# Lecture 2: Shifted symplectic structures

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#### Outline

- shifted symplectic and isotropic structures
- examples and constructions

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## Symplectic structures

**Recall:** For X a smooth scheme/ $\mathbb{C}$  is a symplectic structure is an  $\omega \in H^0(X, \Omega_X^{2,cl})$  such that its adjoint  $\omega^{\flat} : T_X \to \Omega_X^1$  is a sheaf isomorphism.

**Note:** Does not work for *X* singular (or stacky or derived):

- *T<sub>X</sub>* and Ω<sup>1</sup><sub>X</sub> are too crude as invariants and get promoted to complexes T<sub>X</sub> and L<sub>X</sub>.
- A form being closed is not just a condition but rather an extra structure.

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**Definition:** X derived Artin stack locally of finite presentation (so that  $\mathbb{L}_X$  is perfect).

- A *n*-shifted 2-form  $\omega : \mathcal{O}_X \to \mathbb{L}_X \land \mathbb{L}_X[n]$  i.e.  $\omega \in \pi_0(\mathcal{A}^2(X; n))$  - is nondegenerate if its adjoint  $\omega^{\flat} : \mathbb{T}_X \to \mathbb{L}_X[n]$  is an isomorphism (in  $D_{qcoh}(X)$ ).
- The space of *n*-shifted symplectic forms *Sympl*(*X*; *n*) on *X*/ℂ is the subspace of  $\mathcal{A}^{2,cl}(X; n)$  of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces

$$\begin{array}{c} Sympl(X,n) \longrightarrow \mathcal{A}^{2,cl}(X,n) \\ \downarrow & \downarrow \\ \mathcal{A}^{2}(X,n)^{nd} \longrightarrow \mathcal{A}^{2}(X,n) \end{array}$$

# Shifted symplectic structures: examples (i)

- Nondegeneracy: a duality between the stacky (positive degrees) and the derived (negative degrees) parts of L<sub>X</sub>.
- $G = GL_n \rightsquigarrow BG$  has a canonical 2-shifted symplectic form whose underlying 2-shifted 2-form is

 $k \to (\mathbb{L}_{BG} \land \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^{\lor}[-1] \land \mathfrak{g}^{\lor}[-1])[2] = Sym^2 \mathfrak{g}^{\lor}$ 

given by the dual of the trace map  $(A, B) \mapsto tr(AB)$ .

- Same as above (with a choice of G-invariant symm bil form on g) for G reductive over k.
- The *n*-shifted cotangent bundle  $T^{\vee}X[n] := \operatorname{Spec}_X(\operatorname{Sym}(\mathbb{T}_X[-n]))$ has a canonical *n*-shifted symplectic form.

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## Shifted symplectic structures: examples (ii)

**Theorem:** [PTVV] Let *F* be a derived Artin stack equipped with an *n*-shifted symplectic form  $\omega \in Symp(F, n)$ . Let *X* be an  $\mathcal{O}$ -compact derived stack equipped with an  $\mathcal{O}$ -orientation  $[X] : \mathbb{H}(X, \mathcal{O}_X) \longrightarrow k[-d]$  of degree *d*. If the derived mapping stack MAP(X, F) is a derived Artin stack locally of finite presentation over *k*, then, MAP(X, F) carries a canonical (n-d)-shifted symplectic structure.

#### Remark:

- 0) Analog to Alexandrov-Kontsevich-Schwarz-Zaboronsky result.
- A *d* O-orientation on X is a kind of Calabi-Yau structure of dimension *d*;
- A compact oriented topological *d*-manifold has an *O*-orientation of degree *d* (Poincaré duality).

#### Lagrangian structures

Let  $(Y, \omega)$  be a *n*-shifted symplectic derived stack. A lagrangian structure on a map  $f : X \to Y$  is

- **a** path  $\gamma$  in  $\mathcal{A}^{2,\mathrm{cl}}(X;n)$  from  $f^*\omega$  to 0
- that is 'non-degenerate' (in a suitable sense), i.e. the induced map  $\theta_{\gamma} : \mathbb{T}_f \to \mathbb{L}_X[n-1]$  is an equivalence.

Examples:

- usual smooth lagrangians L → (Y, ω) where (Y, ω) is a smooth (0)-symplectic scheme.
- there is a bijection between lagrangian structures on the canonical map  $X \rightarrow (\operatorname{Spec} k, \omega_{n+1})$  and *n*-shifted symplectic structures on X (thus lagrangian structures generalize shifted symplectic structures)

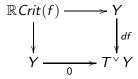
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# Shifted symplectic structures: examples (iii)

**Theorem:** [PTVV] Let  $(F, \omega)$  be *n*-shifted symplectic derived Artin stack, and  $L_i \rightarrow F$  a map of derived stacks equipped with a Lagrangian structure, i = 1, 2. Then the homotopy fiber product  $L_1 \times_F L_2$  is canonically a (n - 1)-shifted derived Artin stack.

In particular, if F = Y is a smooth symplectic Deligne-Mumford stack (e.g. a smooth symplectic variety), and  $L_i \hookrightarrow Y$  is a smooth closed lagrangian substack, i = 1, 2, then the derived intersection  $L_1 \times_F L_2$  is canonically (-1)-shifted symplectic.

**Remark:** An interesting case is the derived critical locus  $\mathbb{R}Crit(f)$  for f a global function on a smooth symplectic Deligne-Mumford stack Y. Here



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**Recall:** In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:** 



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**Recall:** In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:** *a* symplectic structure is locally (in the  $C^{\infty}$  or analytic setting) or formally (in the algebraic setting) isomorphic to the standard symplectic structure on a cotangent bundle.

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In the derived and stacky setting there are two natural incarnations of an n-shifted symplectic cotangent bundle:

- (a) The shifted cotangent bundle  $T_{M}^{\vee}[n] = \mathbf{Spec}_{/M} (\mathrm{Sym}_{\mathcal{O}_{M}}^{\bullet} (T_{M}[-n]))$ , equipped with *n*-th shift of the standard symplectic form;
- (b) The derived critical locus  $\mathbf{Rcrit}(\mathbf{w})$  of an n+1 shifted function  $\mathbf{w} : M \to \mathbb{A}^1[n+1]$ , equipped with the inherited *n*-shifted symplectic form  $\omega_{\mathbf{Rcrit}(\mathbf{w})}$ .

defined for general Lagrangian intersections in **[PTVV'2012]**.

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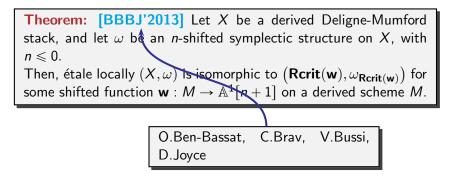
- (a) The shifted cotangent bundle  $T_{M}^{\vee}[n] = \mathbf{Spec}_{/M} \left( \operatorname{Sym}_{\mathcal{O}_{M}}^{\bullet}(T_{M}[-n]) \right)$ , equipped with *n*-th shift of the standard symplectic form;
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**Note:** (a) is a special case of (b) corresponding to the zero shifted function.

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**Remark:** • Shifted cotangent bundles are too restrictive to serve as local models of shifted symplectic structures.

• Derived critical loci of shifted functions have enough flexibility to provide local models. This leads to a remarkable shifted version of the Darboux theorem:



**Theorem:** [BBBJ'2013] Let X be a derived Deligne-Mumford stack, and let  $\omega$  be an *n*-shifted symplectic structure on X, with  $n \leq 0$ . Then, étale locally  $(X, \omega)$  is isomorphic to  $(\text{Rcrit}(\mathbf{w}), \omega_{\text{Rcrit}(\mathbf{w})})$  for some shifted function  $\mathbf{w} : M \to \mathbb{A}^1[n+1]$  on a derived scheme M.

**Question:** Find additional geometric structures that will ensure a global existence of a potential?

**Answer:** Potentials always exist in the presence of isotropic foliations.



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**Theorem:** [CPTVV] Let X be a derived stack, locally of f.p. and let  $\omega$  be an *n*-shifted symplectic structure on X. Assume:

- $\omega$  is exact, i.e.  $[\omega] = 0 \in H^{\bullet}_{DR}(X)$ ;
- $(X, \omega)$  is equipped with an isotropic foliation  $(\mathscr{L}, h)$ .

Then there exists

- a shifted function  $f : [X/\mathscr{L}] \to \mathbb{A}^1[n+1]$ , and
- a symplectic map  $s : X \to \mathbf{Rcrit}(f)$  of *n*-shifted symplectic stacks, i.e.  $s^* \omega_{\mathbf{Rcrit}(f)} = \omega$ . Moreover, if  $(\mathscr{L}, h)$  is Lagrangian, then *s* is étale.