V.A con distribuzione uniforme su un intervallo $[a,b]$ $\quad \mathbb{P}_X = U([a,b])$

$$f(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)$$

V.A con distribuzione gaussiana di parametri $\mu \in \mathbb{R}$, $\sigma^2 > 0$ $\quad \mathbb{P}_X = N(\mu, \sigma^2)$
o distribuzione normale

Distribuzione gaussiana standard $N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \qquad x \in \mathbb{R}$$

$$\mathbb{P}_X = N(0,1) \qquad f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \qquad x \in \mathbb{R}$$

$$\mu \in \mathbb{R} \quad \sigma > 0 \qquad Y := \mu + \sigma X \quad , \quad \mathbb{P}_Y = N(\mu, \sigma^2) \Leftarrow$$

$$\mathbb{P}_Y = N(\mu, \sigma^2) \qquad X := \frac{Y-\mu}{\sigma} \implies \mathbb{P}_X = N(0,1)$$

$$\underline{\mathbb{E}[Y] = \mathbb{E}[\mu + \sigma X] = \mu + \sigma \mathbb{E}[X]} \qquad \text{con } \mathbb{P}_X = N(0,1)$$

$$\mathbb{E}[X] = \int_{\mathbb{R}} \underbrace{x \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)}_{\text{funzione dispari}} dx = 0 \qquad \int_{\mathbb{R}} \left| x \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \right| dx < +\infty$$

$$\implies \mathbb{P}_X = N(0,1) \implies \mathbb{E}[X] = 0 \qquad \mathbb{P}_Y = N(\mu, \sigma^2) \implies \mathbb{E}[Y] = \mu + \sigma \cdot 0 = \mu$$

$$\mathrm{Var}[X] = \mathbb{E}\left[(X - \underbrace{\cancel{\mathbb{E}[X]}}_{=0})^2\right] = \mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\, dx =$$

$$= \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \underset{\downarrow\, g}{(-x)} \underbrace{\left(-x \exp\left(\frac{-x^2}{2}\right)\right)}_{\frac{d}{dx}\exp\left(-\frac{x^2}{2}\right)} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\cancel{-x \exp\left(\frac{-x^2}{2}\right) \Big|_{x \to -\infty}^{x \to +\infty}}}_{=0} + \int_{\mathbb{R}} \exp\left(\frac{-x^2}{2}\right) dx \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(\frac{-x^2}{2}\right) dx \qquad y = \frac{x}{\sqrt{2}}$$

$$x = \sqrt{2}\, y$$

$$= \frac{1}{\sqrt{\cancel{2}\pi}} \int_{\mathbb{R}} \exp(-y^2) \cancel{\sqrt{2}}\, dy = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-y^2) dy = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1 \qquad dx = \sqrt{2}\, dy$$

$$\mathbb{P}_X = N(0,1) \implies \mathrm{Var}[X] = 1$$

$$\mathbb{P}_Y = N(\mu, \sigma^2) \qquad \mathrm{Var}[Y] = \mathrm{Var}[\mu + \sigma X] = \mathbb{E}\left[\left(\mu + \sigma X - \mathbb{E}[\mu + \sigma X]\right)^2\right] =$$

$$= \mathbb{E}\left[(\mu + \sigma X - \mu)^2\right] = \mathbb{E}[\sigma^2 X^2] = \sigma^2 \mathbb{E}[X^2] =$$

$\mathbb{P}_Y = N(\mu, \sigma^2)$ $\quad$ $Var[Y] = Var[\mu+\sigma X] = \mathbb{E}[(\mu+\sigma X - \mathbb{E}[\mu+\sigma X])^2] =$

$= \mathbb{E}[(\mu+\sigma X-\mu)^2] = \mathbb{E}[\sigma^2 X^2] = \sigma^2 \mathbb{E}[X^2] =$

$= \sigma^2 Var[X] = \sigma^2 \cdot 1 = \sigma^2$

V.A. CON DISTRIBUZIONE ESPONENZIALE DI PARAMETRO $\lambda>0$

Una v.a. X ha distribuzione esponenziale di parametro λ se F_X è A.C. con densità

$$f(x) = \begin{cases} 0 & x\le 0 \\ \lambda e^{-\lambda x} & x>0 \end{cases}$$

1) $f(x)\geq 0 \quad \forall x\in\mathbb{R}$ ✓

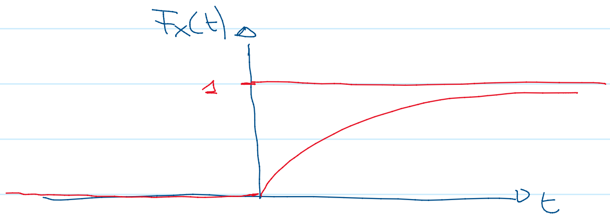
2) $\int_{\mathbb{R}} f(x)dx = 1$

$$\int_{\mathbb{R}} f(x)dx = \underbrace{\cancel{\int_{-\infty}^{0} 0\, dx}}_{=0} + \int_0^{+\infty}\lambda e^{-\lambda x}dx = -e^{-\lambda x}\Big|_{x=0}^{x\to+\infty} = 0-(-1) = 1$$

$$F_X(t) = \int_{-\infty}^{t} f(x)dx = \begin{cases} t<0 & \int_{-\infty}^{t} 0\,dx = 0 \\ t\geq 0 & \underbrace{\cancel{\int_{-\infty}^{0} 0\,dx}}_{=0} + \int_0^t \lambda e^{-\lambda x}dx \end{cases}$$

$t\geq 0 \quad \int_0^t \lambda e^{-\lambda x}dx = -e^{-\lambda x}\Big|_0^t = -e^{-\lambda t}-(-1) = 1-e^{-\lambda t}$

$$F_X(t) = \begin{cases} 0 & t<0 \\ 1-e^{-\lambda t} & t\geq 0 \end{cases} \qquad f(x) = \begin{cases} 0 & x\le 0 \\ \lambda e^{-\lambda x} & x>0 \end{cases}$$



$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)dx = \int_0^{+\infty} x\lambda e^{-\lambda x}dx = \int_0^{+\infty}(-x)\underbrace{(-\lambda e^{-\lambda x})}_{\frac{d}{dx}e^{-\lambda x}}dx =$$

$$= -xe^{-\lambda x}\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} e^{-\lambda x}dx = 0-0+\left(\frac{-1}{\lambda}e^{-\lambda x}\right)\Big|_{x=0}^{x\to+\infty} =$$

$$= -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)dx = \int_0^{+\infty} x^2\lambda e^{-\lambda x}dx = \int_0^{+\infty}(-x^2)\underbrace{(-\lambda e^{-\lambda x})}_{\frac{d}{dx}e^{-\lambda x}}dx$$

$$= \underbrace{\cancel{-x^2 e^{-\lambda x}\Big|_{x=0}^{x\to+\infty}}}_{=0} + \int_0^{+\infty} 2x e^{-\lambda x}dx = \frac{2}{\lambda}\underbrace{\int_0^{+\infty}\lambda x e^{-\lambda x}dx}_{\mathbb{E}[X]=\frac{1}{\lambda}} = \frac{2}{\lambda}\cdot\mathbb{E}[X] = \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$x=0$ $\quad {}_0$ $\quad = \frac{2}{\lambda^2}$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$Var[X] = \frac{1}{\lambda^2}$

Se X ha distribuzione esponenziale

Siano $s, t \geq 0 \Rightarrow \mathbb{P}(X \leq t+s \mid X > t) = \mathbb{P}(X \leq s)$

DIM
$$\mathbb{P}(X \leq t+s \mid X > t) = \frac{\mathbb{P}(X \leq t+s, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X \leq t+s) - \mathbb{P}(X \leq t)}{1 - \mathbb{P}(X \leq t)} =$$

$$\frac{F_X(t+s) - F_X(t)}{1 - F_X(t)} = \frac{(\cancel{1} - e^{-\lambda(t+s)}) - (\cancel{1} - e^{-\lambda t})}{\cancel{1} - (\cancel{1} - e^{-\lambda t})} =$$

$$= \frac{e^{-\lambda t} - e^{-\lambda t - \lambda s}}{e^{-\lambda t}} = \frac{\cancel{e^{-\lambda t}}(1 - e^{-\lambda s})}{\cancel{e^{-\lambda t}}} = 1 - e^{-\lambda s} = F_X(s) = \mathbb{P}(X \leq s)$$

ESERCIZIO Una v.a. X ha distribuzione gaussiana di parametri $\mu = 6$, e $\sigma^2 = 4$. Calcolare $\mathbb{P}(|X-8| \leq 1)$ in termini della legge gaussiana standard $\Phi(t)$

Sia X_0 v.a. t.c. $\mathbb{P}_{X_0} = N(0,1) \Rightarrow$ so che la v.a. $Y := \mu + \sigma X_0$ ha distribuzione $N(\mu, \sigma^2)$. In particolare se scelgo $\mu = 6$ e $\sigma = 2$, ho che $\mathbb{P}_Y = N(6,2) = \mathbb{P}_X \Rightarrow$

$$\mathbb{P}(|X-8| \leq 1) = \mathbb{P}(|Y-8| \leq 1) = \mathbb{P}(|6 + 2X_0 - 8| \leq 1) = \mathbb{P}(|2X_0 - 2| \leq 1)$$

$$= \mathbb{P}\left(|X_0 - 1| \leq \frac{1}{2}\right) = \mathbb{P}\left(-\frac{1}{2} \leq X_0 - 1 \leq \frac{1}{2}\right) = \mathbb{P}\left(\frac{1}{2} \leq X_0 \leq \frac{3}{2}\right) =$$

$$= \mathbb{P}\left(X_0 \leq \frac{3}{2}\right) - \mathbb{P}\left(X_0 < \frac{1}{2}\right) = \mathbb{P}\left(X_0 \leq \frac{3}{2}\right) - \mathbb{P}\left(X_0 \leq \frac{1}{2}\right) = \Phi\left(\frac{3}{2}\right) - \Phi\left(\frac{1}{2}\right)$$

$\mathbb{P}(X > t) = 1 - \mathbb{P}(X \leq t)$

$\mathbb{P}(X \geq t) = \mathbb{P}(X > t) + \mathbb{P}(X = t)$

$\mathbb{P}(X < t) = \mathbb{P}(X \leq t) - \mathbb{P}(X = t)$

$F_X(t) = \int_{-\infty}^{t} f(x)\,dx$

$\mathbb{P}(X = t) = 0 \quad \forall t \in \mathbb{R}$

$$\mathbb{P}\left(X_0 \leq \frac{1}{2}\right) = \cancel{\mathbb{P}\left(X_0 = \frac{1}{2}\right)}^{\,0} + \mathbb{P}\left(X_0 < \frac{1}{2}\right)$$

Esercizio La v.a. X ha distribuzione gaussiana di valore atteso μ. Sapendo che $\mathbb{P}(|X - \mu| \leq 1) \geq \frac{1}{2}$, calcolare una limitazione per la varianza di X

Sia σ^2 la varianza di X. Sia X_0 v.a. con distribuzione $N(0,1)$

Sia σ^2 la varianza di X - Sia X_0 v.a. con distribuzione $N(0,1)$

So che la v.a. $Y := \mu + \sigma X_0$ ha distribuzione $N(\mu, \sigma^2)$ cioè $\mathbb{P}_Y = \mathbb{P}_X$

$$\Rightarrow \quad \mathbb{P}(|X-\mu| \le 1) = \mathbb{P}(|Y-\mu| \le 1) = \mathbb{P}(|\cancel{\mu} + \sigma X_0 - \cancel{\mu}| \le 1)$$

$$\mathbb{P}(\sigma |X_0| \le 1) \geq \frac{1}{2} \qquad\qquad \Phi(t) + \Phi(-t) = 1 \quad \forall t \in \mathbb{R}$$

$$\mathbb{P}\left(|X_0| \le \frac{1}{\sigma}\right) \geq \frac{1}{2}$$

$$\frac{1}{2} \le \mathbb{P}\left(\frac{-1}{\sigma} \le X_0 \le \frac{1}{\sigma}\right) = \mathbb{P}\left(X_0 \le \frac{1}{\sigma}\right) - \mathbb{P}\left(X_0 < \frac{-1}{\sigma}\right) = \Phi\left(\frac{1}{\sigma}\right) - \Phi\left(\frac{-1}{\sigma}\right)$$

$$= \Phi\left(\frac{1}{\sigma}\right) - \left(1 - \Phi\left(\frac{1}{\sigma}\right)\right) = 2\Phi\left(\frac{1}{\sigma}\right) - 1$$

$$2\Phi\left(\frac{1}{\sigma}\right) - 1 \geq \frac{1}{2} \qquad \Phi\left(\frac{1}{\sigma}\right) \geq \frac{3}{4}$$

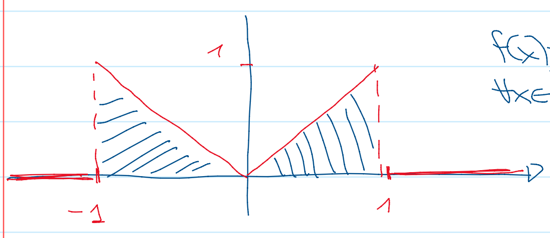
$$\Phi(t) = \int_{-\infty}^{t} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)}_{>0} dx \qquad\qquad \Phi'(t) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) > 0$$

$\Rightarrow \Phi$ strettamente monotona crescente

$$\frac{1}{\sigma} \geq \Phi^{-1}\left(\frac{3}{4}\right) \qquad\qquad \sigma \le \frac{1}{\Phi^{-1}\left(\frac{3}{4}\right)}$$

Esercizio La v.a. X ha legge A.C. con densità $f(x) = |x| \mathbb{1}_{[-1,1]}(x)$

Sia $Y = X^2$. Calcolare la densità di Y



$f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$$\int_{\mathbb{R}} f(x)\,dx = 2 \cdot \frac{1}{2} \cdot 1 \cdot 1 = 1$$

Se indico con $g(y)$ la densità di Y, so che

$$g(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right) & y > 0 \end{cases} = \qquad f(\sqrt{y}) = f(-\sqrt{y}) \quad \forall y \geq 0$$

$$= \begin{cases} 0 & y \le 0 \\ \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right) & y > 0 \end{cases} =$$

$$g(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{\sqrt{y}} f(\sqrt{y}) & y > 0 \end{cases}$$

$\underline{y>0}$

$$f(\sqrt{y}) = \begin{cases} |\sqrt{y}| & -1 \le \sqrt{y} \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

$$f(x) = |x| \mathbb{1}_{[-1,1]}(x) = \begin{cases} |x| & -1 \le x \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

$$= \begin{cases} \sqrt{y} & 0 \le \sqrt{y} \le 1 \\ 0 & \sqrt{y} > 1 \end{cases}$$

$$= \begin{cases} \sqrt{y} & 0 \le y \le 1 \\ 0 & y > 1 \end{cases}$$

$$g(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{\sqrt{y}} \cdot \sqrt{y} = 1 & 0 \le y < 1 \\ \frac{1}{\sqrt{y}} 0 = 0 & y > 1 \end{cases}$$

$g(y) = \mathbb{1}_{[0,1]}(y) \Rightarrow Y$ ha distribuzione uniforme sull'intervallo $[0,1]$

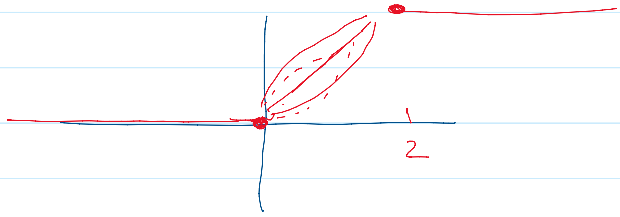
ovvero $\mathbb{P}_Y = U([0,1])$

Esercizio Siano $a, c > 0$ e si consideri la funzione

$$F(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{c}\right)^a & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

Determinare c in modo che F sia una legge A.C., calcolarne la densità associata.

$F(2) = 1 \qquad \left(\frac{2}{c}\right)^a = 1 \qquad \frac{2}{c} = 1 \qquad c = 2$



$$F(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{2}\right)^a & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

$F^{-1}((0,1))$

$$F'(x) = \begin{cases} 0 & x<0 \ , \ x>2 \\ a\left(\frac{x}{2}\right)^{a-1}\cdot\frac{1}{2} & x\in(0,2) \end{cases}$$

$f(x)$ densità associata è $\frac{a}{2}\left(\frac{x}{2}\right)^{a-1} \mathbb{1}_{(0,2)}(x)$

Sia X una v.a. con legge F. Vogliamo calcolare $E[X]$ e $Var[X]$

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)dx = \int_0^2 x\,\frac{a}{2}\,\frac{x^{a-1}}{2^{a-1}}dx = \int_0^2 \frac{a\,x^a}{2^a}dx = \frac{a}{2^a}\,\frac{x^{a+1}}{a+1}\Bigg|_{x=0}^{x=2}$$

$$= \frac{a}{2^a(a+1)}\left(2^{a+1}-0\right) = \frac{2a}{a+1}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)dx = \int_0^2 x^2\cdot\frac{a}{2}\,\frac{x^{a-1}}{2^{a-1}}dx = \frac{a}{2^a}\int_0^2 x^{a+1}dx = \frac{a}{2^a}\,\frac{x^{a+2}}{a+2}\Bigg|_{x=0}^{x=2}$$

$$= \frac{a\cdot 2^{\cancel{a}+2}}{\cancel{2^a}(a+2)} = \frac{4a}{a+2}$$

$$Var[X] = \frac{4a}{a+2} - \left(\frac{2a}{a+1}\right)^2 = \frac{4a}{a+2} - \frac{4a^2}{(a+1)^2} = 4a\,\frac{(a+1)^2 - a(a+2)}{(a+2)(a+1)^2}$$

$$= \frac{4a\left(\cancel{a^2}+\cancel{2a}+1-\cancel{a^2}-\cancel{2a}\right)}{(a+2)(a+1)^2} = \frac{4a}{(a+2)(a+1)^2}$$

— o —

Sia X v.a. t.c. F_X è strettamente monotona crescente in $F^{-1}((0,1))$
$\Rightarrow \exists!\ t\in F^{-1}((0,1))\ :\ F_X(t) = \frac{1}{2}$

Cerco $x\in(0,2)$ t.c. $\left(\frac{x}{2}\right)^a = \frac{1}{2}$ $\qquad \frac{x}{2} = \left(\frac{1}{2}\right)^{1/a} = \frac{1}{2^{1/a}}$

$$x = 2\,\frac{1}{2^{1/a}} = 2^{1-\frac{1}{a}} = 2^{\frac{a-1}{a}}$$

Esercizio Sia X v.a. con distribuzione gaussiana di valore atteso μ e varianza σ^2
Sia $a>0$. Considero $Y := a|X-\mu|$
Calcolare la densità e il valore atteso di Y

$$F_Y(t) = \mathbb{P}(Y\le t) = \mathbb{P}\left(a|X-\mu|\le t\right) =$$

$\boxed{t<0\ \ F_Y(t)=0}$

Sia X_0 v.a. con distribuzione $N(0,1)$ - $\mu + \sigma X_0$ ha distribuzione $N(\mu, \sigma^2)$ cioè la stessa distribuzione di X

$$= \mathbb{P}\left(2\left|\cancel{\mu}+\sigma X_0-\cancel{\mu}\right| \le t\right) = \mathbb{P}\left(\sigma |X_0| \le \frac{t}{2}\right) = \mathbb{P}\left(|X_0| \le \frac{t}{2\sigma}\right) = \quad t \geq 0$$

$$t \geq 0 \quad = \mathbb{P}\left(X_0 \le \frac{t}{2\sigma}\right) - \mathbb{P}\left(X_0 \le \frac{-t}{2\sigma}\right) = \Phi\left(\frac{t}{2\sigma}\right) - \Phi\left(\frac{-t}{2\sigma}\right)$$

$$F_Y(t) = \begin{cases} 0 & t < 0 \\ \Phi\left(\frac{t}{2\sigma}\right) - \Phi\left(\frac{-t}{2\sigma}\right) & t \geq 0 \end{cases} \longleftarrow \quad \Phi(x) + \Phi(-x) = 1 \Rightarrow \Phi\left(\frac{-t}{2\sigma}\right) = 1 - \Phi\left(\frac{t}{2\sigma}\right)$$

$$= \begin{cases} 0 & t < 0 \\ 2\Phi\left(\frac{t}{2\sigma}\right) - 1 & t \geq 0 \end{cases}$$

densità

$$F'_Y(t) = \begin{cases} 0 & t < 0 \\ 2\Phi'\left(\frac{t}{2\sigma}\right) \cdot \frac{1}{2\sigma} & t > 0 \end{cases}$$

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

$$F'_Y(t) = \begin{cases} 0 & t < 0 \\ \frac{2}{2\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{t}{2\sigma}\right)^2\right) & t > 0 \end{cases}$$

$$//$$

densità $g(t) = \begin{cases} 0 & t < 0 \\ \frac{\sqrt{2}}{2\sigma\sqrt{\pi}} \exp\left(\frac{-t^2}{2 \cdot 2^2 \sigma^2}\right) & t > 0 \end{cases}$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} t\, g(t)\, dt = \int_0^{+\infty} t \frac{\sqrt{2}}{2\sigma\sqrt{\pi}} \exp\left(\frac{-t^2}{2 \cdot 2^2 \sigma^2}\right) dt =$$

$$= \frac{\sqrt{2}}{2\sigma\sqrt{\pi}} \int_0^{+\infty} t \exp\left(-\left(\frac{t}{2\sigma\sqrt{2}}\right)^2\right) dt \qquad u = \frac{t}{2\sigma\sqrt{2}}$$

$$t = 2\sigma\sqrt{2}\, u$$

$$= \frac{\sqrt{2}}{\cancel{2\sigma}\sqrt{\pi}} \int_0^{+\infty} \cancel{2\sigma}\sqrt{2}\, u \exp\left(-u^2\right) 2\sigma\sqrt{2}\, du \qquad dt = 2\sigma\sqrt{2}\, du$$

$$= \frac{2\sqrt{2} \cdot \sigma}{(-2)\sqrt{\pi}} \int_0^{+\infty} \underbrace{(-2)u \; e^{-u^2}}_{\frac{d}{du} e^{-u^2}} du = \frac{-\sqrt{2}}{\sqrt{\pi}} \sigma \left(e^{-u^2} \right) \Bigg|_{u=0}^{u \to +\infty} =$$

$$= \frac{-\sqrt{2}}{\sqrt{\pi}} \sigma \left(0 - 1 \right) = \frac{\sqrt{2}}{\sqrt{\pi}} \sigma$$

Esercizio Sia X v.a. con distribuzione esponenziale di parametro $\lambda > 0$

Sia $Y := e^X$

- Densità della distribuzione di Y
- $\mathbb{E}[Y]$ e $Var[Y]$, se esistono

$Y := e^X$ $\quad t \in \mathbb{R}$ $\quad F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(e^X \leq t) =$

$F_Y(t) = \mathbb{P}(\phi) = 0 \qquad \forall t \leq 0$

$t > 0 \quad F_Y(t) = \mathbb{P}(e^X \leq t) = \mathbb{P}(X \leq \ln(t)) = F_X(\ln(t))$

$$F_X(s) = \begin{cases} 0 & s \leq 0 \\ 1 - e^{-\lambda s} & s > 0 \end{cases}$$

$t > 0$ $\quad F_Y(t) = F_X(\ln(t)) = \begin{cases} 0 & \ln(t) \leq 0 \quad \text{cioè } 0 < t \leq 1 \\ 1 - e^{-\lambda \ln(t)} & \ln(t) > 0 \quad \text{cioè } t > 1 \end{cases}$

$$e^{-\lambda \cdot \ln(t)} = \left(e^{\ln(t)} \right)^{-\lambda} = t^{-\lambda}$$

$$F_Y(t) = \begin{cases} 0 & t \leq 1 \\ 1 - t^{-\lambda} & t > 1 \end{cases}$$ è continua e derivabile a tratti

$\Rightarrow$ è A.C. con densità $\quad g(t) = F_Y'(t) = \begin{cases} 0 & t < 1 \\ \lambda t^{-\lambda - 1} & t > 1 \end{cases}$

$$\mathbb{E}[Y] = \int_{\mathbb{R}} t\, g(t)\, dt = \int_1^{+\infty} t \lambda t^{-\lambda-1} dt = \int_1^{+\infty} \lambda t^{-\lambda} dt =$$

$$= \begin{cases} \ln(t) \Big|_{t=1}^{t \to +\infty} & \lambda = 1 \\ \lambda \; t^{-\lambda+1} \Big|^{\to +\infty} & \end{cases} = \begin{cases} +\infty & \lambda = 1 \\ +\infty & -\lambda + 1 > 0 \quad \lambda < 1 \end{cases}$$

$$= \begin{cases} \ldots \Big|_{t=1} \\ \dfrac{\lambda}{-\lambda+1} t^{-\lambda+1} \Big|_{t=1}^{\to+\infty} & \lambda \neq 1 \end{cases} = \begin{cases} +\infty & -\lambda+1>0 \quad \lambda<1 \\ \dfrac{\lambda}{-\lambda+1}(0-1) & -\lambda+1<0 \end{cases}$$

$-\lambda+1>0 \qquad \lim_{t\to+\infty} t^{-\lambda+1} = +\infty$

$-\lambda+1<0 \qquad \lim_{t\to+\infty} t^{-\lambda+1} = 0$

$$= \begin{cases} +\infty & \lambda \leq 1 \\ \dfrac{\lambda}{\lambda-1} & \underline{\lambda>1} \end{cases}$$

Se $\lambda>1$ ha senso calcolare la varianza

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} t^2 g(t)\,dt = \int_1^{+\infty} t^2 \lambda t^{-\lambda-1}\,dt = \lambda \int_1^{+\infty} t^{-\lambda+1}\,dt =$$

$$= \begin{cases} 2\displaystyle\int_1^{+\infty} t^{-1}\,dt = 2\ln(t)\Big|_{t=1}^{t\to+\infty} = +\infty & -\lambda+1=-1 \text{ cioè } \lambda=2 \\ \dfrac{\lambda}{-\lambda+1+1} t^{-\lambda+1+1}\Big|_{t=1}^{t\to+\infty} = \dfrac{\lambda}{-\lambda+2} t^{-\lambda+2}\Big|_{t=1}^{t\to+\infty} & -\lambda+1\neq-1 \text{ cioè } \lambda\neq2 \quad \underline{\lambda>1} \end{cases}$$

$-\lambda+2>0$ ($\lambda<2$) $\qquad \lim_{t\to+\infty} t^{-\lambda+2} = +\infty \qquad \Rightarrow \mathbb{E}[Y^2] = +\infty$

$-\lambda+2<0$ ($\lambda>2$) $\qquad \lim_{t\to+\infty} t^{-\lambda+2} = 0 \qquad \Rightarrow \mathbb{E}[Y^2] = \dfrac{\lambda}{-\lambda+2}(0-1) = \dfrac{\lambda}{\lambda-2}$

$$\mathbb{E}[Y^2] = \begin{cases} +\infty & \lambda \leq 2 \\ \dfrac{\lambda}{\lambda-2} & \lambda>2 \end{cases}$$

$$\mathrm{Var}[Y] = \begin{cases} \text{NON ESISTE} & 0<\lambda\leq1 \\ +\infty & 1<\lambda\leq2 \\ \dfrac{\lambda}{\lambda-2} - \left(\dfrac{\lambda}{\lambda-1}\right)^2 & \lambda>2 \end{cases} \to = \frac{\lambda}{(\lambda-2)(\lambda-1)^2}$$

$$\frac{\lambda}{\lambda-2} - \frac{\lambda^2}{(\lambda-1)^2} = \lambda\left(\frac{1}{\lambda-2} - \frac{\lambda}{(\lambda-1)^2}\right) = \lambda\,\frac{(\cancel{\lambda^2}\cancel{-2\lambda}+1)-(\cancel{\lambda^2}\cancel{-2\lambda})}{(\lambda-2)(\lambda-1)^2}$$

$$= \frac{\lambda}{(\lambda-2)(\lambda-1)^2}$$

Esercizio Sia $f(x) = 6(x-x^2)\mathbb{1}_{(0,1)}(x)$

Provare che è una densità

Sia X una v.a. con legge AC e densità f

Calcolare, se esistono densità, valore atteso e varianza della v.a. $\frac{1}{X}$