Due: Thursday February 23 at the end of class. A portion of the homework will be graded (by Sam Carp) and returned to you at the end of the next class. Remember to staple your homework and put your name on it.

1. Let S be a regular surface and fix a point $p_0 \in \mathbb{R}^3$ such that $p_0 \notin S$. Define the function $f : S \to \mathbb{R}$ by $f(p) = d(p, p_0)^2$ where $p \in S$ and d is the standard Euclidean distance in $\mathbb{R}^3$. Then, if p is a critical point of f iff $p_0 = p + \lambda N(p)$ for some λ (similar to Ex. 4 in Homework 4). Prove that the Hessian of f at a critical point p is given by

$$H_f(p)(v) = 2(||v||^2 - \lambda II_p(v))$$

   Where II_p denotes the second fundamental form of S at the point p.

2. Consider the two families of matrices

$$A_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix}, \qquad B_s = \begin{pmatrix} \cos(s) & \sin(s) & 0 \\ -\sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Then $A_t, B_s \in O(3)$ and the maps $f_t : p \to A_t \cdot p$, $g_s p \to B_s \cdot p$ are isometries of $\mathbb{R}^3$. Let $p \in S = S^2(1)$ be a point of the unit sphere centered in the origin. Then (since $f_t(0) = g_t(0) = 0$) S is preserved by f_t and g_s for every t and s.

   a) For $p = (x, y, z) \in S$ consider the curves $f_t(p)$ and $g_s(p)$ and determine the tangent vectors $X(p)$ and $Y(p)$ to these curves at p. We have then two well defined vector fields X and Y on S.

   b) Let $p = (0, 1, 0) \in S$ and let $\alpha(t) = f_t(p)$. Determine the values of the vector field Y at the points of α.

   c) Determine $D_X Y(p)$.

3. Let $U = \{(u, v) \in \mathbb{R}^2 : u > 0, v \in (0, 2\pi)\}$ and consider the surfaces S_1 and S_2 parametrized by

$$X_1(u, v) = (u\cos(v), u\sin(v), \log(u)), \qquad X_2(u, v) = (u\cos(v), u\sin(v), v)$$

   show that S_1 and S_2 have the same Gauss curvature at $X_1(u, v)$ and $X_2(u, v)$ but $X_2 \circ X_1^{-1}$ is not an isometry between the two surfaces.

4. Compute the mean curvature of the catenoid and the helicoid.