Quaternionic geometry in 8 dimensions

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Differential Geometry in the Large

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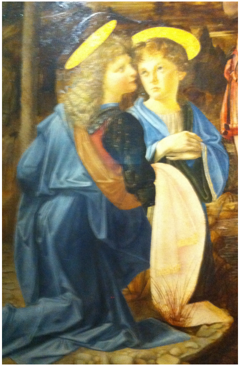


Four categories of manifolds

. . . all equipped with an action of I, J, K on each tangent space $T_x M^{4n}$ $(n \geqslant 2)$ and a torsion-free G connection:

Hyperkähler $Sp(n)$	Hypercomplex $GL(n, \mathbb{H})$
Quaternion-kähler $Sp(n)Sp(1)$	Quaternionic $GL(n, \mathbb{H})Sp(1)$

'Hypercomplex' implies that I, J, K are complex structures.

'Quaternionic' implies that the tautological complex structure on the 2-sphere bundle $Z(\to M)$ is integrable, and the (e.g.) Fueter operator can be defined.

One could also add $SL(n, \mathbb{H})U(1)$ structures to the 2nd column.

Hyperkähler manifolds

We shall soon focus on non-integrable $Sp(2)Sp(1)$ structures, but by way of introduction:

For a hyperkähler manifold, the holonomy of the Levi-Civita connection lies in $Sp(n)$, and the Ricci tensor vanishes.

Calabi gave explicit complete examples on (e.g.) $T^*\mathbb{CP}^n$.

Many HK metrics can be constructed via the HKLR quotient construction, and abound on moduli spaces (e.g. $8k-3 \rightsquigarrow 8k$).

Any K3 surface admits a HK metric by Yau's theorem.

Beauville described two families K^{4n} and A^{4n} of compact HK manifolds, arising from Hilbert schemes of points on a K3 or Abelian surface. They satisfy $24 \,|\, (n\chi)$.

Quaternion-kähler manifolds

This time, the holonomy of the Levi-Civita connection lies in $Sp(n)Sp(1)$. QK manifolds are Einstein, we assume not Ricci-flat. Curvature-wise, they are 'nearly hyperkähler'.

Wolf showed that there is a QK symmetric space (and its dual) for each compact simple Lie group G, and that its twistor spaces has a complex contact structure. *This talk will focus on $G_2/SO(4)$.*

These spaces are the only known complete QK manifolds with $s > 0$, but there is an incomplete metric defined by any pair $\mathfrak{su}(2) \subset \mathfrak{g}$, combining work of Kronheimer and Swann (next slide).

Alekseevsky and Cortés have constructed families of complete non symmetric/homogeneous examples with $s < 0$. LeBrun had shown that there is an infinite-dimensional moduli space.

The miraculous case of $G = SU(3)$

Up to conjugacy, $\mathfrak{su}(3)$ has two TDA's: $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

The first gives rise to the Wolf space $\mathbb{CP}^2 = \dfrac{SU(3)}{S(U(1)\times U(2))}$.

The second gives rise to the Grassmannian $\mathbb{L}$ of special Lagrangian subspaces $\mathbb{R}^3 \subset \mathbb{C}^3$, and there are $SU(3)$-equivariant maps:

$$\begin{array}{ccc} \dfrac{G_2}{SO(4)} \setminus \mathbb{CP}^2 & \cong & \mathbb{V} \\ \downarrow & & \downarrow \ \text{obvious VB with fibre } \mathbb{R}^3 \\ SU(3)/SO(3) & = & \mathbb{L} \end{array}$$

Now, $\mathbb{Z}_3$ acts freely on $G_2/SO(4) \setminus \mathbb{CP}^2$, and the quotient is a submanifold U of $\mathbb{G}\mathrm{r}_3(\mathfrak{su}(3))$ invariant under a Nahm flow. Its Swann bundle is $\mathscr{N} = \{A \in \mathfrak{sl}(3,\mathbb{C}) : A^3 = 0,\ A^2 \neq 0\}$.

Coassociative submanifolds

The Wolf space $G_2/SO(4)$ parametrizes coassociative subspaces $i\colon \mathbb{R}^4 \subset \mathbb{R}^7$. These are subspaces for which $i^*\varphi = 0$, where

$$\varphi = e^{125} - e^{345} + e^{136} - e^{426} + e^{147} - e^{237} + e^{567}$$

is the standard 3-form with stabilizer G_2.

The space $\mathbb{L}$ parametrizes some special coassociative submanifolds $L^\perp \subset \pi^{-1}(\mathbb{RP}^2)$ of the 7-dimensional total space

$$\begin{array}{ccc} & & \Lambda^2_- T^*\mathbb{CP}^2 \\ & & \big\downarrow \pi \\ \mathbb{RP}^2 \subset & & \mathbb{CP}^2 \end{array}$$

with the Bryant-S metric with holonomy G_2 [Karigiannis-MinOo].
Moreover, $G_2/SO(4)$ is intimately connected with this total space.

Groups containing Sp(2)

On $\mathbb{R}^8 = \mathbb{H}^2 \ni (p,q)$, define a 'hyperkähler triple'

$$\tfrac{1}{2}(dp \wedge dp + dq \wedge dq) = \omega_1 i + \omega_2 j + \omega_3 k$$

of 2-forms

$$\begin{cases} \omega_1 = e^{12} + e^{34} + e^{56} + e^{78} \\ \omega_2 = e^{13} + e^{42} + e^{57} + e^{86} \\ \omega_3 = e^{14} + e^{23} + e^{58} + e^{67}. \end{cases}$$

The stabilizer of

$$\Omega_\lambda = \tfrac{1}{2}(\lambda\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3)$$

is $Sp(2)U(1) \subset SU(4)$ except that:

- $\mathrm{stab}(\Omega_1) = Sp(2)Sp(1)$.
- $\mathrm{stab}(\Omega_{-1}) = Spin(7)$.

Closed versus parallel

A holonomy reduction occurs when $\nabla\Omega = 0$.

For the Levi-Civita connection, obviously $\nabla\Omega = 0 \;\Rightarrow\; d\Omega = 0$.

- If $\Omega = \Omega_{-1}$ has stabilizer $Spin(7)$ then $d\Omega = 0 \Rightarrow \nabla\Omega = 0$ [Fernández-Gray]. In this case,

$$\nabla\Omega \;\in\; \Lambda^1 \otimes \mathfrak{g}^\perp \;\cong\; \Lambda^3 \;\cong\; \Lambda^5.$$

- If $\Omega = \Omega_1$ has stabilizer $Sp(2)Sp(1)$, by contrast, $d\Omega$ does not determine $\nabla\Omega$ [Swann]. It is therefore natural to generalize the class of QK manifolds to those almost-QH ones with

$$\Omega \in \Lambda^4_+$$

closed (so harmonic) but not parallel.

Instrinic torsion for Sp(2)Sp(1)

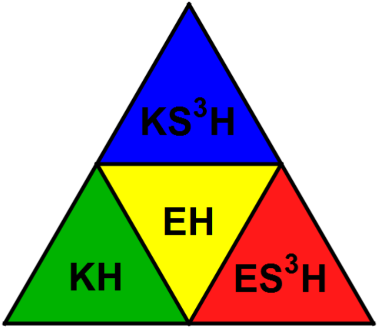
The space $\Lambda^1 \otimes (\mathfrak{sp}(2) + \mathfrak{sp}(1))^\perp$ has 4 components:

If $\nabla\Omega$ lies in. . .

blue then $d\Omega = 0$

red then 'ideal':
$d\omega_i = \sum \alpha_i^j \wedge \omega_j$,
work by Macía

green then quaternionic.



Corollary. (Ideal or quaternionic) and $d\Omega = 0 \;\Rightarrow\; \nabla\Omega = 0$

Harmonic Sp(2)Sp(1) reductions

A first example was found on $M^8 = M^6 \times T^2$ where $M^6 = \Gamma\backslash N$ is a symplectic nilmanifold with a pair of simple closed 3-forms, defining a 'tri-Lagrangian geometry'. The structure group of M^6 reduces to a diagonal $SO(3)$.

There are many more examples of the form $M^7 \times S^1$ obtained by setting $\Omega = \alpha \wedge e^8 + \beta$ and using the fact that

$$Sp(2)Sp(1) \cap SO(7) = G^*_{2\alpha} \cap G^*_{2\beta} = SO(4).$$

[Conti-Madsen classify 11 nilmanifolds and find solvmanifolds].
Are there simply-connected examples?

Theorem [CMS]. The parallel QK 4-form on $G_2/SO(4)$ can be 'freely' deformed to a closed form with stabilizer $Sp(2)Sp(1)$ invariant by the cohomogenous-one action by $SU(3)$.

Symmetric spaces in 8 dimensions

Apart from $\mathbb{CP}^4$ (whose holonomy $U(4)$ is not so special), there are 4 compact models which all admit a cohomogenous-one action, with principal orbits $SU(3)/U(1)_{1,-1}$ and two ends chosen from

$$S^5, \qquad \mathbb{CP}^2, \qquad \mathbb{L} = SU(3)/SO(3)$$

[Gambioli]. The first three are quaternion-kähler:

$\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$	$SU(4)/U(2)Sp(1)$	$\mathbb{CP}^2,\ \mathbb{CP}^2$
$\mathbb{HP}^2$	$Sp(3)/Sp(2)Sp(1)$	$\mathbb{CP}^2,\ S^5$
$G_2/SO(4)$	$G_2/SU(2)Sp(1)$	$\mathbb{CP}^2,\ \mathbb{L}$
$SU(3)$	$SU(3)^2/\Delta$	$S^5,\ \mathbb{L}$

Topological remarks

The Wolf spaces $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$, $\mathbb{HP}^2$, $G_2/SO(4)$ are all spin with $s > 0$. They satisfy $\widehat{A}_2 = 0$ and

$$8\,\chi = 4p_2 - p_1^2.$$

The latter is also valid for any 8-manifold whose structure group reduces to $Spin(7)$, and the Wolf spaces all have such structures, but not holonomy equal to $Spin(7)$ as this would require $\widehat{A}_2 = 1$.

Nonetheless one can search for closed non-parallel 4-forms on the Wolf spaces [with motivation from Foscolo-Haskins' construction of new nearly-kähler metrics on S^6 and $S^3 \times S^3$].

Quest for the QK 4-form on $\mathbb{V}$

The construction of exceptional metrics on vector bundles over 3- and 4-manifolds made use of ‘dictionaries’ of tautological differential forms. It was natural to use similar techniques to identity the parallel 4-form Ω over $\mathbb{V}$, but this took a few years:

Proposition [CM]. The parallel QK 4-form Ω can be expressed $SU(3)$-equivariantly on $\mathbb{V}$ as

$$\frac{3\sin^2(r)\cos^2(r)}{r^2}\mathbf{bb}\boldsymbol{\beta} + \frac{\sqrt{3}\sin(2r)}{r}\mathbf{b}\tilde{\boldsymbol{\beta}} + \frac{\sin^2(r)\cos^2(r)}{r^2}\mathbf{a}\tilde{\boldsymbol{\beta}}\boldsymbol{\epsilon} - \frac{-5\sin(2r)+\sin(6r)+4r\cos(2r)}{128\sqrt{3}r^3}\boldsymbol{\gamma}\boldsymbol{\epsilon}\boldsymbol{\epsilon}$$
$$+\frac{\sin^4(r)(\cos(2r)+\cos(4r)+1)}{2\sqrt{3}r^4}\mathbf{bbb}\,\mathbf{a}\boldsymbol{\epsilon} + \frac{\sqrt{3}(2r\cos(2r)-\sin(2r))}{8r^3}\mathbf{b}\boldsymbol{\beta}\,\mathbf{a}\boldsymbol{\epsilon}$$
$$+\frac{3(2r\sin(4r)+\cos(4r)-1)}{4r^4}\mathbf{ab}\,\mathbf{ab}\boldsymbol{\beta} + \frac{\sin^2(r)(5r-6\sin(2r)-3\sin(4r)+r(13\cos(2r)+5\cos(4r)+\cos(6r)))}{96\sqrt{3}r^5}\mathbf{ab}\,\boldsymbol{\epsilon}\boldsymbol{\epsilon}\boldsymbol{\epsilon}$$
$$+\frac{\sin^3(2r)(\sin(2r)-2r\cos(2r))}{32r^6}\mathbf{abb}\,\mathbf{a}\boldsymbol{\epsilon}\boldsymbol{\epsilon} - \frac{\sin^3(2r)\cos(2r)}{8r^3}\mathbf{a}\boldsymbol{\gamma}\,\mathbf{a}\boldsymbol{\gamma}$$

and equals $3\,\mathbf{bb}\boldsymbol{\beta} + 2\sqrt{3}\,\mathbf{b}\tilde{\boldsymbol{\beta}}$ when $r=0$.

Letters, syllables and words

The $SU(3)$-invariant differential forms on $\mathbb{V}$ arise from forms defined on the fibre with values in the exterior algebra of the base, everything invariant by $SO(3)$. Syllables arise by contracting letters using the inner product or the volume form on $\mathbb{R}^3$. Examples:

- the syllable **aa** equals $r = \sum(a_i)^2$;
- $\Lambda^2(T_x^*\mathbb{L}) \cong \mathfrak{so}(5) \cong \mathbb{R}^3 \oplus \mathbb{R}^7$, and the value of the syllable $\mathbf{a}\boldsymbol{\beta}$ is the pullback of the 2-form in $\mathbb{R}^3$ it represents:

$$a_1(-e^{12}+2e^{34}) + a_2(e^{13}-e^{24}-\sqrt{3}e^{25}) + a_3(e^{14}+\sqrt{3}e^{15}-e^{56});$$

- differentiating the a_i gives $b_i = da_i +$ connection forms, then $\mathbf{bbb} = b_1 \wedge b_2 \wedge b_3$ and $\mathbf{bb}\boldsymbol{\beta} = \mathfrak{S}\ b_i \wedge b_j \wedge \beta_k$;
- words like $\mathbf{bb}\boldsymbol{\beta}$ and $\mathbf{bbb}\,\mathbf{a}\boldsymbol{\epsilon}$ of degree 4 can be formed by wedging 1 or 2 syllables together.

More invariant 4-forms

A generic $SU(3)$-invariant 4-form on $\mathbb{V}$ is

$$k_1\mathbf{bb}\beta + k_2\mathbf{b}\tilde{\beta} + k_3\mathbf{ab}\tilde{\beta} + k_4\mathbf{b}\gamma\epsilon + k_5\mathbf{a}\tilde{\beta}\epsilon + k_6\gamma\epsilon\epsilon$$
$$+k_7\mathbf{bbb}\,\mathbf{a}\epsilon + k_8\mathbf{b}\beta\,\mathbf{a}\epsilon + k_9\mathbf{ab}\,\mathbf{ab}\beta + k_{10}\mathbf{ab}\,\mathbf{a}\gamma\epsilon$$
$$+k_{11}\mathbf{ab}\,\epsilon\epsilon\epsilon + k_{12}\mathbf{abb}\,\mathbf{a}\epsilon\epsilon + k_{13}\mathbf{ab}\beta\,\mathbf{a}\epsilon + k_{14}\mathbf{a}\gamma\,\mathbf{a}\gamma.$$

It extends smoothly across $\mathbb{L}$ iff k_i are smooth even functions of r.
It is closed if and only if

$$\begin{cases} k_1' = rk_9, \quad k_2' = 8rk_8, \\ k_3 = k_{13} = 0, \quad k_5 = \frac{1}{3}k_1, \\ 2rk_{12}' + 12k_{12} - \frac{1}{r}k_{14}' = 0, \\ 24\frac{1}{r}k_6' + \frac{1}{r}k_8' - 72k_{11} - 6k_7 = 0 \end{cases}$$

Group parameters

In order to express the parallel 4-form relative to the standard basis (e_i, b_j), we need to express the k_i in terms of parameters

$$e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_{13}}, e^{i\lambda_{14}}, \begin{pmatrix} \lambda_8 & \lambda_9 \\ \lambda_{10} & \lambda_{11} \end{pmatrix}, e^{i\lambda_3}, \begin{pmatrix} \lambda_4 & \lambda_5 \\ \lambda_6 & \lambda_7 \end{pmatrix}, \lambda_{12}$$

for the group $U(1)^4 \rtimes GL(2,\mathbb{R}) \times U(1) \times GL(2,\mathbb{R}) \times \mathbb{R}^*$ that commutes with the $U(1)$ stabilizer of each $SU(3)$ orbit. Closure imposes ODE's on the λ_i (**but not $\boldsymbol{\lambda_7}$**), and we find a solution

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_9 = \lambda_{10} = 0,\ \lambda_{12} = -1,$$
$$\lambda_5 = -\cos(2r),\ \lambda_6 = \sqrt{3},\ \lambda_8 = \tfrac{1}{2}(3 - 2\cos^2 r)\cos r,$$
$$\lambda_{11} = \tfrac{\cos(2r)\sqrt{3}\sin r}{2r},\ \lambda_{13} = \tfrac{(1+2\cos^2 r)\sin r}{2r},$$
$$\lambda_{14} = \tfrac{\sqrt{3}}{2}(-1 + 2\cos^2 r)\cos r.$$

Linear deformation

Problem. To preserve the stabilizer by solving

$$\Omega + t\phi = g(t)\Omega, \qquad g(t) \in GL(8, \mathbb{R}).$$

NB. If $A \in \mathfrak{gl}(8, \mathbb{R})$ satisfies $A \cdot (A \cdot \Omega) = 0$ then $\phi = A \cdot \Omega$ works.

Surprisingly, this can be applied in the $SU(3)$-equivariant case with $A = e_{56}$ to obtain a new triple with $\widetilde{\omega}_2 = \omega_2 - \lambda e^{58}$, $\widetilde{\omega}_3 = \omega_3 + \lambda e^{57}$. It is the interpretation of what happens if $\lambda_7 \neq 0$.

Theorem. The closed 4-form

$$\widetilde{\Omega} = \Omega + f(r)(\mathbf{aa}\,\mathbf{b}\boldsymbol{\gamma}\boldsymbol{\epsilon} + 3\mathbf{ab}\,\mathbf{a}\boldsymbol{\gamma}\boldsymbol{\epsilon})$$

defines a metric on $G_2/SO(4)$ with an $Sp(2)Sp(1)$-structure that is not QK, for any smooth non-zero function $f\colon [0, \pi/4] \to \mathbb{R}$ vanishing on neighbourhoods of the endpoints.

Other geometrical structures

Theorem [Gauduchon-Moroianu-Semmelmann]. Apart from the Grassmannians $\mathbb{G}\mathrm{r}_2(\mathbb{C}^n)$, the Wolf spaces (including $E_8/E_7Sp(1)$) do not admit almost-complex structures even stably.

NB. $\mathbb{V} \setminus \mathbb{L}$ does admit an $SU(3)$-invariant almost Hermitian structure of generic type defined by ω_1.

Proposition. There does not exist an $SU(3)$-invariant $Spin(7)$ structure on $\mathbb{V}$: only $Sp(2)Sp(1)$ is possible.

Work is in progress to:

- solve the ODE's on the λ_i parameters to find other harmonic structures on $G_2/SO(4)$
- establish the existence or otherwise of harmonic $Sp(2)Sp(1)$ structures on $\mathbb{HP}^2$ and $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$.

The fourth 8-manifold SU(3)

The compact 8-manifold underlying $SU(3)$ admits a host of geometrical structures, including:

- a left-invariant hypercomplex structure [Joyce]
- an invariant $Sp(2)Sp(1)$ metric that is 'ideal' [Macía]
- a $PSU(3)$ structure defined by the stable 3-form

$$\gamma(x, y, z) = \langle\, [x, y], z\,\rangle \quad \text{on} \quad T_x SU(3) \cong \mathfrak{su}(3),$$

which is harmonic: $d\gamma = 0 = d{*_\gamma}\gamma$ [obvious].

Harmonic $PSU(3)$ metrics have been found on nilmanifolds [Witt].
Do there exist new simply-connected examples?

Consimilarity in SU(3)

The cohomogeneity-one action of $SU(3)$ on itself is twisted conjugation:

$$X \mapsto PX\overline{P}^{-1} = PXP^{\top}, \qquad X,\, P \in SU(3).$$

The stabilizer of the identity is $SO(3)$ and its orbit is

$$\{PP^{\top} : P \in SU(3)\} = \{X \in SU(3) : X\overline{X} = I\}.$$

In fact, $f\colon X \mapsto X\overline{X}$ maps $SU(3)$ 'con-Ad' equivariantly onto the hypersurface

$$\mathscr{H} = \{P \in SU(3) : \mathrm{tr}\, P \in \mathbb{R}\},$$

which can be identified with the Thom space of the VB $\Lambda^2_- T^*\mathbb{CP}^2$.

Back to G_2 holonomy

The action of $SU(3)$ on $\mathbb{HP}^2$ commutues with S^1, so there's a residual quotient to be performed:

$$\begin{array}{ccccc} & & \mathbb{L} & \stackrel{f}{\longrightarrow} & \mathrm{pt} \\ \mathbb{HP}^2 \setminus \mathbb{CP}^2 & \cong & SU(3) \setminus \mathbb{L} & \to & \Lambda^2_- T^*\mathbb{CP}^2 \\ \downarrow & & \downarrow & & \downarrow \\ S^5 & = & S^5 & \to & \mathbb{CP}^2 \end{array}$$

S^5 is the zero set of a QK moment map, and $\mathbb{CP}^2 = \mathbb{HP}^2 /\!/\!/ S^1$. The 7-dimensional quotent can be identified with $S^7 \setminus \mathbb{CP}^2$ [Atiyah-Witten, Miyaoka] as well as $\mathscr{H} \setminus \mathrm{pt} \subset SU(3)$.

Problem: Clarify the relationship of these S^1 quotients and between metrics in 7 and 8 dimensions with reduced holonomy.