Lecture 3: Shifted deformation quantization

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Outline

- based on joint works with D.Calaque, B.Toën, G.Vezzosi, M.Vaquié
- shifted deformation quantization
- shifted Poisson geometry

Weak and strong quantization (i)

Fundamental statement:

Theorem: [Kontsevich,Yekutiely] Let (X, π) be a smooth Poisson scheme/$\mathbb{C}$. Then

- There exists a formal deformation quantization of (X, π) as a stack of algebroids. That is, there exists a stack of algebroids $\mathcal{X}/\mathbb{C}[[\hbar]]$ with $\mathcal{X}/\hbar = X$ and infinitesimal π.
- The formal deformation quantizations of (X, π) are classified by Poisson deformations of (X, π) over $\mathbb{C}[[\hbar]]$.

Weak and strong quantization (ii)

Shifted Poisson structures arise when we study deformations of X in which we allow only partial non-commutativity in the deformed structure.

To understand this it is useful to view Kontsevich's deformation quantization as a two step process:

(weak quantization): Deform the symmetric monoidal dg category $(L_{\mathsf{qcoh}}(X), \otimes)$ of sheaves on X to a $\mathbb{C}[[\hbar]]$-linear dg category $\mathcal{L}$.

(strong quantization): Deform the structure sheaf $\mathcal{O}_X$ to a sheaf $\mathcal{A}_{\mathcal{X}}$ of associative $\mathbb{C}[[\hbar]]$ algebras.

Remark: If the strong quantization exists, then we can take $\mathcal{L} = \mathcal{A}_{\mathcal{X}} - \text{mod}$.

Shifted weak and strong quantization (i)

Fix $n \in \mathbb{Z}$.

Conventions:

- An n-shifted Poisson bracket on a cdga A is a graded Lie bracket on A of degree $(-n)$ which is a graded derivation of the product structure.
- $\mathbb{P}_n$ will denote the operad controlling $n-1$-shifted (unbounded) Poisson cdga.
- $\mathbb{E}_n$ will denote the topological operad of little n-dimensional disks.

Remark: With these conventions $\mathbb{P}_n$ is the homology of $\mathbb{E}_n$ for $n \geqslant 2$, and in particular the homology of an $\mathbb{E}_n$ algebra is naturally a $(n-1)$-shifted Poisson cdga.

Shifted weak and strong quantization (ii)

Recall: Typically a family of $\mathbb{E}_n$ algebras (for $n \geqslant 2$) over the formal disk will specialize to a $\mathbb{P}_n$-algebra at the closed point.

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Caution: One needs conditions on an $\mathbb{E}_n$-algebra over $\mathbb{C}((\hbar))$ to ensure a $\mathbb{P}_n$-algebra specialization at $\hbar = 0$. For instance we may require that the $\hbar = 0$ specialization is a $\mathbf{cdga}^{\leqslant 0}$. More generally: the deformation space is graded and the deformations that have $(n-1)$-shifted Poisson limits are the ones of degree 2.

Shifted weak and strong quantization (ii)

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Note: The operad $\mathbb{E}_n$ is naturally filtered, and the associated graded is $\mathbb{P}_n$. The Rees construction applied to this filtration then gives an operad $BD_n \to \mathbb{A}^1$ which interpolates between $\mathbb{E}_n$ ($=$ the fiber of BD_n over $t \neq 0$) and $\mathbb{P}_n$ ($=$ the fiber of BD_n at $t = 0$.

Using BD_n we can now formulate the weak and strong shifted quantization problem.

Affine version of the quantization problem

(strong)$_n$ Show that every $\mathbb{P}_{n+1}$ algebra $A/\mathbb{C}$ lifts to a BD_{n+1}-algebra.

(weak)$_n$ Show that for every $\mathbb{P}_{n+1}$ algebra $A/\mathbb{C}$ the category $A-$mod deforms as a $\mathbb{E}_n$-monoidal category.

Note: For $n \geqslant 1$ this follows immediately from the formality of $\mathbb{E}_{n+1}$: choosing a formality isomorphism $\mathbb{E}_{n+1} \cong \mathbb{P}_{n+1}$ trivializes BD_n over $\mathbb{A}^1$ so we can promote $\mathbb{P}_{n+1}$-algebras to $\mathbb{E}_{n+1}$.

In particular: the strong quantization problem has a solution.

General shifted quantization

Problem: Let (X, ω) be an n-shifted symplectic derived stack. Construct a canonical, formal 1-parameter quantization of (X, ω), where this means:

- $n > 0$: A deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar]]$ as a sheaf of $\mathbb{E}_{n+1}$ algebras.
- $n = 0$: A deformation of $L_{\text{perf}}(X)$ as a dg category over $\mathbb{C}[[\hbar]]$.
- $n < 0$: (red shift trick) A deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar_{2n}]]$ as a sheaf of $\mathbb{E}_{1-n}$-algebras, where $|\hbar_{2n}| = 2n$.

Note: The notion of a formal deformation of a dg category$/\mathbb{C}$ is still under development. A good proxy for such a deformation is a $\mathbb{C}[u]$-linear structure with $|u| = 2$ ($\mathbb{C}[u]$ is the $\mathbb{E}_2$ Koszul dual of $\mathbb{C}[[\hbar]]$).

Red shift trick (i)

$\mathrm{compl}^{\mathrm{gr}}_{\mathbb{C}}$ - the category $\mathbb{Z}$-graded complexes of $\mathbb{C}$-vector spaces.

$\otimes$ - the usual symmetric monoidal structure on $\mathrm{compl}^{\mathrm{gr}}_{\mathbb{C}}$ (with **no** grading signs in the symmetry of in the external grading and with the usual grading signs in the symmetry for the homological grading).

Consider $\Phi : \mathrm{compl}^{\mathrm{gr}}_{\mathbb{C}} \to \mathrm{compl}^{\mathrm{gr}}_{\mathbb{C}}$ given by

$$(\Phi(E))(n) = E(n)[2n].$$

Note: This is a monoidal auto equivalence. In particular if $\mathbb{O}$ is any operad in $\mathrm{compl}^{\mathrm{gr}}_{\mathbb{C}}$, then $\Phi(\mathbb{O})$ is also a well defined operad in $\mathrm{compl}^{\mathrm{gr}}_{\mathbb{C}}$.

Red shift trick (ii)

One checks that $\Phi(\mathbb{P}_n) = \mathbb{P}_{n+2}$.

Consequences:

- Formality of $\mathbb{P}_n$ implies formality of $\mathbb{P}_{n\pm 2}$.
- Φ gives an equivalence between the category of graded $\mathbb{P}_n$ algebras and the category of graded $\mathbb{P}_{n+2}$ algebras.

Shifted polyvectors

X - derived stack/$\mathbb{C}$, locally of finite presentation.
$n \in \mathbb{Z}$
Consider the global **$n+1$-shifted polyvector fields on X**:

$$\boxed{\mathrm{Pol}(X, n+1) := \mathbb{R}\Gamma\left(X, \mathrm{Sym}\, \mathbb{T}_X[-1-n]\right)}$$

When equipped with the Schouten-Nijenhuis bracket this is graded Poisson dg algebra which after a shift by $n+1$ becomes a graded dgLie algebra. Thus

$$\mathrm{Pol}(X, n+1)[n+1] \in \mathrm{dgLie}^{\mathrm{gr}}_{\mathbb{C}}.$$

Shifted Poisson structures

Definition:

(a) An ***n*-shifted Poisson structure on X** is a morphism of graded dg Lie algebras $\pi : \mathbb{C}[-1](2) \to \mathrm{Pol}(X, n+1)[n+1]$.

(b) π is **non-degenerate** if the associated element in cohomology π_0 induces a quasi-isomorphism $\pi_0^\flat : \mathbb{L}_X \xrightarrow{\sim} \mathbb{T}_X[-n]$.

$\pi : \mathbb{C}(2) \to \mathrm{Pol}(X, n+1)[n+2]$ gives rise to an element $\pi_0 \in \mathbb{H}^{-n}(X, \Phi_n^{(2)}(\mathbb{T}_X))$, where

$$\Phi_n^{(2)}(\mathbb{T}_X) := \begin{cases} \mathrm{Sym}^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if } n \text{ is odd} \\ \wedge^2_{\mathcal{O}_X} \mathbb{T}_X, & \text{if } n \text{ is even.} \end{cases}$$

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Notation:

- $\mathsf{Poiss}(X, n) = \mathsf{Map}_{\mathsf{dgLie}^{gr}_{\mathbb{C}}}(\mathbb{C}[-1](2), \mathsf{Pol}(X, n+1)[n+1])$ will denote the space of n-shifted Poisson structures.
- $\mathsf{Poiss}(X, n)^{\mathrm{nd}} \subset \mathsf{Poiss}(X, n)$ will denote the space of non-degenerate n-shifted Poisson structures

Equivalence theorems (i)

To quantize all the interesting shifted symplectic structures on moduli spaces we need two comparison results. The first allows us to pass from symplectic to Poisson structures:

Theorem: [CPTVV] Let X be a derived Artin stack locally of finite presentation. Then there exists a natural map of spaces

$$\sigma : \mathsf{Poiss}(X, n)^{\mathsf{nd}} \to \mathsf{Sympl}(X, n)$$

which is a weak homotopy equivalence.

Remark: A version of this theorem for Deligne-Mumford derived stacks was recently proven by J. Pridham by a different method.

Equivalence theorems (ii)

Theorem: [Melani,CPTVV] Let X be a derived Artin stack. Then there exists a natural map of spaces **Details**

$$\mu : \mathrm{Poiss}(X, n) \to \mathbb{P}_{n+1}(X).$$

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Gaitsgory: X is 1-affine if the global section functor

$$\Gamma(X, -) : \mathcal{Q}coh_X-\mathbf{mod} \to \mathsf{Qcoh}(X)-\mathbf{mod}$$

is an equivalence.

Equivalence theorems (ii)

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Remark

- The two comparison results together convert any n-shifted symplectic structure ω into a $\mathbb{P}_{n+1}$ structure $\mu \circ \sigma^{-1}(\omega)$ on X.
- This $\mathbb{P}_{n+1}$-structure combined with the formality of $\mathbb{E}_{n+1}$ then give rise to a shifted quantization of X.

From Poisson to symplectic

Goal: Explain the geometry leading to the equivalence

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- Such a derived scheme X can be represented by a pair $(t_0X, \mathcal{O}_X)$

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algebraic space

sheaf of $\mathbf{cdga}^{\leqslant 0}$

$\mathbb{P}_{n+1}$-structures

Definition: A $\mathbb{P}_{n+1}$-structure on a derived scheme X is a pair $(\mathcal{O}'_X, \alpha)$, where

- $\mathcal{O}'_X$ is a sheaf of strict $\mathbb{P}_{n+1}$-algebras on t_0X;
- $\alpha : \mathcal{O}'_X \to \mathcal{O}_X$ is a quasi-isomorphism of sheaves of **cdga**$^{\leqslant 0}$.

Goal: Define a map of spaces

$$\begin{array}{ccc} \mathbb{P}_{n+1}(X)^{\mathrm{nd}} & \longrightarrow & \mathrm{Sympl}(X, n) \\ & & \cap \\ & & A^{2,cl}(X, n) \end{array}$$

$\mathbb{P}_{n+1}$-structures (i)

Note: Defining $\mathbb{P}_{n+1}$-structure is straightforward for derived schemes or derived Deligne-Mumford stacks but is somewhat subtle for derived Artin stacks.

$\mathbb{P}_{n+1}$-structures (i)

Given $A \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$, the space of $\mathbb{P}_{n+1}$-structures on A is the mapping space

$$\mathbb{P}_{n+1}(A) = \mathrm{Map}_{\mathrm{dgOp}_{\mathbb{C}}}\left(\mathbb{P}_{n+1}, \mathrm{End}_{\mathbb{E}_1}(A)\right).$$

The comparison between n-shifted Poisson structures and $\mathbb{P}_{n+1}$-structures in the affine case is provided by Melani's theorem:

Theorem: [Melani] For any $A \in \mathrm{cdga}_{\mathbb{C}}^{\leqslant 0}$, there is a natural map of spaces

$$\mu : \mathrm{Poiss}(A, n) \to \mathbb{P}_{n+1}(A),$$

which is a weak equivalence if $\mathbb{L}_A$ is perfect.

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$\mathbb{P}_{n+1}$-structures (ii)

Note:

- Melani's proof works in greater generality, and specifically in the relative situation (i.e. for families of algebras), for graded, or mixed graded families of algebras, etc.
- The statement is functorial: both $\mathrm{Poiss}(-, n)$ and $\mathbb{P}_{n+1}(-)$ are stacks on the small étale site of derived schemes, and Melani's construction gives an equivalence of spaces

$$\mathrm{Poiss}(X, n) \cong \mathbb{P}_{n+1}(X)$$

  for any X which is a derived DM stack which is locally of finite type.

$\mathbb{P}_{n+1}$-structures (iii)

This reasoning does not work for derived Artin stacks. Descent problem: we can not pull back shifted Poisson structures or $\mathbb{P}_{n+1}$-structures by smooth maps. So we need to properly understand the definition of $\mathbb{P}_{n+1}(X)$ when X is a derived Artin stack.

We use relative formal geometry (relative geometry of X over X_{dR}) to define $\mathbb{P}_{n+1}(X)$ and to define the map

$$\mu : \mathrm{Poiss}(X, n) \to \mathbb{P}_{n+1}(X)$$

in complete generality.

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