Due: Thursday February 9 at the end of class. A portion of the homework will be graded (by Sam Carp) and returned to you at the end of the next class. Remember to staple your homework and put your name on it.

1. a) Let S^2 be the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$ and let $N=(0,0,1)$, $S=(0,0,-1)$ be the north and the south poles. For $(u,v)\in U=\mathbb{R}^2$ let $\mathbf{x}_1(u,v)$ be the intersection (different from N) of the line through $(u,v,0)$ and N with S^2. Write explicity $\mathbf{x}_1(u,v)$ and the inverse $\mathbf{x}_1^{-1}(x,y,z)$ and conclude that $\mathbf{x}_1$ is a local parametrization that covers all the points of $S^2\setminus\{N\}$. Construct a similar parametrization $\mathbf{x}_2(u,v)$ by replacing N with S and conclude that the sphere can be covered with 2 coordinate neighborhoods.

   b) Write the transition map $\mathbf{x}_{12}(u,v):\mathbb{R}^2\to\mathbb{R}^2$ and the differential of $\mathbf{x}_{12}(u,v)$. Verify that $\mathbf{x}_{12}(u,v)$ is a diffeomorphism using the inverse function theorem.

2. Let a,b,c be positive real numbers. Find an explicit diffeomorphism between the sphere S^2 and the ellipsoid
$$S=\{(x,y,z)\in\mathbb{R}^3:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\}.$$

3. Let $f:\mathbb{R}^3\to\mathbb{R}$ be defined by $f(x,y,z)=z^2$. Show that 0 is not a regular value for f but $f^{-1}(0)$ is a regular surface.

4. Find a parametrization of a neighborhood of $p=(1,0,0)$ for the surface $S=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2-z^2=1\}$ and determine a basis of T_pS. Find a vector perpendicular to T_pS.

5. Show that the cone $S=\{(x,y,z)\in\mathbb{R}^3:x^2+y^2=z^2\}$ is not a regular surface.

6. Let $S=f^{-1}(a)$, where $f:\mathbb{R}^3\to\mathbb{R}$ is smooth and a is a regular value.

   a) Show that, for $p=(x_0,y_0,z_0)\in S$, the gradient of f at p is perpendicular to T_pS (hint: consider curve $\alpha(t):I\to S$ such that $\alpha(0)=p$). Conclude that the equation of the tangent space at p is
$$\frac{\partial f}{dx}(p)(x-x_0)+\frac{\partial f}{dy}(p)(y-y_0)+\frac{\partial f}{dz}(p)(z-z_0)=0.$$

   b) Let S be the surface parametrized by $\mathbf{x}(u,v)=(u^3,v^3,uv)$ for $(u,v)\in\mathbb{R}^2$. Find the regular points of S.

   c) Show that x is a bijection onto $\{(x,y,z)\in\mathbb{R}^3, xy-z^3=0\}$ and use the formula above to determine the tangent space at $p=(1,1,1)$.

   d) Show that $(1,2,1)$ belongs to T_pS and find a curve α on S such that $\alpha(0)=p$ and $\alpha'(0)=v$. (Hint: try to find a plane curve)