Min-max approach to Yau's conjecture

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  - The topological condition is very mild: fails for manifolds with the homotopy type of a CROSS (Sullivan–Vigué-Poirrier).
- Rademacher, (1989) *Assume closed* M^n *and simply connected. For "almost every" metric* (M^n, g) *admits an infinite number of closed geodesics.*

Yau's conjecture

Just like geodesics are critical points for the length functional, Minimal surfaces are critical points for the volume functional.

Yau's Conjecture '82

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Yau's Conjecture '82

Every compact 3-dimensional manifold admits an infinite number of immersed minimal surfaces.

- Simon–Smith, (1982) *Every (S^3, g) admits a smooth embedded minimal sphere.*
- Pitts (1981), Schoen–Simon, (1982) *Every compact manifold (M^{n+1}, g) admits an embedded minimal hypersurface smooth outside a set of codimension 7.*
- Khan–Markovic, (2012) *Closed hyperbolic 3-manifolds admit an infinite number of minimal immersed surfaces for any metric.*

Almgren Pitts Min-max Theory

- (M^{n+1}, g) closed compact Riemannian n-manifold, $2 \leq n \leq 6$.
- $\mathcal{Z}_n(M; \mathbb{Z}_2) = \{$integral mod 2 currents T with $\partial T = 0\}$
  $= $ "$\{$all compact hypersurfaces in $M\}$".

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- $[\Phi_k] = \{\text{all } \Psi \text{ homotopic to } \Phi_k\}$;
- The k-width is

$$\omega_k(M) := \inf_{\{\Phi \in [\Phi_k]\}} \sup_{x \in \mathbb{RP}^k} vol(\Phi(x)).$$

Compare with

$$\lambda_k(M) = \inf_{\{(k+1) \text{ plane } P \subset W^{1,2}\}} \sup_{f \in P - \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Almgren Pitts Min-max theory

Theorem (Pitts, '81, Schoen–Simon, '82) *For all* $k \in \mathbb{N}$ *there is an embedded minimal hypersurface* Σ_k *(with multiplicities) so that*

$$\omega_k(M) = \inf_{\{\Phi \in [\Phi_k]\}} \sup_{x \in \mathbb{RP}^k} vol(\Phi(x)) = vol(\Sigma_k).$$

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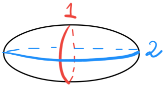
Theorem (Marques–N., '13) *Assume* (M, g) *has positive Ricci curvature. Then* M *admits an infinite number of distinct embedded minimal hypersurfaces.*

To handle the general case, need more information on the minimal surfaces Σ_k...

Index estimates



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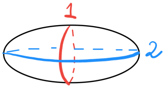

Theorem (Marques–N., '15) *For every $k \in \mathbb{N}$, one can find a minimal embedded hypersurface Σ_k with*

- $\omega_k(M) = vol(\Sigma_k) = \inf_{\{\Phi \in [\Phi_k]\}} \sup_{x \in \mathbb{RP}^k} vol(\Phi(x))$;
- *index of support of $\Sigma_k \leq k$.*

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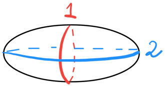

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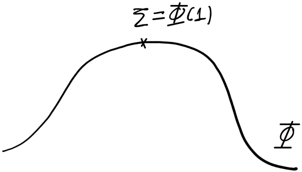

- *Suppose $\Phi : [0,2] \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ with $\max_t vol(\Phi(t)) = vol(\Phi(1))$ and $\Sigma = \Phi(1)$ minimal with index 2.*

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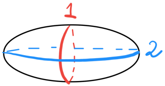

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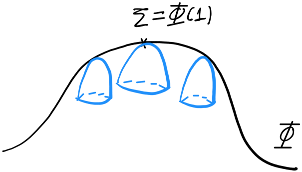

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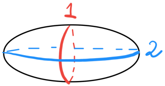

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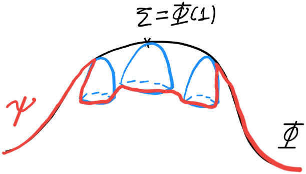

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- *Near Σ, there is a disc of deformations whose volume is a parabola.*
- *Find Ψ homotopic to Φ with* $\max_t vol(\Psi(t)) < vol(\Phi(1))$.

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- $\omega_1(M) = vol(\Sigma_1)$;
- index of $\Sigma_1 = 1$;
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Rmk: Σ_1 can be j(index 0) + (index 1) but neither j(index 1) nor (index 0).

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Basic approach to rule out multiplicity: Suppose there is $\Phi : [0, 2] \to \mathcal{Z}_n(M)$ with $\max_t vol(\Phi(t)) = vol(\Phi(1))$ and for $|t - 1| < \varepsilon$, $\Phi(t) = 2S_t$ where

- S_1 is minimal surface with index one;
- $vol(S_t) < vol(S_1)$ if $t \neq 1$.

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There is path $\{L_t\}$ connecting $2S_{1-\varepsilon}$ to $S_{1-\varepsilon} + S_{1+\varepsilon}$ and then to $2S_{1+\varepsilon}$ so that

$$vol(L_t) < 2vol(S_1) = vol(\Phi(1)) \quad \text{for all } |t - 1| \leq \varepsilon.$$

Multiplicity one Conjecture

Conjecture (Marques–N, '15) *For bumpy metrics* (M^{n+1}, g)*,* $2 \leq n \leq 6$*, unstable components in min-max hypersurfaces obtained with multi-parameters have multiplicity one.*

- The previous theorem confirms the conjecture for one parameter.
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Theorem (Marques–N) *Assuming the multiplicity one Conjecture, for every* $k \in \mathbb{N}$ *there is an embedded minimal hypersurface* Σ_k *such that*

- *index of* $\Sigma_k = k$ *and unstable components have multiplicity one;*
- $\omega_k(M) = vol(\Sigma_k)$.

Corollary *The minimal hypersurfaces* $\{\Sigma_k\}_{k\in\mathbb{N}}$ *are all distinct and so a stronger version of Yau's conjecture holds.*

Non-linear Spectrum

For $k \in \mathbb{N}$, $\omega_k(M) = \inf_{\{\Phi \in [\Phi_k]\}} \sup_{x \in \mathbb{RP}^k} vol(\Phi(x))$.

The sequence $\{\omega_k(M)\}_{k \in \mathbb{N}}$ is a non-linear spectrum of (M, g). Recall

$$\lambda_k(M) = \inf_{\{(k+1) \text{ plane } P \subset W^{1,2}\}} \sup_{f \in P - \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

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Theorem (Gromov, 80's, Guth, '07) *$\omega_k(M)$ grows like $k^{1/(n+1)}$ as k tends to infinity.*

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Theorem (Gromov, 80's, Guth, '07) $\omega_k(M)$ *grows like* $k^{1/(n+1)}$ *as* k *tends to infinity.*

Weyl Law states that

$$\lim_{k\to\infty} \frac{\lambda_k(M)}{k^{\frac{2}{n+1}}} = \frac{4\pi^2}{(\omega_{n+1}\,\mathrm{vol}\,M)^{\frac{2}{n+1}}}.$$

Conjecture (Gromov): $\{\omega_k(M)\}_{k\in\mathbb{N}}$ *also satisfies a Weyl Law.*

Non-linear Spectrum

Weyl Law (Liokumovich–Marques–N, '16) *Weyl Law holds meaning that there is* $\alpha(n)$ *such that for all compact* (M^{n+1}, g) *(with possible* $\partial M \neq 0$*)*

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Can we estimate $\alpha(n)$?

- $P_d = \operatorname{span}\{\text{spherical harmonics on } S^3 \text{ with degree} \leq d\}$ and $\mathbb{RP}^k = (P_d - \{0\})/\{f \sim cf\}$, where k grows like d^3,
- $\Phi_k : \mathbb{RP}^k \to \mathcal{Z}_2(S_3)$, $\quad \Phi_k([f]) = \partial\{f < 0\}$. From Crofton formula we know that

  $$\sup_{[f]\in\mathbb{RP}^k} vol(\Phi_k([f])) \leq 4\pi d$$

  and we estimate $\alpha(2) \leq (48/\pi)^{1/3}$. Is this sharp?

Weyl Law – Approach when $M^{n+1} \subset \mathbb{R}^{n+1}$

Assume $\mathrm{vol}(M) = 1$. With C the unit cube, find $\{C_i\}_{i=1}^N$ disjoint cubes in M so that $\mathrm{vol}(M \setminus \cup_{i=1}^N C_i)$ is very small.

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Conversely, we can find disjoint regions $\{M_i\}_{i=1}^N$ in C so that every M_i is similar to M and $\mathrm{vol}(C \setminus \cup_{i=1}^N M_i)$ is very small and we show

$$\liminf_{k\to\infty} \frac{\omega_k(C)}{k^{\frac{1}{n+1}}} \geq \liminf_{k\to\infty} \frac{\omega_k(M)}{k^{\frac{1}{n+1}}}.$$

This shows that the liminf of $\frac{\omega_k(M)}{k^{\frac{1}{n+1}}}$ is universal.

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- Compactness properties of minimal hypersurfaces with bounded index: Sharp, Buzano–Sharp, Carlotto, Chodosh–Ketover–Maximo, Li-Zhou;

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- Ketover studied genus estimates for min-max in the surface case;
- Nurser computed the first 9 widths of S^3 and Aix did it for S^2;
- Guaraco did min-max for Allen-Cahn equation;
- Song showed that the least area minimal surface is always embedded;
- Compactness properties of minimal hypersurfaces with bounded index: Sharp, Buzano–Sharp, Carlotto, Chodosh–Ketover–Maximo, Li-Zhou;
- Beck–Hanin–Hughes studied min-max families given by nodal sets of eigenfunctions.