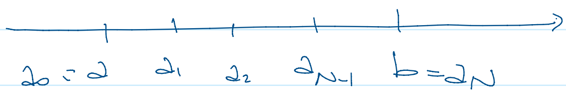
TEOREMA Sia $f:[a,b]\to\mathbb{R}$ limitata, non negativa e integrabile secondo Riemann. Allora f è $\mathcal{L}^1$-misurabile e

$$\underbrace{\int_{[a,b]} f(x)\,dx}_{\text{integrale di Lebesgue}} = \underbrace{\int_a^b f(x)\,dx}_{\text{integrale di Riemann}}$$

DIM ① $\varphi:[a,b]\to\mathbb{R}$ non negativa e costante a tratti



c_i = valore assunto da φ nell'intervallo $(a_{i-1}, a_i]$ $i=1 \dots N$

$$\varphi(x) = \sum_{i=1}^N c_i \mathbb{1}_{(a_{i-1},a_i]}(x) \qquad I_R(\varphi) := \sum_{i=1}^N c_i (a_i - a_{i-1})$$

φ funzione costante a tratti è $\mathcal{L}^1$-misurabile $\Rightarrow$ è una funzione semplice

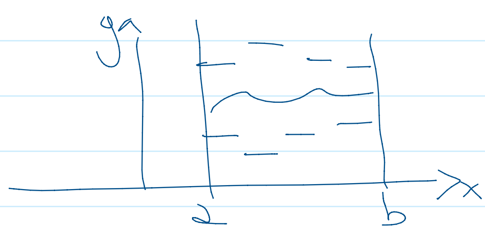
$$\Rightarrow I_{\mathcal{L}}(\varphi) = \sum_{i=1}^N c_i\, \mathcal{L}^1\Big(\underbrace{\{x\in[a,b] : \varphi(x) = c_i\}}_{=(a_i - a_i]}\Big) = \sum_{i=1}^N c_i (a_i - a_{i-1})$$

$\Rightarrow$ Per ogni funzione costante a tratti $I_{\mathcal{L}}(\varphi) = I_R(\varphi) = \mathcal{L}^2(SG_{\varphi,[a,b]})$

② Sia $f:[a,b]\to\mathbb{R}$ non negativa e integrabile secondo Riemann
$\forall \varepsilon > 0$ $\exists\, \varphi, \psi : [a,b]\to\mathbb{R}$ funzioni costanti a tratti t.c.

$$\begin{cases} 0 \le \varphi(x) \le f(x) \le \psi(x) & \forall x\in[a,b] \\ I_R(\psi) - I_R(\varphi) < \varepsilon \end{cases}$$

$$0 \le \varphi(x) \le f(x) \le \psi(x)$$



$$SG_\varphi \subseteq SG_f \subseteq SG_\psi$$

$$0 \le \mathcal{L}^2(SG_\psi \setminus SG_\varphi) = \mathcal{L}^2(SG_\psi) - \mathcal{L}^2(SG_\varphi) =$$

$$= I_R(\psi) - I_R(\varphi) < \varepsilon$$

$\forall \varepsilon > 0 \quad \exists \varphi, \psi$ funzioni semplici $\quad 0 \le \varphi(x) \le f(x) \le \psi(x)$

e $\quad \mathcal{L}^2(SG_\psi \setminus SG_\varphi) < \varepsilon$

SG_ψ è $\mathcal{L}^2$-misurabile $\Rightarrow \exists A \subseteq \mathbb{R}^2$ aperto $\quad A \supseteq SG_\psi \quad \mathcal{L}^2(A \setminus SG_\psi) < \varepsilon$

$$SG_\varphi \subseteq SG_f \subseteq SG_\psi \subseteq A$$

$$A \setminus SG_\varphi \supseteq A \setminus SG_f \supseteq A \setminus SG_\psi$$

$\mathcal{L}^{2*}(A \setminus SG_f) \le \mathcal{L}^{2*}(A \setminus SG_\varphi) = \qquad A \setminus SG_\varphi = (A \setminus SG_\psi) \cup (SG_\psi \setminus SG_\varphi)$

$$= \mathcal{L}^2(A \setminus SG_\varphi) = \underbrace{\mathcal{L}^2(A \setminus SG_\psi)}_{<\varepsilon} + \underbrace{\mathcal{L}^2(SG_\psi \setminus SG_\varphi)}_{<\varepsilon}$$

$\forall \varepsilon > 0 \quad \exists A$ aperto $\quad A \supseteq SG_f \quad$ t.c. $\quad \mathcal{L}^{2*}(A \setminus SG_f) < 2\varepsilon$

cioè $\quad SG_f$ è misurabile

$\Downarrow$

f è $\mathcal{L}^1$ misurabile in $[a,b]$ e $\displaystyle\int_{[a,b]} f(x)\,dx = \mathcal{L}^2(SG_f)$

$$I_R(\varphi) = \mathcal{L}^2(SG_\varphi) \le \int_{[a,b]} f(x)\,dx \le \mathcal{L}^2(SG_\psi) = I_R(\psi)$$

$$\int_a^b f(x)\,dx = \sup_\varphi I_R(\varphi) \le \int_{[a,b]} f(x)\,dx \le \inf_\psi I_R(\psi) = \int_a^b f(x)\,dx$$

$$\Rightarrow \int_{[a,b]} f(x)\,dx = \int_a^b f(x)\,dx$$

OSSERVAZIONE $\qquad f(x) = \mathbb{1}_{\mathbb{Q} \cap [0,1]}(x)$

Non è integrabile secondo Riemann

$f(x) = 0 \quad \mathcal{L}^1$-q.o. $x \in [0,1] \quad \Rightarrow \displaystyle\int_{[0,1]} f(x)\,dx = 0$

TEOREMA DI CONVERGENZA DOMINATA DI LEBESGUE

Sia $E \subseteq \mathbb{R}^n$ misurabile e sia $(f_k)_{k \ldots}$ successione di funzioni.

TEOREMA DI CONVERGENZA DOMINATA DI LEBESGUE

Sia $E \subseteq \mathbb{R}^n$ misurabile e sia $(f_k)_{k\geq 1}$ successione di funzioni, $f_k : E \to \overline{\mathbb{R}}$, misurabili e t.c.

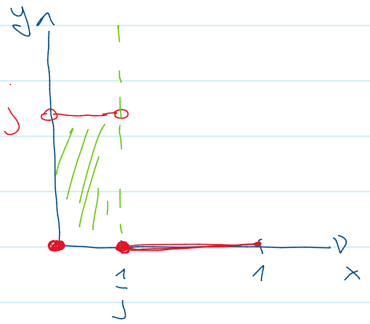
- per q.o. $\underline{x} \in E$ $\quad \exists \lim\limits_{k\to\infty} f_k(\underline{x}) =: f(\underline{x})$
- $\exists \varphi : E \to \overline{\mathbb{R}}$ sommabile t.c. $|f_j(\underline{x})| \leq \varphi(\underline{x})$ q.o. $\underline{x} \in E$

$\int_E \varphi(\underline{x})d\underline{x}$ finito $\forall j \in \mathbb{N}$

Allora
$$\lim_{j\to\infty} \int_E |f_j(\underline{x}) - f(\underline{x})|\, d\underline{x} = 0$$
e, in particolare $\quad \lim\limits_{j\to\infty} \int_E f_j(\underline{x})d\underline{x} = \int_E f(\underline{x})d\underline{x}$

CONTROESEMPIO

$E = [0,1]$
$$f_j(x) = \begin{cases} 0 & x=0 \vee x \in [\frac{1}{j}, 1] \\ j & x \in (0, \frac{1}{j}) \end{cases}$$



$$\int_{[0,1]} f_j(\underline{x})d\underline{x} = j \cdot \frac{1}{j} + 0\left(1 - \frac{1}{j}\right) = 1 \quad \forall j$$

$\lim\limits_{j\to+\infty} f_j(x) = ?$ $\qquad f_j(0) = 0 \quad \forall j$

$\lim\limits_{j\to+\infty} f_j(0) = 0$

Se $x > 0 \Rightarrow \exists \overline{j}$ t.c. $\frac{1}{\overline{j}} < x \Rightarrow \forall j > \overline{j} \quad \frac{1}{j} < \frac{1}{\overline{j}} < x$

$\Rightarrow$ Se $x > 0 \quad \exists \overline{j} = \overline{j}(x)$ t.c. $f_j(x) = 0 \quad \forall j > \overline{j}$

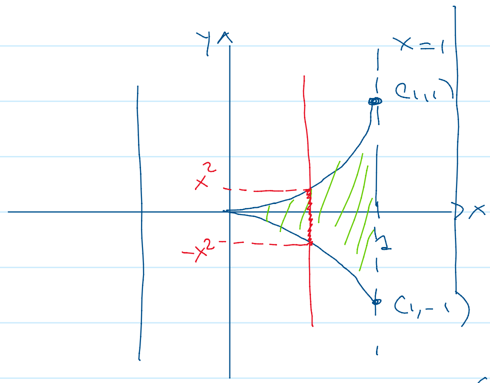
$$\Rightarrow \lim_{j\to+\infty} f_j(x) = \lim_{j\to\infty} 0 = 0$$

$f_j(x)$ converge a 0 $\forall x \in [0,1]$ $\qquad \int_{[0,1]} 0 \, dx = 0 \neq \lim \int_{[0,1]} f_j(x)dx$

ESERCIZI

$E = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ |y| \leq x^2\}$ $\qquad f(x,y) = xe^y$

$$\int_E xe^y \, d(x,y) \qquad \iint_E xe^y \, dx\,dy$$



$$-x^2 \leq y \leq x^2$$

$$E_x = \begin{cases} \phi & x<0 \vee x>1 \\ [-x^2, x^2] & \forall x \in [0,1] \end{cases}$$

$$\iint_E xe^y \, dx\,dy = \int_{\mathbb{R}} \left(\int_{E_x} xe^y \, dy \right) dx = \int_{[0,1]} \left(\int_{[-x^2,x^2]} xe^y \, dy \right) dx$$

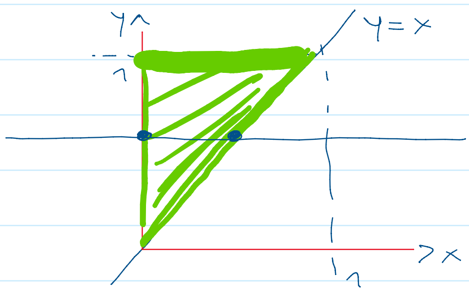
$$= \int_0^1 \left(\int_{-x^2}^{x^2} xe^y \, dy \right) dx = \int_0^1 xe^y \Big|_{y=-x^2}^{y=x^2} dx = \int_0^1 \left(xe^{x^2} - xe^{-x^2} \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} 2x \cdot e^{x^2} + \frac{1}{2} \left(-2xe^{-x^2} \right) \right) dx =$$

$$= \left(\frac{1}{2} e^{x^2} + \frac{1}{2} e^{-x^2} \right) \Big|_{x=0}^{x=1} = \frac{1}{2} \left(e^1 + e^{-1} - e^0 - e^0 \right) =$$

$$= \frac{1}{2} \left(e + e^{-1} - 2 \right) = \frac{1}{2} \left(e^{1/2} - e^{-1/2} \right)^2 = 2 \left(\frac{e^{1/2} - e^{-1/2}}{2} \right)^2 = 2 \left(\sinh\left(\frac{1}{2}\right) \right)^2$$

ESERCIZIO $\quad f(x,y) = x^2 y \qquad E = \{ (x,y) \in \mathbb{R}^2 : \; 0 \leq x \leq y \leq 1 \}$



$$E_y = \begin{cases} \phi & y<0 \vee y>1 \\ [0, y] & y \in [0,1] \end{cases}$$

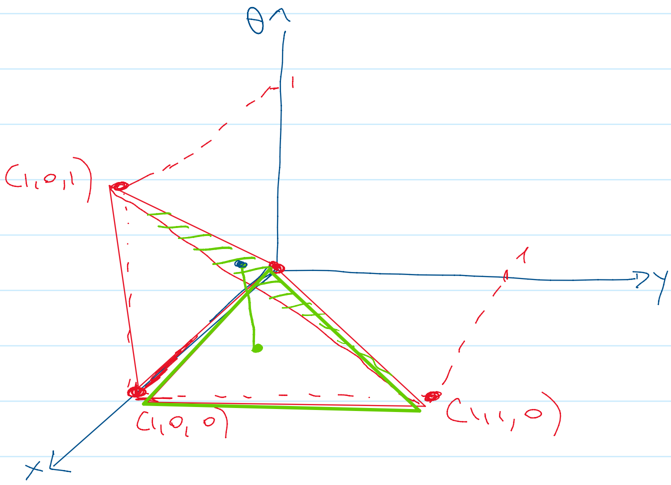
$$\int_E x^2 y \, dx\,dy = \int_{\mathbb{R}} \left(\int_{E_y} x^2 y \, dx \right) dy = \int_{[0,1]} \left(\int_{[0,y]} x^2 y \, dx \right) dy =$$

$$= \int_0^1 \left(\int_0^y x^2 y \, dx \right) dy = \int_0^1 \frac{y}{3} x^3 \Big|_{x=0}^{x=y} dy = \int_0^1 \frac{y}{3} \left(y^3 - 0 \right) dy =$$

$$= \int_0^1 \frac{1}{3} y^4 \, dy = \frac{1}{15} y^5 \Big|_{y=0}^{y=1} = \frac{1}{15} (1-0) = \frac{1}{15}$$

$$= \int_0^1 \frac{1}{3} y^4 \, dy = \frac{1}{15} y^5 \Big|_{y=0}^{y=1} = \frac{1}{15}(1-0) = \frac{1}{15}$$

ESERCIZIO Sia E il Tetraedro di vertici
$(0,0,0)$ $(1,0,0)$ $(1,1,0)$ $(1,0,1)$
Disegnare E e calcolare l'integrale su E di $f(x,y,z) = y + \sin(z)$



$(x,y) \in \mathbb{R}^2$
Sia T il Triangolo nel piano Oxy di vertici $(0,0)$, $(1,0)$ e $(1,1)$
Allora

$$E_{(x,y)} = \begin{cases} \emptyset & (x,y) \notin T \\ [0, x-y] & (x,y) \in T \end{cases}$$

π piano passante per $(0,0,0)$, $(1,0,1)$ e $(1,1,0)$

$$ax + by + cz + d = 0 \qquad (a,b,c) \neq (0,0,0)$$

passaggio per $(0,0,0)$ $\Rightarrow$ $d = 0$ $\qquad a = 1$
passaggio per $(1,0,1)$ $\Rightarrow$ $a + c = 0$ $\qquad b = c = -1$
passaggio per $(1,1,0)$ $\Rightarrow$ $a + b = 0$

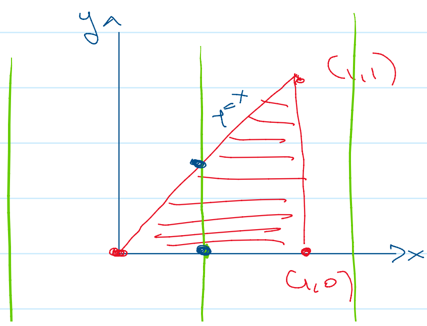
$$x - y - z = 0 \qquad \boxed{z = x - y}$$

$$\int_{\mathbb{R}^2} \left(\int_{E_{(x,y)}} (y + \sin(z)) \, dz \right) dx\,dy = \int_T \left(\int_0^{x-y} (y + \sin(z)) \, dz \right) dx\,dy$$

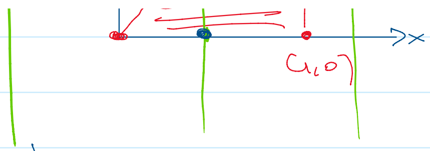
$$= \int_T (yz - \cos(z)) \Big|_{z=0}^{z=x-y} dx\,dy = \int_T \left(y(x-y) - \cos(x-y) + 1 \right) dx\,dy =$$

$(0,0)$, $(1,0)$ $(1,1)$

$$E_x = \begin{cases} \emptyset & x < 0 \lor x > 1 \\ [0, x] & x \in [0,1] \end{cases}$$



$]0,x]$ $\qquad x\in]0,1]$



$$= \int_{\mathbb{R}} \left(\int_{E_x} \left(y(x-y) - \cos(x-y) + 1 \right) dy \right) dx =$$

$$\int_0^1 \left(\int_0^x \left(yx - y^2 - \cos(x-y) + 1 \right) dy \right) dx = \qquad -\cos(x-y) = -\cos(y-x)$$

$$\int_0^1 \left(\frac{x y^2}{2} - \frac{1}{3} y^3 - \underset{x-x=0}{\sin(y-x)} + y \right) \Bigg|_{y=0}^{y=x} dx =$$

$$= \int_0^1 \left(\frac{1}{2} x^3 - \frac{1}{3} x^3 + x + \sin(-x) \right) dx \qquad \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$= \int_0^1 \left(\frac{1}{6} x^3 + x - \sin(x) \right) x = \frac{1}{24} x^4 + \frac{x^2}{2} + \cos(x) \Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{24} + \frac{1}{2} + \cos(1) - 1 = \frac{1+12-24}{24} + \cos(1) = \frac{-11}{24} + \cos(1)$$

FORMULA PER IL CAMBIAMENTO DI VARIABILE

Sia $\varphi : [a,b] \to [c,d]$ derivabile, invertibile (strettamente monotone)

$$\int_c^d f(x)\,dx = \int_{\varphi^{-1}(c)}^{\varphi^{-1}(d)} (f \circ \varphi)(t)\, \varphi'(t)\,dt$$

$$\begin{cases} \varphi^{-1}(c) = a \\ \varphi^{-1}(d) = b \end{cases} \quad \text{se } \varphi \text{ è strettamente crescente}$$

$$\begin{cases} \varphi^{-1}(c) = b \\ \varphi^{-1}(d) = a \end{cases} \quad \text{se } \varphi \text{ è strettamente decrescente}$$

$$\int_E f(x)\,dx \qquad \begin{array}{l} E \subseteq \mathbb{R}^n \text{ è } \mathcal{L}^n\text{-misurabile} \\ f : E \to \overline{\mathbb{R}} \text{ è integrabile} \end{array}$$

$F \subseteq \mathbb{R}^n$ misurabile $\qquad \Phi : F \to E$ invertibile $\Rightarrow \begin{array}{l} \Phi(F) = E \\ \Phi^{-1}(E) = F \end{array}$

$$\Phi : \underline{t} = (t_1 \ldots t_n) \longmapsto \left(\Phi_1(t_1 \ldots t_n) \ldots \Phi_n(t_1 \ldots t_n) \right)$$

t.c. ciascuna componente $\varphi_1 \ldots \varphi_n : E \to \mathbb{R}$ è una funzione C^1

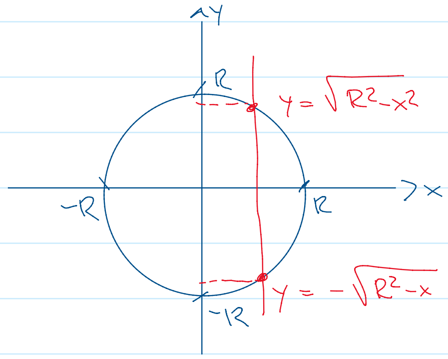
t.c. ciascuna componente $\varphi_1 \dots \varphi_n : E \to \mathbb{R}$ è una funzione C^1

Considero la matrice $n \times n$ $\left(\frac{\partial \varphi_i}{\partial t_j}(\underline{t}) \right)_{i,j=1\dots n}$

Essa si indica $J_{\Phi}(\underline{t})$ e si chiama MATRICE JACOBIANA di Φ

Allora $\displaystyle\int_E f(\underline{x})d\underline{x} = \int_{F=\Phi^{-1}(E)} f(\Phi(\underline{t})) \, |\det J_{\Phi}(\underline{t})| \, d\underline{t}$ (NO DIM)

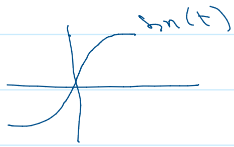
ESEMPIO $C = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \le R^2 \}$



$$C_x = \begin{cases} \emptyset & x<-R \vee x>R \\ [-\sqrt{R^2-x^2}, \sqrt{R^2-x^2}] & x \in [-R,R] \end{cases}$$

$$\mathcal{L}^2(C) = \int_{\mathbb{R}^2} \mathbb{1}_C(x,y)dxdy = \int_C 1 \, dxdy = \int_{\mathbb{R}} \left(\int_{C_x} 1 \, dy \right) dx =$$

$$= \int_{-R}^{R} \left(\int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} 1 \, dy \right) dx = \int_{-R}^{R} 2\sqrt{R^2-x^2} \, dx$$



$x = R\sin(t)$ $dx = R\cos(t)dt$ $\sqrt{R^2-x^2} = \sqrt{R^2\cos^2(t)} =$

$t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$= R|\cos(t)| =$

$R\cos(t)$

$$= \int_{-\pi/2}^{\pi/2} 2 R\cos(t) \cdot R\cos(t) dt =$$

$\cos(2t) = 2\cos^2(t) - 1$

$$= R^2 \int_{-\pi/2}^{\pi/2} 2\cos^2(t) dt$$

$2\cos^2(t) = 1 + \cos(2t)$

$$= R^2 \int_{-\pi/2}^{\pi/2} (1+\cos(2t))dt = R^2 \left(t + \frac{1}{2}\sin(2t) \right) \Big|_{t=-\pi/2}^{t=\pi/2} =$$

$$= R^2 \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \pi R^2$$

$\Phi : (r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$

$\Psi : (r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$

$C = \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \le R^2\}$

$x = r\cos\theta \qquad x^2+y^2 = r^2 \qquad r^2 \le R^2$

$y = r\sin\theta$

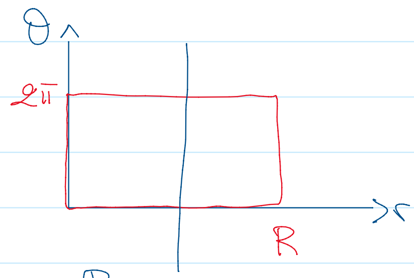
$\boxed{\begin{array}{l} r \in [0,R] \\ \theta \in [0,2\pi] \end{array}}$

$F = \Psi^{-1}(C)$

$$J_\Psi(r,\theta) = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{pmatrix}(r,\theta) =$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \qquad \det J_\Psi(r,\theta) = r\cos^2\theta + r\sin^2\theta = r > 0$$

$$\int_C 1 \, dx\,dy = \int_{\Psi^{-1}(C)} 1 \cdot |\det J_\Psi(r,\theta)| \, dr\,d\theta = \int_{[0,R]\times[0,2\pi]} r \, dr\,d\theta =$$

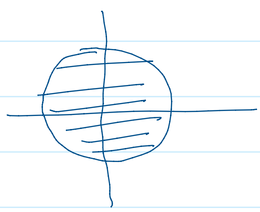


$$E_r = \begin{cases} \phi & r > R \\ [0,2\pi] & r \in [0,R] \end{cases}$$

$$= \int_0^R \left(\int_0^{2\pi} r \, d\theta \right) dr = \int_0^R r\theta \Big|_{\theta=0}^{\theta=2\pi} dr = \int_0^R 2\pi r \, dr =$$

$$= \pi r^2 \Big|_0^R = \pi R^2$$

$$\int_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy$$



$$\exp(-x^2-y^2) = \lim_{n\to\infty} \exp(-x^2-y^2)\,\mathbb{1}_{D_n}(x,y)$$

$$D_n = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le n^2\}$$

$$f_n(x,y) = \exp(-x^2-y^2)\,\mathbb{1}_{D_n}(x,y) \le f_{n+1}(x,y)$$

$$\int_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \lim_{n\to\infty} \int_{\mathbb{R}^2} \exp(-x^2-y^2)\,\mathbb{1}_{D_n}(x,y)\,dx\,dy =$$

$$\int_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \lim_{n\to\infty} \int_{\mathbb{R}^2} \exp(-x^2-y^2)\,\mathbb{1}_{D_n}(x,y)\,dx\,dy =$$

$$= \lim_{n\to\infty} \int_{D_n} \exp(-x^2-y^2)\,dx\,dy = \lim_{n\to\infty} \pi\left(1-e^{-n^2}\right) = \pi$$

$$\int_{D_n} \exp(-x^2-y^2)\,dx\,dy \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \Phi(r,\theta)$$

$$D_n = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le n^2\}$$

$$x^2+y^2 \le n^2$$

$$r^2 \le n^2 \qquad r\in[0,n]$$

$$\theta \in [0,2\pi]$$

$$= \int_{[0,n]\times[0,2\pi]} \exp(-r^2)\, r\,dr\,d\theta =$$

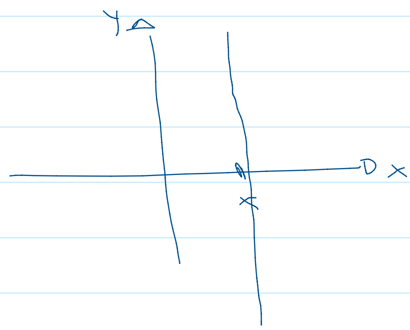
$$= \int_0^{2\pi}\left(\int_0^n \exp(-r^2)\cdot r\,dr\right)d\theta = \int_0^{2\pi}\left(-\frac{1}{2}\int_0^n -2r\exp(-r^2)\,dr\right)d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\left(\exp(-r^2)\right)\Big|_{r=0}^{r=n} d\theta = \int_0^{2\pi} -\frac{1}{2}\left(\exp(-n^2)-1\right)d\theta =$$

$$\frac{1}{2}\left(1-\exp(-n^2)\right)\cdot 2\pi = \pi\left(1-\exp(-n^2)\right)$$

$$\Rightarrow \int_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \pi$$

$$\mathbb{R}^2_x = \mathbb{R} \quad \forall x\in\mathbb{R}$$



$$= \int_{\mathbb{R}^2} \exp(-x^2)\cdot\exp(-y^2)\,dx\,dy =$$

$$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \exp(-x^2)\exp(-y^2)\,dy\right)dx$$

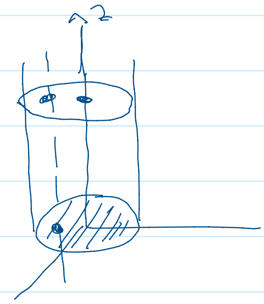
$$= \int_{\mathbb{R}} \exp(-x^2)\underbrace{\left(\int_{\mathbb{R}} \exp(-y^2)\,dy\right)}\,dx =$$

$$= \underbrace{\int_{\mathbb{R}} \exp(-y^2)\,dy} \cdot \underbrace{\int_{\mathbb{R}} \exp(-x^2)\,dx} = \left(\int_{\mathbb{R}} \exp(-t^2)\,dt\right)^2$$

$$\left(\int_{\mathbb{R}} e^{-t^2}\,dt\right)^2 = \pi \qquad \Rightarrow \boxed{\int_{\mathbb{R}} e^{-t^2}\,dt = \sqrt{\pi}}$$

$$\left(\int_{\mathbb{R}} e^{-t^2}\,dt\right)^2 = \pi \quad \Rightarrow \quad \boxed{\int_{\mathbb{R}} e^{-t^2}\,dt = \sqrt{\pi}}$$

$$C = \{(x,y,z) \in \mathbb{R}^3 : \quad x^2+y^2 \le R^2 \quad 0 \le z \le h\}$$



$$\mathcal{L}^3(C) = \int_C 1 \, dx\,dy\,dz$$

$$E_{(x,y)} = \begin{cases} [0,h] & x^2+y^2 \le R^2 \\ \emptyset & \text{altrimenti} \end{cases}$$

$$D := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le R^2\}$$

$$\mathcal{L}^3(C) = \int_D \left(\int_0^h 1\,dz\right) dx\,dy = \int_D h \, dx\,dy = h \underbrace{\int_D 1\,dx\,dy}_{\pi R^2} = \pi h R^2$$

$$\Phi\,(r,\theta,t) \longmapsto (r\cos\theta, r\sin\theta, t)$$

$$\boxed{r \ge 0 \quad \theta \in [0,2\pi] \quad t \in \mathbb{R}}$$

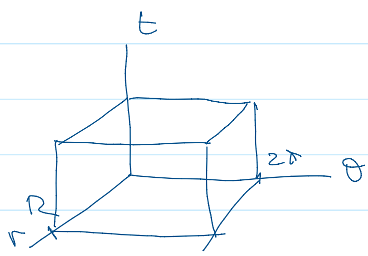
$$x^2+y^2 \le R^2 \qquad r^2 \le R^2$$

$$0 \le z \le h \qquad \boxed{\begin{array}{l} r \in [0,R] \\ t \in [0,h] \\ \theta \in [0,2\pi] \end{array}}$$

$$J_\Phi = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{dt} \\ \frac{dy}{dr} & \frac{dy}{d\theta} & \frac{dy}{dt} \\ \frac{dz}{dr} & \frac{dz}{d\theta} & \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det J_\Phi(r,\theta,t) = 1 \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$



$$\int_0^{2\pi}\left(\int_0^R\left(\int_0^h 1 \cdot r\,dt\right)dr\right)d\theta =$$

$$= \int_0^{2\pi}\left(\int_0^R rt\Big|_{t=0}^{t=h} dr\right)d\theta = \int_0^{2\pi}\left(\int_0^R rh\,dr\right)d\theta =$$

$$= \int_0^{2\pi} \frac{h}{2} r^2 \Big|_{r=0}^{r=R} d\theta = \int_0^{2\pi} \frac{hR^2}{2}\,d\theta = \frac{hR^2}{2}\,2\pi = \pi h R^2$$

$$D = \{(x,y,z) \in \mathbb{R}^3 : \quad x^2+y^2+z^2 \le R^2\}$$

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2\}$$

$$\Psi(r,\theta,\varphi) = (\underbrace{r\sin\theta\cos\varphi}_{x}, \underbrace{r\sin\theta\sin\varphi}_{y}, \underbrace{r\cos\theta}_{z})$$

$r \geq 0 \quad \theta \in [0,\pi], \quad \varphi \in [0, 2\pi]$

$$J_\Psi(r,\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}$$

$$= \cos\theta\left(r^2\cos\theta\sin\theta\cos^2\varphi + r^2\cos\theta\sin\theta\sin^2\varphi\right) + r\sin\theta\left(r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi\right)$$

$$= \cos\theta \cdot r^2\cos\theta\sin\theta + r\sin\theta \cdot r\sin^2\theta = r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

$$|\det J_\Psi(r,\theta,\varphi)| = |r^2\sin\theta| = r^2\sin\theta \qquad \text{perché } \theta \in [0,\pi]$$

Retroimmagine della palla $x^2+y^2+z^2 \leq R^2$

$$\underbrace{(r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2}_{r^2\sin^2\theta} + (r\cos\theta)^2 \leq R^2$$

$$r^2\sin^2\theta + r^2\cos^2\theta \leq R^2$$

$r^2 \leq R^2, \ r \geq 0 \Rightarrow r \in [0,R]$
$\varphi \in [0, 2\pi], \quad \theta \in [0,\pi]$

$$\int_D 1\, dx\,dy\,dz = \int_P 1\ r^2\sin\theta\ dr\,d\theta\,d\varphi \qquad P = \{(r,\theta,\varphi) : r \in [0,R],\ \theta \in [0,\pi],\ \varphi \in [0,2\pi]\}$$

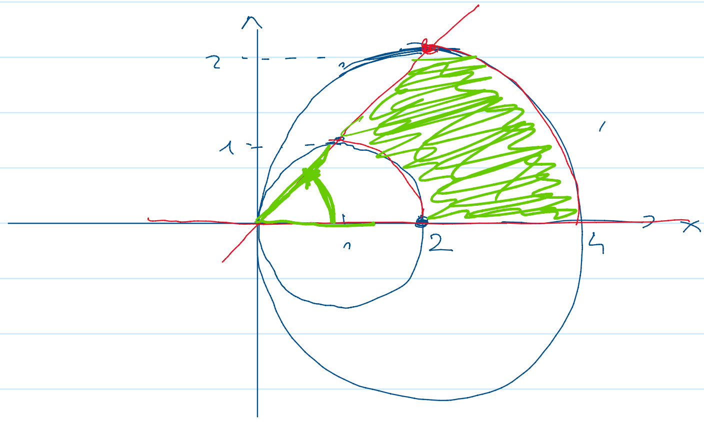
$$= \int_0^R \left(\int_0^\pi \left(\int_0^{2\pi} r^2\sin\theta\, d\varphi\right) d\theta\right) dr =$$

$$= \int_0^R \left(\int_0^\pi r^2\sin\theta\, 2\pi\right) d\theta \Big) dr = \int_0^R \left(-2\pi r^2\cos\theta \Big|_{\theta=0}^{\theta=\pi}\right) dr =$$

$$= \int_0^R -2\pi r^2(-1-1)\, dr = \int_0^R 4\pi r^2\, dr = \frac{4}{3}\pi r^3 \Big|_{r=0}^{r=R} = \frac{4}{3}\pi R^3$$

Sia D la parte di piano delimitata da $x^2-2x+y^2=0$, da $x^2-4x+y^2=0$, dalla bisettrice del 1° e 3° quadrante e dall'asse delle ascisse.

Disegnare D e calcolarne l'Area



$x^2-2x+y^2=0$

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$

circonferenza centrata in $(1,0)$ e raggio 1

$x^2-4x+y^2=0$

$x^2-4x+4+y^2=4$

$(x-2)^2+y^2=4$

circonferenza centrata in $(2,0)$ e raggio 2

$\Psi(r,\varphi)=(r\cos\varphi,\ r\sin\varphi)$

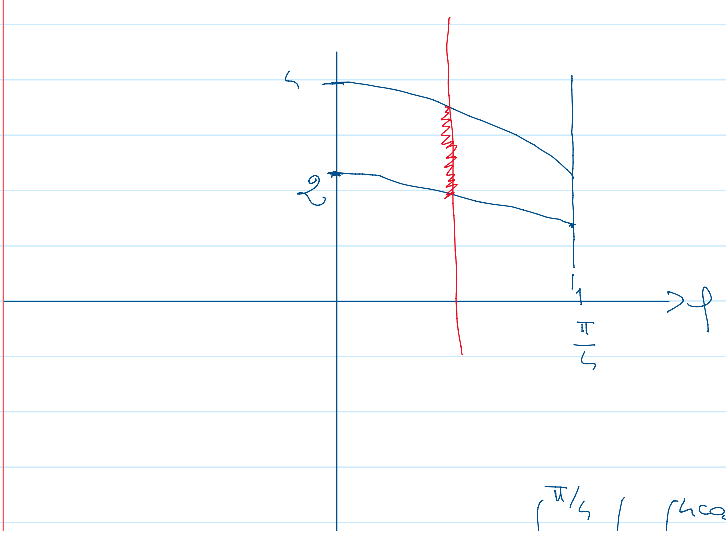
$r\in[0,+\infty)$

$\varphi\in[0,2\pi]$

$$\begin{cases} 0\le y\le x \\ (x-1)^2+y^2\ge 1 \\ (x-2)^2+y^2\le 4 \end{cases} \qquad \begin{cases} 0\le r\sin\varphi\le r\cos\varphi \\ r^2-2r\cos\varphi+\cancel{1}\ge \cancel{1}\,0 \\ r^2-4r\cos\varphi+\cancel{4}\le\cancel{4}\,0 \end{cases}$$

$$\begin{cases} 0\le \sin\varphi\le\cos\varphi \\ r-2\cos\varphi\ge 0 \\ r-4\cos\varphi\le 0 \end{cases} \longrightarrow \varphi\in[0,\pi/4] \implies 2\cos\varphi\le r\le 4\cos\varphi$$

$E=\Psi^{-1}(D)=\{(r,\varphi)\ :\ \varphi\in[0,\frac{\pi}{4}],\ 2\cos\varphi\le r\le 4\cos\varphi\}$



$$E_\varphi \begin{cases} \emptyset & \varphi<0 \vee \varphi>\frac{\pi}{4} \\ [2\cos\varphi, 4\cos\varphi] & \varphi\in[0,\pi/4] \end{cases}$$

$$\text{Area}=\int_D 1\,dx\,dy=$$

$$=\int_E 1\cdot r\,dr\,d\varphi=$$

$$\int^{\pi/4} \int^{4\cos\varphi}$$

$$= \int_E$$

$$= \int_0^{\pi/4} \left(\int_{2\cos\varphi}^{4\cos\varphi} r \, dr \right) d\varphi =$$

$$= \int_0^{\pi/4} \left(\frac{r^2}{2} \Bigg|_{r=2\cos\varphi}^{r=4\cos\varphi} \right) d\varphi = \int_0^{\pi/4} \frac{1}{2} \left(16\cos^2\varphi - 4\cos^2\varphi \right) d\varphi =$$

$$= 6 \int_0^{\pi/4} \cos^2\varphi \, d\varphi$$

$$\cos(2\varphi) = 2\cos^2(\varphi) - 1$$
$$2\cos^2(\varphi) = 1 + \cos(2\varphi)$$

$$= 3 \int_0^{\pi/4} \left(1 + \cos(2\varphi) \right) d\varphi$$

$$= 3 \left(\varphi + \frac{1}{2} \sin(2\varphi) \right) \Bigg|_{\varphi=0}^{\varphi=\pi/4} = 3 \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{3}{4}\pi + \frac{3}{2}$$