BRANCHES OF FORCED OSCILLATIONS IN DEGENERATE SYSTEMS OF SECOND ORDER ODEs

MARTA LEWICKA AND MARCO SPADINI

1. INTRODUCTION

This paper is devoted to studying the set of oscillations of a mass point, constrained to a smooth manifold, and forced by an autonomous vector field $G$ with a periodic perturbation $F$. We focus on a class of systems where $G$ is “degenerate”: its set of zeros being a noncompact submanifold of the constraint. There seem to be no results in the literature for this general case while the “extreme” cases (i.e., when $G \equiv 0$ or $G^{-1}(0)$ is compact) are well understood. For instance, in [2] there are studied branches of $T$-periodic solutions to second order differential equations of the form

$$
\xi = \lambda F(t, x, \dot{x}) \quad \lambda \geq 0,
$$

where $F$ is tangent to a given differentiable manifold $X$ and is $T$-periodic in $t$, under the assumption that the averaged vector field

$$
p \mapsto \int_0^T F(t, p, 0) \, dt := \frac{1}{T} \int_0^T F(t, p, 0) \, dt
$$

is admissible for the degree (that is, the set of its zeros is compact). In [4], $T$-periodic solutions to equations of the form

$$
\xi = G(\xi, \dot{\xi}) + \lambda F(t, x, \dot{x}) \quad \lambda \geq 0,
$$

are studied under the assumption that $G$ is admissible for the degree. In this case, the average of $F$ plays no role. As we said, little is known about the case when $G^{-1}(0)$ is noncompact.

In this paper we wish to address, at least partially, this problem. We examine the case when $X$ is the Cartesian product of two manifolds and $G$ is constantly zero on one of them. In particular, this approach allows us to recover known results about (E1) and (E2).

Let $M$ and $N$ be two smooth manifolds in $\mathbb{R}^k$. Consider the following system of two coupled second order ODEs:

$$
\begin{cases}
\ddot{x}_M = \lambda f(t, x, \dot{x}, \dot{y}), \\
\ddot{y}_N = g(x, \dot{x}, \dot{y}) + \lambda h(t, x, \dot{x}, \dot{y}),
\end{cases}
$$

under the following assumptions on vector fields $f, h, g$:

(i) $f : \mathbb{R} \times TM \times TN \to \mathbb{R}^k$ is continuous, $T$-periodic in $t$ and tangent to $M$, that is: $f(t, p, v, q, w) \in T_p M$ for all $t \in \mathbb{R}$, $p \in M$, $v \in T_p M$, $q \in N$, $w \in T_q N$,

(ii) $h : \mathbb{R} \times TM \times TN \to \mathbb{R}^k$ is continuous, $T$-periodic in $t$ and tangent to $N$,

(iii) $g : TM \times TN \to \mathbb{R}^k$ is continuous and tangent to $N$.

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In (1.1) \( \lambda \) is a nonnegative parameter and the subscripts \( \pi_M, \pi_N \) denote the projections on the tangent spaces to \( M \) and \( N \), respectively. That is, for example, \( \ddot{x}_{\pi_M}(t) \) denotes the orthogonal projection of the acceleration \( \ddot{x}(t) \in \mathbb{R}^k \) onto \( T_{\dot{x}(t)}M \).

In studying (1.1), the following vector field, tangent to \( M \times N \), is of major importance:

\[
\nu : M \times N \rightarrow \mathbb{R}^k \times \mathbb{R}^k, \quad \nu(p, q) = \left( \int_0^T f(t, p, 0, q, 0) \, dt, \, g(p, 0, q, 0) \right).
\]

Given a manifold \( X \subset \mathbb{R}^s \), by \( C^1_T(X) \) we denote the space of \( T \)-periodic \( C^1 \) functions from \( \mathbb{R} \) to \( X \), with the topology inherited from the Banach space \( C^1([0,T], \mathbb{R}^s) \).

We will also identify points on \( X \) with constant functions from \( \mathbb{R} \) to \( X \). Thus, if \( S \) is a subset of \( C^1_T(X) \), by \( S \cap X \) we mean the set of those points of \( X \), that regarded as constant maps belong to \( S \).

Our main result is the following:

**Theorem 1.1.** Assume (A1) and let \( \Omega \) be an open subset of \([0,\infty) \times C^1_T(M \times N)\) such that

\[
\deg(\nu, \Omega \cap (M \times N))
\]

is well defined and nonzero. Then there exists a connected set \( \Gamma \subset \Omega \) enjoying the properties:

(i) every triple \((\lambda, x, y) \in \Gamma\) is a solution to (1.1),

(ii) if \((\lambda, x, y) \in \Gamma\) then the parameter \( \lambda > 0 \) or \((x, y) \notin M \times N \) (that is, \((x, y)\) is not constant),

(iii) \( \Gamma \cap \left( \{0\} \times \nu^{-1}(0) \right) \cap \Omega \neq \emptyset \), where \( \overline{\Gamma} \) stands for the closure of \( \Gamma \) in \([0,\infty) \times C^1_T(M \times N)\),

(iv) \( \Gamma \cap \Omega \) is not contained in any compact subset of \( \Omega \).

In particular, if \( M \times N \) is closed in \( \mathbb{R}^{2k} \) and \( \Omega = [0,\infty) \times C^1_T(M \times N) \) then \( \Gamma \) is unbounded.

When either \( N \) or \( M \) is a singleton, our result reduces to Theorem 2.2 of [2] and Theorem 4.2 of [4], respectively.

The structure of this short paper is as follows. In Section 2 we compute the fixed point index of the \( T \)-translation operator associated to the reduced first order system, which is a version of (1.1) on the tangent bundle \( T(M \times N) \). Section 3 contains the proof of Theorem 1.1 and an example illustrating the theory.

The results presented here are in the spirit of [5] where the first order case is discussed. The techniques we use are close to those of, e.g. [2, 4], therefore we describe only the main new ingredients and refer to those papers for a more detailed exposition.

### 2. Reduction to a first order system.

Towards a proof of Theorem 1.1, we conveniently express the system (1.1) in the first order form. Given a manifold \( M \), one can prove (see, e.g. [1]) that there exists a unique smooth map \( r_M : TM \rightarrow \mathbb{R}^k \) such that for any \( C^2 \) curve \( x : \mathbb{R} \rightarrow M \), \( r_M(x(t), \dot{x}(t)) \) is the orthogonal projection of \( \ddot{x}(t) \) onto \( T_{\dot{x}(t)}M \). The map \( r_M \) satisfies, in particular, \( r_M(p, v) \in (T_pM)^\perp \) and:

\[
|r_M(x(t), \dot{x}(t))| = \kappa_M(x(t), \dot{x}(t)) \cdot |\dot{x}(t)|^2,
\]
where \( \kappa_M(p,v) \) is the normal curvature of \( M \) at \( p \) in the direction of \( v \). Hence (1.1) can be equivalently written as a first order system on \( TM \times TN \):

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= r_M(x_1,x_2) + \lambda f(t,x_1,x_2,y_1,y_2), \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= r_N(y_1,y_2) + g(x_1,x_2,y_1,y_2) + \lambda h(t,x_1,x_2,y_1,y_2).
\end{align*}
\]

(2.2)

For \( t \geq 0 \), denote by \( P^\lambda_t(p,v,w) \) the value at time \( t \) (when defined) of the solution to (2.2) which takes as initial values:

\[
x_1(0) = p, \quad x_2(0) = v, \quad y_1(0) = q, \quad y_2(0) = w.
\]

Lemma 2.1. Let \( f,g,h \) be \( C^1 \) vector fields satisfying (A1). Assume that for some relatively compact open subset \( U \) of \( TM \times TN \) we have that:

(i) \( P^0_t \) is well defined on \( U \),

(ii) every fixed point of \( P^0_t \) on \( \partial U \) corresponds to a constant solution of (1.1), \((x_i,y_i)(t) = (p,q)\),

(iii) \( \nu \) has no zeros on the boundary (in \( M \times N \)) of the set \( U \cap (M \times N) \).

Then for \( \lambda > 0 \) sufficiently small:

\[\text{ind}(P^\lambda_t, U) = \deg (\nu, U \cap (M \times N)).\]

Proof. For a given \( \lambda \geq 0 \) and \( \mu \in [0,1] \), let \( H(\lambda,\mu,v,q,w,\mu) \in TM \times TN \) be the value at time \( T \) of the solution to:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= r_M(x_1,x_2) + \lambda \left( \mu f(t,x_1,x_2,y_1,y_2) + (1-\mu) \int_0^T f(t,x_1,x_2,y_1,y_2) \, dt \right), \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= r_N(y_1,y_2) + g(x_1,x_2,y_1,y_2) + \mu h(t,x_1,x_2,y_1,y_2),
\end{align*}
\]

(2.4)

satisfying (2.3).

1. We first claim that for every small \( \lambda \), the mapping \( H(\lambda,\cdot) : U \times [0,1] \rightarrow TM \times TN \) is an admissible homotopy for the fixed point index. We argue by contradiction and assume that there are sequences \( \lambda_n \rightarrow 0, \mu_n \rightarrow \mu_0 \in [0,1], (p_i,v_i,q_i,w_i) \rightarrow (p_0,v_0,q_0,w_0) \in \partial U \) such that the corresponding solutions \((x_i^1,x_i^2,y_i^1,y_i^2)\) of (2.4) satisfy \( x_i^1(T) = p_i, x_i^2(T) = v_i, y_i^1(T) = q_i, y_i^2(T) = w_i \). Clearly the sequence \((x_1^1,x_2^1,y_1^1,y_2^1)\) converges uniformly on \([0,T]\) to a \( T \)-periodic solution of (2.2) with \( \lambda = 0 \). In view of (ii), there must be \( v_0 = 0, w_0 = 0 \) and

\[
g(p_0,0,0,0) = 0.
\]

We will now show that also \( \nu(p_0,q_0) = 0 \) and hence obtain a contradiction with (iii). By (2.4) and in view of the periodicity of \((x_1^1,x_2^1)\) we have:

\[
\int_0^T |r_M(x_1^1(t),x_2^1(t))| \, dt \leq C_1 \int_0^T |x_2^1(t)|^2 \, dt \leq C_2 \int_0^T \left( \int_0^T |\dot{x}_2^1(s)| \, ds \right)^2 \, dt \\
\leq C_3 T^2 \int_0^T |\dot{x}_2^1(t)|^2 \, dt \leq C_4 \int_0^T |r_M(x_1^1(t),x_2^1(t))|^2 \, dt + C_5 \lambda_1,
\]

(2.6)

where \( C_1, C_2 \) and \( C_3 \) are positive constants which may depend on \( T, f \) and the geometry of \( M \) but are independent of \( i \). To see the second inequality in (2.6), notice that \( x_2^1 \) is the derivative of a periodic function \( x_1^1 \), and thus any component of \( x_2^1 \) must have a zero in \([0,T]\).
The last inequality in (2.6) follows from (2.4) and the following simple calculation:
\[
\int_0^T \left| \mu f(t, x_1, x_2, y_1, y_2) + (1 - \mu) \int_0^T f(s, x_1, x_2, y_1, y_2) \, ds \right|^2 \, dt
\]
\[
= \mu^2 \int_0^T |f|^2 + (1 - \mu^2)T \cdot \left| \int_0^T f(t, x_1, x_2, y_1, y_2) \, dt \right|^2 \leq \int_0^T |f(t, x_1, x_2, y_1, y_2)|^2 \, dt
\]
The last quantity above is clearly bounded, independently of \(i\), because all trajectories \((x_1^i, x_2^i, y_1^i, y_2^i)\) are contained in a compact region of \(TM \times TN\).

Using (2.1) again and since \(x_2^i\) converges to 0, we obtain for sufficiently large \(i\):
\[
C_3 \int_0^T |r_M(x_1^i(t), x_2^i(t))|^2 \, dt \leq \frac{1}{2} \int_0^T |r_M(x_1^i(t), x_2^i(t))| \, dt,
\]
Thus, by (2.6):
\[
\int_0^T |r_M(x_1^i(t), x_2^i(t))| \, dt \leq 2C_3\lambda_1^2.
\]
Integrating on \([0, T]\) the second equation in (2.4) we get:
\[
\left| \int_0^T f(t, x_1, x_2, y_1, y_2) \, dt \right| = \frac{1}{\lambda_1} \left| \int_0^T r_M(x_1^i(t), x_2^i(t)) \, dt \right| \leq 2C_3\lambda_1,
\]
which after passing to the limit implies: \(0 = \int_0^T f(t, p_0, 0, q_0, 0) \, dt\). Hence by (2.5) we obtain \(\nu(p_0, q_0) = 0\).

2. By the homotopy invariance of the fixed point index, we conclude that for every small \(\lambda > 0\) there holds:
\[
\text{ind} (P_{2\lambda}^T, U) = \text{ind} (H(\lambda, \cdot, \mu = 0), U).
\]
The last index above is by Theorem 2.1 [3] equal to \(\text{deg} (-\nu_{\lambda}, U)\), where
\[
\nu_{\lambda}(p, v, q, w) = \left( v, r_M(p, v) + \lambda \int_0^T f(t, p, v, q, w) \, dt, w, r_N(q, w) + g(p, v, q, w) \right).
\]
Further, Lemma 3.2 [4] implies that:
\[
\text{deg} (-\nu_{\lambda}, U) = \text{deg} (\tilde{\nu}_{\lambda}, U \cap (M \times N))
\]
where \(\tilde{\nu}_{\lambda}(p, q) = (\lambda \int_0^T f(t, p, 0, q, 0) \, dt, g(p, 0, q, 0))\). On the other hand, clearly:
\[
\text{deg} (\tilde{\nu}_{\lambda}, U \cap (M \times N)) = \text{deg} (\nu, U \cap (M \times N))
\]
which ends the proof of the Lemma.

3. A PROOF OF THEOREM 1.1 AND AN EXAMPLE

We will use the following abstract result from [2]:

**Lemma 3.1.** Let \(Y\) be a locally compact metric space and let \(K\) be a nonempty, compact subset of it. Assume that any compact subset of \(Y\) containing \(K\) has nonempty boundary. Then \(Y \setminus K\) contains a connected set whose closure intersects \(K\) and is not compact.

**Proof of Theorem 1.1.** We prove the result under the additional assumption that \(f, g, h\) are \(C^1\). The extension to nonsmooth vector fields follows in a straightforward manner, as in [5].

Let \(W\) be the subset of \([0, \infty) \times TM \times TN\) given by:
\[
W = \{(\lambda, x_1(0), x_2(0), y_1(0), y_2(0)); (\lambda, x_1, x_2, y_1, y_2) \in \Omega\},
\]
and set
\[ S = W \cap \{(\lambda, x_1(0), x_2(0), y_1(0), y_2(0)); (\lambda, x_1, x_2, y_1, y_2) \text{ solves (2.2)}\}, \]
\[ K = S \cap \{(0) \times \nu^{-1}(0)\). \]

We will prove that the set \( S \setminus K \) has a connected subset which meets \( K \) and whose closure is not compact. This will be done by checking the assumptions of Lemma 3.1 for the pair \((Y, K)\) with:
\[ Y = S \setminus \{(0, p, q, 0); g(p, 0, q, 0) = 0 \text{ and } \nu(p, q) \neq 0\}. \]

In the sequel, given any set \( A \subset [0, \infty) \times TM \times TN \) and \( \lambda \geq 0 \), we will denote \( A_{\lambda} = \{(p, v, q, w); (\lambda, p, v, q, w) \in A\}. \)

Firstly, since by assumption we have \( \deg (\nu, W_0) \neq 0 \), we conclude that \( K \) must be nonempty. Because of the regularity of \( g \), arguing as in the first part of the proof of Lemma 2.1 one can show that any sequence \((\lambda, p_i, v_i, q_i, w_i) \in S\) with \( \lambda_i \to 0^+ \) converges to a point in \( Y \), and conclude that \( Y \) is locally compact.

Assume now, by contradiction, that \( Y \) has a compact subset \( C \), containing \( K \) and with empty boundary in \( Y \). Choose an open set \( A \subset W \) so that \( A \cap Y = C \) and \( \partial A \cap S = \emptyset \). In particular \( \partial A_0 \cap K = \emptyset \). Now, by Lemma 2.1 we see that for a sufficiently small \( \lambda > 0 \):
\[
\text{ind} (P_0^\lambda, A_\lambda) = \deg (\nu, A_\lambda \cap (M \times N)) = \deg (\nu, W_0) \neq 0.
\]

On the other hand, the map \( \delta \mapsto \text{ind} (P_0^\delta, A_\delta) \) is constant in view of the generalized homotopy invariance of the fixed point index. Recalling the compactness of \( C \), its value must equal 0 for some \( \delta > 0 \), when \( P_0^\delta \) has no fixed points in \( A_\delta \). This, however, contradicts (3.1) and ends the proof of the theorem.

Observe that the connected set \( \Gamma \) in Theorem 1.1 might be contained in the slice \( \{(0) \times C^1_\lambda (M \times N)\) as in the system:
\[
\begin{align*}
\dot{x} &= \lambda f(t, x, y), \\
\dot{y} &= -y + \lambda \sin t,
\end{align*}
\]
where we put \( M = N = \mathbb{R}, T = 2\pi. \)

**Example.** Let \( n \in \mathbb{N} \) be an odd number and consider the two coupled ODEs:
\[
\begin{align*}
\dot{x} &= -x - \alpha \dot{x} + \mu (y - x)^n, \\
\dot{y} &= \mu (x - y)^n + f(t),
\end{align*}
\]

(3.2)
describing the mechanical system as in the figure below.

There are two equal masses \( P_1 \) and \( P_2 \) and a fixed point \( O \) confined to a linear rail and connected by two springs: a nonlinear spring \( S_2 \) (whose elastic force is proportional to the \( n \)-th power of the displacement) and a linear spring \( S_1 \). Moreover, \( P_1 \) is subject to friction and \( P_2 \) to a \( T \)-periodic force \( f \) with nonzero average. In (3.2) \( \alpha > 0 \) is the friction coefficient and \( \mu > 0 \) is a parameter used to control the stiffness of \( S_2 \).
We apply Theorem 1.1 to show that for small $\mu > 0$, (3.2) admits a $T$-periodic solution. With the change of variable $\lambda = \mu^n$, $\xi = \lambda x$, $\eta = \lambda y$ the system becomes:

$$
\begin{cases}
\ddot{\xi} = -\xi - \alpha \dot{\xi} + \lambda (\eta - \xi)^n, \\
\ddot{\eta} = \lambda \left( (\xi - \eta)^n + f(t) \right).
\end{cases}
$$

Take $\Omega = [0, \infty) \times C^1_0(\mathbb{R}^2)$, and notice that the degree of the vector field

$$
\nu(p, q) = \left( (p - q)^n + \int_0^T f(t) \, dt, -q \right)
$$

relative to $\Omega \cap \mathbb{R}^2$ is nonzero. By Theorem (1.1), (3.3) has an unbounded connected set of $T$-periodic solutions that branches from 

$$
\left( 0, - \left( \int_0^T f(t) \, dt \right)^{1/n}, 0 \right).
$$

This proves the claim.

References


University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455, USA
E-mail address: lewicka@math.umn.edu

Dipartimento di Matematica Applicata, Via S. Marta 3, I-50139 Firenze, Italy
E-mail address: marco.spadini@unifi.it