Lecture 1: Forms and closed forms

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Outline

- tangent and cotangent complex
- shifted forms and closed forms
- examples

Tangent complex

$X \in \mathrm{dSt}_{\mathbb{C}}, \quad x : \mathrm{Spec}(\mathbb{C}) \to X$ a point

$$\begin{pmatrix}\text{Stalk } \mathbb{T}_{X,x} \text{ of the} \\ \text{tangent complex}\end{pmatrix} = \begin{pmatrix}\text{normalized chain complex} \\ \text{of the homotopy fiber of} \\ X(\mathbb{C}[\varepsilon]) \to X(\mathbb{C}) \text{ over } x\end{pmatrix}$$

simplicial abelian group

Tangent complex

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When X is a moduli stack:

$H^{-1}(\mathbb{T}_{X,x}) =$ infinitesimal automorphisms of x;

$H^{0}(\mathbb{T}_{X,x}) =$ infinitesimal deformations of x;

$H^{1}(\mathbb{T}_{X,x}) \supseteq$ obstructions of x.

Examples (i):

- $X = BG = [\mathrm{pt}/G] \Rightarrow \mathbb{T}_{X,\mathrm{pt}} = \mathfrak{g}[1]$.
- $X =$ moduli of vector bundles E on a smooth projective $Y \Rightarrow \mathbb{T}_{X,E} = R\Gamma(Y, \mathrm{End}(E))[1]$.
- $X =$ moduli of maps f from C to $Y \Rightarrow \mathbb{T}_{X,f} = R\Gamma(C, f^*T_Y)$.
- $X =$ moduli of local systems $\mathbb{E}$ on a compact manifold $Y \Rightarrow \mathbb{T}_{X,\mathbb{E}} = R\Gamma(Y, \mathrm{End}(\mathbb{E}))[1]$.

Examples (ii):

- X = derived intersection

$$L_1 \overset{h}{\underset{M}{\times}} L_2 = \left(L_1 \cap L_2, \mathcal{O}_{L_1} \overset{L}{\underset{\mathcal{O}_M}{\bigotimes}} \mathcal{O}_{L_2} \right)$$

of smooth subvarieties $L_1, L_2 \subset M$ in a smooth $M \Rightarrow$

$$\mathbb{T}_{X,x} = \left[\underset{0}{T_{L_1,x} \oplus T_{L_2,x}} \longrightarrow \underset{1}{T_{M,x}} \right],$$

$H^0(\mathbb{T}_{X,x}) =$ $T_{L_1 \cap L_2, x}$;
$H^1(\mathbb{T}_{X,x}) =$ failure of transversality.

Examples (iii):

Special case: $X =$ derived zero locus $\mathrm{Rzero}(s)$ of $s \in H^0(L, E)$.

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an algebraic vector bundle on a smooth variety L

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Thus

$$X = L \overset{h}{\underset{M}{\times}} L = \left(Z, \, i_L^{-1}(\mathrm{Sym}^\bullet(E^\vee[1]), s^\flat) \right),$$

where:

- $Z = t_0X = \mathrm{zero}(s)$ is the scheme theoretic zero locus of s,
- $i_L : Z \to L$ is the natural inclusion, and
- $s^\flat$ is the contraction with s.

Examples (iv):

In particular $\mathbb{T}_X = \underset{0}{\left[\ i_L^* T_L \oplus i_L^* T_L\right.} \xrightarrow{i_L^* do + i_L^* ds} \underset{1}{\left. i_M^* T_M\ \right]},$

where

- $M = \mathrm{tot}(E)$, and
- i_M, o, and s are the natural maps

$$\begin{tikzcd} & L \arrow[dr, "o"] & \\ Z \arrow[ur, "i_L"] \arrow[rr, "i_M"] \arrow[dr, "i_L"'] & & M. \\ & L \arrow[ur, "s"'] & \end{tikzcd}$$

Examples (iv):

In particular $\mathbb{T}_X = \left[\, i_L^* T_L \oplus i_L^* T_L \xrightarrow{i_L^* do + i_L^* ds} i_M^* T_M \,\right],$

$$\begin{array}{cc} 0 & 1 \end{array}$$

where

- $M = \mathrm{tot}(E)$, and
- i_M, o, and s are the natural maps

Exercise: Show that there is a natural quasi-isomorphism

$$\mathbb{T}_X = \left[i_L^* T_L \xrightarrow{(\nabla s)^\flat} i_L^* E \right] = \left[T_L \xrightarrow{(\nabla s)^\flat} E \right]_{|Z}.$$

Note: $\nabla : E \to E \otimes \Omega^1_L$ is an algebraic connection which exists only locally and is not unique. However $(\nabla s)_{|Z}$ is well defined globally and independent of the choice of ∇.

Cotangent complex

$A \in \mathrm{cdga}_{\mathbb{C}}$, $\quad X = \mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$QA \to A$ a cofibrant (quasi-free) replacement

$$\begin{pmatrix} \text{cotangent} & \text{complex} \\ \mathbb{L}_X = \mathbb{L}_A & \end{pmatrix} = \begin{pmatrix} \text{Kähler} & \text{differentials} \\ \Omega^1_{QA} \text{ of } QA & \end{pmatrix}$$

If $X \in \mathrm{dSt}_{\mathbb{C}}$ is a general derived Artin stack, then
$X = \mathrm{hocolim}\{\mathbf{Spec}\, A \to X\}$ (in the model category $\mathrm{dSt}_{\mathbb{C}}$) and

$$\mathbb{L}_X = \mathrm{holim}_{\mathbf{Spec}\, A \to X} \mathbb{L}_A$$

Note:

- $\mathbb{L}_X \in L_{qcoh}(X)$ - the dg category of quasi-coherent $\mathcal{O}_X$ modules.
- X is locally of finite presentation iff $\mathbb{L}_X$ is perfect. In this case $\mathbb{T}_X = \mathbb{L}_X^{\vee} = \mathrm{Hom}(\mathbb{L}_X, \mathcal{O}_X)$.

p-forms

$A \in \mathrm{cdga}_{\mathbb{C}}$, $X = \mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$QA \to A$ a cofibrant (quasi-free) replacement. Then:

$\oplus_{p \geqslant 0} \wedge_A^p \mathbb{L}_A = \oplus_{p \geqslant 0} \Omega_{QA}^p$ - a fourth quadrant bicomplex with
vertical differential $d : \Omega_{QA}^{p,i} \to \Omega_{QA}^{p,i+1}$ induced by d_{QA}, and
horizontal differential $d_{DR} : \Omega_{QA}^{p,i} \to \Omega_{QA}^{p+1,i}$ given by the de Rham differential.

Hodge filtration: $F^q(A) := \oplus_{p > q} \Omega_{QA}^p$: still a fourth quadrant bicomplex.

(shifted) closed p-forms

Motivation: If X is a smooth scheme$/\mathbb{C}$, then $\Omega_X^{p,cl} \cong \left(\Omega_X^{\geqslant p}[p], d_{DR}\right)$. Use $(\Omega_X^{\geqslant p}[p], d_{DR})$ as a model for closed p forms in general.

Definition:

- **complex of closed p-forms on $X = \mathrm{Spec}\, A$:** $\mathbf{A}^{p,cl}(A) := \mathrm{tot}^{\prod}(F^p(A))[p]$
- **complex of n-shifted closed p-forms on $X = \mathrm{Spec}\, A$:** $\mathbf{A}^{p,cl}(A; n) := \mathrm{tot}^{\prod}(F^p(A))[n+p]$
- **Hodge tower:** $\cdots \to \mathbf{A}^{p,cl}(A)[-p] \to \mathbf{A}^{p-1,cl}(A)[1-p] \to \cdots \to \mathbf{A}^{0,cl}(A)$

(shifted) closed p-forms (ii)

Explicitly an n-shifted closed p-form ω on $X = \mathbf{Spec}\, A$ is an infinite collection

$$\omega = \{\omega_i\}_{i\geqslant 0}\,, \qquad \omega_i \in \Omega_A^{p+i,n-i}$$

satisfying

$$d_{DR}\omega_i = -d\omega_{i+1}.$$

Note: The collection $\{\omega_i\}_{i\geqslant 1}$ is the key closing ω.

p-forms and closed p-forms

Note:

- The complex $\mathbf{A}^{0,cl}(A)$ of closed 0-forms on $X = \mathbf{Spec}\, A$ is exactly Illusie's derived de Rham complex of A.
- There is an underlying p-form map

$$\mathbf{A}^{p,cl}(A; n) \to \wedge^p \mathbb{L}_{A/k}[n]$$

  inducing

$$H^0(\mathbf{A}^{p,cl}(A)[n]) \to H^n(X, \wedge^p \mathbb{L}_{A/k}).$$

- The homotopy fiber of the underlying p-form map can be very complicated (complex of keys): being closed is *not* a property but rather a list of coherent data.

Functoriality and gluing:

- the *n*-shifted *p*-forms ∞-functor
  $\mathcal{A}^p(-;n) : \mathbf{cdga}_{\mathbb{C}} \to \mathbf{SSets} : A \mapsto |\,\Omega^p_{QA}[n] \simeq (\wedge^p_A \mathbb{L}_A)[n]\,|$
- the *n*-shifted closed *p*-forms ∞-functor
  $\mathcal{A}^{p,\mathrm{cl}}(-;n) : \mathbf{cdga}_{\mathbb{C}} \to \mathbf{SSets} : A \mapsto |\,\mathbf{A}^{p,cl}(A)[n]\,|$
- $\mathcal{A}^p(-;n)$ and $\mathcal{A}^{p,\mathrm{cl}}(-;n)$ are derived stacks for the étale topology.
- underlying *p*-form map (of derived stacks)

$$\mathcal{A}^{p,\mathrm{cl}}(-;n) \to \mathcal{A}^p(-;n)$$

Notation: $|-|$ denotes $Map_{\mathbb{C}-dgMod}(\mathbb{C},-)$ i.e. Dold-Kan applied to the $\leqslant 0$-truncation - all our dg-modules have cohomological differential.

global forms and closed forms

For a derived Artin stack X (locally of finite presentation $/\mathbb{C}$) we have

Definition:

- $\mathcal{A}^p(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-))$ is the space of p-forms on X;
- $\mathcal{A}^{p,\mathrm{cl}}(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,\mathrm{cl}}(-))$ is the space of closed p-forms on X;
- the corresponding n-shifted versions :
  $\mathcal{A}^p(X;n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-;n))$
  $\mathcal{A}^{p,\mathrm{cl}}(X;n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,\mathrm{cl}}(-;n))$
- an n-shifted (respectively closed) p-form on X is an element in $\pi_0\mathcal{A}^p(X;n)$ (respectively in $\pi_0\mathcal{A}^{p,\mathrm{cl}}(X;n)$)

global forms and closed forms (ii)

Note:

1) If X is a smooth scheme there are no negatively shifted forms.
2) When $X = \mathbf{Spec}\, A$ then there are no positively shifted forms. For general X they might exist for any $n \in \mathbb{Z}$.

global forms and closed forms (ii)

- underlying p-form map (of simplicial sets)

$$\mathcal{A}^{p,\mathrm{cl}}(X;n) \to \mathcal{A}^{p}(X;n)$$

- this map is not a monomorphism for general X, its homotopy fiber at a given p-form ω_0 is the space of keys of ω_0.
- If X is a smooth and proper scheme then this map is indeed a mono (homotopy fiber is either empty or contractible) $\Rightarrow$ no new phenomena in this case.
- **Theorem** (PTVV): for X derived Artin, $\mathcal{A}^{p}(X;n) \simeq \mathrm{Map}_{\mathrm{L}_{\mathrm{qcoh}}(X)}(\mathcal{O}_X, (\wedge^p \mathbb{L}_X)[n])$ (smooth descent)
- in particular an n-shifted p-form on X is an element in $H^n(X, \wedge^p \mathbb{L}_X)$

global forms and closed forms (iii)

Remark: If $A \in cdga$ is quasi-free, and $X = \mathbf{Spec}\, A$, then

$$\mathcal{A}^{p,cl}(X;n) = \left| \prod_{i \geqslant 0} \left(\Omega_A^{p+1}[n-i], d + d_{DR} \right) \right|$$

$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= |NC(A)(p)[n+p]|$$

global forms and closed forms (iii)

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negative cyclic complex of weight p

global forms and closed forms (iii)

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$$= \left| \mathrm{tot}^{\Pi}(F^p(A))[n] \right|$$

$$= |NC(A)(p)[n+p]|$$

Hence

$$\boxed{\pi_0 \mathcal{A}^{p,cl}(X;n) = HC_-^{n-p}(A)(p).}$$

global forms and closed forms (iv)

Definition: Given a higher Artin derived stack the n-th algebraic de Rham cohomology of X is defined to be $H^n_{DR}(X) = \pi_0 \mathcal{A}^{0,cl}(X; n)$.

Remark:

- agrees with Illusie's definition in the affine case.
- if X is a higher Artin derived stack locally f.p., then $H^\bullet_{DR}(X) \cong H^\bullet_{DR}(t_0 X) =$ algebraic de Rham cohomology of the underived higher stack $t_0 X$ defined by Hartschorne's completion formalism.

global forms and closed forms (iv)

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- agrees with Illusie's definition in the affine case.
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Corollary: Let X be a locally f.p. derived stack and let ω be an n-shifted closed p-form on X woth $n < 0$. Then ω is exact, i.e. $[\omega] = 0 \in H^{n+p}_{DR}(X)$.

Examples (i):

(1) If $X = \mathrm{Spec}(A)$ is an usual (underived) smooth affine scheme, then

$$\mathcal{A}^{p,cl}(X;n) = (\tau_{\leqslant n}(\underset{0}{\Omega_A^p} \xrightarrow{d_{DR}} \underset{1}{\Omega_A^{p+1}} \xrightarrow{d_{DR}} \cdots))[n],$$

and hence

$$\pi_0\mathcal{A}^{p,cl}(X;n) = \begin{cases} 0, & n < 0 \\ \Omega_A^{p,cl}, & n = 0 \\ H_{DR}^{n+p}(X), & n > 0 \end{cases}$$

e.g. if $X = \mathbb{C}^\times$, then $dz/z \in \pi_0\mathcal{A}^{1,cl}(X;0)$ and also $dz/z \in \pi_0\mathcal{A}^{0,cl}(X;1)$.

Examples (ii):

(2) If X is a smooth and proper scheme, then $\pi_i \mathcal{A}^{p,cl}(X;n) = F^p H^{n+p-i}_{DR}(X)$.

(3) If X is a (underived) higher Artin stack, and $X_\bullet \to X$ is a smooth affine simplicial groupoid presenting X, then $\pi_0 \mathcal{A}^p(X;n) = H^n(\Omega^p(X_\bullet), \delta)$ with $\delta =$ Čech differential.
In particular if G is a complex reductive group, then

$$\pi_0 \mathcal{A}^p(BG;n) = \begin{cases} 0, & n \neq p \\ \left(\mathrm{Sym}^\bullet \mathfrak{g}^\vee\right)^G, & n = p. \end{cases}$$

Examples (iii):

(4) Similarly

$$\mathcal{A}^{p,cl}(BG;n) = \left| \prod_{i \geqslant 0} \left(\mathrm{Sym}^{p+i}\,\mathfrak{g}^{\vee}\right)^{G} [n+p-2i] \right|,$$

and so

$$\pi_0 \mathcal{A}^{p,cl}(BG;n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \left(\mathrm{Sym}^{p}\,\mathfrak{g}^{\vee}\right)^{G}, & \text{if } n \text{ is even.} \end{cases}$$

Examples (iv):

(5) If $X = \mathrm{Rzero}(s)$ for $s \in H^0(L, E)$ on a smooth L, then

$$\Omega^1_X = \underset{-1}{E^\vee_{|Z}} \xrightarrow{(\nabla s)^\flat} \underset{0}{\Omega^1_{L|Z}},$$

and if we choose ∇ local flat algebraic connection on E we can rewrite Ω^1_X as a module over the Koszul complex:

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^1_L \arrow[r, "s^\flat"] & E^\vee \otimes \Omega^1_L \arrow[r, "s^\flat"] & \Omega^1_L \arrow[r] & \Omega^1_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \arrow[r, "s^\flat"] \arrow[u] & E^\vee \otimes E^\vee \arrow[r, "s^\flat"] \arrow[u] & E^\vee \arrow[r] \arrow[u, "{[\nabla, s^\flat]}"'] & E^\vee_{|Z} \arrow[u, "(\nabla s)^\flat"'] & -1
\end{tikzcd}$$

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$
\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^2_L \arrow[r] & E^\vee \otimes \Omega^2_L \arrow[r] & \Omega^2_L \arrow[r] & \Omega^2_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & (E^\vee \otimes \Omega^1_L)_{|Z} \arrow[u] & -1 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee | Z \arrow[u] & -2
\end{tikzcd}
$$

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^2_L \arrow[r] & E^\vee \otimes \Omega^2_L \arrow[r] & \Omega^2_L \arrow[r] & \Omega^2_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & (E^\vee \otimes \Omega^1_L)_{|Z} \arrow[u] & -1 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee | Z \arrow[u] & -2
\end{tikzcd}$$

2 forms of degree -1

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \wedge^2 E^\vee \otimes \Omega^2_L & \longrightarrow & E^\vee \otimes \Omega^2_L & \longrightarrow & \Omega^2_L & \longrightarrow & \Omega^2_{L|Z} & \quad 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L & \to & E^\vee \otimes E^\vee \otimes \Omega^1_L & \to & E^\vee \otimes \Omega^1_L & \to & (E^\vee \otimes \Omega^1_L)_{|Z} & \quad -1 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & \wedge^2 E^\vee \otimes S^2 E^\vee & \longrightarrow & E^\vee \otimes S^2 E^\vee & \longrightarrow & S^2 E^\vee & \longrightarrow & S^2 E^\vee | Z & \quad -2
\end{array}$$

Note: The de Rham differnetial $d_{DR} : \Omega^1_X \to \Omega^2_X$ is the sum $d_{DR} = \nabla + \kappa$, where κ is the Koszul contraction

$$\kappa : \wedge^a E^\vee \otimes S^b E^\vee \to \wedge^{a-1} E^\vee \otimes S^{b+1} E^\vee .$$

Examples (vi):

Important Remark: **[Behrend]** If $E = \Omega^1_L$ and so s is a 1-form, then a 2-form of degree -1 corresponds to a pair of elements

$\alpha \in (\Omega^1_L)^\vee \otimes \Omega^2_L$ and $\phi \in (\Omega^1_L)^\vee \otimes \Omega^1_L$ such that $[\nabla, s^\flat](\phi) = s^\flat(\alpha)$.

Take $\phi = \mathrm{id} \in (\Omega^1_L)^\vee \otimes \Omega^1_L$. Suppose the local ∇ is chosen so that $\nabla(\mathrm{id}) = 0$ (i.e. ∇ is torsion free). Then $[\nabla, s^\flat](\mathrm{id}) = ds$.

Conclusion: The pair (α, id) gives a 2-form of degree -1 iff $ds = s^\flat(\alpha)$, or equivalently $ds_{|Z} = 0$, i.e. is an almost closed 1-form on L.

Exercise: Suppose s is almost closed and let (α, id) be an associated 2-form of degree -1. Describe the complex of keys for (α, id) if it exists.