega \in \mathrm{Sympl}(A, n))$ is equivalent to a pair $(\widetilde{A}, \widetilde{\omega})$, where $\widetilde{\omega}$ is strictly closed and strictly non-degenerate. In particular $\widetilde{\omega}^{-1}$ is a strict n-shifted Poisson structure on A.

The equivalence theorem (ii)

Full faithfulness: have to compute σ on mapping spaces.

Have to show: for any two non-degenerate $\mathbb{P}_{n+1}$ structures on X, σ will identify the stack of paths between these structures (which is a stack over X) with the stack of paths between the corresponding n-shifted symplectic forms.

The case of loops (i)

Fix $(\mathcal{O}'_X, \alpha)$ with π - a strict $n+1$ structure on $\mathcal{O}'_X$. Consider the completed (product) total complexes of forms and polyvector fields on $\mathcal{O}'_X$. Contraction with π gives a natural map

$$\begin{array}{c} \left(\mathrm{Sym}^{\Pi}(\mathbb{L}'_X[-1])[n+1], d + d_{DR}\right) \\ \Big\downarrow{\scriptstyle \pi^\flat} \\ \left(\mathrm{Sym}^{\Pi}(\mathbb{T}'_X[-1-n])[n+1], d + [\pi, \bullet]\right) \end{array}$$

which is a filtered quasi-isomorphism of complexes which respects the stupid filtrations.

The case of loops (ii)

In particular we have a quasi-iso

$$\mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{L}'_X[-1])[n+1] \xrightarrow{\pi^\flat} \mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1]$$

But

$$\begin{pmatrix}\text{stack of loops in} \\ \mathbb{P}_{n+1}(X) \ \ \text{based} \\ \text{at } \pi\end{pmatrix} \overset{\text{Dold-Kan}}{\longleftrightarrow} \left(\mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1], d + [\pi, \bullet]\right)$$

and

$$\begin{pmatrix}\text{stack of loops} \\ \text{in } \mathrm{Sympl}(X,n) \\ \text{based at } \sigma(\pi)\end{pmatrix} \overset{\text{Dold-Kan}}{\longleftrightarrow} \left(\mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{L}'_X[-1])[n+1], d + d_{DR}\right).$$

The case of loops (iii)

In particular the map $\widehat{\sigma}_\pi$ that σ induces on the loop stacks and $\pi^\flat$ are maps between the same complexes but going in opposite directions:

$$\begin{tikzcd} \mathsf{Sym}^{\Pi,\geqslant 2}(\mathbb{L}'_X[-1])[n+1] \arrow[r, bend left, "\pi^\flat"] & \mathsf{Sym}^{\Pi,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1] \arrow[l, bend left, "\widehat{\sigma}_\pi"] \end{tikzcd}$$

By the construction of σ the composition $\pi^\flat \circ \widehat{\sigma}_\pi = \mathrm{id}$ and thus σ is fully faithful on loops.

The case of paths (i)

We must show that every homotopy class of paths between $\sigma(\pi_1)$ and $\sigma(\pi_2)$ in $\mathsf{Sympl}(X, n)$ can be lifted to a path between π_1 and π_2 in $\mathbb{P}_{n+1}(X)$.

To simplify the discussion suppose π_1 and π_2 are two n-shifted Poisson structures that give rise to the same shifted symplectic structure $\omega = \sigma(\pi_1) = \sigma(\pi_2)$.

The case of paths (ii)

As we saw, $\widehat{\sigma}_{\pi_1}$ gives a quasi-iso of complexes which is in fact a quasi-iso of dg Lie algebras

$$\begin{array}{c}\left(\mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1], d+[\pi_1,\bullet],[\bullet,\bullet]\right)\\ \Big\downarrow{\scriptstyle \widehat{\sigma}_{\pi_1}}\\ \left(\mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{L}'_X[-1])[n+1], d+d_{DR},0\right)\end{array}$$

This follows, since $\widehat{\sigma}_{\pi_1}$ is the derivative at π_1 of the map of functors σ, and a map of functors induces a dg Lie map on tangent complexes.

The case of paths (iii)

$\sigma(\pi_2)$ is a cocycle in the de Rham complex so $\widehat{\sigma}_{\pi_1}^{-1}(\sigma(\pi_2))$ is a cocycle in the complex $(\mathrm{Sym}^{\Pi,\geqslant 2}(\mathbb{T}'_X[-1-n])[n+1], d + [\pi_1, \bullet])$.

If we replace X by a formal neighborhood of a point in X and use the fact that the de Rham cohomology of a formal neighborhood is contractible, we can find a vector field ξ in $\mathbb{T}'_X$ so that

$$d\xi + [\pi_1, \xi] = \widehat{\sigma}_{\pi_1}^{-1}(\sigma(\pi_2)).$$

Since $\sigma(\pi_2) = \sigma(\pi_1) = \omega$ we get that $\mathrm{Lie}_\xi(\omega) = 0$ is locally Hamiltonian.
Therefore on a formal neighborhood of a point we have $\xi = \pi^\flat(df)$ for some function f. Thus the formal automorphism given by ξ fixes π_1 which implies $\pi_1 = \pi_2$ and gives the desired path.

Formal geometry (i)

Question: Why working formally at a point is relevant to our original question?

Let X be a derived scheme¿ the natural map $X \to X_{DR}$ realizes X as a family of formal derived schemes over X_{DR}. **Details**

The previous argument actually help us prove the following formal equivalence theorem:

Theorem: [CPTVV] Let X be a derived DM stack locally of finite presentation. Then there exists a natural equivalence of stacks:

$$\mathrm{Poiss}(X/X_{DR}, n)^{\mathrm{nd}} \to \mathrm{Sympl}(X/X_{DR}, n).$$

Formal geometry (ii)

Key remark: The moduli stacks of Poisson and symplectic structures on $X/\mathbb{C}$ are isomorphic to the moduli stacks of Poisson and symplectic structures on X/X_{DR}.

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The **key remark** reduces the global equivalence theorem to the formal equivalence theorem. The latter identifies two stacks over X_{DR} so we can prove it locally over X_{DR}.

Formal geometry (iii)

Need to show that for any $A \in \mathbf{cdga}^{\leqslant 0}$ and any formal stack $Z \to S = \mathbb{R}\mathbf{Spec}(A)$ such that $Z_{\mathrm{red}} = S_{\mathrm{red}}$ we have an isomorphism

$$\mathrm{Poiss}(Z/S, n)^{\mathrm{nd}} \tilde{\to} \mathrm{Sympl}(Z/S, n)$$

of stacks over S.

Formal geometry (iii)

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of stacks over S.

Note: This is easier since $Z \to S$ is given by **algebra data**. More precisely: the map $Z \to S$ gives rise to a sheaf $\mathbb{D}_Z$ of mixed graded cdga on S ($=$ a mixed graded cdga linear over A), where

$$\mathbb{D}_Z = \begin{pmatrix} \text{the relative de Rham complex of } S_{\text{red}} \to Z \\ \text{with a mixed structure given by } d_{DR} \end{pmatrix}$$

Formal geometry (iv)

Remark:

- Expect that $Z \to \mathbb{D}_Z$ gives an isomorphism of the stack parametrizing $Z \to S$ with $Z \in \mathrm{dforSt}_{\mathbb{C}}$, $Z_{\mathrm{red}} = S_{\mathrm{red}}$ with the stack of A-linear mixed graded cdga.
- We prove that there is a symmetric monoidal equivalence $L_{\mathrm{perf}}(Z)$ and $\varepsilon - \mathrm{perf}(\mathbb{D})$.
- We prove that shifted forms, closed forms, and Poisson structures on Z/S are the same as shifted forms, closed forms, and Poisson structures on $\mathbb{D}_Z/S$.

Formal geometry (v)

Thus the formal comparison theorem reduces to showing that

$$\mathrm{Poiss}(\mathbb{D}_Z/S, n)^{\mathrm{nd}} \to \mathrm{Sympl}(\mathbb{D}_Z/S, n)$$

is an equivalence.

Step 1: Prove this for a reduced A. In this case $Z \to S$ has a section $S = Z_{\mathrm{red}} \hookrightarrow Z$ which gives an augmentation for the mixed graded cdga $\mathbb{D}_Z/A$.

Formal geometry (vi)

The exact triangle for the relative cotangent complexes of the maps $S \to Z \to S$ gives an identification

$$\mathbb{L}_{S/Z} \cong \mathbb{L}_{Z/S}[1]_{|S}.$$

Thus

$$\mathbb{D}_Z = \mathsf{Sym}^{\Pi}(\mathbb{L}_{Z/S})_{|S}.$$

Since S is reduced we have that $Z \to S$ is of finite type and so $\mathbb{L}_{Z/S}$ is perfect.
Dualization converts the mixed structure on $\mathbb{D}_Z$ into a dgLie structure on $\mathbb{T}_{Z/S}$, which is easily seen to be the usual bracket.

Formal geometry (vii)

Thus the problem reduces to comparing nd Poisson brackets and forms on the dg Lie $(\mathbb{T}_{Z/S}, [\bullet, \bullet])$.

For this:

- Use Costello-Gwilliam formal Darboux lemma to identify nd forms with invariant pairings and nd Poisson structures with invariant copairings on the dgLie $(\mathbb{T}_{Z/S}, [\bullet, \bullet])$.
- Pass to the minimal model of the dgLie where the pairings and copairings become strict and are manifestly the same.

Note: Minimal models do not necessarily exist over A so we have to localize and check automorphisms over the localizations.

Formal geometry (viii)

Step 2: If S is not reduced, then $Z \to S$ is locally of almost finite presentation and so $\mathbb{L}_{Z/S}$ will not be perfect. In this case we can not dualize and use the dgLie argument.

Instead: extend the formal equivalence statement to derived or nilpotent thickenings by using deformation theory.

Strategy:

- Use Postnikov induction to decompose $S_{\mathrm{red}} \to S$ into a sequence of square zero extensions;
- Analyze the map of stacks $\mathrm{Poiss}(-, n)^{\mathrm{nd}} \to \mathrm{Sympl}(-, n)$. By **Step 1** this map is an iso over the reduction, so we only need to show that the map is an iso on tangent complexes and that both sides have obstruction theories (explicit calculation).

$\mathbb{P}_{n+1}$-structures (i)

Note: Defining $\mathbb{P}_{n+1}$-structure is straightforward for derived schemes or derived Deligne-Mumford stacks but is somewhat subtle for derived Artin stacks.

$\mathbb{P}_{n+1}$-structures (i)

Given $A \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$, the space of $\mathbb{P}_{n+1}$-structures on A is the mapping space

$$\mathbb{P}_{n+1}(A) = \mathrm{Map}_{\mathrm{dgOp}_{\mathbb{C}}}\left(\mathbb{P}_{n+1}, \mathrm{End}_{\mathbb{E}_1}(A)\right).$$

The comparison between n-shifted Poisson structures and $\mathbb{P}_{n+1}$-structures in the affine case is provided by Melani's theorem:

Theorem: [Melani] For any $A \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$, there is a natural map of spaces

$$\mu : \mathrm{Poiss}(A, n) \to \mathbb{P}_{n+1}(A),$$

which is a weak equivalence if $\mathbb{L}_A$ is perfect.

Back

$\mathbb{P}_{n+1}$-structures (ii)

Note:

- Melani's proof works in greater generality, and specifically in the relative situation (i.e. for families of algebras), for graded, or mixed graded families of algebras, etc.
- The statement is functorial: both $\mathrm{Poiss}(-, n)$ and $\mathbb{P}_{n+1}(-)$ are stacks on the small étale site of derived schemes, and Melani's construction gives an equivalence of spaces

$$\mathrm{Poiss}(X, n) \simeq \mathbb{P}_{n+1}(X)$$

for any X which is a derived DM stack which is locally of finite type.

$\mathbb{P}_{n+1}$-structures (iii)

This reasoning does not work for derived Artin stacks. Descent problem: we can not pull back shifted Poisson structures or $\mathbb{P}_{n+1}$-structures by smooth maps. So we need to properly understand the definition of $\mathbb{P}_{n+1}(X)$ when X is a derived Artin stack.

We use relative formal geometry (relative geometry of X over X_{dR}) to define $\mathbb{P}_{n+1}(X)$ and to define the map

$$\mu : \mathrm{Poiss}(X, n) \to \mathbb{P}_{n+1}(X)$$

in complete generality.

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Technical subtlety

Note: $\mathcal{O}_X$ is locally f.p. but $\mathcal{O}'_X$ will not be locally f.p. as a cdga. In fact, since $\mathcal{O}'_X$ is chosen to be cofibrant as a $\mathbb{P}_{n+1}$-algebra, it will not be locally f.p. as a cdga but will only be **weakly** locally f.p..

This only guarantees that $\mathbb{T}'_X$ and $\mathbb{L}'_X$ are weakly perfect, i.e. are complexes of $\mathcal{O}_X$-modules of possibly infinite rank (the cohomology sheaves are of course of finite rank). In particular we can not view a shifted Poisson structure as a map to $(\operatorname{Sym} \mathbb{T}'_X[-1-n])[n+1]$ but rather as a map to shifted polylinear map $\mathbb{L}'_X \times \cdots \times \mathbb{L}'_X \to \mathcal{O}'_X$.

Note: We deal with this carefully in the paper. To simplify the exposition, I will pretend here that this issue does not arise.

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Family of formal neighborhoods

Let X be a derived scheme, $X \to X_{DR}$ the natural map, and S an usual underived affine scheme.

Suppose $S \to X_{DR}$ is a given morphism. By definition this is the same as a morphism $f : S_{\mathrm{red}} \to X$. The choice of f gives us a morphism

$$f \times i : S_{\mathrm{red}} \to X \times S,$$

where $i : S_{\mathrm{red}} \to S$ is the natural closed immersion. Now using the formal groupoid presentation of X_{DR} we can compute the fiber of X over $S \to X_{DR}$ to get

$$X \times_{X_{DR}} S = \begin{pmatrix} \text{the formal completion of } S_{\mathrm{red}} \\ \text{inside } X \times S \end{pmatrix}$$

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