Lecture 2: Shifted symplectic structures

Tony Pantev

University of Pennsylvania

Summer School in Derived Geometry
Pavia, September 2015

Outline

- shifted symplectic and isotropic structures
- examples and constructions

Symplectic structures

Recall: For X a smooth scheme$/\mathbb{C}$ is a **symplectic structure** is an $\omega \in H^0(X, \Omega_X^{2,cl})$ such that its adjoint $\omega^\flat : T_X \to \Omega_X^1$ is a sheaf isomorphism.

Note: Does not work for X singular (or stacky or derived):

- T_X and Ω_X^1 are too crude as invariants and get promoted to complexes $\mathbb{T}_X$ and $\mathbb{L}_X$.
- A form being closed is not just a condition but rather an extra structure.

Definition: X derived Artin stack locally of finite presentation (so that $\mathbb{L}_X$ is perfect).

- A n-shifted 2-form $\omega : \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ - i.e. $\omega \in \pi_0(\mathcal{A}^2(X;n))$ - is nondegenerate if its adjoint $\omega^\flat : \mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism (in $D_{qcoh}(X)$).
- The space of n-shifted symplectic forms $Sympl(X;n)$ on $X/\mathbb{C}$ is the subspace of $\mathcal{A}^{2,cl}(X;n)$ of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces

$$\begin{array}{ccc} Sympl(X,n) & \longrightarrow & \mathcal{A}^{2,cl}(X,n) \\ \downarrow & & \downarrow \\ \mathcal{A}^2(X,n)^{nd} & \longrightarrow & \mathcal{A}^2(X,n) \end{array}$$

Shifted symplectic structures: examples (i)

- Nondegeneracy: a duality between the stacky (positive degrees) and the derived (negative degrees) parts of $\mathbb{L}_X$.
- $G = GL_n \rightsquigarrow BG$ has a canonical 2-shifted symplectic form whose underlying 2-shifted 2-form is

$$k \to (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee[-1] \wedge \mathfrak{g}^\vee[-1])[2] = Sym^2 \mathfrak{g}^\vee$$

  given by the dual of the trace map $(A, B) \mapsto tr(AB)$.
- Same as above (with a choice of G-invariant symm bil form on $\mathfrak{g}$) for G reductive over k.
- The n-shifted cotangent bundle $T^\vee X[n] := \mathrm{Spec}_X(\mathrm{Sym}(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form.

Shifted symplectic structures: examples (ii)

Theorem: [PTVV] Let F be a derived Artin stack equipped with an n-shifted symplectic form $\omega \in Symp(F, n)$. Let X be an $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation $[X] : \mathbb{H}(X, \mathcal{O}_X) \longrightarrow k[-d]$ of degree d. If the derived mapping stack $MAP(X, F)$ is a derived Artin stack locally of finite presentation over k, then, $MAP(X, F)$ carries a canonical $(n-d)$-shifted symplectic structure.

Remark:

0) Analog to Alexandrov-Kontsevich-Schwarz-Zaboronsky result.
1) A d $\mathcal{O}$-orientation on X is a kind of Calabi-Yau structure of dimension d;
2) A compact oriented topological d-manifold has an $\mathcal{O}$-orientation of degree d (Poincaré duality).

Lagrangian structures

Let (Y, ω) be a n-shifted symplectic derived stack. A lagrangian structure on a map $f : X \to Y$ is

- path γ in $\mathcal{A}^{2,\mathrm{cl}}(X; n)$ from $f^*\omega$ to 0
- that is 'non-degenerate' (in a suitable sense), i.e. the induced map $\theta_\gamma : \mathbb{T}_f \to \mathbb{L}_X[n-1]$ is an equivalence.

Examples:

- usual smooth lagrangians $L \hookrightarrow (Y, \omega)$ where (Y, ω) is a smooth (0)-symplectic scheme.
- there is a bijection between lagrangian structures on the canonical map $X \to (\mathrm{Spec}\, k, \omega_{n+1})$ and n-shifted symplectic structures on X (thus lagrangian structures generalize shifted symplectic structures)

Shifted symplectic structures: examples (iii)

Theorem: [PTVV] Let (F, ω) be n-shifted symplectic derived Artin stack, and $L_i \to F$ a map of derived stacks equipped with a Lagrangian structure, $i = 1, 2$. Then the homotopy fiber product $L_1 \times_F L_2$ is canonically a $(n-1)$-shifted derived Artin stack.

In particular, if $F = Y$ is a smooth symplectic Deligne-Mumford stack (e.g. a smooth symplectic variety), and $L_i \hookrightarrow Y$ is a smooth closed lagrangian substack, $i = 1, 2$, then the derived intersection $L_1 \times_F L_2$ is canonically (-1)-shifted symplectic.

Remark: An interesting case is the derived critical locus $\mathbb{R}Crit(f)$ for f a global function on a smooth symplectic Deligne-Mumford stack Y. Here

$$\begin{array}{ccc} \mathbb{R}Crit(f) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle df} \\ Y & \xrightarrow[0]{} & T^{\vee}Y \end{array}$$

Local models (i)

Recall: In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:**

Local models (i)

Recall: In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:** *a symplectic structure is locally (in the C^∞ or analytic setting) or formally (in the algebraic setting) isomorphic to the standard symplectic structure on a cotangent bundle.*

Local models (i)

In the derived and stacky setting there are two natural incarnations of an n-shifted symplectic cotangent bundle:

(a) The shifted cotangent bundle $T^{\vee}_{M}[n] = \mathbf{Spec}_{/M}\left(\mathrm{Sym}^{\bullet}_{\mathcal{O}_M}\left(T_M[-n]\right)\right)$, equipped with n-th shift of the standard symplectic form;

(b) The derived critical locus $\mathbf{Rcrit}(\mathbf{w})$ of an $n+1$ shifted function $\mathbf{w} : M \to \mathbb{A}^1[n+1]$, equipped with the inherited n-shifted symplectic form $\omega_{\mathbf{Rcrit}(\mathbf{w})}$.

defined for general Lagrangian intersections in **[PTVV'2012]**.

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Note: **(a)** is a special case of **(b)** corresponding to the zero shifted function.

Local models (ii)

Remark: • Shifted cotangent bundles are too restrictive to serve as local models of shifted symplectic structures.

• Derived critical loci of shifted functions have enough flexibility to provide local models. This leads to a remarkable shifted version of the Darboux theorem:

Local models (ii)

Theorem: **[BBBJ'2013]** Let X be a derived Deligne-Mumford stack, and let ω be an n-shifted symplectic structure on X, with $n \leqslant 0$.
Then, étale locally (X, ω) is isomorphic to $\left(\mathbf{Rcrit}(\mathbf{w}), \omega_{\mathbf{Rcrit}(\mathbf{w})}\right)$ for some shifted function $\mathbf{w} : M \to \mathbb{A}^1[n+1]$ on a derived scheme M.

O.Ben-Bassat, C.Brav, V.Bussi, D.Joyce

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Question: Find additional geometric structures that will ensure a global existence of a potential?

Local models (iii)

Answer: Potentials always exist in the presence of isotropic foliations.

Local models (iii)

Theorem: **[CPTVV]** Let X be a derived stack, locally of f.p. and let ω be an n-shifted symplectic structure on X. Assume:

- ω is exact, i.e. $[\omega] = 0 \in H^{\bullet}_{DR}(X)$;
- (X, ω) is equipped with an isotropic foliation $(\mathscr{L}, h)$.

Then there exists

- a shifted function $f : [X/\mathscr{L}] \to \mathbb{A}^1[n+1]$, and
- a symplectic map $s : X \to \mathbf{Rcrit}(f)$ of n-shifted symplectic stacks, i.e. $s^*\omega_{\mathbf{Rcrit}(f)} = \omega$.

Moreover, if $(\mathscr{L}, h)$ is Lagrangian, then s is étale.