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DISTRIBUZIONE GEOMETRICA MODIFICATA DI PARAMETRO $p \in (0,1)$

Una v.a. X che questa distribuzione, e si scrive $\mathbb{P}_X = G'(p)$ se

- X è distribuita sugli interi non negativi
- $\mathbb{P}(X=k) = p(1-p)^k \quad \forall k = 0,1,2,\dots$

PROPRIETÀ DELLA MANCANZA DI MEMORIA

Sia X una v.a. distribuita sugli interi non negativi
Dico che X gode della proprietà di mancanza di memoria se
$\forall i,j \geq 0 \quad \mathbb{P}(X \leq i+j \mid X \geq j) = \mathbb{P}(X \leq i)$

LEMMA Se X è una v.a. con $\mathbb{P}_X = G'(p)$, allora X manca di memoria

DIM
$$\mathbb{P}(X \leq i+j \mid X \geq j) = \frac{\mathbb{P}(X \leq i+j, X \geq j)}{\mathbb{P}(X \geq j)} =$$

$$= \frac{\mathbb{P}(X \leq i+j) - \mathbb{P}(X < j)}{1 - \mathbb{P}(X < j)} = \frac{\mathbb{P}(X \leq i+j) - \mathbb{P}(X \leq j-1)}{1 - \mathbb{P}(X \leq j-1)}$$

$k \in \mathbb{N}$
$$\mathbb{P}(X \leq k) = \sum_{\ell=0}^{k} \mathbb{P}(X=\ell) = \sum_{\ell=0}^{k} p(1-p)^\ell = p \sum_{\ell=0}^{k} x^\ell \Big|_{x=1-p} =$$

$$= p \cdot \frac{1-x^{k+1}}{1-x}\Big|_{x=1-p} = \cancel{p}\, \frac{1-(1-p)^{k+1}}{\cancel{1-(1-p)}}$$

$$\mathbb{P}(X \leq i+j \mid X \geq j) = \frac{(\cancel{1} - (1-p)^{i+j+1}) - (\cancel{1} - (1-p)^{j-1+1})}{\cancel{1} - (\cancel{1} - (1-p)^{j-1+1})} =$$

$$= \frac{(1-p)^j - (1-p)^{i+j+1}}{(1-p)^j} = 1 - (1-p)^{i+1} = \mathbb{P}(X \leq i) \quad \boxtimes$$

LEMMA Sia X v.a. distribuita sugli interi non negativi t.c. $\mathbb{P}(X=0) \in (0,1)$
Se X gode della proprietà di mancanza di memoria, allora X segue la distribuzione geometrica modificata di parametro $p := \mathbb{P}(X=0)$

DIM So che
$$\mathbb{P}(X \leq i+j \mid X \geq j) = \mathbb{P}(X \leq i) \qquad \forall i,j = 0,1,2,\dots$$

Scelgo $i=0$
$$\mathbb{P}(X \leq j \mid X \geq j) = \mathbb{P}(X \leq 0) \qquad \{X \leq 0\} \quad \{X = 0\}$$

$$\frac{\mathbb{P}(X\le j,\ X\ge j)}{\mathbb{P}(X\ge j)} = \mathbb{P}(X=0) \qquad \{X\le j, X\ge j\} = \{X=j\}$$

$$p_0 := \mathbb{P}(X=0) \qquad \mathbb{P}(X=j) = p_0\,\mathbb{P}(X\ge j)$$

$$\mathbb{P}(X=j+1) = p_0\,\mathbb{P}(X\ge j+1)$$

$$\mathbb{P}(X=j)-\mathbb{P}(X=j+1) = p_0\Big(\underbrace{\mathbb{P}(X\ge j)-\mathbb{P}(X\ge j+1)}_{=\mathbb{P}(X=j)}\Big)$$

$$\{X=j\} = \{X\ge j\}\setminus\{X\ge j+1\}$$

$$\mathbb{P}(X=j)-\mathbb{P}(X=j+1) = p_0\,\mathbb{P}(X=j)$$

$$\mathbb{P}(X=j+1) = (1-p_0)\,\mathbb{P}(X=j) \Leftarrow \qquad p_0 := \mathbb{P}(X=0)\in(0,1)$$

$j=0 \quad \mathbb{P}(X=1) = (1-p_0)\,p_0$

$j=1 \quad \mathbb{P}(X=2) = (1-p_0)\,\mathbb{P}(X=1) = (1-p_0)^2\,p_0$

$j=2 \quad \mathbb{P}(X=3) = (1-p_0)\,\mathbb{P}(X=2) = (1-p_0)^3\,p_0$

$\vdots$

Per induzione si verifica che $\mathbb{P}(X=k) = p_0(1-p_0)^k$

ovvero che $\mathbb{P}_X = G'(p_0)$

— o —

Si dice che una v.a. X ha distribuzione A.C. con densità $f(x)$ se

- $f:\mathbb{R}\to[0,+\infty]$ Lebesgue-misurabile t.c. $\int_{\mathbb{R}} f(x)\,dx = 1$
- $\forall A\in\mathcal{B}(\mathbb{R})$

$$\mathbb{P}(X\in A) = \int_A f(x)\,dx$$

— o —

Sia X v.a. con distribuzione A.C. e densità $f(x)$

Sia $Y = X^2$

$Y = \varphi\circ X \qquad \varphi: s\in\mathbb{R}\mapsto s^2\in\mathbb{R}$

Se ψ è una funzione Borel misurabile nonnegativa, allora

$$\int_{\mathbb{R}} \psi(t)\,\mathbb{P}_{\varphi\circ X}(dt) = \int_{\mathbb{R}} (\psi\circ\varphi)(s)\,\mathbb{P}_X(ds) \qquad \forall\varphi \text{ di Borel}$$

$A \in \mathcal{B}(\mathbb{R}) \qquad \psi(t) = \mathbb{1}_A(t)$

$$\left\{\begin{array}{l} \int_{\mathbb{R}} \mathbb{1}_A(t)\, \mathbb{P}_{\varphi\circ X}(dt) = \int_{\mathbb{R}} \mathbb{1}_A(\varphi(s))\, \mathbb{P}_X(ds) \\ = \mathbb{P}(\varphi\circ X \in A) = \mathbb{P}_{\varphi\circ X}(A) \end{array}\right.$$

Con $\varphi(s) = s^2$ $\qquad \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi\circ X}(dt) = \int_{\mathbb{R}} \psi(\varphi(s))\, \mathbb{P}_X(ds)$ $\quad \forall \psi$ di Borel nonnegative

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi\circ X}(dt) = \int_{\mathbb{R}} \psi(s^2)\, \mathbb{P}_X(ds) = \int_{\mathbb{R}} \psi(s^2) f(s)\, ds =$$

$t = s^2$

$$= \int_0^{+\infty} \psi(s^2) f(s)\, ds + \int_{-\infty}^0 \psi(s^2) f(s)\, ds = 1 + 2$$

1 $\int_0^{+\infty} \psi(s^2) f(s)\, ds$ $\qquad s^2 = t \quad s = \sqrt{t} = t^{1/2} \quad ds = \frac{1}{2\sqrt{t}}\, dt$

$s = 0 \quad t = 0$

$s \to +\infty \quad t \to +\infty$

$$= \int_0^{+\infty} \psi(t)\, f(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}}\, dt$$

2 $\int_{-\infty}^0 \psi(s^2) f(s)\, ds$ $\qquad s^2 = t \quad s = -\sqrt{t} \quad ds = \frac{-1}{2\sqrt{t}}\, dt$

$s = 0 \quad t = 0$

$s \to -\infty \quad t \to +\infty$

$$= \int_{+\infty}^0 \psi(t) f(-\sqrt{t}) \cdot \frac{-1}{2\sqrt{t}}\, dt$$

$$= \int_0^{+\infty} \psi(t)\, f(-\sqrt{t}) \cdot \frac{1}{2\sqrt{t}}\, dt$$

$$\Rightarrow \int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{X^2}(dt) = \int_0^{+\infty} \psi(t) f(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}}\, dt + \int_0^{+\infty} \psi(t) f(-\sqrt{t}) \cdot \frac{1}{2\sqrt{t}}\, dt =$$

$$= \int_0^{+\infty} \psi(t)\, \frac{1}{2\sqrt{t}} \left(f(\sqrt{t}) + f(-\sqrt{t}) \right) dt = \int_{\mathbb{R}} \psi(t)\, g(t)\, dt$$

$$\boxed{g(t) := \begin{cases} \frac{1}{2\sqrt{t}} \left(f(\sqrt{t}) + f(-\sqrt{t}) \right) & t > 0 \\ 0 & t \le 0 \end{cases}}$$

$A \in \mathcal{B}(\mathbb{R}) \qquad \psi(t) = \mathbb{1}_A(t)$

$A \in \mathcal{B}(\mathbb{R}) \qquad \varphi(t) = \mathbb{1}_A(t)$

$$\int_{\mathbb{R}} \mathbb{1}_A(t)\, \mathbb{P}_{X^2}(dt) = \int_{\mathbb{R}} \mathbb{1}_A(t)\, g(t)\, dt = \int_A g(t)\, dt$$

$$= \mathbb{P}_{X^2}(A) \qquad \Rightarrow \mathbb{P}_{X^2} \text{ è A.C. con densità } g$$

Sia X v.a. con distribuzione AC. e densità $f(x)$

$Y := \alpha X + \beta \qquad \beta \in \mathbb{R},\ \alpha \neq 0$

$Y = \varphi \circ X \qquad \varphi(s) = \alpha s + \beta$

Sia ψ funzione di Borel non negative

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\varphi \circ X}(dt) = \int_{\mathbb{R}} \psi(\varphi(s))\, \underline{\mathbb{P}_X(ds)}$$

$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\alpha X + \beta}(dt) = \int_{\mathbb{R}} \psi(\alpha s + \beta) f(s)\, ds \qquad t = \alpha s + \beta \quad s = \frac{t-\beta}{\alpha}$$

$ds = \frac{1}{\alpha} dt \qquad s \to -\infty \quad \begin{cases} t \to -\infty & \alpha > 0 \\ t \to +\infty & \alpha < 0 \end{cases}$

$\qquad s \to +\infty \quad \begin{cases} t \to +\infty & \alpha > 0 \\ t \to -\infty & \alpha < 0 \end{cases}$

$$= \int_{\mathbb{R}} \underline{\operatorname{sgn}(\alpha)}\, \psi(t) f\left(\frac{t-\beta}{\alpha}\right) \underline{\frac{1}{\alpha}}\, dt = \int_{\mathbb{R}} \psi(t) \frac{1}{|\alpha|} f\left(\frac{t-\beta}{\alpha}\right) dt$$

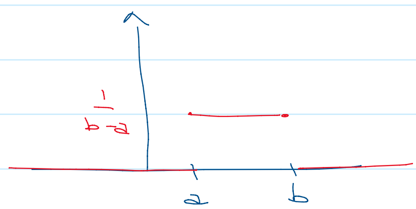
$$\int_{\mathbb{R}} \psi(t)\, \mathbb{P}_{\alpha X + \beta}(dt) = \int_{\mathbb{R}} \psi(t) \left(\frac{1}{|\alpha|} f\left(\frac{t-\beta}{\alpha}\right)\right) dt \qquad \forall \psi \text{ di Borel non negative}$$

$$\Rightarrow \mathbb{P}_{\alpha X + \beta} \text{ è AC. con densità } g(t) = \frac{1}{|\alpha|} f\left(\frac{t-\beta}{\alpha}\right)$$

DISTRIBUZIONE UNIFORME SU UN INTERVALLO

Dico che X è distribuita uniformemente sull'intervallo $[a,b]$ e scrivo $\mathbb{P}_X = U([a,b])$ se $\mathbb{P}_X$ è A.C. con densità

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$



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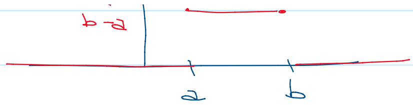
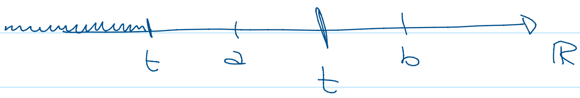
$$f(x) = \begin{cases} \dots \\ 0 & x \notin [a,b] \end{cases}$$



$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx$$

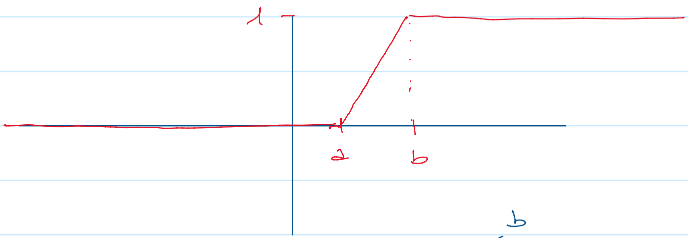


$t < a \qquad F_X(t) = \int_{-\infty}^{t} 0\,dx = 0$

$t \in [a,b) \qquad F_X(t) = \int_{-\infty}^{a} 0\,dx + \int_a^t \frac{1}{b-a}\,dx = \frac{t-a}{b-a}$

$t \geq b \qquad F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_a^b \frac{1}{b-a}\,dx + \int_b^t 0\,dx$

$$= 0 + \frac{1}{b-a}(b-a) + 0 = 1$$



$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_{x=a}^{x=b} = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \left.\frac{x^3}{3}\right|_{x=a}^{x=b} = \frac{b^3-a^3}{3(b-a)} =$$

$$= \frac{a^2+ab+b^2}{3}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

DISTRIBUZIONE ESPONENZIALE DI PARAMETRO $\lambda > 0$

È la distribuzione AC. associata alla densità

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$f(x) \geq 0 \quad \forall x \in \mathbb{R} \qquad f(x) > 0$ SSE $x > 0 \quad \longrightarrow \quad X > 0 \quad \mathbb{P}$-q.c.

$$\int_{\mathbb{R}} f(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_{x=0}^{x\to+\infty} = 0 - (-1) = 1$$

Supponiamo che una v.a. X abbia questa distribuzione (si scrive $\mathbb{P}_X = Exp(\lambda)$)

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^{+\infty} x \lambda e^{-\lambda x}\,dx = \int_0^{+\infty} \underline{\underline{(-x)}}\, \underbrace{\boxed{\left(-\lambda e^{-\lambda x}\right)}}_{\frac{d}{dx} e^{-\lambda x}}\,dx$$

$$= -x e^{-\lambda x}\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} +1 \cdot e^{-\lambda x}\,dx$$

$$\int f\, g'$$

$$g'(x) = -\lambda e^{-\lambda x}$$

$$g(x) = + e^{-\lambda x}$$

$$= \int_0^{+\infty} e^{-\lambda x}\,dx = \frac{-1}{\lambda}\int_0^{+\infty} -\lambda e^{-\lambda x}\,dx$$

$$= \frac{-1}{\lambda}\left(e^{-\lambda x}\right)\Big|_{x=0}^{x\to+\infty} = -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\,dx = \int_0^{+\infty} (-x^2)\underbrace{\left(-\lambda e^{-\lambda x}\right)}_{\frac{d}{dx} e^{-\lambda x}}\,dx$$

$$= -x^2 e^{-\lambda x}\Big|_{x=0}^{x\to+\infty} + \int_0^{+\infty} +2x e^{-\lambda x}\,dx = \int_0^{+\infty} 2x e^{-\lambda x}\,dx =$$

$$= \frac{2}{\lambda}\underbrace{\int_0^{+\infty} x \lambda e^{-\lambda x}\,dx}_{= \int_{\mathbb{R}} x f(x)\,dx = E[X] = \frac{1}{\lambda}} = \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

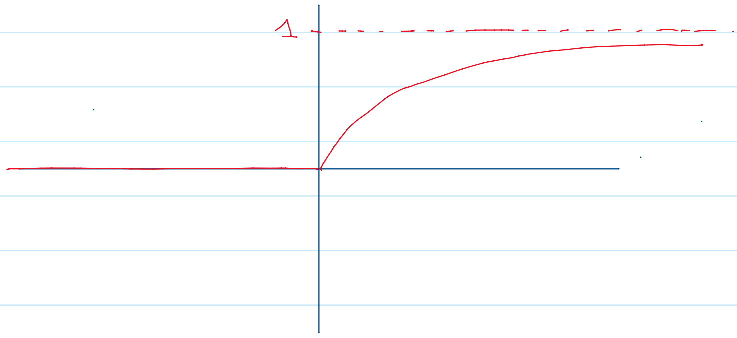
$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \begin{cases} 0 & t<0 \\ \int_0^t \lambda e^{-\lambda x}\,dx & t \geq 0 \end{cases}$$

$$t>0 \quad \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^t \lambda e^{-\lambda x}\,dx$$

$$t \geq 0 \qquad \int_0^t \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_{x=0}^{x=t} =$$

$$= -e^{-\lambda t} - (-1) = 1 - e^{-\lambda t}$$

$$f(x) = \begin{cases} 0 & x<0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases} \qquad F_X(t) = \begin{cases} 0 & t<0 \\ 1-e^{-\lambda t} & t \geq 0 \end{cases} \leftarrow$$



PROPRIETÀ DI MANCANZA DI MEMORIA

Dico che una v.a. gode della proprietà di mancanza di memoria se $\forall t, s \in [0, +\infty)$ $\quad \mathbb{P}(X \le t+s \mid X \ge t) = \mathbb{P}(X \le s)$

LEMMA Se X è una v.a. con $\mathbb{P}_X = Exp(\lambda)$, $\lambda > 0$, allora X manca di memoria

DIM

$t, s \ge 0$ $\quad \mathbb{P}(X \le t+s \mid X \ge t) = \dfrac{\mathbb{P}(X \le t+s,\ X \ge t)}{\mathbb{P}(X \ge t)} = \dfrac{\mathbb{P}(X \le t+s) - \mathbb{P}(X \le t)}{1 - \mathbb{P}(X \le t)} =$

$$= \frac{F_X(t+s) - F_X(t)}{1 - F_X(t)} = \frac{(\cancel{1} - e^{-\lambda(t+s)}) - (\cancel{1} - e^{-\lambda t})}{\cancel{1} - (\cancel{1} - e^{-\lambda t})} = \frac{e^{-\lambda t} - e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= 1 - e^{-\lambda s} = F_X(s) = \mathbb{P}(X \le s) \qquad \square$$

TEOREMA (no dim) Sia X una v.a. non negativa t.c. $\mathbb{P}(X=0) < 1$. Allora se $\mathbb{P}(X \le t+s \mid X \ge t) = \mathbb{P}(X \ge s)$ $\forall t, s \in [0, +\infty)$, si ha che X ha distribuzione esponenziale

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DISTRIBUZIONE GAUSSIANA (O NORMALE) DI PARAMETRI $\mu \in \mathbb{R}$ E $\sigma^2 > 0$

Si indica $N(\mu, \sigma^2)$ ed è la distribuzione AC associata alla densità

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} \right) \qquad \forall x \in \mathbb{R}$$

- $f(x) > 0 \quad \forall x \in \mathbb{R}$; $\displaystyle\int_{\mathbb{R}} f(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \exp\left(-\left(\frac{x-\mu}{\sigma\sqrt{2}} \right)^2 \right) dx$

$y = \dfrac{x-\mu}{\sigma\sqrt{2}}$ $\qquad x = \mu + y\underset{\vee\, 0}{\sigma}\sqrt{2}$ $\qquad dx = \sigma\sqrt{2}\, dy$

$x \to -\infty \quad y \to -\infty$

$x \to +\infty \quad y \to +\infty$

$\sigma\sqrt{2}$

$x \to -\infty \quad y \to -\infty$

$x \to +\infty \quad y \to +\infty$

$$= \frac{1}{\sqrt{\cancel{2}\pi\cancel{\sigma^2}}}\int_{\mathbb{R}} \exp(-y^2)\,\cancel{\sigma\sqrt{2}}\,dy = \frac{1}{\sqrt{\pi}}\underbrace{\int_{\mathbb{R}} \exp(-y^2)\,dy}_{=\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

Se $\mu = 0$, $\sigma^2 = 1$, la distribuzione gaussiana si dice DISTRIBUZIONE GAUSSIANA STANDARD ($N(0,1)$)

$$f_0(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \qquad \forall x \in \mathbb{R}$$

Sia X v.a. t.c. $P_X = N(0,1)$

$$E[|X|] = \int_{\mathbb{R}} |x| f_0(x)\,dx = \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx = -2\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}(-x)\exp\left(-\frac{x^2}{2}\right)dx$$

$$= \frac{-2}{\sqrt{2\pi}}\int_0^{+\infty}\underbrace{-x\exp\left(-\frac{x^2}{2}\right)}_{\frac{d}{dx}\exp\left(-\frac{x^2}{2}\right)}dx = -\frac{2}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\Bigg|_{x=0}^{x\to+\infty} = -\frac{2}{\sqrt{2\pi}}(0-1)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} < +\infty$$

$\Rightarrow E[X]$ esiste ed è finito

$$E[X] = \int_{\mathbb{R}} x f_0(x)\,dx \qquad f_0(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

è una funzione pari

$$= \int_0^{+\infty} x f_0(x)\,dx + \int_{-\infty}^{0}\underbrace{x f_0(x)\,dx}_{y=-x}$$

$$= \int_0^{+\infty} x f_0(x)\,dx + \int_{+\infty}^{0} +y f_0(-y)\,dy = \int_0^{+\infty} x f_0(x)\,dx - \int_0^{+\infty} y\,\underbrace{f_0(-y)}_{f_0(y)}\,dy$$

$$= L - L = 0 \quad \Rightarrow \boxed{E[X] = 0}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\cancel{\mathbb{E}[X]})^2 = \int_{\mathbb{R}} x^2 f_0(x)\,dx = \int_{\mathbb{R}} x^2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-x) \underbrace{\left(-x \exp\left(\frac{-x^2}{2}\right)\right)}_{\frac{d}{dx}\exp\left(\frac{-x^2}{2}\right)} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\cancel{\left. -x \exp\left(\frac{-x^2}{2}\right)\right|_{x\to-\infty}^{x\to+\infty}}^{=0} + \int_{\mathbb{R}} +1 \exp\left(\frac{-x^2}{2}\right) dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\left(\frac{x}{\sqrt{2}}\right)^2\right) dx \qquad y = \frac{x}{\sqrt{2}} \quad x = y\sqrt{2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-y^2)\sqrt{2}\, dy = \frac{1}{\sqrt{2\pi}} \sqrt{2}\sqrt{\pi} = 1$$

$\Rightarrow$ se $\mathbb{P}_X = N(0,1) \Rightarrow \mathbb{E}[X] = 0, \quad \mathrm{Var}[X] = 1$

Se $\mathbb{P}_Y = N(\mu, \sigma^2)$?

Sia X_0 t.c. $\mathbb{P}_{X_0} = N(0,1)$ Considero $X := \mu + \sigma X_0$

Sappiamo che X ha distribuzione A.C. con densità

$$g(x) = \frac{1}{|\sigma|} f_0\left(\frac{x-\mu}{\sigma}\right) \qquad \text{dove } f_0(x) \text{ è la densità di } X_0$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \quad \forall x \in \mathbb{R}$$

$$\Rightarrow g(x) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \mathbb{P}_{\mu+\sigma X_0} = N(\mu, \sigma^2) = \mathbb{P}_Y$$

$$\Rightarrow \mathbb{E}[Y] = \mathbb{E}[\mu + \sigma X_0] = \mu + \sigma \cancel{\mathbb{E}[X_0]}^{=0} = \mu$$

$$\text{e} \quad \mathrm{Var}[Y] = \mathrm{Var}[\mu + \sigma X_0] = \sigma^2 \cancel{\mathrm{Var}[X_0]}^{=1} = \sigma^2$$

$$F_Y(t) = F_{\mu+\sigma X_0}(t) = \mathbb{P}(\mu + \sigma X_0 \le t) = \mathbb{P}\left(X_0 \le \frac{t-\mu}{\sigma}\right) = F_{X_0}\left(\frac{t-\mu}{\sigma}\right)$$

$$F_{X_0}(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad \to \quad \Phi(t)$$ si usa per indicare la legge associata a $N(0,1)$

$$F_Y(t) = \Phi\left(\frac{t-\mu}{\sigma}\right) \qquad \text{se } P_Y = N(\mu, \sigma^2)$$

PROPRIETÀ $\quad \forall t \in \mathbb{R} \quad \Phi(t) + \Phi(-t) = 1$

DIM

$$\Phi(-t) = \int_{-\infty}^{-t} f_0(x)dx \qquad y = -x \qquad dx = -1\,dy$$

$$= \int_{+\infty}^{t} -f_0(-y)\,dy = \int_{t}^{+\infty} \underbrace{f_0(-y)}_{f_0 \text{ è pari}}\,dy = \qquad f_0(x) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)$$

$$= \int_{t}^{+\infty} f_0(y)\,dy = \int_{\mathbb{R}} f_0(y)\,dy - \int_{-\infty}^{t} f_0(y)\,dy$$

$$= 1 - \Phi(t) \qquad \Rightarrow \qquad \Phi(-t) = 1 - \Phi(t) \qquad \boxtimes$$

ESERCIZIO $\quad \underline{P_X = U([0,a]),\ a > 0} \qquad Y := \sqrt{X}$

distribuzione, valore atteso e varianza di $Y = ?$

$$Y = \sqrt{X} = \varphi \circ X \qquad \text{P-q.c. } \underline{X \in [0,a]}$$

$$\varphi(s) = \begin{cases} \sqrt{s} & s \geq 0 \\ 0 & s < 0 \end{cases}$$

$$\int_{\mathbb{R}} \psi(t)\, P_Y(dt) = \int_{\mathbb{R}} \psi(\varphi(s))\, P_X(ds)$$

$$P_X = f(x)\,dx \quad \text{con} \quad f(x) = \begin{cases} \frac{1}{a} & x \in [0,a] \\ 0 & x \notin [0,a] \end{cases}$$

$$\to \int_{\mathbb{R}} \psi(t) P_Y(dt) = \int_{\mathbb{R}} \psi(\varphi(s)) f(s)\,ds = \int_0^a \psi(\underbrace{\varphi(s)}_{=\sqrt{s}}) \frac{1}{a}\,ds = \int_0^a \psi(\sqrt{s}) \frac{1}{a}\,ds$$

$$\sqrt{s} = t \qquad s = t^2 \qquad ds = 2t\,dt$$

$$= \int_0^{\sqrt{a}} \psi(t) \frac{1}{a} 2t\,dt = \int_0^{\sqrt{a}} \psi(t) \frac{2t}{a}\,dt = \int_{\mathbb{R}} \psi(t) g(t)\,dt \leftarrow$$

$$g(t) = \begin{cases} \frac{2t}{2} & t \in [0, \sqrt{2}] \\ 0 & t \notin [0, \sqrt{2}] \end{cases} \Rightarrow \mathbb{P}_{\sqrt{X}} = g(t)\,dt$$

$$E[\sqrt{X}] = \int_{\mathbb{R}} t\,g(t)\,dt = \int_0^{\sqrt{2}} t \cdot \frac{2t}{2}\,dt = \frac{2}{2}\frac{t^3}{3}\Big|_{t=0}^{t=\sqrt{2}} = \frac{2}{3}\sqrt{2}$$

$$\frac{2}{3\,2}\left(2\sqrt{2} - 0\right)$$

$$E[(\sqrt{X})^2] = E[X] = \frac{0+2}{2} = \frac{2}{2}$$

$$Var[\sqrt{X}] = E[(\sqrt{X})^2] - (E[\sqrt{X}])^2 = \frac{2}{2} - \frac{4}{9}2 = 2\left(\frac{1}{2} - \frac{4}{9}\right) = 2\,\frac{9-8}{18} = \frac{2}{18}$$

ESERCIZIO

X e Y v.a. distribuite sull'insieme degli interi non negativi

$$P(Y=i \mid X+Y=k) = \begin{cases} \binom{k}{i} p^i (1-p)^{k-i} & \forall i = 0, \dots k \\ 0 & i > k \end{cases} \quad \forall k$$

$$\mathbb{P}_{X+Y} = \mathcal{P}(\lambda)$$

$$\mathbb{P}_X = ? \quad \mathbb{P}_Y = ?$$

$\forall k = 0, 1, 2 \dots$ $\quad P(Y=k) = ?$ $\quad P(X=k) = ?$

$P(Y=i) = ?$

$X+Y$ è distribuita sugli interi non negativi

$\{X+Y=k\}_{k=0,1\dots}$ è una partizione misurabile dell'evento certo

$$P(Y=i) = \sum_{k \geq 0} P(Y=i \mid X+Y=k)\,P(X+Y=k)$$

$$= \sum_{k \geq i} P(Y=i \mid X+Y=k)\,\underline{P(X+Y=k)}$$

$$= \sum_{k \geq i} \binom{k}{i} p^i (1-p)^{k-i} \cdot e^{-\lambda}\frac{\lambda^k}{k!} =$$

$$= \sum_{k \geq i} \frac{\cancel{k!}}{i!(k-i)!} p^i (1-p)^{k-i} e^{-\lambda} \frac{\lambda^k}{\cancel{k!}} =$$

$$= e^{-\lambda}\frac{p^i}{i!} \sum_{k \geq i} \frac{(1-p)^{k-i}}{(k-i)!} \lambda^{k-i} \cdot \lambda^i =$$

$$= e^{-\lambda}\frac{(\lambda p)^i}{} \sum \frac{(\lambda(1-p))^{k-i}}{} \qquad j = k-i$$

$$= e^{-\lambda} \frac{(\lambda p)^i}{i!} \sum_{k \geq i} \frac{(\lambda(1-p))^{k-i}}{(k-i)!}$$

$j = k - i$

$k \geq i \iff j \geq 0$

$$= e^{-\lambda} \frac{(\lambda p)^i}{i!} \underbrace{\boxed{\sum_{j \geq 0} \frac{(\lambda(1-p))^j}{j!}}}_{e^{\lambda(1-p)}} = \cancel{e^{-\lambda}} \cdot e^{\lambda(\cancel{1}-p)} \frac{(\lambda p)^i}{i!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

$$\Rightarrow P(Y = i) = e^{-\lambda p} \frac{(\lambda p)^i}{i!} \quad \forall i = 0, 1, 2, \ldots$$ cioè $P_Y = \mathcal{P}(\lambda p)$

$j = 0, 1,$ $$P(X = j) = \sum_{k \geq 0} P(X = j \mid X + Y = k) P(X + Y = k) =$$

$P(X = j, X + Y = k)$ $\qquad \{X = j, X + Y = k\} = \{Y = k - j, X + Y = k\}$

$$= \sum_{k \geq 0} \underline{P(Y = k - j \mid X + Y = k)} P(X + Y = k) =$$

$0 \leq k - j \leq k$ sse $k \geq j$

$$= \sum_{k \geq j} \binom{k}{k-j} p^{k-j} (1-p)^{k-(k-j)} e^{-\lambda} \frac{\lambda^k}{k!} =$$

$$= \sum_{k \geq j} \frac{\cancel{k!}}{(k-j)!\, j!} p^{k-j} (1-p)^j e^{-\lambda} \frac{\lambda^k}{\cancel{k!}} =$$

$\lambda^k = \lambda^{k-j} \cdot \lambda^j$

$$= \frac{e^{-\lambda} (1-p)^j \lambda^j}{j!} \sum_{k \geq j} \frac{1}{(k-j)!} p^{k-j} \lambda^{k-j}$$

$i = k - j$

$k \geq j \iff i \geq 0$

$$= \frac{e^{-\lambda} (\lambda(1-p))^j}{j!} \cdot \underbrace{\sum_{i \geq 0} \frac{(\lambda p)^i}{i!}}_{e^{\lambda p}} = \frac{e^{-\lambda} (\lambda(1-p))^j}{j!} e^{\lambda p}$$

$$= e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}$$

cioè $$P(X = j) = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!} \quad \forall j = 0, 1, 2, \ldots$$

$$P_X = \mathcal{P}(\lambda(1-p))$$