1. Let S be a surface of revolution: $S = (f(v)\cos(u), f(v)\sin(u), g(v))$, where $(f(v), 0, g(v))$ is a regular curve in the xz plane parametrized by arclength. Show that the Gaussian curvature of S is
$$K = -\frac{f''(v)}{f(v)}$$
and use this fact to construct a surface that has $K = 1$ but it is not contained in the unit sphere.

2. Let $\alpha : [0, 2\pi] \to \mathbb{R}^3$ be the curve defined by
$$\alpha(t) = (\cos(t)^2 - \frac{1}{2}, \sin(t)\cos(t), \sin(t))$$
Determine the singular points, the curvature and the torsion of α. Show that the trace of the curve lies on a sphere S and on a cylinder C (the axis of the cylinder is the z axis). Compute the first and the second fundamental form of S and C.

3. Show that the surface $(u + uv^2 - \frac{1}{3}u^3, v + vu^2 - \frac{1}{3}v^3, u^2 - v^2)$ is minimal and that the coordinates curves are lines of curvature.

4. Let $\alpha(v) : I \to \mathbb{R}^3$ be a bi-regular curve in $\mathbb{R}^3$ and let $w(v)$ be a unit vector field along α such that $\alpha' \perp w'$. Consider the surface
$$S = \{\alpha(v) + u\, w(v), u \in \mathbb{R}, v \in I\}$$
determine a relation between $\beta' \wedge w$ and w' and use it to describe the regular points of S. Show that the regular points of S are not elliptic.

5. Show that a closed (compact without boundary) surface of genus $g > 0$ in $\mathbb{R}^3$ has elliptic, parabolic and hyperbolic points.

1. We have

$$X_u = (-f(v)\sin(u), f(v)\cos(u), 0), \qquad X_v = (f'(v)\cos(u), f'(v)\sin(u), g'(v))$$
$$X_{uu} = (-f(v)\cos(u), -f(v)\sin(u), 0), \qquad X_{vv} = (f''(v)\cos(u), f''(v)\sin(u), g''(v))$$
$$X_{uv} = (-f'(v)\sin(u), f'(v)\cos(u), 0)$$

A unit vector field orthogonal to X_u and X_v can be obtained by normalization of $X_u \times X_v$. Using the fact that the curve is parametrized by arclength this simplifies to:

$$N = (g'(v)\cos(u), g'(v)\sin(u), -f'(v)).$$

We can compute the coefficients of the first fundamental form

$$E = f(v)^2, \qquad F = 0, \qquad G = 1$$

for the second fundamental form

$$e = -f(v)g'(v), \qquad f = 0, \qquad g = -f''(v)g'(v) + g''(v)f'(v)$$

then the ratio of the determinants of the two forms is

$$K = \frac{g'(v)}{f(v)}(-f''(v)g'(v) + g''(v)f'(v))$$

since $f'(v)^2 + g'(v)^2 = 1$ we have $2f'(v)f'(v) + 2g'(v)g''(v) = 0$ hence

$$g''(v) = -\frac{f'(v)f''(v)}{g'(v)}$$

if we replace in the expression of the Gauss curvature we get the requested equality.

It follows that the Gauss curvature is equal to 1 if and only if

$$f''(v) + f(v) = 0$$

hence $f(v) = a\cos(v) + b\sin(v)$, then $g(v)$ is determined, up to a constant, by the condition

$$g'(v)^2 = 1 - f'(v)^2$$

The surface is rotationally symmetric and it lies in a unit sphere if and only if, up to isometries of $\mathbb{R}^3$, the generating plane curve $(f(v), 0, g(v))$ is $(\cos(v), 0, \sin(v))$. It is clear that by choosing a, b, one can find surfaces that do not satisfy this condition.

2. One may notice that $\alpha(t) - (\frac{1}{2}, 0, 0)$ has unit norm or, doing a longer computation, consider $\alpha(t) - (a, b, c)$ and determine a, b, c so that the norm is constant (e.g. by taking the power series). Hence the curve lies in the sphere of radius 1 and center in $(-\frac{1}{2}, 0, 0$. This can be parametrized by

$$(\cos(v)\cos(u) - \frac{1}{2}, \cos(v)\sin(u), \sin(v))$$

it is clear that α is the image of the curve $(u(t), v(t)) = (t, t)$. Since this sphere is isometric to the standard sphere under a translation (i.e. an isometry of $\mathbb{R}^3$) we have that the first and the second fundamental forms coincide with the one of the standard sphere (see the notes). If the curve lies on a cylinder whose axis is parallel to the y axis then we have to find a point $(a, b, 0)$ such that $(\cos(t)^2 - \frac{1}{2}, \sin(t)\cos(t), 0) - (a, b, 0)$ has constant norm. It is easy to check that $(0, 0, 0)$ satisfies this condition and the norm is equal to $\frac{1}{2}$. Hence the curve lies on the cylinder

$$x^2 + y^2 = \frac{1}{4}$$

that can be parametrized by

$$(\frac{1}{2}\cos(u), \frac{1}{2}\sin(u), v)$$

here we omit the computation of the fundamental forms.

3. We have

$$X_u = (1 + v^2 - u^2, 2uv, 2u), \qquad X_v = (2uv, 1 + u^2 - v^2, -2v)$$

$$X_{uu} = (-2u, 2v, 2), \quad X_{uv} = (2v, 2u, 0), \quad X_{vv} = (2u, -2v, -2)$$

Moreover

$$X_u \times X_v = (-2u - 2u^3 - 2uv^2, 2v + 2v^3 + 2u^2v, 1 - u^4 - v^4 - 2u^2v^2)$$

denote by λ the norm of this vector. Then

$$N = \frac{1}{\lambda} X_u \times X_v.$$

...

4. Since w has constant norm, we have $w' \perp w$, moreover we assumed $\alpha' \perp w'$. Hence α' and w' are both orthogonal to w'. It follows that

$$\alpha' \times w = \lambda(v) w'$$

for some function λ. Denote by

$$X(u, v) = \alpha(v) + uw(v)$$

the parametrization of S, then

$$X_u = w, \qquad X_v = \alpha' + uw'$$

hence

$$X_u \times X_v = w \times \alpha' + uw \times w' = -\lambda w' + uw \times w'$$

since $w' \perp w \times w'$ this is the sum of two orthogonal vectors and it is zero if and only if both of them vanish. It follows that

$$X_u \times X_v = 0 \iff \lambda(v) = 0, u = 0$$

hence the singular points of s are the points of α such that $\lambda(t) = 0$.

If S is elliptic then the maximum and the minimum of the Gauss curvature are positive. On the other hand through each point of S we have a line that lies on S, parallel to w. Hence there is a normal section of S that contains a line, i.e. has curvature equal to 0.

5. A compact surface is $\mathbb{R}^3$ is orientable. Hence we can apply the Gauss Bonnet theorem and we have that

$$\int_S K = 2\pi\chi(S) = 2\pi(2 - 2g) < 0.$$

Hence there are points where the curvature is negative. On the other hand, since S is compact, there is at least one point where the Gauss curvature is positive. By continuity there are points on S where the curvature vanishes.