Due: Thursday March 23 at the end of class. A portion of the homework will be graded (by Sam Carp) and returned to you at the end of the next class. Remember to staple your homework and put your name on it.

1. Let α be a bi-regular curve on a surface S, parametrized by arclength and denote by $\{\underline{t}, \underline{n}, \underline{b}\}$ the Frenet frame of α. If α is a geodesic then $\underline{n} = N$ is normal to the surface. α is a line of curvature if it is tangent to a principal direction at every point.

   a) If α is a line of curvature then $N' = \lambda(t)\underline{t}$ for some function $\lambda(t)$.

   b) Show that if α is both a line of curvature and a geodesic then α is a plane curve.

2. Let $\alpha : I \to \mathbb{R}^3$ be a bi-regular curve parametrized by arclength. Denote by $\{\underline{t}, \underline{n}, \underline{b}\}$ the Frenet frame of α. Then $\beta(s) = \underline{n}(s)$ is a curve whose trace lies on the unit sphere S^2. Compute $\beta'(s)$ and show that β is a regular curve. Denote by t the arclength parameter of β so that $\beta(s(t))$ is parametrized by arclength. Show that the geodesic curvature of β is then given by
$$k_g = \frac{d}{ds}\left(\arctan\left(\frac{\tau}{k}\right)\right)\frac{ds}{dt}$$
   where k, τ are the curvature and the torsion of the curve α.

3. Let S^2 be the unit sphere in $\mathbb{R}^3$, centered in the origin. and let $N = (0,0,1)$. Let $v \in T_N S^2$ be a unit vector and denote by γ_v the geodesic on S^2 such that $\gamma(0) = N$ and $\gamma'(0) = v$. Denote by $N(p) = p$ the unit normal field on S^2

   a) Show that $W(t) = \gamma_v'(t) \times N$ is a parallel vector field along γ_v (no computations, just use the properties of the parallel transport).

   b) Prove that if $W_1(t)$ and $W_2(t)$ are parallel along a curve α on a surface S then any parallel vector field along α is a linear combination with constant coefficients $W(t) = aW_1(t) + bW_2(t)$ of W_1 and W_2.

   c) Let $v = (\cos(\theta), \sin(\theta), 0) \in T_N S^2$, determine the parallel transport $E_1(t)$, $E_2(t)$ of $e_1 = (1,0,0)$, $e_2 = (0,1,0) \in T_N S^2$ along $\gamma_v(t)$.

   d) Let $X(u,v) = (\cos(v)\cos(u), \cos(v)\sin(u), \sin(v))$ be a local parametrization of S^2, with $u \in (0, 2\pi)$ and $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Write E_1 and E_2 as linear combinations of X_u and X_v.

   e) Compute the bracket $[E_1, E_2]$ of E_1 and E_2 and conclude that E_1 and E_2 cannot be coordinate fields of a local parametrization of S^2.

4. Determine the area of the surface of a torus.